A Note on the Approximation of Weak Probabilistic Bisimulation

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Outline

• Why approximating equivalence checking
• Pseudometrics
• Approximating weak probabilistic bisimulation
• Computing the approximating relation
• Future work
Why Approximating Equivalence Checking

Applications of equivalence checking:

- relating a process model to a reference model
- verifying substitutions/transformations/reductions that are expected to preserve dependability properties of systems
- noninterference analysis
Why (Weak) Bisimulation

- bisimulation semantics is sensitive to deadlock and to properties that depend on the branching structure of the models
- weak bisimulation abstracts from unnecessary details and internal behaviors
- minimization modulo bisimulation (e.g. it coincides with lumping for Markov chains)
Bisimulation Semantics

Nondeterministic Case

- exact analysis: 0/1 result

Probabilistic Case

- exact analysis: 0/1 result
  - strong (Larsen and Skou, ...)
  - weak (Baier and Hermanns, Philippou et al., ...)

- approximate analysis: [0; 1] result
  - pseudometrics: (Desharnais et al., van Breugel et al., ...)
  - approx. relations: (DiPierro et al.)

strong and weak bisimulation (Milner)
Pseudometrics [Desharnais et al.]

- **Model** labelled Markov processes (and not only . . .)
- **Semantics** logical characterization of bisimulation:
  \[ \mathcal{L} := \top \mid \phi_1 \land \phi_2 \mid \langle a \rangle_q \phi \]
- From the logic-based characterization to the functional expressions based characterization:
  \[ f := 1 \mid 1 - f \mid \langle a \rangle f \mid \min(f_1, f_2) \mid \sup_{i \in \mathbb{N}} f_i \mid f \ominus q \]
  
  * \( 1(s) = 1 \)
  * \( (1 - f)(s) = 1 - f(s) \)
  * \( \langle a \rangle f(s) = c \int_s f(t) \tau_a(s, dt) \)
  * \( (f \ominus q)(s) = \max(f(s) - q, 0) \)
Pseudometrics [Desharnais et al.]

- **Model** labelled Markov processes (and not only . . . )
- **Semantics** logical characterization of bisimulation:
  \[ L := \top | \phi_1 \land \phi_2 | \langle a \rangle_q \phi \]
- From the logic-based characterization to the functional expressions based characterization:
  \[ f := 1 | 1 - f | \langle a \rangle f | \min(f_1, f_2) | \sup_{i \in \mathbb{N}} f_i | f \ominus q \]
- \( s \) and \( s' \) are bisimilar iff they satisfy the same logical formulas iff they have the same values for each functional expression
- pseudometric: \[ d^c(P, Q) = \sup_{f \in F^c} \left| f_P(p_0) - f_Q(q_0) \right| \]
Pseudometrics [Desharnais et al.]

\[ \langle a \rangle \cdot \langle a \rangle 1 \text{ evaluates to } 3c^2/4 \text{ at state } s_0 \]
and to 0 elsewhere.

\[ \langle a \rangle \cdot \langle a \rangle 1 \text{ evaluates to } 3c^2/8 \text{ at state } s'_0 \]
Pseudometrics [Desharnais et al.]

- If $c = 1$ then $s_3 \ (s'_3)$ is as important as $s_0 \ (s'_0)$
- No functional expression reveals that the probability of reaching $s_3 \ (s'_3)$ is 1
- The pseudometric takes into account the distance between $s_0$ and $s'_0$
evaluating the distance between states in the same class – obviously \((s_0, s_0') \in R\)

• employing a notion of distance that takes into account the probability of being at the states under comparison
Why Approximating Relation

- (weak) bisimulation $\rightarrow (\tau -)$ lumping

  A lumpable partition of the states of the MC allows the generation of an aggregated MC that is smaller than the original one, but leads to several results for the original DTMC without any error.

- $\?? \rightarrow$ nearly- (or quasi-) lumpability

  A threshold corresponding to the maximum difference among elements of the same partition group is used to determine bounds on the error made when calculating results for the original MC from the aggregated, approximate one.
Probabilistic Transition Systems and $\approx_{\text{PB}}$

A PTS is a tuple $(S, \text{Act}, T, s_0)$ such that:

- $S$ is a finite set of states
- $s_0 \in S$ is the initial state
- $\text{Act}$ is a non-empty finite set of actions
- $T \subseteq S \times \text{Act} \times [0, 1] \times S$ is a finite transition relation such that $\forall s \in S$ it holds that:

$$\sum \{|p| \exists a \in \text{Act}, t \in S. (s, a, p, t) \in T \} \in \{0, 1\}$$
Probabilistic Transition Systems and $\approx_{\text{PB}}$

**Definition.** An equivalence relation $R \subseteq S \times S$ is a weak probabilistic bisimulation if and only if, whenever $(s, s') \in R$, then for all $C$ in the quotient set $S/R$ and $\forall a \in \text{Act}$:

$$\text{Prob}(s, \tau^*a, C) = \text{Prob}(s', \tau^*a, C)$$

The union of all the weak probabilistic bisimulations is the largest weak probabilistic bisimulation, called weak probabilistic bisimilarity and denoted by $\approx_{\text{PB}}$. 
**Approximating \( \approx_{PB} \) (1)**

**Definition.** A relation \( R \subseteq S \times S \) is a weak probabilistic bisimulation with \( \varepsilon \)-precision, where \( \varepsilon \in (0, 1) \), if and only if, whenever \( (s, s') \in R \), then for all \( C \) in the quotient set \( S/R \) and \( \forall a \in Act \):

\[
| \text{Prob}(s, \tau^*a, C) - \text{Prob}(s', \tau^*a, C) | \leq \varepsilon
\]

Two states \( s, s' \in S \) are weakly probabilistically bisimilar with \( \varepsilon \)-precision, denoted \( s \approx_{PB\varepsilon} s' \), if there exists a weak probabilistic bisimulation with \( \varepsilon \)-precision including the pair \( (s, s') \).
Approximating $\approx_{PB}$ (2)

- the evaluation of $\varepsilon$ is local
- it does not take into account the probability of being in $s$ and in $s'$

The weighted distance between two states $s$ and $s'$ with respect to $a \in \text{Act}$ and a set of states $C$ is defined as:

$$d(s, s', a, C) = \text{Reach}(s) \cdot \text{Reach}(s') \cdot | \text{Prob}(s, \tau^*a, C) - \text{Prob}(s', \tau^*a, C) |$$

where:
$$\text{Reach}(s) = 1 \text{ if } s \text{ is the initial state } s_0, \text{ else } \text{Reach}(s) \text{ is the aggregate probability of reaching } s \text{ from } s_0 \text{ via states different from } s \text{ itself}$$
Approximating $\approx_{PB}$ (2)

**Definition.** A relation $R \subseteq S \times S$ is a weak probabilistic bisimulation with $\varepsilon$ precision if and only if, whenever $(s, s') \in R$, then for all $C$ in the partition induced by $R$ and $\forall a \in \text{Act}$:

$$d(s, s', a, C) \leq \varepsilon$$

- this definition is conservative with respect to $\approx_{PB}$
- given $R \subseteq S \times S$ and $P_R$ the partition induced by $R$, the closeness of $R$ with respect to $\approx_{PB}$ is defined as:

$$\delta_R = \max\{d(s, s', a, C) \mid (s, s') \in R, a \in \text{Act}, C \in P_R\}$$

- among all the possible relations, the most interesting one is given by the closest approximation of $\approx_{PB}$:

$$\min_{R \in \mathcal{R}} \delta_R$$
Algorithm for Approximating $\approx_{PB}$

- **Idea:** given a threshold, the algorithm calculates, if possible, a candidate relation $R$ whose estimation $\delta_R$ of the closeness to $\approx_{PB}$ is less than the threshold.

- **Complexity:** time complexity $O(n^5)$ with respect to the number $n$ of states.

- **Approach:** differently from the Paige and Tarjan partition refinement algorithm, which relies on the transitivity of the relation, the approximating algorithm starts from the singletons and from the pair of initial states and then performs the aggregations that minimize the distances.
Algorithm for Approximating $\approx_{PB}$

1. $R = \{s_0, s'_0\} \cup \bigcup_{s \in (S - \{s_0\} \cup S' - \{s'_0\})} \{s\}$
2. $\delta^R = 0$ and $(s_0, s'_0)$ is unmarked
3. WHILE $\exists (s, s') \in R. \text{Reach}(s) \cdot \text{Reach}(s') = \max\{|\text{Reach}(t) \cdot \text{Reach}(u)| (t, u) \in R \text{ is unmarked} |\} > \delta^R$ DO
   
   (a) mark $(s, s')$
   
   (b) $d^R(s, s') = \max_{a \in \text{Act}, C \in R}\{d(s, s', a, C)\}$
       Let $d^R(s, s') = d(s, s', \neg C)$
   
   (c) IF $\exists C \in R$, such that
       1. $\forall a \in \text{Act}. d(s, s', a, \neg C \cup C) < d^R(s, s') \land$
       2. $\forall$ marked $(t, u) \in R, \forall a \in \text{Act}. d(t, u, a, \neg C \cup C) < \max(\delta^R, d^R(s, s'))$
       THEN
           • merge $\neg C$ and $C$
           • goto (b)
   
   (d) IF $d^R(s, s') > \delta^R$ THEN $\delta^R = d^R(s, s')$
Algorithm for Approximating $\approx_{PB}$

- If no threshold is used, the algorithms converges towards a local optimum.

- In order to pass from a local optimum to another one, a backtracking mechanism is implemented to disaggregate partition groups.

- The backtracking can be guided by meta-heuristic techniques in order to speed up the convergence towards the optimum.
Future Work

- comparing the approximate version of $\approx_{PB}$ with other approximating relations
- testing the efficiency of the algorithm in the best/average/worst case
- relating the closeness $\delta_R$ of the approximating relation $R$ with the threshold parameterizing the nearly-lumpability