Revisiting True Concurrency Bisimilarities: On the Role of Backward Ready Multisets and Why They Are Not Enough for HPB and HHPB

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Abstract

Bisimilarities over stable configuration structures can be divided into three families. In the first one – including interleaving, step, pomset, and forward-reverse bisimilarities – no isomorphism is required between the events matched during the bisimulation game. In the second one – including weak history-preserving, weak history-preserving pomset, and weak hereditary history-preserving bisimilarities – a labeling- and causality-preserving isomorphism is required between matched events, which is specific to each pair of configurations related by the bisimulation relation and hence can vary for a matched event from pair to pair. In the third one – including history-preserving and hereditary history-preserving bisimilarities – a single isomorphism is built incrementally, which is therefore fixed for all matched events. We revisit true concurrency bisimilarities by introducing variants that additionally check that the backward ready multisets of related configurations coincide. While the distinguishing power of the bisimilarities of the second and third families does not change, the power of the revised bisimilarities of the first family is equal to that of the bisimilarities of the second family. The latter bisimilarities can thus be characterized by replacing variable isomorphisms with simply counting incoming transitions. In contrast, backward ready multisets are not enough to characterize the third family in the simultaneous presence of autoconcurrency and non-local conflicts. We show that a further check for the existence of diamond and half-diamond substructures is necessary in that case to achieve the same distinguishing power as incremental isomorphisms.

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1 Introduction

Bisimilarity is a central notion in concurrency theory. It expresses the capability of two processes of mimicking each other's behavior stepwise, yielding a behavioral equivalence that is extremely sensitive to branching points. Over labeled transition systems [20], i.e., state-transition graphs whose transitions are labeled with actions, its classical definition [22, 21] complies with the *interleaving view* of concurrency based on the *expansion law* [18]. This means that the parallel composition of two processes reduces to a nondeterministic choice among all possible sequencings of the actions individually executable by the two processes.

Several variants have been subsequently introduced, which are *truly concurrent* in the sense that they do not meet the expansion law. Their spectrum, depicted in Figure 1, has been investigated in [14, 12, 25], where their definitions have been formulated over stable configuration structures [15]. These are variants of event structures [30] akin to labeled transition systems in which causality, concurrency, and conflicts can be retrieved a posteriori.

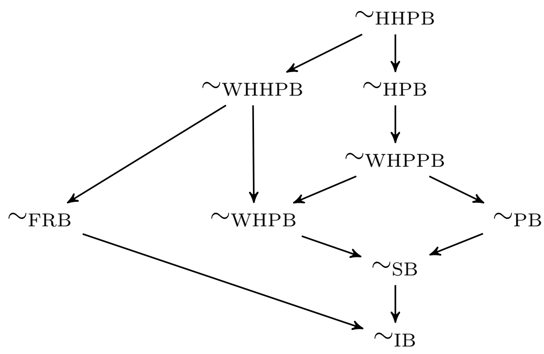

Figure 1 Main bisimilarities over stable configuration structures examined in [14, 12, 25].

We can distinguish among three families of bisimilarities in the true concurrency spectrum (whose member definitions will be recalled in Section 2):

1. The bisimilarities of the first family do not require any isomorphism between the events matched so far during the bisimulation game. The *no-iso family* includes:
   - *Interleaving bisimilarity* $\sim_{\mathrm{IB}}$ [22, 21], which considers transitions labeled with individual actions.
   - *Step bisimilarity* $\sim_{\mathrm{SB}}$ [27], which considers transitions labeled with multisets of actions corresponding to concurrent events.
   - *Pomset bisimilarity* $\sim_{\mathrm{PB}}$ [6], which considers transitions labeled with partially ordered multisets (pomsets for short) of actions corresponding to events that are causally related or concurrent.
   - *Forward-reverse bisimilarity* $\sim_{\mathrm{FRB}}$ [23], which examines not only outgoing transitions (forward direction) but also incoming ones (backward direction) labeled with actions.
2. The bisimilarities of the second family require a labeling- and causality-preserving isomorphism between the events matched so far, which is specific to the pair of configurations at hand and hence can vary for a matched event from pair to pair. The *var-iso family* includes:
   - *Weak history-preserving bisimilarity* $\sim_{\mathrm{WHPB}}$ [11], which considers transitions labeled with individual actions.
   - *Weak history-preserving pomset bisimilarity* $\sim_{\mathrm{WHPPB}}$ [14], which considers transitions labeled with pomsets.
   - *Weak hereditary history-preserving bisimilarity* $\sim_{\mathrm{WHHPB}}$ [25], which examines incoming transitions too.
3. The bisimilarities of the third family incrementally build a single labeling- and causality-preserving isomorphism between the events matched so far, with their underlying bisimulations being ternary relations and their distinguishing power not depending on the kind (i.e., action/step/pomset) of considered transitions [14]. The *incr-iso family* includes:
   - *History-preserving bisimilarity* $\sim_{\mathrm{HPB}}$ [28], which examines only outgoing transitions.
   - *Hereditary history-preserving bisimilarity* $\sim_{\mathrm{HHPB}}$ [3], which examines incoming transitions as well.

The two bisimilarities of the last family are especially important because they are the coarsest and the finest equivalences, respectively, that are preserved under action refinement and are capable of respecting causality, branching, and their interplay while abstracting from choices between identical alternatives [14]. Moreover, $\sim_{\mathrm{HHPB}}$ can be obtained as a special case of a categorical definition of bisimilarity over concurrency models [19].

While $\sim_{\mathrm{HPB}}$ is known to coincide with causal bisimilarity [8, 9], characterizing $\sim_{\mathrm{HHPB}}$ is more complicated. To be precise, in the absence of *autoconcurrency*, i.e., identically labeled concurrent events, $\sim_{\mathrm{HHPB}}$ (which is based on *ternary* bisimulation relations) coincides with $\sim_{\mathrm{FRB}}$ (which is based on *binary* bisimulation relations). This was established in [3] for configuration graphs of prime event structures and later in [24] by requiring the absence of repeated, identically labeled events along forward computations.

In [25] it was then shown that, on stable configuration structures, it suffices to impose the absence of equidepth autoconcurrency, i.e., the absence of identically labeled concurrent events occurring at the same depth within a configuration (where the depth of an event is defined as the length of the longest causal chain of events up to and including the considered event). Another characterization was provided in [1] by relying on an extension of stable configuration structures that somehow encodes the past behavior by uniquely identifying events. No restriction is needed in this case, but the characterizing bisimilarity is based again on ternary bisimulation relations, witnessing once more that the forward and backward bisimulation games alone are not enough to capture $\sim_{\mathrm{HHPB}}$.

A different approach has been recently proposed in [4]. Instead of focusing on the unique identification of identically labeled events, the idea is to count them. A variant $\sim_{\mathrm{FRB:brm}}$ of $\sim_{\mathrm{FRB}}$ has been introduced there, which additionally checks whether pairs of related configurations have the same *backward ready multiset*, intended as the multiset of actions labeling incoming transitions.

Let us consider for example a process that can perform a in parallel with a – which we can represent as $a \parallel_{\emptyset} a$ by using a CSP-like parallel composition [7] – and its interleaving expansion – which we can represent as $a \,.\, a + a \,.\, a$ by using CCS-like action prefix and nondeterministic choice [21]. The two processes are told apart by $\sim_{\mathrm{HHPB}}$ because the two a-labeled events in the former process are concurrent while the two a-labeled events along either branch of the latter process are causally related, hence no causality-preserving isomorphism can be established among these events. The two processes are distinguished by $\sim_{\mathrm{FRB:brm}}$ too. Indeed, the process reached from $a \parallel_{\emptyset} a$ after performing both a-actions – which we can represent as $a^{\dagger} \parallel_{\emptyset} a^{\dagger}$ according to the notation introduced in [5, 4] with $\dagger$ decorating executed actions – has $\{\!|\, a, a \,|\!\}$ as backward ready multiset, whereas the two processes reached from $a \,.\, a + a \,.\, a$ after performing both a-actions along either branch – which we can respectively represent as $a^{\dagger} \,.\, a^{\dagger} + a \,.\, a$ and $a \,.\, a + a^{\dagger} \,.\, a^{\dagger}$ – have $\{\!|\, a \,|\!\}$ as backward ready multiset.

From the example above it is clear that the backward ready multiset of a configuration is a measure of the degree of concurrency of the events in the configuration itself. Moreover, comparing the backward ready multisets of two configurations is much simpler than building a possibly incremental, labeling- and causality-preserving isomorphism between the events matched so far. Therefore, it is interesting to investigate whether backward ready multisets can concur to provide an alternative characterization of isomorphisms for all bisimilarities in the var-iso and incr-iso families. This is the objective of the present paper.

Our first contribution (Section 3) is twofold. Let us call *brm-variant* of a bisimilarity the variant that is obtained by additionally requiring that any two related configurations possess the same backward ready multiset. On the one hand, we show that the brm-variants of var-iso and incr-iso bisimilarities coincide with the corresponding original bisimilarities, meaning that for those two families the backward ready multiset equality check does not increase the distinguishing power. On the other hand, we show that the brm-variants of no-iso bisimilarities are strictly finer than the corresponding original bisimilarities. Interestingly, $\sim_{\mathrm{IB:brm}}$ and $\sim_{\mathrm{SB:brm}}$ coincide with $\sim_{\mathrm{WHPB}}$ (and $\sim_{\mathrm{WHPB:brm}}$), while $\sim_{\mathrm{PB:brm}}$ coincides with $\sim_{\mathrm{WHPPB}}$ (and $\sim_{\mathrm{WHPPB:brm}}$) and $\sim_{\mathrm{FRB:brm}}$ coincides with $\sim_{\mathrm{WHHPB}}$ (and $\sim_{\mathrm{WHHPB:brm}}$).

In other words, the var-iso bisimilarities are characterized by the brm-variants of the no-iso bisimilarities, i.e., variable isomorphisms can be replaced by simply counting incoming transitions (see the forthcoming Figure 2).

Our second contribution (Section 4) is the development of an extension of Hennessy-Milner logic [18] that includes backward ready multisets [4], a pomset-based forward modality [13, 17, 29, 6], and an action-based backward modality [10]. We show that its fragments characterize all bisimilarities of the no-iso and var-iso families, including their brm-variants.

Our third contribution (Section 5) is an investigation of why backward ready multisets are not enough to capture incr-iso bisimilarities in the simultaneous presence of autoconcurrency and *non-local conflicts*, i.e., conflicting events occurring in distinct processes that are composed in parallel. What we observe from a number of significant examples – most of which are taken from [25] – is that, when trying to relate two configurations, in addition to the bisimulation game – which is the baseline – and backward ready multisets – which measure concurrent events by counting *incoming* transitions – we should also consider the existence of diamond-shaped configuration substructures – which indicate the presence of concurrent events by looking at *outgoing* transitions – along with half-diamond-shaped configuration substructures – which reveal the presence of *conflicting* events. We show that the resulting diamond-based variants of $\sim_{\mathrm{IB:brm}}$ and $\sim_{\mathrm{FRB:brm}}$, denoted by $\sim_{\mathrm{IB:brm:diam}}$ and $\sim_{\mathrm{FRB:brm:diam}}$, respectively coincide with $\sim_{\mathrm{HPB}}$ and $\sim_{\mathrm{HHPB}}$ (see the forthcoming Figure 2).

Section 6 concludes the paper by outlining some directions for future work.

2 Background Definitions and Results

In this section we recall the stable configuration structure model (Section 2.1) and the three families of bisimilarities defined over it that have been studied in [14, 12, 25] (Sections 2.2, 2.3, 2.4). The spectrum of the considered bisimilarities has already been shown in Figure 1. Every arrow stands for the strict inclusion of the source bisimilarity in the target one, while the absence of a chain of arrows between two equivalences means that those two equivalences are incomparable.

We will use the following process algebraic notation to express examples: a stands for an action (i.e., the label of an event), $a\,.\,\_$ is the action prefix operator (the a-labeled event is a cause of the subsequent ones), $\_ + \_$ is the alternative composition operator (events in different branches are in conflict with each other), and $\_ \parallel_L \_$ is the parallel composition operator (events in different branches are concurrent, i.e., independent of each other, unless they have to synchronize because their labels are identical and belong to L). When an action is decorated with †, it means that it has already been executed.

2.1 Stable Configuration Structures

Configuration structures [15] are variants of event structures [30] expressing their underlying labeled transition system view. A configuration, which represents a state, is the set of non-conflicting events that occurred so far. Every configuration is finite and left-closed with respect to causality, i.e., it contains all the events causing the ones in the configuration itself. Therefore, given two arbitrary events in a configuration, they are either causally related, i.e., can occur only in a specific order, or concurrent, i.e., can occur in any order.

In the following definitions taken from [14], we assume a countable set $\mathcal{A}$ of action labels and we denote by $\mathcal{P}_{\mathrm{fin}}(\mathcal{E})$ the set of finite subsets of set $\mathcal{E}$.

▶ **Definition 1.** *A* configuration structure *is a triple* $C = (\mathcal{E}, \mathcal{C}, \ell)$ *where:*
- $\mathcal{E}$ *is a set of* events.
- $\mathcal{C} \subseteq \mathcal{P}_{\mathrm{fin}}(\mathcal{E})$ *is a set of* configurations.
- $\ell : \bigcup_{X \in \mathcal{C}} X \to \mathcal{A}$ *is a* labeling function *for events occurring in configurations.* ⌟

As shown in [14], the configurations of a prime event structure [30] – based on causality and conflict relations – form a configuration structure, while those of a stable event structure [30] – based on consistency and enabling notions – form a configuration structure that is stable in the following sense.

▶ **Definition 2.** *A configuration structure* $C = (\mathcal{E}, \mathcal{C}, \ell)$ *is said to be* stable *iff it is:*
- *Rooted:* $\emptyset \in \mathcal{C}$.
- *Connected:* $\forall X \in \mathcal{C} \setminus \{\emptyset\}. \exists e \in X. X \setminus \{e\} \in \mathcal{C}$.
- *Closed under bounded unions/intersections:* $\forall X,Y,Z \in \mathcal{C}. X \cup Y \subseteq Z \Longrightarrow X \cup Y, X \cap Y \in \mathcal{C}$. ⌟

The notions of causality and concurrency for events can be defined locally to configurations, while the notion of conflict has to be defined globally in a stable configuration structure.

▶ **Definition 3.** *Let* $C = (\mathcal{E}, \mathcal{C}, \ell)$ *be a stable configuration structure and* $X \in \mathcal{C}$*:*
- *The* causality relation *over* $X \in \mathcal{C}$ *is defined by letting* $e_1 \leq_X e_2$*, for two events* $e_1, e_2 \in X$*, iff* $e_2 \in Y$ *implies* $e_1 \in Y$ *for all* $Y \in \mathcal{C}$ *such that* $Y \subseteq X$*. We write* $e_1 <_X e_2$ *when* $e_1 \leq_X e_2$ *and* $e_1 \neq e_2$.
- *Two distinct events* $e_1, e_2 \in X$ *are* concurrent *in* X*, written* $e_1 \, co_X \, e_2$*, iff* $e_1 \not<_X e_2$ *and* $e_2 \not<_X e_1$.
- *Two distinct events* $e_1, e_2 \in \mathcal{E}$ *are* conflicting*, written* $e_1 \# e_2$*, iff there exists no* $Z \in \mathcal{C}$ *such that* $e_1, e_2 \in Z$. ⌟

A notion of transition between configurations can be naturally introduced.

▶ **Definition 4.** *Let* $C = (\mathcal{E}, \mathcal{C}, \ell)$ *be a stable configuration structure,* $a \in \mathcal{A}$*, and* $X, X' \in \mathcal{C}$*. We say that there exists an* a*-labeled* transition *from* X *to* X'*, written* $X \xrightarrow{a}_C X'$*, iff* $X \subseteq X'$*,* $X' \setminus X = \{e\}$*, and* $\ell(e) = a$. ⌟

2.2 Bisimilarities with No Event Isomorphisms

The bisimilarities of the first family do not require any isomorphism between the events matched so far during the bisimulation game. We start by recalling the standard bisimilarity of [22, 21], in which a single action at a time is considered. It can be termed interleaving because it complies with the view according to which the parallel composition of two processes is equivalent to a nondeterministic choice among all possible sequencings of the actions executable by the two processes themselves.

▶ **Definition 5.** *We say that two stable configuration structures* $C_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$*,* $i \in \{1, 2\}$*, are* interleaving bisimilar*, written* $C_1 \sim_{\mathrm{IB}} C_2$*, iff there exists an interleaving bisimulation between them, i.e., a relation* $\mathcal{B} \subseteq \mathcal{C}_1 \times \mathcal{C}_2$ *such that* $(\emptyset, \emptyset) \in \mathcal{B}$ *and, whenever* $(X_1, X_2) \in \mathcal{B}$*, then:*
- *For each* $X_1 \xrightarrow{a}_{C_1} X'_1$ *there exists* $X_2 \xrightarrow{a}_{C_2} X'_2$ *such that* $(X'_1, X'_2) \in \mathcal{B}$*, and vice versa.* ⌟

Secondly, we have step bisimilarity [27], which considers transitions labeled with multisets of actions corresponding to concurrent events. Formally $X \xrightarrow{A}_C X'$ for $A : \mathcal{A} \to \mathbb{N}$ iff $X \subseteq X'$, $X' \setminus X = E$ with $e_1 \, co_{X'} \, e_2$ for all $e_1, e_2 \in E$, and $\ell(E) \triangleq \{\!| \ell(e) \mid e \in E |\!\} = A$, with $\{\!|$ and $|\!\}$ being multiset delimiters.

▶ **Definition 6.** *We say that two stable configuration structures $\mathcal{C}_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, are* step bisimilar*, written $\mathcal{C}_1 \sim_{\mathrm{SB}} \mathcal{C}_2$, iff there exists a step bisimulation between them, i.e., a relation $\mathcal{B} \subseteq \mathcal{C}_1 \times \mathcal{C}_2$ such that $(\emptyset, \emptyset) \in \mathcal{B}$ and, whenever $(X_1, X_2) \in \mathcal{B}$, then:*

- *For each $X_1 \xrightarrow{A}_{\mathcal{C}_1} X_1'$ there exists $X_2 \xrightarrow{A}_{\mathcal{C}_2} X_2'$ such that $(X_1', X_2') \in \mathcal{B}$, and vice versa.* ⌟

$\sim_{\mathrm{SB}}$ is strictly finer than $\sim_{\mathrm{IB}}$: for example, $a \parallel_\emptyset b$ and $a \,.\, b + b \,.\, a$ are $\sim_{\mathrm{IB}}$-equivalent but not $\sim_{\mathrm{SB}}$-equivalent, because a and b in either branch of the second process are not concurrent and hence the $\{\!|\, a, b \,|\!\}$-transition of the first process cannot be matched by the second one.

Thirdly, we have pomset bisimilarity [6], which considers transitions labeled with partially ordered multisets of actions corresponding to events that are causally related or concurrent. Formally $X \xrightarrow{U}_{\mathcal{C}} X'$ iff $X \subseteq X'$ and $X' \setminus X = H$ with $U = [(H, <_{X'} \cap (H \times H), \ell \cap (H \times \mathcal{A})]_{\cong}$, where $\cong$ is a labeling- and causality-preserving isomorphism over events and U is the $\cong$-class of equivalence for event set H.

▶ **Definition 7.** *We say that two stable configuration structures $\mathcal{C}_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, are* pomset bisimilar*, written $\mathcal{C}_1 \sim_{\mathrm{PB}} \mathcal{C}_2$, iff there exists a pomset bisimulation between them, i.e., a relation $\mathcal{B} \subseteq \mathcal{C}_1 \times \mathcal{C}_2$ such that $(\emptyset, \emptyset) \in \mathcal{B}$ and, whenever $(X_1, X_2) \in \mathcal{B}$, then:*

- *For each $X_1 \xrightarrow{U}_{\mathcal{C}_1} X_1'$ there exists $X_2 \xrightarrow{U}_{\mathcal{C}_2} X_2'$ such that $(X_1', X_2') \in \mathcal{B}$, and vice versa.* ⌟

$\sim_{\mathrm{PB}}$ is strictly finer than $\sim_{\mathrm{SB}}$: for instance, $a \parallel_\emptyset b$ and $(a \parallel_\emptyset b) + a \,.\, b$ are identified by $\sim_{\mathrm{SB}}$ but told apart by $\sim_{\mathrm{PB}}$, because the transition of the second process labeled with a pomset containing the two causally-related actions a and b to the right of $+$ cannot be matched by the first process as its actions a and b are concurrent and hence originate a different pomset.

Finally, we have forward-reverse bisimilarity [23], which considers transitions labeled with individual actions like $\sim_{\mathrm{IB}}$ but examines incoming transitions (backward direction of computation) in addition to outgoing ones (forward direction of computation).

▶ **Definition 8.** *We say that two stable configuration structures $\mathcal{C}_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, are* forward-reverse bisimilar*, written $\mathcal{C}_1 \sim_{\mathrm{FRB}} \mathcal{C}_2$, iff there exists a forward-reverse bisimulation between them, i.e., a relation $\mathcal{B} \subseteq \mathcal{C}_1 \times \mathcal{C}_2$ such that $(\emptyset, \emptyset) \in \mathcal{B}$ and, whenever $(X_1, X_2) \in \mathcal{B}$, then:*

- *For each $X_1 \xrightarrow{a}_{\mathcal{C}_1} X_1'$ there exists $X_2 \xrightarrow{a}_{\mathcal{C}_2} X_2'$ such that $(X_1', X_2') \in \mathcal{B}$, and vice versa.*
- *For each $X_1' \xrightarrow{a}_{\mathcal{C}_1} X_1$ there exists $X_2' \xrightarrow{a}_{\mathcal{C}_2} X_2$ such that $(X_1', X_2') \in \mathcal{B}$, and vice versa.* ⌟

$\sim_{\mathrm{FRB}}$ is strictly finer than $\sim_{\mathrm{IB}}$: as an example, $a \parallel_\emptyset b$ and $a \,.\, b + b \,.\, a$ are $\sim_{\mathrm{IB}}$-equivalent but not $\sim_{\mathrm{FRB}}$-equivalent, because $a^\dagger \parallel_\emptyset b^\dagger$ has two differently labeled incoming transitions whereas $a^\dagger . b^\dagger + b \,.\, a$ and $a \,.\, b + b^\dagger . a^\dagger$ have only one. $\sim_{\mathrm{FRB}}$ is incomparable with $\sim_{\mathrm{SB}}$ and $\sim_{\mathrm{PB}}$. On the one hand, $a \,.\, (b + c) + (a \parallel_\emptyset b)$ and $a \,.\, (b + c) + (a \parallel_\emptyset b) + a \,.\, b$ are identified by $\sim_{\mathrm{PB}}$ and hence $\sim_{\mathrm{SB}}$ but told apart by $\sim_{\mathrm{FRB}}$, because $a \,.\, (b + c) + (a \parallel_\emptyset b) + a^\dagger . b$ can only be matched by $a \,.\, (b + c) + (a^\dagger \parallel_\emptyset b)$ – as $a^\dagger . (b + c) + (a \parallel_\emptyset b)$ enables both b and c – but then $a \,.\, (b + c) + (a \parallel_\emptyset b) + a^\dagger . b^\dagger$ has a single incoming transition whilst $a \,.\, (b + c) + (a^\dagger \parallel_\emptyset b^\dagger)$ has two differently labeled ones. On the other hand, $a \parallel_\emptyset a$ and $a \,.\, a$ are equated by $\sim_{\mathrm{FRB}}$ but distinguished by $\sim_{\mathrm{SB}}$ and hence $\sim_{\mathrm{PB}}$, because the two occurrences of a in the second process are not concurrent and hence the $\{\!|\, a, a \,|\!\}$-transition of the first process cannot be matched by the second one.

2.3 Bisimilarities with Variable Event Isomorphisms

The bisimilarities of the second family build suitable isomorphisms between the events matched so far during the bisimulation game. The isomorphisms are *specific* to the pairs of configurations at hand, hence can *vary* for a matched event from pair to pair. The resulting bisimilarities are called *weak history-preserving* in [14]. Unlike [21], the word weak is not to be intended as abstracting from unobservable actions, but as preserving history in a weak sense.

▶ **Definition 9.** *Given two stable configuration structures $\mathcal{C}_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$ and two configurations $X_i \in \mathcal{C}_i$, $i \in \{1, 2\}$, a* labeling- and causality-preserving isomorphism *between X_1 and X_2 is induced by a bijection $f : X_1 \to X_2$ such that:*
- *$\ell_1(e) = \ell_2(f(e))$ for all $e \in X_1$ (labeling preservation).*
- *$e \leq_{X_1} e' \iff f(e) \leq_{X_2} f(e')$ for all $e, e' \in X_1$ (causality preservation).* ⌟

We start by recalling the one of these bisimilarities that considers transitions labeled with individual actions [11].

▶ **Definition 10.** *We say that two stable configuration structures $\mathcal{C}_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, are* weak history-preserving bisimilar*, written $\mathcal{C}_1 \sim_{\rm WHPB} \mathcal{C}_2$, iff there exists a weak history-preserving bisimulation between them, i.e., a relation $\mathcal{B} \subseteq \mathcal{C}_1 \times \mathcal{C}_2$ such that $(\emptyset, \emptyset) \in \mathcal{B}$ and, whenever $(X_1, X_2) \in \mathcal{B}$, then:*
- *There exists a labeling- and causality-preserving bijection $f : X_1 \to X_2$.*
- *For each $X_1 \xrightarrow{a}_{\mathcal{C}_1} X'_1$ there exists $X_2 \xrightarrow{a}_{\mathcal{C}_2} X'_2$ such that $(X'_1, X'_2) \in \mathcal{B}$, and vice versa.* ⌟

$\sim_{\rm WHPB}$ is strictly finer than $\sim_{\rm SB}$: for example, $a \,\|_\emptyset\, b$ and $(a \,\|_\emptyset\, b) + a \,.\, b$ are identified by $\sim_{\rm SB}$ but told apart by $\sim_{\rm WHPB}$, because a and b to the right of $+$ in the second process are causally related and hence cannot be mapped to the concurrent a and b in the first process. $\sim_{\rm WHPB}$ is incomparable with $\sim_{\rm PB}$ and $\sim_{\rm FRB}$. On the one hand, $a \,.\, (b + c) + (a \,\|_\emptyset\, b)$ and $a \,.\, (b + c) + (a \,\|_\emptyset\, b) + a \,.\, b$ are identified by $\sim_{\rm PB}$ but told apart by $\sim_{\rm WHPB}$, while $a \,\|_\emptyset\, a$ and $a \,.\, a$ are equated by $\sim_{\rm FRB}$ but distinguished by $\sim_{\rm WHPB}$. On the other hand, the two complex processes in the forthcoming Figure 3 (where parallel transitions have the same action label) are identified by $\sim_{\rm WHPB}$ but told apart by $\sim_{\rm PB}$, while $(a \,\|_\emptyset (b + c)) + (a \,\|_\emptyset\, b) + ((a + c) \,\|_\emptyset\, b)$ and $(a \,\|_\emptyset (b + c)) + ((a + c) \,\|_\emptyset\, b)$ are equated by $\sim_{\rm WHPB}$ but distinguished by $\sim_{\rm FRB}$, because $(a \,\|_\emptyset (b+c)) + (a^\dagger \,\|_\emptyset\, b^\dagger) + ((a+c) \,\|_\emptyset\, b)$ has two incoming transitions departing from configurations having no alternative outgoing transitions whilst one of the two incoming transitions of $(a^\dagger \,\|_\emptyset (b^\dagger + c)) + ((a + c) \,\|_\emptyset\, b)$ and $(a \,\|_\emptyset (b + c)) + ((a^\dagger + c) \,\|_\emptyset\, b^\dagger)$ departs from a configuration having an outgoing c-transition too.

Then we have the one that considers transitions labeled with pomsets [14].

▶ **Definition 11.** *We say that two stable configuration structures $\mathcal{C}_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, are* weak history-preserving pomset bisimilar*, written $\mathcal{C}_1 \sim_{\rm WHPPB} \mathcal{C}_2$, iff there exists a weak history-preserving pomset bisimulation between them, i.e., a relation $\mathcal{B} \subseteq \mathcal{C}_1 \times \mathcal{C}_2$ such that $(\emptyset, \emptyset) \in \mathcal{B}$ and, whenever $(X_1, X_2) \in \mathcal{B}$, then:*
- *There exists a labeling- and causality-preserving bijection $f : X_1 \to X_2$.*
- *For each $X_1 \xrightarrow{U}_{\mathcal{C}_1} X'_1$ there exists $X_2 \xrightarrow{U}_{\mathcal{C}_2} X'_2$ such that $(X'_1, X'_2) \in \mathcal{B}$, and vice versa.* ⌟

$\sim_{\rm WHPPB}$ is strictly finer than $\sim_{\rm WHPB}$ and $\sim_{\rm PB}$: for instance, the two complex processes in the forthcoming Figure 3 are identified by $\sim_{\rm WHPB}$ but told apart by $\sim_{\rm WHPPB}$, while $a \,.\, (b + c) + (a \,\|_\emptyset\, b)$ and $a \,.\, (b + c) + (a \,\|_\emptyset\, b) + a \,.\, b$ are equated by $\sim_{\rm PB}$ but distinguished by $\sim_{\rm WHPPB}$, because the rightmost a in the second process can only be matched by the rightmost a in the first process but then the rightmost causally-related a and b in the second process cannot be mapped to the rightmost concurrent a and b in the first process. $\sim_{\rm WHPPB}$ is incomparable with $\sim_{\rm FRB}$: as an example, $a \,\|_\emptyset\, a$ and $a \,.\, a$ are identified by $\sim_{\rm FRB}$ but told apart by $\sim_{\rm WHPPB}$, whereas $(a \,\|_\emptyset (b+c)) + (a \,\|_\emptyset\, b) + ((a+c) \,\|_\emptyset\, b)$ and $(a \,\|_\emptyset (b+c)) + ((a+c) \,\|_\emptyset\, b)$ are equated by $\sim_{\rm WHPPB}$ but distinguished by $\sim_{\rm FRB}$.

Finally, we have the one that considers transitions labeled with individual actions like $\sim_{\rm WHPB}$ but examines incoming transitions in addition to outgoing ones [25].

▶ **Definition 12.** *We say that two stable configuration structures* $\mathcal{C}_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1,2\}$, *are* weak hereditary history-preserving bisimilar*, written* $\mathcal{C}_1 \sim_{\mathrm{WHHPB}} \mathcal{C}_2$*, iff there exists a weak hereditary history-preserving bisimulation between them, i.e., a relation* $\mathcal{B} \subseteq \mathcal{C}_1 \times \mathcal{C}_2$ *such that* $(\emptyset, \emptyset) \in \mathcal{B}$ *and, whenever* $(X_1, X_2) \in \mathcal{B}$*, then:*

- *There exists a labeling- and causality-preserving bijection* $f : X_1 \to X_2$.
- *For each* $X_1 \xrightarrow{a}_{\mathcal{C}_1} X_1'$ *there exists* $X_2 \xrightarrow{a}_{\mathcal{C}_2} X_2'$ *such that* $(X_1', X_2') \in \mathcal{B}$*, and vice versa.*
- *For each* $X_1' \xrightarrow{a}_{\mathcal{C}_1} X_1$ *there exists* $X_2' \xrightarrow{a}_{\mathcal{C}_2} X_2$ *such that* $(X_1', X_2') \in \mathcal{B}$*, and vice versa.* ⌟

$\sim_{\mathrm{WHHPB}}$ is strictly finer than $\sim_{\mathrm{WHPB}}$ and $\sim_{\mathrm{FRB}}$: for example, $(a \parallel_\emptyset (b+c)) + (a \parallel_\emptyset b) + ((a+c) \parallel_\emptyset b)$ and $(a \parallel_\emptyset (b+c)) + ((a+c) \parallel_\emptyset b)$ are identified by $\sim_{\mathrm{WHPB}}$ but told apart by $\sim_{\mathrm{WHHPB}}$, while $a \parallel_\emptyset a$ and $a \,.\, a$ are equated by $\sim_{\mathrm{FRB}}$ but distinguished by $\sim_{\mathrm{WHHPB}}$. $\sim_{\mathrm{WHHPB}}$ is incomparable with $\sim_{\mathrm{WHPPB}}$: for instance, the two complex processes in the forthcoming Figure 3 are identified by $\sim_{\mathrm{WHHPB}}$ but told apart by $\sim_{\mathrm{WHPPB}}$, whilst $(a \parallel_\emptyset (b+c)) + (a \parallel_\emptyset b) + ((a+c) \parallel_\emptyset b)$ and $(a \parallel_\emptyset (b+c)) + ((a+c) \parallel_\emptyset b)$ are equated by $\sim_{\mathrm{WHPPB}}$ but distinguished by $\sim_{\mathrm{WHHPB}}$.

2.4 Bisimilarities with Incremental Event Isomorphisms

The bisimilarities of the third family additionally require that a *single* labeling- and causality-preserving isomorphism between the events matched so far be built *incrementally*. As a consequence, the underlying bisimulations become *ternary* relations because they must include the bijection inducing the isomorphism. These bisimilarities are termed *history-preserving* in a strong sense. In the following, $f \upharpoonright X$ denotes the restriction of function f to set X.

The first of these bisimilarities considers transitions labeled with individual actions [28]. Its distinguishing power would not change in the case that step or pomset transitions were considered [14].

▶ **Definition 13.** *We say that two stable configuration structures* $\mathcal{C}_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1,2\}$, *are* history-preserving bisimilar*, written* $\mathcal{C}_1 \sim_{\mathrm{HPB}} \mathcal{C}_2$*, iff there exists a history-preserving bisimulation between them, i.e., a relation* $\mathcal{B} \subseteq \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{P}(\mathcal{E}_1 \times \mathcal{E}_2)$ *such that* $(\emptyset, \emptyset, \emptyset) \in \mathcal{B}$ *and, whenever* $(X_1, X_2, f) \in \mathcal{B}$*, then:*

- f *is a labeling- and causality-preserving bijection from* X_1 *to* X_2.
- *For each* $X_1 \xrightarrow{a}_{\mathcal{C}_1} X_1'$ *there exist* $X_2 \xrightarrow{a}_{\mathcal{C}_2} X_2'$ *and* $f' \subseteq \mathcal{E}_1 \times \mathcal{E}_2$ *such that* $(X_1', X_2', f') \in \mathcal{B}$ *and* $f' \upharpoonright X_1 = f$*, and vice versa.* ⌟

$\sim_{\mathrm{HPB}}$ is strictly finer than $\sim_{\mathrm{WHPPB}}$: as an example, the two complex processes in the forthcoming Figure 5 are identified by $\sim_{\mathrm{WHPPB}}$ but told apart by $\sim_{\mathrm{HPB}}$. $\sim_{\mathrm{HPB}}$ is incomparable with $\sim_{\mathrm{WHHPB}}$: for instance, the two complex processes in the forthcoming Figure 3 are identified by $\sim_{\mathrm{WHHPB}}$ but told apart by $\sim_{\mathrm{HPB}}$, whilst $(a \parallel_\emptyset (b+c)) + (a \parallel_\emptyset b) + ((a+c) \parallel_\emptyset b)$ and $(a \parallel_\emptyset (b+c)) + ((a+c) \parallel_\emptyset b)$ are equated by $\sim_{\mathrm{HPB}}$ but distinguished by $\sim_{\mathrm{WHHPB}}$.

The second one examines incoming transitions in addition to outgoing ones [3]. Like the first one, its distinguishing power would not change in the case that step or pomset transitions were considered [14].

▶ **Definition 14.** *We say that two stable configuration structures* $\mathcal{C}_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1,2\}$, *are* hereditary history-preserving bisimilar*, written* $\mathcal{C}_1 \sim_{\mathrm{HHPB}} \mathcal{C}_2$*, iff there exists a hereditary history-preserving bisimulation between them, i.e., a relation* $\mathcal{B} \subseteq \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{P}(\mathcal{E}_1 \times \mathcal{E}_2)$ *such that* $(\emptyset, \emptyset, \emptyset) \in \mathcal{B}$ *and, whenever* $(X_1, X_2, f) \in \mathcal{B}$*, then:*

- f *is a labeling- and causality-preserving bijection from* X_1 *to* X_2.
- *For each* $X_1 \xrightarrow{a}_{\mathcal{C}_1} X_1'$ *there exist* $X_2 \xrightarrow{a}_{\mathcal{C}_2} X_2'$ *and* $f' \subseteq \mathcal{E}_1 \times \mathcal{E}_2$ *such that* $(X_1', X_2', f') \in \mathcal{B}$ *and* $f' \upharpoonright X_1 = f$*, and vice versa.*

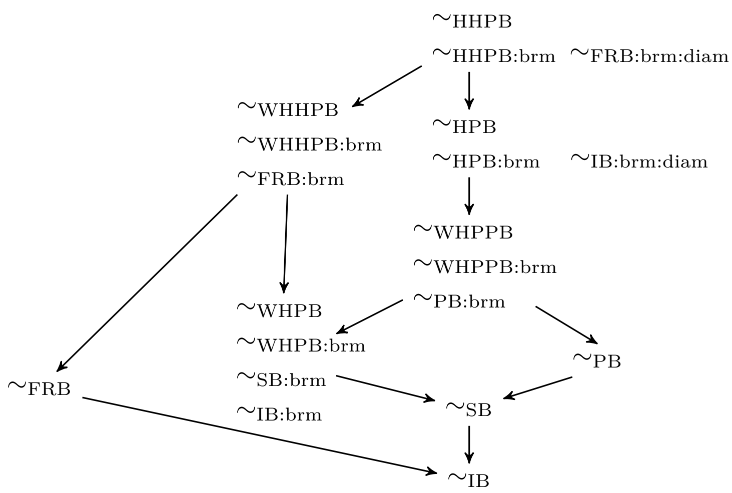

Figure 2 Spectrum of the main bisimilarities examined in [14, 12, 25] and their brm-variants.

- *For each* $X'_1 \xrightarrow{a}_{C_1} X_1$ *there exist* $X'_2 \xrightarrow{a}_{C_2} X_2$ *and* $f' \subseteq \mathcal{E}_1 \times \mathcal{E}_2$ *such that* $(X'_1, X'_2, f') \in \mathcal{B}$ *and* $f \upharpoonright X'_1 = f'$, *and vice versa.* ⌟

$\sim_{\rm HHPB}$ is strictly finer than $\sim_{\rm HPB}$ and $\sim_{\rm WHHPB}$: for example, $(a \parallel_\emptyset (b + c)) + (a \parallel_\emptyset b) + ((a + c) \parallel_\emptyset b)$ and $(a \parallel_\emptyset (b + c)) + ((a + c) \parallel_\emptyset b)$ are identified by $\sim_{\rm HPB}$ but told apart by $\sim_{\rm HHPB}$, while the two complex processes in the forthcoming Figure 3 are equated by $\sim_{\rm WHHPB}$ but distinguished by $\sim_{\rm HHPB}$.

3 Extending Bisimilarities with Backward Ready Multiset Equality

Forward-reverse bisimilarity was extended in [4] by additionally checking that the backward ready multisets of related configurations are equal. The purpose was to approach the distinguishing power of hereditary history-preserving bisimilarity also in the presence of autoconcurrency, i.e., identically labeled concurrent events. We recall from Section 1 that the two $\sim_{\rm HHPB}$-inequivalent processes $a \parallel_\emptyset a$ and $a \,.\, a$ are identified by $\sim_{\rm FRB}$ but told apart by $\sim_{\rm FRB:brm}$, because $a^\dagger \parallel_\emptyset a^\dagger$ has two incoming a-labeled transitions while $a^\dagger \,.\, a^\dagger$ has only one.

▶ **Definition 15.** *Let* $C = (\mathcal{E}, \mathcal{C}, \ell)$ *be a stable configuration structure and* $X \in \mathcal{C}$. *The* backward ready multiset *of* X *is* $brm(X) = \{\!|\, a \in \mathcal{A} \mid X' \xrightarrow{a}_C X \,|\!\}$. *For each bisimilarity* $\sim_B$ *in Figure 1 we denote by* $\sim_{B:\rm brm}$ *its backward ready multiset variant obtained by adding the clause* $brm(X_1) = brm(X_2)$ *for each pair* $(X_1, X_2) \in \mathcal{B}$. ⌟

In this section we investigate the discriminating power of any $\sim_{B:\rm brm}$. We first examine the bisimilarities of the no-iso family (Section 3.1), then those of the var-iso and incr-iso families (Section 3.2). The resulting extended spectrum is shown in Figure 2.

3.1 No-Iso Bisimilarities with Backward Ready Multiset Equality

Adding the backward ready multiset equality check to $\sim_{\rm IB}$ yields a discriminating power greater than the one of $\sim_{\rm SB}$. The intuition is that $brm(X)$ coincides with the label of the incoming step transition of X with the largest label. A counterexample witnessing the strict inclusion of $\sim_{\rm IB:brm}$ in $\sim_{\rm SB}$ is given by $a \parallel_\emptyset b$ and $a \parallel_\emptyset b + a \,.\, b$, which are identified by $\sim_{\rm SB}$ but told apart by $\sim_{\rm IB:brm}$ because $a^\dagger \parallel_\emptyset b^\dagger$ has two incoming transitions while $a \parallel_\emptyset b + a^\dagger \,.\, b^\dagger$ has only one.

▶ **Theorem 16.** *Let $C_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, be two stable configuration structures. If $C_1 \sim_{\mathrm{IB:brm}} C_2$ then $C_1 \sim_{\mathrm{SB}} C_2$.* ⌟

Surprisingly, adding the backward ready multiset equality check to $\sim_{\mathrm{SB}}$ yields the same distinguishing power as $\sim_{\mathrm{IB:brm}}$.

▶ **Theorem 17.** *Let $C_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, be two stable configuration structures. Then $C_1 \sim_{\mathrm{IB:brm}} C_2$ iff $C_1 \sim_{\mathrm{SB:brm}} C_2$.* ⌟

As far as $\sim_{\mathrm{SB:brm}}$ and $\sim_{\mathrm{PB:brm}}$ are concerned, we notice that they are finer than their original counterparts and that the latter is finer than the former:

- $a \parallel_\emptyset b$ and $a \parallel_\emptyset b + a \,.\, b$ are identified by $\sim_{\mathrm{SB}}$ but told apart by $\sim_{\mathrm{SB:brm}}$, because $a^\dagger \parallel_\emptyset b^\dagger$ has two incoming transitions while $a \parallel_\emptyset b + a^\dagger \,.\, b^\dagger$ has only one.
- $a \,.\, (b + c) + (a \parallel_\emptyset b)$ and $a \,.\, (b + c) + (a \parallel_\emptyset b) + a \,.\, b$ are equated by $\sim_{\mathrm{PB}}$ but distinguished by $\sim_{\mathrm{PB:brm}}$, because $a \,.\, (b + c) + a \parallel_\emptyset b + a^\dagger \,.\, b^\dagger$ can only be matched by $a \,.\, (b + c) + (a^\dagger \parallel_\emptyset b^\dagger)$ – otherwise c would be enabled after performing a – but the former has a single incoming transition whereas the latter has two.
- The two complex processes in the forthcoming Figure 3 (where parallel transitions have the same action label) are identified by $\sim_{\mathrm{SB:brm}}$ but told apart by $\sim_{\mathrm{PB:brm}}$, because in the former after executing a it is always possible to perform a pomset given by a and b causally related to each other, while in the latter this is not the case due to configuration 6_F.

The main result about no-iso bisimilarities is that the backward ready multiset equality check turns them into var-iso bisimilarities. In the following, given a stable configuration structure $C = (\mathcal{E}, \mathcal{C}, \ell)$ and $X \in \mathcal{C}$, we denote by $brs_{\mathrm{e}}(X) = \{e \in \mathcal{E} \mid X' \xrightarrow{\ell(e)}_C X\}$ the backward ready set of X expressed in terms of events (which are all distinct); note that $brm(X) = \{\!|\, \ell(e) \mid e \in brs_{\mathrm{e}}(X) \,|\!\}$. Moreover, we denote by $|M| = \sum_{a \in M} M(a)$ the cardinality of a multiset $M : \mathcal{A} \to \mathbb{N}$, where $M(a)$ is the multiplicity of a in M; note that $|brm(X)| = |brs_{\mathrm{e}}(X)|$. Given a transition $X \xrightarrow{\ell(e)}_C X'$, the set of immediate causes of e is given by $causes_{X'}(e) = brs_{\mathrm{e}}(X) \setminus brs_{\mathrm{e}}(X')$. The intuition is that, since $brs_{\mathrm{e}}(X)$ collects the concurrent events that originate the incoming transitions of X, any event in $brs_{\mathrm{e}}(X) \setminus brs_{\mathrm{e}}(X')$ must have been a cause of the newly added event e, otherwise it would originate another incoming transition of X'. Some preliminary results are in order – where the possibly occurring bisimulations or bisimilarities are not necessarily no-iso – which highlight the relationships that exist among brs_{e} cardinality, brs_{e} inclusion, and causality.

▶ **Lemma 18.** *Let $C = (\mathcal{E}, \mathcal{C}, \ell)$ be a stable configuration structure and $X \xrightarrow{\ell(e)}_C X'$ for $X, X' \in \mathcal{C}$ and $e \in \mathcal{E}$. Then $|brs_{\mathrm{e}}(X)| < |brs_{\mathrm{e}}(X')|$ iff $brs_{\mathrm{e}}(X) \subsetneq brs_{\mathrm{e}}(X')$.* ⌟

▶ **Lemma 19.** *Let $C_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, be two stable configuration structures and $X_i \xrightarrow{a}_{C_i} X'_i$ for $X_i, X'_i \in \mathcal{C}_i$ such that $(X_1, X_2), (X'_1, X'_2) \in \mathcal{B}$ for some brm-bisimulation $\mathcal{B}$. Then $brs_{\mathrm{e}}(X_1) \subsetneq brs_{\mathrm{e}}(X'_1)$ iff $brs_{\mathrm{e}}(X_2) \subsetneq brs_{\mathrm{e}}(X'_2)$.* ⌟

▶ **Lemma 20.** *Let $C = (\mathcal{E}, \mathcal{C}, \ell)$ be a stable configuration structure and $X \xrightarrow{\ell(e)}_C X'$ for $X, X' \in \mathcal{C}$ and $e \in \mathcal{E}$ such that $brs_{\mathrm{e}}(X) \not\subseteq brs_{\mathrm{e}}(X')$. Then $e' <_{X'} e$ for all $e' \in causes_{X'}(e)$.* ⌟

▶ **Theorem 21.** *Let $C_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, be two stable configuration structures. If $C_1 \sim_{B:\mathrm{brm}} C_2$ then for each pair of related configurations $X_1 \in \mathcal{C}_1$ and $X_2 \in \mathcal{C}_2$ there exists a labeling- and causality-preserving bijection $f : X_1 \to X_2$.* ⌟

▶ **Corollary 22.** $\sim_{\mathrm{IB:brm}} = \sim_{\mathrm{WHPB}}$, $\sim_{\mathrm{PB:brm}} = \sim_{\mathrm{WHPPB}}$, $\sim_{\mathrm{FRB:brm}} = \sim_{\mathrm{WHHPB}}$. ⌟

3.2 Iso Bisimilarities with Backward Ready Multiset Equality

Adding the backward ready multiset equality check to var-iso and incr-iso bisimilarities does not increase their distinguishing power. The reason is that the aforementioned check can be derived from the existing isomorphism.

▶ **Theorem 23.** *Let $\mathcal{C}_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, be two stable configuration structures and $B \in \{\mathrm{WHPB}, \mathrm{WHPPB}, \mathrm{WHHPB}, \mathrm{HPB}, \mathrm{HHPB}\}$. Then $\mathcal{C}_1 \sim_B \mathcal{C}_2$ iff $\mathcal{C}_1 \sim_{B:\mathrm{brm}} \mathcal{C}_2$.* ⌟

4 Logical Characterizations of No-Iso and Var-Iso Bisimilarities

In this section we introduce an extension of the backward ready multiset logic of [4], which is in turn an extension of Hennessy-Milner logic [18] comprising a backward modality [10], denoted by $\langle a^\dagger \rangle$, and a backward ready multiset proposition, denoted by M. We use its fragments to characterize the bisimilarities of the no-iso and var-iso families, including their brm-variants that allow us to get rid of variable isomorphisms.

What we add is the capability of expressing pomsets, because in this way we can deal with all the three kinds of transitions appearing in the equivalences of Section 2: single action, step, and pomset. This is accomplished by replacing the usual single-action modality $\langle a \rangle$ with a more expressive modality $\langle \alpha \rangle$, where α is a *pomset term* generated by the following syntax in which $a \in \mathcal{A}$, "$;$" represents causality, and "$\|$" represents concurrency:

$$\alpha \ ::= \ a \mid \alpha \, ; \alpha \mid \alpha \parallel \alpha$$

The pomset $U_\alpha = [(E_\alpha, <_\alpha, \ell_\alpha)]_\cong$ associated with α is defined by induction on the syntactical structure of α as follows where $\uplus$ denotes disjoint union:

$$\begin{array}{rcl} U_a & = & [(\{e\}, \emptyset, \{(e, a)\})]_\cong \\ U_{\alpha_1 \, ; \, \alpha_2} & = & [(E_{\alpha_1} \uplus E_{\alpha_2}, <_{\alpha_1} \cup <_{\alpha_2} \cup (E_{\alpha_1} \times E_{\alpha_2}), \ell_{\alpha_1} \cup \ell_{\alpha_2})]_\cong \\ U_{\alpha_1 \, \| \, \alpha_2} & = & [(E_{\alpha_1} \uplus E_{\alpha_2}, <_{\alpha_1} \cup <_{\alpha_2}, \ell_{\alpha_1} \cup \ell_{\alpha_2})]_\cong \end{array}$$

The resulting pomsets are called *series-parallel pomsets* in [13] as they can be constructed by starting from atomic actions and then combining them via sequential and parallel compositions. In [17, 29] it has been proven that a finite pomset is series-parallel iff it is N-free, i.e., it has no subpomsets of the form $N = [(\{e_1, e_2, e_3, e_4\}, \{(e_1, e_3), (e_2, e_4), (e_1, e_4)\}, \{(e_i, a) \mid 1 \le i \le 4\})]_\cong$. As shown in [17, 13] and remarked in [6], N-free pomsets can be completely characterized by the syntax of pomset terms. In particular, they constitute the kind of pomsets that label the transitions considered by pomset bisimilarity [6].

The syntax of our modal logic $\mathcal{L}_{\mathrm{brm,pom}}$ is the following:

$$\phi \ ::= \ \mathsf{true} \mid M \mid \neg \phi \mid \phi \wedge \phi \mid \langle \alpha \rangle \phi \mid \langle a^\dagger \rangle \phi$$

with $M : \mathcal{A} \to \mathbb{N}$, α being a pomset term, and $a \in \mathcal{A}$. Given a stable configuration structure $\mathcal{C} = (\mathcal{E}, \mathcal{C}, \ell)$, we let $\mathcal{C} \models \phi$ iff $\emptyset \models_{\mathcal{C}} \phi$, where $X \models_{\mathcal{C}} \phi$ for $X \in \mathcal{C}$ is defined by induction on the syntactical structure of $\phi \in \mathcal{L}_{\mathrm{brm,pom}}$ as follows (when $\mathcal{C}$ is clear from the context, we will simply write $\models$):

$$\begin{array}{lcll} X & \models_{\mathcal{C}} & \mathsf{true} & \\ X & \models_{\mathcal{C}} & M & \text{iff } \ brm(X) = M \\ X & \models_{\mathcal{C}} & \neg \phi' & \text{iff } \ X \not\models_{\mathcal{C}} \phi' \\ X & \models_{\mathcal{C}} & \phi_1 \wedge \phi_2 & \text{iff } \ X \models_{\mathcal{C}} \phi_1 \text{ and } X \models_{\mathcal{C}} \phi_2 \\ X & \models_{\mathcal{C}} & \langle \alpha \rangle \phi' & \text{iff there exists } X \xrightarrow{U_\alpha}_{\mathcal{C}} X' \text{ such that } X' \models_{\mathcal{C}} \phi' \\ X & \models_{\mathcal{C}} & \langle a^\dagger \rangle \phi' & \text{iff there exists } X' \xrightarrow{a}_{\mathcal{C}} X \text{ such that } X' \models_{\mathcal{C}} \phi' \end{array}$$

Let us reconsider the counterexamples right after Theorem 17. It holds that: $a \parallel_\emptyset b + a \, . \, b$ satisfies $\langle a \rangle \langle b \rangle \{\!| \, b \, |\!\}$ while $a \parallel_\emptyset b$ does not; $a \, . \, (b + c) + (a \parallel_\emptyset b) + a \, . \, b$ satisfies $\langle a \rangle (\neg \langle c \rangle \mathsf{true} \wedge \langle b \rangle \{\!| \, b \, |\!\})$ while $a \, . \, (b + c) + (a \parallel_\emptyset b)$ does not; $\mathcal{E} \models \neg \langle a \rangle \neg \langle a \, ; b \rangle \mathsf{true}$ in the forthcoming Figure 3

Table 1 Fragments of $\mathcal{L}_{\text{brm,pom}}$ characterizing no-iso and var-iso bisimilarities.

	true	M	$\neg$	$\wedge$	$\langle a \rangle$	$\langle \alpha_{\text{S}} \rangle$	$\langle \alpha \rangle$	$\langle a^{\dagger} \rangle$	
$\mathcal{L}_{\text{IB}}$	✓		✓	✓	✓				$\sim_{\text{IB}}$
$\mathcal{L}_{\text{SB}}$	✓		✓	✓		✓			$\sim_{\text{SB}}$
$\mathcal{L}_{\text{PB}}$	✓		✓	✓			✓		$\sim_{\text{PB}}$
$\mathcal{L}_{\text{FRB}}$	✓		✓	✓	✓			✓	$\sim_{\text{FRB}}$
$\mathcal{L}_{\text{IB:brm}}$	✓	✓	✓	✓	✓				$\sim_{\text{IB:brm}}, \sim_{\text{WHPB}}, \sim_{\text{WHPB:brm}}$
$\mathcal{L}_{\text{SB:brm}}$	✓	✓	✓	✓		✓			$\sim_{\text{SB:brm}}, \sim_{\text{WHPB}}, \sim_{\text{WHPB:brm}}$
$\mathcal{L}_{\text{PB:brm}}$	✓	✓	✓	✓			✓		$\sim_{\text{PB:brm}}, \sim_{\text{WHPPB}}, \sim_{\text{WHPPB:brm}}$
$\mathcal{L}_{\text{FRB:brm}}$	✓	✓	✓	✓	✓			✓	$\sim_{\text{FRB:brm}}, \sim_{\text{WHHPB}}, \sim_{\text{WHHPB:brm}}$

(where only different actions with the same subscript are causally related) while F does not.

Over stable configuration structures that are *image finite*, i.e., such that each of its configurations has finitely many outgoing transitions labeled with the same action, the no-iso and var-iso bisimilarities are characterized by the fragments of $\mathcal{L}_{\text{brm,pom}}$ shown in Table 1, where α_{S} is a pomset term in which ";" does not occur. This can be proven by directly addressing only the no-iso ones along with their brm-variants and then exploiting Corollary 22. Note that true, negation, and conjunction are common to all fragments, while M is needed only for the brm-variants. The used modalities depend on the specific bisimilarity.

▶ **Theorem 24.** *Let $C_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, be two image-finite stable configuration structures and $B \in \{$IB, SB, PB, FRB, IB:brm, SB:brm, PB:brm, FRB:brm$\}$. Then $C_1 \sim_B C_2$ iff $C_1 \models \phi \Longleftrightarrow C_2 \models \phi$ for all $\phi \in \mathcal{L}_B$.* ⌟

▶ **Theorem 25.** *Let $C_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, be two image-finite stable configuration structures with $VIB \in \{$WHPB, WHPB:brm$\} \wedge NIbrmB \in \{$IB:brm, SB:brm$\}$ or $VIB \in \{$WHPPB, WHPPB:brm$\} \wedge NIbrmB \in \{$PB:brm$\}$ or $VIB \in \{$WHHPB, WHHPB:brm$\} \wedge NIbrmB \in \{$FRB:brm$\}$. Then $C_1 \sim_{VIB} C_2$ iff $C_1 \models \phi \Longleftrightarrow C_2 \models \phi$ for all $\phi \in \mathcal{L}_{NIbrmB}$.* ⌟

5 Backward Ready Multisets and Diamonds for the Incr-Iso Family

In this section we discuss how to pass from the no-iso to the incr-iso family of bisimilarities. The intuition is that, when relating two configurations, in addition to verifying backward ready multiset equality – which is sufficient to pass from the no-iso to the var-iso family – in the simultaneous presence of autoconcurrency and non-local conflicts we also need to verify the existence of suitable diamond-shaped and half-diamond-shaped subconfiguration structures. When two transitions depart from a configuration thus opening a shape resembling the initial part of a diamond, whether it closes (diamond) or not (half diamond) gives us information about concurrency or conflict, respectively, among events not belonging to that configuration. Working with diamonds and half diamonds we obtain the same distinguishing power achieved by building an incremental isomorphism between the events matched so far.

Let us formalize what we mean by diamond and half diamond. The former is the result of interleaving two concurrent events starting from a given configuration. In the case of a half diamond of a configuration X, we not only require the existence of two transitions from X that open something that never closes, but we also require that at least one of these transitions does not belong to any diamond opening at the same configuration X.

▶ **Definition 26.** *Let $\mathsf{C} = (\mathcal{E}, \mathcal{C}, \ell)$ be a stable configuration structure. Given two configurations $X, Y \in \mathcal{C}$, the* diamond *that opens at X and closes at Y is defined by* $d(X,Y) = \{(X, X', X'', Y) \mid X \xrightarrow{a}_{\mathsf{C}} X', X \xrightarrow{b}_{\mathsf{C}} X'', X' \xrightarrow{b}_{\mathsf{C}} Y, X'' \xrightarrow{a}_{\mathsf{C}} Y\}$*. We denote by* $\mathit{diam}(X) = \{d(X,Y) \mid Y \in \mathcal{C}\}$ *the set of diamonds opening at X.* ⌟

▶ **Definition 27.** *Let $\mathsf{C} = (\mathcal{E}, \mathcal{C}, \ell)$ be a stable configuration structure. Given a configuration $X \in \mathcal{C}$, a* half diamond *that opens at X is a triple (X, X', X'') such that $X \xrightarrow{a}_{\mathsf{C}} X'$, $X \xrightarrow{b}_{\mathsf{C}} X''$, $\nexists Y \in \mathcal{C}. (X' \xrightarrow{b}_{\mathsf{C}} Y \wedge X'' \xrightarrow{a}_{\mathsf{C}} Y)$, $\nexists D_1, D_2 \in \mathit{diam}(X). (X' \in D_1 \wedge X'' \in D_2)$. We denote by $\mathit{hdiam}(X)$ the set of half diamonds opening at X.* ⌟

Let us indicate with $\sim_{\mathrm{IB:brm:diam}}$ and $\sim_{\mathrm{FRB:brm:diam}}$ the variants of $\sim_{\mathrm{IB:brm}}$ and $\sim_{\mathrm{FRB:brm}}$ where related configurations satisfy the following additional condition:

- If $\mathit{diam}(X_1) \neq \emptyset \wedge \mathit{diam}(X_2) \neq \emptyset$ then $\mathit{hdiam}(X_1) \neq \emptyset \iff \mathit{hdiam}(X_2) \neq \emptyset$.

We now provide some examples that illustrate the effect of this condition. In the forthcoming Figures 3–6, configurations are represented by numbers, parallel transitions have the same action label, the subscripts associated with identical actions identify the events originating those actions, and only different actions with the same subscript are causally related.

▶ **Example 28.** We start by noting that premise $\mathit{diam}(X_1) \neq \emptyset \wedge \mathit{diam}(X_2) \neq \emptyset$ is necessary for avoiding to rule out pairs of configurations that are trivially equivalent like for example a and $a + a$. If we were to check only the existence of half diamonds, then we would distinguish a and $a + a$, as the latter opens one half diamond while the former does not. By first requiring that both of them open at least one diamond each, the overall condition is vacuously satisfied and hence they are not told apart. ⌟

▶ **Example 29.** A more interesting example is given by the configuration structures E and F in Figure 3, taken from [25, Example 4.7], which are described by the following processes according to [16]:

$$((a_1 \,.\, b_1) \,\|_{\emptyset} (a_2 \,.\, b_2) \,\|_{\emptyset} (a_3 \,.\, b_3)) \,\|_{\{a_1, a_3, b_1, b_2, b_3\}} ((a_1 + a_3) \,\|_{\emptyset} (b_1 + b_2) \,\|_{\{b_2\}} (b_2 + b_3))$$

and:

$$((a_1 \,.\, b_1) \,\|_{\emptyset} (a_2 \,.\, b_2) \,\|_{\emptyset} (a_3 \,.\, b_3)) \,\|_{\{a_1, a_3, b_1, b_2\}} ((a_1 + a_3) \,\|_{\{a_3\}} (a_3 + b_2) \,\|_{\{b_2\}} (b_1 + b_2))$$

where a relabeling mapping a_1, a_2, a_3 to a and b_1, b_2, b_3 to b should then be applied so as to get rid of action subscripts in the bisimulation game.

It is easy to see that E and F – where the latter features autoconcurrency as well as a conflict between actions a_3 and b_2 occurring in different parallel subprocesses – are not equated by any of the bisimilarities of the incr-iso family because:

- If 1_{F} performs a_3 thus evolving to 4_{F}, then 1_{E} can respond by performing a_3 (or equivalently a_1) thus evolving to 4_{E}. The initial bijection is $\{(a_3, a_3)\}$.
- If 4_{F} performs a_2 thus evolving to 6_{F}, then 4_{E} can respond by performing a_2 thus evolving to 6_{E}. The bijection becomes $\{(a_3, a_3), (a_2, a_2)\}$.
- If 6_{E} performs b_2 thus evolving to 13_{E} – note that E has become the challenger in the bisimulation game – then 6_{F} can only respond by performing b_3 thus evolving to 12_{F}. The bijection becomes $\{(a_3, a_3), (a_2, a_2), (b_2, b_3)\}$ but causality is not preserved because a_2 causes b_2 in E whereas a_2 does not cause b_3 in F.

It is worth noting that $\sim_{\mathrm{PB}}$ and $\sim_{\mathrm{WHPPB}}$ also distinguish these two configuration structures, because in E it is always possible to perform a pomset transition $a \,; b$ after any a-labeled transition while in F we cannot perform $a \,; b$ after a_3. On the other hand, if we were to use a brm-based equivalence like $\sim_{\mathrm{IB:brm}}$ or $\sim_{\mathrm{FRB:brm}}$, we would not be able to distinguish E and F because 13_{E} and 12_{F} have the same backward ready multisets. However, we can distinguish

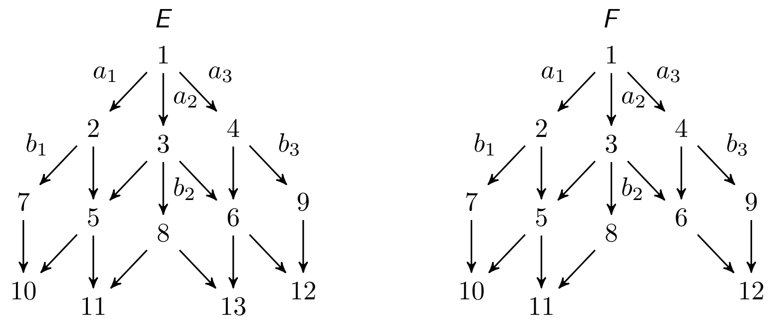

Figure 3 E and F are not $\sim_{\mathrm{HPB/HHPB}}$-equivalent but are $\sim_{\mathrm{WHPB/WHHPB}}$-equivalent.

them by using our diamond-based condition. It is easy to check that 1_F can perform an a-labeled transition and reach 3_F, which opens a diamond and a half diamond where only the transition from 3_F to 8_F belong to another diamond of 3_F, while all the configurations a-reachable from 1_E can only open diamonds. ⌟

▶ **Example 30.** A reasoning similar to the one of the previous example applies to the configuration structures G and H in Figure 4, taken from [25, Example 4.8], whose corresponding processes are:

$$((a_1 \,.\, b_1) \parallel_{\emptyset} (a_2 \,.\, b_2) \parallel_{\emptyset} (a_3 \,.\, b_3) \parallel_{\emptyset} (a_4 \,.\, b_4)) \parallel_{\{a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4\}}$$
$$((a_1 + a_3) \parallel_{\{a_1\}} (a_1 + a_4) \parallel_{\{a_4\}} (a_2 + a_4) \parallel_{\emptyset} (b_1 + b_2) \parallel_{\{b_2\}} (b_2 + b_3) \parallel_{\{b_3\}} (b_3 + b_4))$$

and:

$$((a_1 \,.\, b_1) \parallel_{\emptyset} (a_2 \,.\, b_2) \parallel_{\emptyset} (a_3 \,.\, b_3) \parallel_{\emptyset} (a_4 \,.\, b_4)) \parallel_{\{a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4\}}$$
$$((a_1 + a_3) \parallel_{\{a_1\}} (a_1 + a_4) \parallel_{\{a_4\}} (a_2 + a_4) \parallel_{\{a_2\}} (a_2 + b_3) \parallel_{\{b_3\}} (b_3 + b_4) \parallel_{\{\emptyset\}} (b_1 + b_2))$$

where H features autoconcurrency and a non-local conflict. These configuration structures are distinguished by $\sim_{\mathrm{PB}}$, $\sim_{\mathrm{WHPPB}}$, $\sim_{\mathrm{HPB}}$, $\sim_{\mathrm{HHPB}}$, but identified by $\sim_{\mathrm{IB:brm}}$ and $\sim_{\mathrm{FRB:brm}}$. ⌟

▶ **Example 31.** Consider the configuration structures I and J in Figure 5, taken from [25, Example 3.16], whose corresponding processes are:

$$((a_1 \,.\, b_1) \parallel_{\emptyset} (a_2 \,.\, b_2) \parallel_{\emptyset} (a_3 \,.\, b_3) \parallel_{\emptyset} a_4) \parallel_{\{a_1,a_2,a_3,a_4,b_2,b_3\}}$$
$$((b_2 + b_3) \parallel_{\{b_2\}} (a_1 + b_2) \parallel_{\{a_1\}} (a_1 + a_3) \parallel_{\{a_1\}} (a_1 + a_4) \parallel_{\{a_4\}} (a_2 + a_4))$$

and:

$$((a_1 \,.\, b_1) \parallel_{\emptyset} (a_2 \,.\, b_2) \parallel_{\emptyset} (a_3 \,.\, b_3) \parallel_{\emptyset} a_4) \parallel_{\{a_1,a_2,a_3,a_4,b_2,b_3\}}$$
$$((a_1 + b_2) \parallel_{\{a_1\}} (a_1 + a_3) \parallel_{\{a_1\}} (a_1 + a_4) \parallel_{\{a_4\}} (a_2 + a_4) \parallel_{\{a_2\}} (a_2 + b_3))$$

I and J can be distinguished only by the bisimilarities of the incr-iso family because:

- If 1_I performs a_3 thus evolving to 4_I, then 1_J can respond by performing a_3 (or a_1, a_2, or a_4) thus evolving to 4_J. The initial bijection is $\{(a_3, a_3)\}$.
- If 4_J performs a_2 thus evolving to 9_J, then 4_I can respond by performing a_2 thus evolving to 9_I. The bijection becomes $\{(a_3, a_3), (a_2, a_2)\}$.
- If 9_I performs b_3 thus evolving to 13_I, then 9_J can only respond by performing b_2 thus evolving to 12_J. The bijection becomes $\{(a_3, a_3), (a_2, a_2), (b_3, b_2)\}$ but causality is not preserved because a_3 causes b_3 in I whereas a_3 does not cause b_2 in J.

On the other hand, I and J are identified by $\sim_{\mathrm{PB}}$ and $\sim_{\mathrm{WHPPB}}$ and their backward ready multisets variants because 1_I and 1_J can both perform a single a-transition followed by either $a \,;\, b$ or $a \parallel b$. However, we can distinguish them with our diamond-based condition because if 1_J performs a_3 it evolves to 4_J, which opens a diamond and a half diamond, while all the configurations a-reachable from 1_I only open diamonds. ⌟

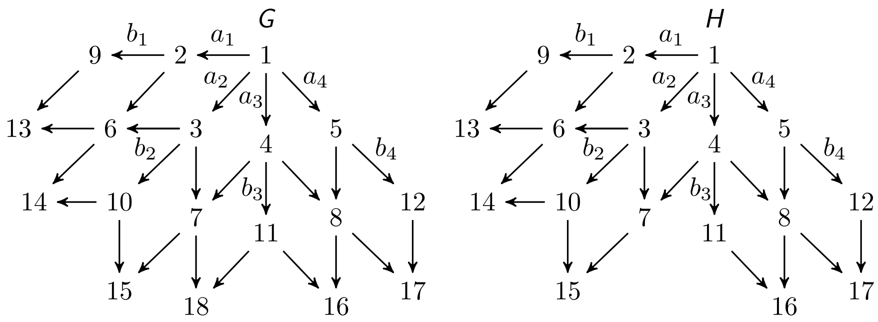

Figure 4 G and H are not $\sim_{\mathrm{HPB/HHPB}}$-equivalent but are $\sim_{\mathrm{WHPB/WHHPB}}$-equivalent.

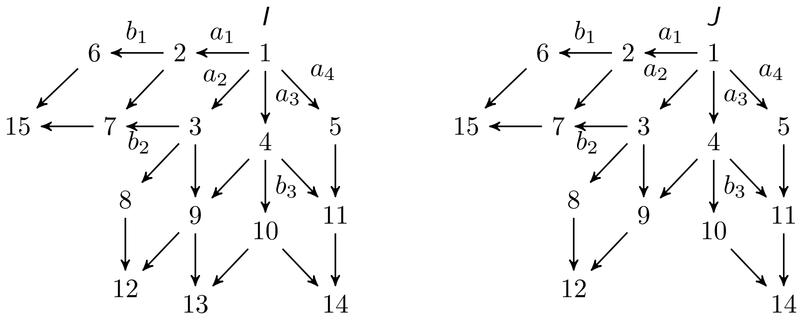

Figure 5 I and J are not $\sim_{\mathrm{HPB/HHPB}}$-equivalent but are $\sim_{\mathrm{PB/WHPPB}}$-equivalent.

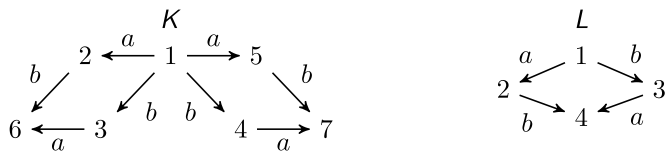

Figure 6 K and L are identified by all the considered bisimilarities.

▶ **Example 32.** We must make sure that our diamond-based condition is not too discriminating. As an example, consider the configuration structures K and L in Figure 6, whose corresponding processes are $(a \parallel_{\emptyset} b) + (a \parallel_{\emptyset} b)$ and $(a \parallel_{\emptyset} b)$. The configurations 1_K and 1_L both open diamonds, but the former also present the triples $(1_K, 2_K, 5_K)$, $(1_K, 2_K, 4_K)$, $(1_K, 3_K, 5_K)$, $(1_K, 3_K, 4_K)$ that resemble half diamonds. However, they do not comply with Definition 27 because all the configurations $2_K, 3_K, 4_K$, 5_K appear in diamonds of 1_K. ⌟

▶ **Theorem 33.** *Let $C_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, be two stable configuration structures. Then $C_1 \sim_{\mathrm{IB:brm:diam}} C_2$ iff $C_1 \sim_{\mathrm{HPB}} C_2$.* ⌟

▶ **Theorem 34.** *Let $C_i = (\mathcal{E}_i, \mathcal{C}_i, \ell_i)$, $i \in \{1, 2\}$, be two stable configuration structures. Then $C_1 \sim_{\mathrm{FRB:brm:diam}} C_2$ iff $C_1 \sim_{\mathrm{HHPB}} C_2$.* ⌟

6 Conclusions

In this paper we have revisited the spectrum of bisimilarities over stable configuration structures by dividing it into the no-iso, var-iso, and incr-iso families. We have shown that an additional backward ready multiset equality check – which is simpler to work with than isomorphisms – causes the discriminating power of no-iso bisimilarities to coincide with that of var-iso bisimilarities. Both families have been logically characterized via $\mathcal{L}_{\rm brm,pom}$.

In the presence of autoconcurrency, backward ready multisets may not enough to obtain the discriminating power of incr-iso bisimilarities too. More precisely, recalling that $\sim_{\rm IB:brm}$ and $\sim_{\rm FRB:brm}$ are capable of distinguishing, e.g., between $a \,\|_{\emptyset}\, a$ and $a \,.\, a + a \,.\, a$ like $\sim_{\rm HPB}$ and $\sim_{\rm HHPB}$, the examples in Figures 3–5 clearly indicate that backward ready multisets are not enough in the simultaneous presence of autoconcurrency and non-local conflicts.

We have proven that, in order to reach the discriminating power of incr-iso bisimilarities also in the aforementioned case, it is necessary to examine diamonds and half diamonds. This yields an alternative characterization of ($\sim_{\rm HPB}$ and) $\sim_{\rm HHPB}$ that, unlike the one in [1], uses binary relations instead of ternary ones and does not rely on event identification. The structural properties of diamonds and half diamonds clarify the boundary between the var-iso and incr-iso families, i.e., between weak and strong history preservations, thus potentially simplifying equivalence checking in a true concurrency setting.

As for future work, similar to [2, 26] we would like to find an extension of $\mathcal{L}_{\rm brm,pom}$ whose fragments characterize the entire spectrum in Figure 2, i.e., also incr-iso bisimilarities.

References

1. C. Aubert and I. Cristescu. How reversibility can solve traditional questions: The example of hereditary history-preserving bisimulation. In *Proc. of the 31st Int. Conf. on Concurrency Theory (CONCUR 2020)*, volume 171 of *LIPIcs*, pages 7:1–7:23, 2020.
2. P. Baldan and S. Crafa. A logic for true concurrency. *Journal of the ACM*, 61:24:1–24:36, 2014.
3. M.A. Bednarczyk. Hereditary history preserving bisimulations or what is the power of the future perfect in program logics. Technical report, Polish Academy of Sciences, Gdansk, 1991.
4. M. Bernardo, A. Esposito, and C.A. Mezzina. Alternative characterizations of hereditary history-preserving bisimilarity via backward ready multisets. In *Proc. of the 28th Int. Conf. on Foundations of Software Science and Computation Structures (FOSSACS 2025)*, volume 15691 of *LNCS*, pages 67–87. Springer, 2025.
5. M. Bernardo and S. Rossi. Reverse bisimilarity vs. forward bisimilarity. In *Proc. of the 26th Int. Conf. on Foundations of Software Science and Computation Structures (FOSSACS 2023)*, volume 13992 of *LNCS*, pages 265–284. Springer, 2023.
6. G. Boudol and I. Castellani. Concurrency and atomicity. *Theoretical Computer Science*, 59:25–84, 1988.
7. S.D. Brookes, C.A.R. Hoare, and A.W. Roscoe. A theory of communicating sequential processes. *Journal of the ACM*, 31:560–599, 1984.
8. Ph. Darondeau and P. Degano. Causal trees. In *Proc. of the 16th Int. Coll. on Automata, Languages and Programming (ICALP 1989)*, volume 372 of *LNCS*, pages 234–248. Springer, 1989.
9. Ph. Darondeau and P. Degano. Causal trees: Interleaving + causality. In *Proc. of the LITP Spring School on Theoretical Computer Science: Semantics of Systems of Concurrent Processes*, volume 469 of *LNCS*, pages 239–255. Springer, 1990.
10. R. De Nicola and F. Vaandrager. Three logics for branching bisimulation. *Journal of the ACM*, 42:458–487, 1995.
11. P. Degano, R. De Nicola, and U. Montanari. Observational equivalences for concurrency models. In *Proc. of the 3rd IFIP Working Conf. on Formal Description of Programming Concepts*, pages 105–134. North-Holland, 1986.
12. H. Fecher. A completed hierarchy of true concurrent equivalences. *Information Processing Letters*, 89:261–265, 2004.

13 J.L. Gischer. The equational theory of pomsets. *Theoretical Computer Science*, 61:199–224, 1988.
14 R.J. van Glabbeek and U. Goltz. Refinement of actions and equivalence notions for concurrent systems. *Acta Informatica*, 37:229–327, 2001.
15 R.J. van Glabbeek and G.D. Plotkin. Configuration structures, event structures and Petri nets. *Theoretical Computer Science*, 410:4111–4159, 2009.
16 R.J. van Glabbeek and F. Vaandrager. Bundle event structures and CCSP. In *Proc. of the 14th Int. Conf. on Concurrency Theory (CONCUR 2003)*, volume 2761 of *LNCS*, pages 57–71. Springer, 2003.
17 J. Grabowski. On partial languages. *Fundamenta Informaticae*, 4:427–498, 1981.
18 M. Hennessy and R. Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM*, 32:137–162, 1985.
19 A. Joyal, M. Nielsen, and G. Winskel. Bisimulation from open maps. *Information and Computation*, 127:164–185, 1996.
20 R.M. Keller. Formal verification of parallel programs. *Communications of the ACM*, 19:371–384, 1976.
21 R. Milner. *Communication and Concurrency*. Prentice Hall, 1989.
22 D. Park. Concurrency and automata on infinite sequences. In *Proc. of the 5th GI Conf. on Theoretical Computer Science (GI-TCS 1981)*, volume 104 of *LNCS*, pages 167–183. Springer, 1981.
23 I. Phillips and I. Ulidowski. Reversing algebraic process calculi. *Journal of Logic and Algebraic Programming*, 73:70–96, 2007.
24 I. Phillips and I. Ulidowski. Reversibility and models for concurrency. In *Proc. of the 4th Int. Workshop on Structural Operational Semantics (SOS 2007)*, volume 192(1) of *ENTCS*, pages 93–108. Elsevier, 2007.
25 I. Phillips and I. Ulidowski. A hierarchy of reverse bisimulations on stable configuration structures. *Mathematical Structures in Computer Science*, 22:333–372, 2012.
26 I. Phillips and I. Ulidowski. Event identifier logic. *Mathematical Structures in Computer Science*, 24(2):1–51, 2014.
27 L. Pomello. Some equivalence notions for concurrent systems. an overview. In *Advances in Petri Nets*, volume 222 of *LNCS*, pages 381–400. Springer, 1985.
28 A.M. Rabinovich and B.A. Trakhtenbrot. Behavior structures and nets. *Fundamenta Informaticae*, 11:357–404, 1988.
29 J. Valdes, R.E. Tarjan, and E.L. Lawler. The recognition of series parallel digraphs. *SIAM Journal on Computing*, 11:298–313, 1982.
30 G. Winskel. Event structures. In *Advances in Petri Nets*, volume 255 of *LNCS*, pages 325–392. Springer, 1986.

A Proofs of Results

Proof of Theorem 16. We show that any backward-ready-multiset interleaving bisimulation $\mathcal{B}$ witnessing $\mathcal{C}_1 \sim_{\mathrm{IB:brm}} \mathcal{C}_2$ is a step bisimulation too. Observing that $(\emptyset, \emptyset) \in \mathcal{B}$, let us consider $(X_1, X_2) \in \mathcal{B}$:

- If $X_1 \xrightarrow{A}_{\mathcal{C}_1} X'_1$ then there is a sequence of transitions $X_1 \xrightarrow{a_1}_{\mathcal{C}_1} \dots \xrightarrow{a_n}_{\mathcal{C}_1} X'_1$ where $a_1, \dots, a_n$ are the actions occurring in multiset A counted with their multiplicities. From $(X_1, X_2) \in \mathcal{B}$ it follows that there exists a sequence of transitions $X_2 \xrightarrow{a_1}_{\mathcal{C}_2} \dots \xrightarrow{a_n}_{\mathcal{C}_2} X'_2$ such that $(X'_1, X'_2) \in \mathcal{B}$ and $\mathit{brm}(X'_1) = \mathit{brm}(X'_2)$. Since A contains actions labeling events that are all concurrent in X'_1, it holds that $A \subseteq \mathit{brm}(X'_1)$. From $\mathit{brm}(X'_1) = \mathit{brm}(X'_2)$ we derive that A contains actions labeling events that are all concurrent in X'_2, which are all enabled by X_2. Therefore, there exists a step transition $X_2 \xrightarrow{A}_{\mathcal{C}_2} X'_2$ such that $(X'_1, X'_2) \in \mathcal{B}$.
- If $X_2 \xrightarrow{A}_{\mathcal{C}_2} X'_2$ then we reason in a similar way. ∎

Proof of Theorem 17. The proof is divided into two parts:

- Suppose that $\mathcal{C}_1 \sim_{\mathrm{IB:brm}} \mathcal{C}_2$. Then from Theorem 16 and $\mathit{brm}(X_1) = \mathit{brm}(X_2)$ for every pair of related configurations, it follows that $\mathcal{C}_1 \sim_{\mathrm{SB:brm}} \mathcal{C}_2$.
- We prove that any backward-ready-multiset step bisimulation $\mathcal{B}$ witnessing $\mathcal{C}_1 \sim_{\mathrm{SB:brm}} \mathcal{C}_2$ is a backward-ready-multiset interleaving bisimulation too. Observing that $(\emptyset, \emptyset) \in \mathcal{B}$, let us consider $(X_1, X_2) \in \mathcal{B}$:
  - If $X_1 \xrightarrow{a}_{\mathcal{C}_1} X'_1$ then there exists $X_2 \xrightarrow{a}_{\mathcal{C}_2} X'_2$ such that $(X'_1, X'_2) \in \mathcal{B}$ because a single-action transition is a special case of step transition.
  - If $X_2 \xrightarrow{a}_{\mathcal{C}_2} X'_2$ then we reason in a similar way.
  - $\mathit{brm}(X_1) = \mathit{brm}(X_2)$ by definition of $\mathcal{B}$. ∎

Proof of Lemma 18. The proof is divided into two parts:

- Suppose that $|\mathit{brs}_{\mathrm{e}}(X)| < |\mathit{brs}_{\mathrm{e}}(X')|$. We proceed by contradiction and assume that $\mathit{brs}_{\mathrm{e}}(X) \not\subseteq \mathit{brs}_{\mathrm{e}}(X')$. Since $X \xrightarrow{\ell(e)}_{\mathcal{C}} X'$, it cannot be $X \cap X' = \emptyset$ and hence there must exist $e' \in X$ such that $e' \in \mathit{brs}_{\mathrm{e}}(X)$ but $e' \notin \mathit{brs}_{\mathrm{e}}(X')$; in this case there must also exist $e'' \in X'$ such that $e'' \notin \mathit{brs}_{\mathrm{e}}(X)$ but $e'' \in \mathit{brs}_{\mathrm{e}}(X')$ as $|\mathit{brs}_{\mathrm{e}}(X)| < |\mathit{brs}_{\mathrm{e}}(X')|$. We assume e, e', e'' to be all distinct in order for $|\mathit{brs}_{\mathrm{e}}(X)| < |\mathit{brs}_{\mathrm{e}}(X')|$ to hold. Since $e, e'' \in \mathit{brs}_{\mathrm{e}}(X')$, these events are concurrent in X'. Therefore, there exist $\bar{X} = X' \setminus \{e\} = X$, $X'' = X' \setminus \{e''\}$, and such that $e'' \in \mathit{brs}_{\mathrm{e}}(\bar{X}) = \mathit{brs}_{\mathrm{e}}(X)$ and $e' \in \mathit{brs}_{\mathrm{e}}(X'')$. This leads to a contradiction because e'' cannot be in $\mathit{brs}_{\mathrm{e}}(X)$.
- Suppose that $\mathit{brs}_{\mathrm{e}}(X) \subsetneq \mathit{brs}_{\mathrm{e}}(X')$. Then obviously $|\mathit{brs}_{\mathrm{e}}(X)| < |\mathit{brs}_{\mathrm{e}}(X')|$. ∎

Proof of Lemma 19. From Lemma 18 and the fact that $\mathcal{B}$ is a brm-bisimulation it follows that: $\mathit{brs}_{\mathrm{e}}(X_1) \subsetneq \mathit{brs}_{\mathrm{e}}(X'_1)$ iff $|\mathit{brs}_{\mathrm{e}}(X_1)| < |\mathit{brs}_{\mathrm{e}}(X'_1)|$ iff $|\mathit{brm}(X_1)| < |\mathit{brm}(X'_1)|$ iff $|\mathit{brm}(X_2)| < |\mathit{brm}(X'_2)|$ iff $|\mathit{brs}_{\mathrm{e}}(X_2)| < |\mathit{brs}_{\mathrm{e}}(X'_2)|$ iff $\mathit{brs}_{\mathrm{e}}(X_2) \subsetneq \mathit{brs}_{\mathrm{e}}(X'_2)$. ∎

Proof of Lemma 20. From $\mathit{brs}_{\mathrm{e}}(X) \not\subseteq \mathit{brs}_{\mathrm{e}}(X')$ it follows that there exist $e_1, \dots, e_n \in \mathit{brs}_{\mathrm{e}}(X)$ with $n > 0$ such that $e_1, \dots, e_n \notin \mathit{brs}_{\mathrm{e}}(X')$ and hence $e_1, \dots, e_n \in \mathit{causes}_{X'}(e)$. We need to prove that $e \in Y$ implies $e_i \in Y$ with $1 \le i \le n$ for all $Y \in \mathcal{C}$ such that $Y \subseteq X'$. The first, and only, configuration among the considered ones in which the event e appears is X'. From $\mathit{causes}_{X'}(e) \subseteq \mathit{brs}_{\mathrm{e}}(X) \subseteq X \subseteq X'$ it follows that $e_i \in X'$ and hence $e_i <_{X'} e$. ∎

Proof of Theorem 21. Suppose that $\mathcal{C}_1 \sim_{B:\mathrm{brm}} \mathcal{C}_2$ due to some brm-bisimulation $\mathcal{B}$. Then, given $(X_1, X_2) \in \mathcal{B}$, the existence in $\mathcal{C}_1$ of a sequence of transitions $\emptyset \xrightarrow{\ell_1(e_{1,n})}_{\mathcal{C}_1} X_{1,1} \dots X_{1,n-1} \xrightarrow{\ell_1(e_{1,n})}_{\mathcal{C}_1} X_{1,n} = X_1$ implies the existence in $\mathcal{C}_2$ of a sequence of transitions $\emptyset \xrightarrow{\ell_2(e_{2,1})}_{\mathcal{C}_2}$

$X_{2,1} \dots X_{2,n-1} \xrightarrow{\ell_2(e_{2,n})}_{C_2} X_{2,n} = X_2$ such that $\ell_1(e_{1,h}) = \ell_2(e_{2,h})$ and $(X_{1,h}, X_{2,h}) \in \mathcal{B}$ for all $h = 1, \dots, n$, and vice versa. Note that $e_{1,h} \neq e_{1,k}$ and $e_{2,h} \neq e_{2,k}$ for all $h \neq k$ because in each transition the source configuration and the target configuration differ by one event, which is the executed event (see Definition 4).

We now show that it is possible to construct a labeling- and causality-preserving bijection $f : X_1 \to X_2$. We proceed by induction on the number n of transitions between $\emptyset$ and X_1:

- If $n = 1$ then we have $\emptyset \xrightarrow{\ell(e_{1,1})}_{C_1} X_{1,1}$ and $\emptyset \xrightarrow{\ell(e_{2,1})}_{C_2} X_{2,1}$ with $\ell(e_{1,1}) = \ell(e_{2,1})$. We can take $f = \{(e_{1,1}, e_{2,1})\}$, which trivially preserves labeling and causality.
- Let $n > 1$ then consider $X_{1,n-1} \xrightarrow{\ell(e_{1,n})}_{C_1} X_{1,n}$ and $X_{2,n-1} \xrightarrow{\ell(e_{2,n})}_{C_2} X_{2,n}$. By the induction hypothesis there exists a labeling- and causality-preserving isomorphism $f : X_{1,n-1} \to X_{2,n-1}$. Moreover, $\mathit{brm}(X_{1,k}) = \mathit{brm}(X_{2,k})$ for $k \in \{n-1, n\}$. There are two cases:
  - If $\mathit{brs}_{\mathrm{e}}(X_{1,n-1}) \subseteq \mathit{brs}_{\mathrm{e}}(X_{1,n})$ then $e_{1,n}$ is concurrent with the events in $\mathit{brs}_{\mathrm{e}}(X_{1,n-1})$. From Lemma 19 it follows that $\mathit{brs}_{\mathrm{e}}(X_{2,n-1}) \subseteq \mathit{brs}_{\mathrm{e}}(X_{2,n})$ and hence $e_{2,n}$ is concurrent with the events in $\mathit{brs}_{\mathrm{e}}(X_{2,n-1})$ too. Since $e_{1,n}$ and $e_{2,n}$ do not have any known causes, we can simply update the bijection by letting $f' = f \cup \{(e_{1,n}, e_{2,n})\}$, which preserves labeling and causality.
  - If $\mathit{brs}_{\mathrm{e}}(X_{1,n-1}) \not\subseteq \mathit{brs}_{\mathrm{e}}(X_{1,n})$ then some of the events in $\mathit{brs}_{\mathrm{e}}(X_{1,n-1})$ have caused $e_{1,n}$. From Lemma 20 it follows that $\bar{e}_1 <_{X_{1,n}} e_{1,n}$ for all $\bar{e}_1 \in \mathit{causes}_{X_{1,n}}(e_{1,n})$. Notice that $\mathit{causes}_{X_{1,n}}(e_{1,n})$ is a subset of the domain of f. From Lemma 19 we have that $\mathit{brs}_{\mathrm{e}}(X_{2,n-1}) \not\subseteq \mathit{brs}_{\mathrm{e}}(X_{2,n})$ and hence $\bar{e}_2 <_{X_{2,n}} e_{2,n}$ for all $\bar{e}_2 \in \mathit{causes}_{X_{2,n}}(e_{2,n})$. Notice that $\mathit{causes}_{X_{2,n}}(e_{2,n})$ is a subset of the codomain of f. Notice also that $|\mathit{causes}_{X_{1,n}}(e_{1,n})| = |\mathit{causes}_{X_{2,n}}(e_{2,n})|$. If each element of $\mathit{causes}_{X_{1,n}}(e_{1,n})$ is mapped by f to an element of $\mathit{causes}_{X_{2,n}}(e_{2,n})$, then we are done because f preserves labeling and causality and we can simply let $f' = f \cup \{(e_{1,n}, e_{2,n})\}$, otherwise we can construct f' by removing from f all the pairs such that $\bar{e}_1 <_{X_{1,n}} e_{1,n}$ but $f(\bar{e}_1) \not<_{X_{2,n}} f(e_{1,n})$ and then update the resulting function with pairs that respect labeling and causality. We can do this bijectively because $|\mathit{causes}_{X_{1,n}}(e_{1,n})| = |\mathit{causes}_{X_{2,n}}(e_{2,n})|$ and $(X_{1,n}, X_{2,n}) \in \mathcal{B}$. ∎

Proof of Corollary 22. A straightforward consequence of Theorem 21.

Proof of Theorem 23. The proof is divided into two parts:

- Suppose that $\mathcal{C}_1 \sim_B \mathcal{C}_2$ with $\mathcal{B}$ being a B-bisimulation. To prove that $\mathcal{C}_1 \sim_{B:\mathrm{brm}} \mathcal{C}_2$ it is sufficient to show that whenever $(X_1, X_2) \in \mathcal{B}$ for $X_1 \in \mathcal{C}_1$ and $X_2 \in \mathcal{C}_2$, then $\mathit{brm}(X_1) = \mathit{brm}(X_2)$. Let us assume, towards a contradiction, that $\mathit{brm}(X_1) \neq \mathit{brm}(X_2)$. Suppose that X_1 has fewer incoming a-transitions than X_2. Without loss of generality, we can assume that X_1 has one incoming a-transition while X_2 has two. Then in $\mathcal{C}_2$ there is a diamond closing into X_2, i.e., there exist three configurations Y_2, X_2', and X_2'' and two a-labeled events e_2' and e_2'' such that $Y_2 \xrightarrow{\ell_2(e_2')}_{C_2} X_2'$, $Y_2 \xrightarrow{\ell_2(e_2'')}_{C_2} X_2''$, $X_2' \xrightarrow{\ell_2(e_2'')}_{C_2} X_2$, and $X_2'' \xrightarrow{\ell_2(e_2')}_{C_2} X_2$, with e_2' and e_2'' concurrent in X_2.
  Due to $(X_1, X_2) \in \mathcal{B}$, in $\mathcal{C}_1$ there must be two configurations Y_1 and X_1' and two a-labeled events e_1' and e_1'' such that $Y_1 \xrightarrow{\ell_1(e_1')}_{C_1} X_1' \xrightarrow{\ell_1(e_1'')}_{C_1} X_1$, with $e_1' \leq_{X_1} e_1''$ because X_1 has a single incoming a-transition. Since $\mathcal{B}$ is a B-bisimulation, f should relate $e_1', e_1'' \in X_1$ with $e_2', e_2'' \in X_2$ in a causality-preserving way, but this is not possible because $f(e_1') \not\leq_{X_2} f(e_1'')$ where $f(e_1') \in \{e_2', e_2''\}$ and $f(e_1'') \in \{e_2', e_2''\} \setminus \{f(e_1')\}$.
- If $\mathcal{C}_1 \sim_{B:\mathrm{brm}} \mathcal{C}_2$ then it trivially holds that $\mathcal{C}_1 \sim_B \mathcal{C}_2$. ∎

Proof of Theorem 24. Since the proof proceeds by induction on the depth of a formula of $\mathcal{L}_{\text{brm,pom}}$, we start with the inductive definition of *depth* intended as an upper bound to the depth of the syntax tree of the considered formula:

$$\begin{array}{rcl} depth(\mathsf{true}) & = & 0 \\ depth(M) & = & 0 \\ depth(\neg\phi') & = & 1 + depth(\phi') \\ depth(\phi_1 \wedge \phi_2) & = & 1 + \max(depth(\phi_1), depth(\phi_2)) \\ depth(\langle\alpha\rangle\phi') & = & 1 + depth(\phi') \\ depth(\langle a^\dagger\rangle\phi') & = & 1 + depth(\phi') \end{array}$$

We consider each of the eight bisimilarities in turn:

- Let $B = \text{IB}$. The proof is divided into two parts:
  - Assuming that $\mathsf{C}_1 \sim_{\text{IB}} \mathsf{C}_2$, $X_i \in \mathcal{C}_i$ for $i \in \{1, 2\}$ are related by an interleaving bisimulation $\mathcal{B}$, and $X_1 \models \phi$ for an arbitrary formula $\phi \in \mathcal{L}_{\text{IB}}$, we prove that $X_2 \models \phi$ too by proceeding by induction on $k = depth(\phi)$:
    - If $k = 0$ then ϕ must be true, which is trivially satisfied by X_2 too.
    - If $k \geq 1$ there are three cases:
      - If ϕ is $\neg\phi'$, then from $X_1 \models \neg\phi'$ we derive that $X_1 \not\models \phi'$. If it were $X_2 \models \phi'$ then by the induction hypothesis it would hold that $X_1 \models \phi'$, which is not the case. Therefore $X_2 \not\models \phi'$ and hence $X_2 \models \neg\phi'$ too.
      - If ϕ is $\phi_1 \wedge \phi_2$, then from $X_1 \models \phi_1 \wedge \phi_2$ we derive that $X_1 \models \phi_1$ and $X_1 \models \phi_2$. From the induction hypotesis it follows that $X_2 \models \phi_1$ and $X_2 \models \phi_2$ and hence $X_2 \models \phi_1 \wedge \phi_2$ too.
      - If ϕ is $\langle a\rangle\phi'$ then from $X_1 \models \langle a\rangle\phi'$ we derive that there exists $X_1 \xrightarrow{a}_{\mathsf{C}_1} X_1'$ such that $X_1' \models \phi'$. From $(X_1, X_2) \in \mathcal{B}$ we then derive that there exists $X_2 \xrightarrow{a}_{\mathsf{C}_2} X_2'$ such that $(X_1', X_2') \in \mathcal{B}$. By applying the induction hypothesis on ϕ' we derive that $X_2' \models \phi'$ and hence $X_2 \models \langle a\rangle\phi'$ too.
  - Assuming that C_1 and C_2 satisfy the same formulas in $\mathcal{L}_{\text{IB}}$, we prove that the symmetric relation $\mathcal{B} = \{(X_1, X_2) \in \mathcal{C}_1 \times \mathcal{C}_2 \mid X_1 \text{ and } X_2 \text{ satisfy the same formulas in } \mathcal{L}_{\text{IB}}\}$ is an interleaving bisimulation.
    Given $(X_1, X_2) \in \mathcal{B}$ such that $X_1 \xrightarrow{a}_{\mathsf{C}_1} X_1'$, suppose by contradiction that there is no X_2' satisfying the same formulas as X_1' such that $X_2 \xrightarrow{a}_{\mathsf{C}_2} X_2'$, i.e., $(X_1', X_2') \in \mathcal{B}$ for no X_2' a-reachable from X_2. Since X_2 has finitely many outgoing transitions, the set of configurations that X_2 can reach by performing an a-transition is finite, say $\{X_{2_1}', \dots, X_{2_n}'\}$ with $n \geq 0$. Since none of the configurations in the set satisfies the same formulas as X_1', for each $1 \leq i \leq n$ there exists a formula $\phi_i \in \mathcal{L}_{\text{IB}}$ such that $X_1' \models \phi_i$ but $X_{2_i}' \not\models \phi_i$.
    We can then construct the formula $\langle a\rangle \bigwedge_{i=1}^{n} \phi_i$ which is satisfied by X_1 but not by X_2; if $n = 0$ then it is sufficient to take $\langle a\rangle\mathsf{true}$. This formula violates $(X_1, X_2) \in \mathcal{B}$, hence there must exist at least one X_2' such that $X_2 \xrightarrow{a}_{\mathsf{C}_2} X_2'$ so that $(X_1', X_2') \in \mathcal{B}$.
- Let $B = \text{SB}$. The proof is divided into two parts:
  - Assuming that $\mathsf{C}_1 \sim_{\text{SB}} \mathsf{C}_2$, $X_i \in \mathcal{C}_i$ for $i \in \{1, 2\}$ are related by a step bisimulation $\mathcal{B}$, and $X_1 \models \phi$ for an arbitrary formula $\phi \in \mathcal{L}_{\text{SB}}$, we prove that $X_2 \models \phi$ too by proceeding by induction on $k = depth(\phi)$:
    - If $k = 0$ then we proceed like in the case $B = \text{IB}$.
    - If $k \geq 1$ there are three cases:
      - If ϕ is $\neg\phi'$ then we proceed like in the case $B = \text{IB}$.
      - If ϕ is $\phi_1 \wedge \phi_2$ then we proceed like in the case $B = \text{IB}$.

- If ϕ is $\langle \alpha_{\rm S} \rangle \phi'$ then from $X_1 \models \langle \alpha_{\rm S} \rangle \phi'$ we derive that there exists $X_1 \xrightarrow{U_{\alpha_{\rm S}}}_{\mathcal{C}_1} X'_1$ such that $X'_1 \models \phi'$. From $(X_1, X_2) \in \mathcal{B}$ we then derive that there exists $X_2 \xrightarrow{U_{\alpha_{\rm S}}}_{\mathcal{C}_2} X'_2$ such that $(X'_1, X'_2) \in \mathcal{B}$. By applying the induction hypothesis on ϕ' we derive that $X'_2 \models \phi'$ and hence $X_2 \models \langle \alpha_{\rm S} \rangle \phi'$ too.

- Assuming that $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy the same formulas in $\mathcal{L}_{\rm SB}$, we prove that the symmetric relation $\mathcal{B} = \{(X_1, X_2) \in \mathcal{C}_1 \times \mathcal{C}_2 \mid X_1$ and X_2 satisfy the same formulas in $\mathcal{L}_{\rm SB}\}$ is a step bisimulation.

  Given $(X_1, X_2) \in \mathcal{B}$ such that $X_1 \xrightarrow{A}_{\mathcal{C}_1} X'_1$, suppose by contradiction that there is no X'_2 satisfying the same formulas as X'_1 such that $X_2 \xrightarrow{A}_{\mathcal{C}_2} X'_2$, i.e., $(X'_1, X'_2) \in \mathcal{B}$ for no X'_2 A-reachable from X_2. Since X_2 has finitely many outgoing transitions, the set of configurations that X_2 can reach by performing an A-transition is finite, say $\{X'_{2_1}, \dots, X'_{2_n}\}$ with $n \geq 0$. Since none of the configurations in the set satisfies the same formulas as X'_1, for each $1 \leq i \leq n$ there exists a formula $\phi_i \in \mathcal{L}_{\rm SB}$ such that $X'_1 \models \phi_i$ but $X'_{2_i} \not\models \phi_i$.

  We can then construct the formula $\langle \alpha_{\rm S} \rangle \bigwedge\limits_{i=1}^{n} \phi_i$ such that $U_{\alpha_{\rm S}} = A$ which is satisfied by X_1 but not by X_2; if $n = 0$ then it is sufficient to take $\langle \alpha_{\rm S} \rangle \mathsf{true}$. Notice that the pomset term $\alpha_{\rm S}$ is complete with respect to all pomsets where the causality relation is empty. This formula violates $(X_1, X_2) \in \mathcal{B}$, hence there must exists at least one X'_2 such that $X_2 \xrightarrow{A}_{\mathcal{C}_2} X'_2$ so that $(X'_1, X'_2) \in \mathcal{B}$.

- Let $B = \rm PB$. The proof is divided into two parts:
  - Assuming that $\mathcal{C}_1 \sim_{\rm PB} \mathcal{C}_2$, $X_i \in \mathcal{C}_i$ for $i \in \{1, 2\}$ are related by a pomset bisimulation $\mathcal{B}$, and $X_1 \models \phi$ for an arbitrary formula $\phi \in \mathcal{L}_{\rm PB}$, we prove that $X_2 \models \phi$ too by proceeding by induction on $k = \mathit{depth}(\phi)$:
    - If $k = 0$ then we proceed like in the case $B = \rm IB$.
    - If $k \geq 1$ there are three cases:
      - If ϕ is $\neg \phi'$ then we proceed like in the case $B = \rm IB$.
      - If ϕ is $\phi_1 \wedge \phi_2$ then we proceed like in the case $B = \rm IB$.
      - If ϕ is $\langle \alpha \rangle \phi'$ then from $X_1 \models \langle \alpha \rangle \phi'$ we derive that there exists $X_1 \xrightarrow{U_\alpha} X'_1$ such that $X'_1 \models \phi'$. From $(X_1, X_2) \in \mathcal{B}$ we then derive that there exists $X_2 \xrightarrow{U_\alpha} X'_2$ such that $(X'_1, X'_2) \in \mathcal{B}$. By applying the induction hypothesis on ϕ' we derive that $X'_2 \models \phi'$ and hence $X_2 \models \langle \alpha \rangle \phi'$ too.
  - Assuming that $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy the same formulas in $\mathcal{L}_{\rm PB}$, we prove that the symmetric relation $\mathcal{B} = \{(X_1, X_2) \in \mathcal{C}_1 \times \mathcal{C}_2 \mid X_1$ and X_2 satisfy the same formulas in $\mathcal{L}_{\rm PB}\}$ is a pomset bisimulation.

    Given $(X_1, X_2) \in \mathcal{B}$ such that $X_1 \xrightarrow{U}_{\mathcal{C}_1} X'_1$, suppose by contradiction that there is no X'_2 satisfying the same formulas as X'_1 such that $X_2 \xrightarrow{U}_{\mathcal{C}_2} X'_2$, i.e., $(X'_1, X'_2) \in \mathcal{B}$ for no X'_2 U-reachable from X_2. Since X_2 has finitely many outgoing transitions, the set of configurations that X_2 can reach by performing a U-transition is finite, say $\{X'_{2_1}, \dots, X'_{2_n}\}$ with $n \geq 0$. Since none of the configurations in the set satisfies the same formulas as X'_1, for each $1 \leq i \leq n$ there exists a formula $\phi_i \in \mathcal{L}_{\rm SB}$ such that $X'_1 \models \phi_i$ but $X'_{2_i} \not\models \phi_i$.

    We can then construct the formula $\langle \alpha \rangle \bigwedge\limits_{i=1}^{n} \phi_i$ such that $U_\alpha = U$ which is satisfied by X_1 but not by X_2; if $n = 0$ then it is sufficient to take $\langle \alpha \rangle \mathsf{true}$. Notice that the pomset term α is complete with respect to the series-parallel pomsets considered by pomset bisimilarity [6]. This formula violates $(X_1, X_2) \in \mathcal{B}$, hence there must exists at least one X'_2 such that $X_2 \xrightarrow{A} X'_2$ so that $(X'_1, X'_2) \in \mathcal{B}$.

- For $B = \text{FRB}$ the result was already proven in [4].
- Let $B = \text{IB:brm}$. The proof is divided into two parts:
  - Assuming that $\mathcal{C}_1 \sim_{\text{IB:brm}} \mathcal{C}_2$, $X_i \in \mathcal{C}_i$ for $i \in \{1, 2\}$ are related by a backward ready multiset interleaving bisimulation $\mathcal{B}$, and $X_1 \models \phi$ for an arbitrary formula $\phi \in \mathcal{L}_{\text{IB:brm}}$, we prove that $X_2 \models \phi$ too by proceeding by induction on $k = \mathit{depth}(\phi)$:
    * If $k = 0$ then ϕ is either true or M. In the former case it is trivially satisfied by X_2 too. In the latter case it is satisfied by X_2 too because $(X_1, X_2) \in \mathcal{B}$ with $\mathcal{B}$ being a backward ready multiset interleaving bisimulation and hence $\mathit{brm}(X_1) = \mathit{brm}(X_2)$.
    * If $k > 0$ then we proceed like in the case $B = \text{IB}$.
  - Assuming that $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy the same formulas in $\mathcal{L}_{\text{IB:brm}}$, we prove that the symmetric relation $\mathcal{B} = \{(X_1, X_2) \in \mathcal{C}_1 \times \mathcal{C}_2 \mid X_1 \text{ and } X_2 \text{ satisfy the same formulas in } \mathcal{L}_{\text{IB:brm}}\}$ is a backward ready multiset interleaving bisimulation. Given $(X_1, X_2) \in \mathcal{B}$:
    * If $X_1 \xrightarrow{a}_{\mathcal{C}_1} X_1$ then we proceed like in the case $B = \text{IB}$.
    * $\mathit{brm}(X_1) = \mathit{brm}(X_2)$ follows from the fact that X_1 and X_2 satisfy, in particular, the same formulas of the form M.
- Let $B = \text{SB:brm}$. The proof is divided into two parts:
  - Assuming that $\mathcal{C}_1 \sim_{\text{SB:brm}} \mathcal{C}_2$, $X_i \in \mathcal{C}_i$ for $i \in \{1, 2\}$ are related by a backward ready multiset step bisimulation $\mathcal{B}$, and $X_1 \models \phi$ for an arbitrary formula $\phi \in \mathcal{L}_{\text{SB:brm}}$, we prove that $X_2 \models \phi$ too by proceeding by induction on $k = \mathit{depth}(\phi)$:
    * If $k = 0$ then we proceed like in the case $B = \text{IB}$.
    * If $k > 0$ then we proceed like in the case $B = \text{SB}$.
  - Assuming that $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy the same formulas in $\mathcal{L}_{\text{SB:brm}}$, we prove that the symmetric relation $\mathcal{B} = \{(X_1, X_2) \in \mathcal{C}_1 \times \mathcal{C}_2 \mid X_1 \text{ and } X_2 \text{ satisfy the same formulas in } \mathcal{L}_{\text{SB:brm}}\}$ is a backward ready multiset step bisimulation. Given $(X_1, X_2) \in \mathcal{B}$:
    * If $X_1 \xrightarrow{A}_{\mathcal{C}_1} X_1$ then we proceed like in the case $B = \text{SB}$.
    * $\mathit{brm}(X_1) = \mathit{brm}(X_2)$ follows from the fact that X_1 and X_2 satisfy, in particular, the same formulas of the form M.
- Let $B = \text{PB:brm}$. The proof is divided into two parts:
  - Assuming that $\mathcal{C}_1 \sim_{\text{PB:brm}} \mathcal{C}_2$, $X_i \in \mathcal{C}_i$ for $i \in \{1, 2\}$ are related by a backward ready multiset pomset bisimulation $\mathcal{B}$, and $X_1 \models \phi$ for an arbitrary formula $\phi \in \mathcal{L}_{\text{PB:brm}}$, we prove that $X_2 \models \phi$ too by proceeding by induction on $k = \mathit{depth}(\phi)$:
    * If $k = 0$ then we proceed like in the case $B = \text{IB}$.
    * If $k > 0$ then we proceed like in the case $B = \text{PB}$.
  - Assuming that $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy the same formulas in $\mathcal{L}_{\text{PB:brm}}$, we prove that the symmetric relation $\mathcal{B} = \{(X_1, X_2) \in \mathcal{C}_1 \times \mathcal{C}_2 \mid X_1 \text{ and } X_2 \text{ satisfy the same formulas in } \mathcal{L}_{\text{PB:brm}}\}$ is a backward ready multiset pomset bisimulation. Given $(X_1, X_2) \in \mathcal{B}$:
    * If $X_1 \xrightarrow{U}_{\mathcal{C}_1} X_1$ then we proceed like in the case $B = \text{PB}$.
    * $\mathit{brm}(X_1) = \mathit{brm}(X_2)$ follows from the fact that X_1 and X_2 satisfy, in particular, the same formulas of the form M.
- For $B = \text{FRB:brm}$ the result was already proven in [4]. ∎

Proof of Theorem 25. A straightforward consequence of Corollary 22 as well as Theorems 23 and 24 – and also 17 when $\mathit{VIB} \in \{\text{WHPB, WHPB:brm}\} \wedge \mathit{NIbrmB} \in \{\text{IB:brm, SB:brm}\}$. ∎

Proof of Theorem 33. The proof is divided into two parts:

- Suppose that $C_1 \sim_{\mathrm{HPB}} C_2$ due to some history-preserving bisimulation $\mathcal{B}$. Then $C_1 \sim_{\mathrm{IB:brm:diam}} C_2$ follows by proving that $\mathcal{B}' = \{(X_1, X_2) \mid (X_1, X_2, f) \in \mathcal{B}\}$ is an interleaving brm-diam bisimulation. Observing that the starting clause and the clause for transitions matching of $\sim_{\mathrm{IB:brm:diam}}$ (see Definition 5) are a simplification of those of $\sim_{\mathrm{HPB}}$ (see Definition 13), given $(X_1, X_2) \in \mathcal{B}'$, i.e., $(X_1, X_2, f) \in \mathcal{B}$ for some labeling- and causality-preserving bijection f from X_1 to X_2, we just have to show that $\mathit{brm}(X_1) = \mathit{brm}(X_2)$ and that, if $\mathit{diam}(X_1) \neq \emptyset \neq \mathit{diam}(X_2)$, then $\mathit{hdiam}(X_1) \neq \emptyset$ iff $\mathit{hdiam}(X_2) \neq \emptyset$. We consider the two clauses separately:
  - Suppose that $\mathit{brm}(X_1) \neq \mathit{brm}(X_2)$, say X_1 has fewer incoming a-transitions than X_2. Without loss of generality, we can assume that X_1 has one incoming a-transition while X_2 has two. Then in C_2 there is a diamond closing into X_2, i.e., there exist three configurations Y_2, X'_2, and X''_2 and two a-labeled events e'_2 and e''_2 such that $Y_2 \xrightarrow{\ell_2(e'_2)}_{C_2} X'_2$, $Y_2 \xrightarrow{\ell_2(e''_2)}_{C_2} X''_2$, $X'_2 \xrightarrow{\ell_2(e''_2)}_{C_2} X_2$, and $X''_2 \xrightarrow{\ell_2(e'_2)}_{C_2} X_2$, with e'_2 and e''_2 concurrent in X_2.
    Due to $(X_1, X_2, f) \in \mathcal{B}$, in C_1 there must be two configurations Y_1 and X'_1 and two a-labeled events e'_1 and e''_1 such that $Y_1 \xrightarrow{\ell_1(e'_1)}_{C_1} X'_1 \xrightarrow{\ell_1(e''_1)}_{C_1} X_1$, with $e'_1 \leq_{X_1} e''_1$ because X_1 has a single incoming a-transition. Since $\mathcal{B}$ is a history-preserving bisimulation, f should relate $e'_1, e''_1 \in X_1$ with $e'_2, e''_2 \in X_2$ in a causality-preserving way, but this is not possible because $f(e'_1) \not\leq_{X_2} f(e''_1)$ where $f(e'_1) \in \{e'_2, e''_2\}$ and $f(e''_1) \in \{e'_2, e''_2\} \setminus \{f(e'_1)\}$.
  - Suppose that $\mathit{diam}(X_1) \neq \emptyset \neq \mathit{diam}(X_2)$. Furthermore, suppose that $\mathit{hdiam}(X_1) \neq \emptyset$. We need to show that $\mathit{hdiam}(X_2) \neq \emptyset$; the other directions is similar. From $\mathit{hdiam}(X_1) \neq \emptyset$ it follows that there exist at least two transitions $X_1 \xrightarrow{\ell_1(e'_1)}_{C_1} X'_1$ and $X_1 \xrightarrow{\ell_1(e''_1)}_{C_1} X''_1$ such that it does not exist a configuration Y_1 reached by both X''_1 and X'_1, i.e., there are no $X'_1 \xrightarrow{\ell_1(e''_1)}_{C_1} Y_1$ and $X''_1 \xrightarrow{\ell_1(e'_1)}_{C_1} Y_1$. From $(X_1, X_2, f) \in \mathcal{B}$ it follows that there exist $X_2 \xrightarrow{\ell_2(e'_2)}_{C_2} X'_2$ and $X_2 \xrightarrow{\ell_2(e''_2)}_{C_2} X''_2$ such that $\ell_1(e'_1) = \ell_2(e'_2)$, $\ell_1(e''_1) = \ell_2(e''_2)$, $(X'_1, X'_2, f') \in \mathcal{B}$, and $(X''_1, X''_2, f'') \in \mathcal{B}$ for some appropriate labeling- and causality-preserving bijections $f' = f \upharpoonright X'_1$ and $f'' = f \upharpoonright X''_1$. Let us assume towards a contradiction that $\mathit{hdiam}(X_2) = \emptyset$. This implies that there exists a configuration Y_2 such that $X'_2 \xrightarrow{\ell_2(e''_2)}_{C_2} Y_2$ and $X''_2 \xrightarrow{\ell_2(e'_2)}_{C_2} Y_2$. From $(X'_1, X'_2, f') \in \mathcal{B}$ and $(X''_1, X''_2, f'') \in \mathcal{B}$ it follows that there exist $X'_1 \xrightarrow{\ell_1(\bar{e}_1)}_{C_1} Y_1$ and $X''_1 \xrightarrow{\ell_1(\bar{e}'_1)}_{C_1} Y'_1$ with $Y_1 \neq Y'_1$ because otherwise we would violate $\mathit{hdiam}(X_1) \neq \emptyset$, $\ell_1(\bar{e}_1) = \ell_2(e''_2)$, $\ell_1(\bar{e}'_1) = \ell_2(e'_2)$, $(Y_1, Y_2, f''') \in \mathcal{B}$, and $(Y'_1, Y_2, f'''') \in \mathcal{B}$. We have now reached a contradiction, because $e'_1 <_{Y_1} \bar{e}_1$ and $e''_1 <_{Y'_1} \bar{e}'_1$ (recall that this is due to the fact that $Y_1 \neq Y'_1$) but $f'''(e_1) = f''''(\bar{e}_1) = e'_2 \not<_{Y_2} f'''(e''_1) = f''''(\bar{e}'_1) = e''_2$ and hence both f''' and f'''' do not preserve causality, which is contrary to the hypothesis that $(X_1, X_2, f) \in \mathcal{B}$ for a history-preserving bisimulation $\mathcal{B}$ and hence it must be the case that $\mathit{hdiam}(X_2) \neq \emptyset$.
- Suppose that $C_1 \sim_{\mathrm{IB:brm:diam}} C_2$ due to some maximal interleaving brm-diam bisimulation $\mathcal{B}$. Then, given $(X_1, X_2) \in \mathcal{B}$, the existence in C_1 of a sequence of transitions $X_{1,n} \xrightarrow{\ell_1(e_{1,n})}_{C_1} X_{1,n-1} \ldots X_{1,1} \xrightarrow{\ell_1(e_{1,1})}_{C_1} X_1$ implies the existence in C_2 of a sequence of transitions $X_{2,n} \xrightarrow{\ell_2(e_{2,n})}_{C_2} X_{2,n-1} \ldots X_{2,1} \xrightarrow{\ell_2(e_{2,1})}_{C_2} X_2$ such that $\ell_1(e_{1,h}) = \ell_2(e_{2,h})$ and $(X_{1,h}, X_{2,h}) \in \mathcal{B}$ for all $h = 1, \ldots, n$, and vice versa. Note that $e_{1,h} \neq e_{1,k}$ and $e_{2,h} \neq e_{2,k}$ for all $h \neq k$ because in each transition the source configuration and the target configuration differ by one event, which is the executed event (see Definition 4).

Thus $C_1 \sim_{\mathrm{HPB}} C_2$ follows by proving that $\mathcal{B}' = \{(X_1, X_2, \{(e_{1,h}, e_{2,h}) \mid h \in H\}) \mid (X_1, X_2) \in \mathcal{B} \wedge X_{i,|H|} \xrightarrow{\ell_i(e_{i,|H|})}_{C_i} X_{i,|H|-1} \dots X_{i,1} \xrightarrow{\ell_i(e_{i,1})}_{C_i} X_i$ for $i \in \{1,2\} \wedge \ell_1(e_{1,h}) = \ell_2(e_{2,h})$ for all $h \in H \wedge (X_{1,h}, X_{2,h}) \in \mathcal{B}$ for all $h \in H \wedge X_{1,|H|} = \emptyset = X_{2,|H|}\}$ is a history-preserving bisimulation. Observing that $(\emptyset, \emptyset) \in \mathcal{B}$ implies $(\emptyset, \emptyset, \emptyset) \in \mathcal{B}'$, take $(X_1, X_2, \{(e_{1,h}, e_{2,h}) \mid h \in H\}) \in \mathcal{B}'$, so that $(X_1, X_2) \in \mathcal{B}$:

- If $X_1 \xrightarrow{a}_{C_1} X'_1$ due to some a-labeled event e_1, then $X_2 \xrightarrow{a}_{C_2} X'_2$ due to some a-labeled event e_2 such that $(X'_1, X'_2) \in \mathcal{B}$. Since $e_1 \notin X_1$ and $e_2 \notin X_2$, it holds that $(X'_1, X'_2, \{(e_{1,h}, e_{2,h}) \mid h \in H\} \cup \{(e_1, e_2)\}) \in \mathcal{B}'$.
  If we start from $X_2 \xrightarrow{a}_{C_2} X'_2$, then we reason in the same way.
- $f = \{(e_{1,h}, e_{2,h}) \mid h \in H\}$ certainly is a bijection from X_1 to X_2 – as the events along either computation are different from each other, so the two reached configurations X_1 and X_2 contain the same number of events, and paired in a stepwise manner – that preserves labeling – by definition of $\mathcal{B}'$. If $|H| \leq 1$ then causality is trivially preserved. Suppose that X_1 and X_2 break causality and, among all the pairs of $\mathcal{B}$-related configurations that break causality, they are the closest ones to $\emptyset$ and $\emptyset$ (in terms of number of transitions to be executed from either empty configuration). Breaking causality means that $X_1 \xrightarrow{a}_{C_1} X'_1$ due to an a-labeled event $e'_1 \notin X_1$ such that $e_1 \leq_{X'_1} e'_1$ for some $e_1 \in X_1$, but all the responses $X_2 \xrightarrow{a}_{C_2} X'_2$ complying with $\mathcal{B}$, which is maximal, are such that $g(e_1) \not\leq_{X'_2} g(e'_1)$, where g is f extended with the new pair of events (from now on the proof is also illustrated in Figure 7).
  Let us assume, without loss of generality, that the event e_1 appears for the first time in X_1, i.e., there exist the transitions $X_{1,1} \xrightarrow{\ell_1(e_1)}_{C_1} X_1$ and $X_{2,1} \xrightarrow{\ell_2(e_2)}_{C_2} X_2$ for a $e_2 \in X_2$ such that $g(e_1) = e_2$ and $g(e'_1) = e'_2$. From the fact that $g(e_1) = e_2 \not\leq_{X'_2} e'_2 = g(e'_1)$ it follows that the events e_2 and e'_2 are independent in X'_2, meaning that there exists a diamond opening in $X_{2,1}$ and closing in X'_2 composed of the transitions $X_{2,1} \xrightarrow{\ell_2(e_2)}_{C_2} X_2$, $X_{2,1} \xrightarrow{\ell_2(e'_2)}_{C_2} \bar{X}_2$, $\bar{X}_2 \xrightarrow{\ell_2(e_2)}_{C_2} X_2$, and $X_2 \xrightarrow{\ell_2(e'_2)}_{C_2} X_2$. From $X_{2,1} \xrightarrow{\ell_2(e'_2)}_{C_2} \bar{X}_2$, $\bar{X}_2 \xrightarrow{\ell_2(e_2)}_{C_2} X'_2$, and $(X_{1,1}, X_{2,1}) \in \mathcal{B}$ it follows that there exist $X_{1,1} \xrightarrow{\ell_1(e''_1)}_{C_1} \bar{X}_1$, $\bar{X}_1 \xrightarrow{\ell_1(e_1)}_{C_1} \bar{X}'_1$, with $\ell_1(e_1) = \ell_2(e_2)$, $\ell_1(e''_1) = \ell_2(e'_2)$, $(\bar{X}_1, \bar{X}_2) \in \mathcal{B}$, and $(\bar{X}'_1, X'_2) \in \mathcal{B}$. Furthermore, we notice that in $brm(X'_2)$ there are at least two actions, $\ell_2(e_1)$ and $\ell_2(e'_1)$, and hence, from the fact that $(X'_1, X'_2) \in \mathcal{B}$ it follows that the same actions must appear in $brm(X'_1)$. From this we can infer that there are two events independent in X'_1 with the same labels as e_2 and e'_2 that are part of a diamond opening in some configuration $\bar{X}''_1$ and closing in X'_1. This diamond is composed of the transitions $\bar{X}''_1 \xrightarrow{\ell_1(e'_1)}_{C_1} \bar{X}'''_1$, $\bar{X}''_1 \xrightarrow{\ell_1(e'''_1)}_{C_1} X_1$, $\bar{X}'''_1 \xrightarrow{\ell_1(e'''_1)}_{C_1} X'_1$, $X_1 \xrightarrow{\ell_1(e'_1)}_{C_1} X'_1$, with $\ell_1(e'''_1) = \ell_2(e_2)$. At this point, we have that in $brm(X_1)$ there are two identical actions $\ell_1(e_1)$ and $\ell_1(e'''_1)$ and from the fact that $(X_1, X_2) \in \mathcal{B}$ it follows that the same actions must appear in $brm(X_2)$. From this we can infer that there are two events independent in X_2 with the same labels as e_1 and e'''_1 that are part of a diamond opening in some configuration $\bar{X}'''_2$ and closing in X_2 composed of the transitions $\bar{X}'''_2 \xrightarrow{\ell_2(e_2)}_{C_2} \bar{X}''_2$, $\bar{X}'''_2 \xrightarrow{\ell_2(e''_2)}_{C_2} X_{2,1}$, $\bar{X}''_2 \xrightarrow{\ell_2(e''_2)}_{C_2} X_2$, $X_{2,1} \xrightarrow{\ell_2(e_2)}_{C_2} X_2$ with $\ell_2(e_2) = \ell_1(e_1) = \ell_2(e''_2) = \ell_1(e'''_1)$. We notice that there are two incoming transitions in X_1 labeled $\ell_1(e_1)$ and $\ell_1(e'''_1)$ meaning that there is a diamond opening in some configuration $\bar{X}''''_1$ and closing in X_1. The diamond is composed of the transitions $\bar{X}''''_1 \xrightarrow{\ell_1(e_1)}_{C_1} \bar{X}''_1$, $\bar{X}''''_1 \xrightarrow{\ell_1(e'''_1)}_{C_1} X_{1,1}$, $\bar{X}''_1 \xrightarrow{\ell_1(e'''_1)}_{C_1} X_1$, $X_{1,1} \xrightarrow{\ell_1(e_1)}_{C_1} X_1$. We notice that since $\bar{X}''''_1$ and $\bar{X}'''_2$ reach with a transition $X_{1,1}$ and $X_{2,1}$ respectively,

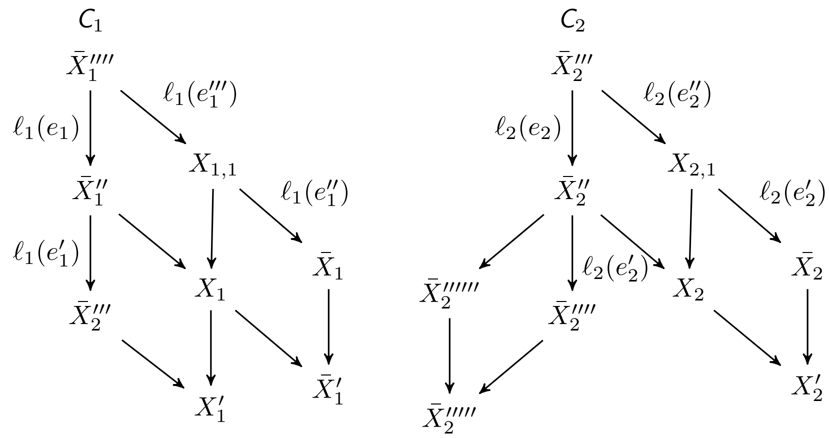

Figure 7 Configurations mentioned in the final part of the proof of Theorem 33

they are part of the history of configurations related by $\mathcal{B}$, and hence we can infer that $(\bar{X}_1'''', \bar{X}_2''') \in \mathcal{B}$. By looking at their transitions we can deduce which other configurations are related by $\mathcal{B}$. The transition $\bar{X}_1'''' \xrightarrow{\ell_1(e_1)}_{C_1} \bar{X}_1''$ is matched by $\bar{X}_2''' \xrightarrow{\ell_2(e_2)}_{C_2} \bar{X}_2''$, and hence $(\bar{X}_1'', \bar{X}_2'') \in \mathcal{B}$. From $\bar{X}_1'' \xrightarrow{\ell_1(e_1')}_{C_1} \bar{X}_1'''$ it follows that there must be a transition $\bar{X}_2'' \xrightarrow{\ell_2(e_2')}_{C_2} \bar{X}_2''''$ with $(\bar{X}_1''', \bar{X}_2'''') \in \mathcal{B}$. Furthermore, we notice that from the fact that $\bar{X}_1''$ opens a diamond it follows that $\bar{X}_2''$ must open a diamond too, otherwise they would not be related by $\mathcal{B}$. Now we observe that the composition of this diamond opens only two cases:

- ∗ If the diamond opens in $\bar{X}_2''$ and closes in X_2', then the events involved would be e_2'' and e_2', and hence the transitions $\bar{X}_1'''' \xrightarrow{\ell_1(e_1)}_{C_1} \bar{X}_1'' \xrightarrow{\ell_1(e_1')}_{C_1} \bar{X}_1'''$ would be matched by $\bar{X}_2''' \xrightarrow{\ell_2(e_2)}_{C_2} \bar{X}_2'' \xrightarrow{\ell_2(e_2')}_{C_2} X_2'$ with $(\bar{X}_1''', X_2') \in \mathcal{B}$ but then we have reached a contradiction because $e_1 \leq_{\bar{X}_1'''} e_1'$ while $e_2 \not\leq_{X_2'} e_2'$ so $\bar{X}_1'''$ and X_2' have broken the causality chain before X_1' and X_2', which is contrary to our hypothesis.
- ∗ If the diamond opens in $\bar{X}_1''$ and closes in some configuration $\bar{X}_2'''''$, passing through $\bar{X}_2''''''$ and $\bar{X}_2''''$, then we would violate the condition about diamonds and half diamonds, because $(\bar{X}_1'', \bar{X}_2'') \in \mathcal{B}$ but while $\mathit{diam}(\bar{X}_1'') \neq \emptyset \neq \mathit{diam}(\bar{X}_1'')$ we have that $\mathit{hdiam}(\bar{X}_2'') \neq \emptyset$ and $\mathit{hdiam}(\bar{X}_1'') = \emptyset$ because only $\bar{X}_2''$ opens an half diamond, thus violating the aforementioned condition. ∎

Proof of Theorem 34. The proof is divided into two parts:

- Suppose that $C_1 \sim_{\mathrm{HHPB}} C_2$ due to some hereditary history-preserving bisimulation $\mathcal{B}$. Then $C_1 \sim_{\mathrm{FRB:brm:diam}} C_2$ follows by proving that $\mathcal{B}' = \{(X_1, X_2) \mid (X_1, X_2, f) \in \mathcal{B}\}$ is a forward-reverse brm-diam bisimulation. Observing that the starting clause and the clauses for transitions matching of $\sim_{\mathrm{FRB:brm:diam}}$ (see Definition 8) are a simplification of those of $\sim_{\mathrm{HPB}}$ (see Definition 14), given $(X_1, X_2) \in \mathcal{B}'$, i.e., $(X_1, X_2, f) \in \mathcal{B}$ for some labeling- and causality-preserving bijection f from X_1 to X_2, we just have to show that $\mathit{brm}(X_1) = \mathit{brm}(X_2)$ and that, if $\mathit{diam}(X_1) \neq \emptyset \neq \mathit{diam}(X_2)$, then $\mathit{hdiam}(X_1) \neq \emptyset$ iff $\mathit{hdiam}(X_2) \neq \emptyset$. To prove this two properties we reason like in the corresponding part of the proof of Theorem 33.
- Suppose that $C_1 \sim_{\mathrm{FRB:brm:diam}} C_2$ due to some maximal forward-reverse brm-diam bisimulation $\mathcal{B}$. Then, given $(X_1, X_2) \in \mathcal{B}$, the existence in C_1 of a sequence of transitions $X_{1,n} \xrightarrow{\ell_1(e_{1,n})}_{C_1} X_{1,n-1} \ldots X_{1,1} \xrightarrow{\ell_1(e_{1,1})}_{C_1} X_1$ implies the existence in C_2 of a sequence

of transitions $X_{2,n} \xrightarrow{\ell_2(e_{2,n})}_{\mathcal{C}_2} X_{2,n-1} \dots X_{2,1} \xrightarrow{\ell_2(e_{2,1})}_{\mathcal{C}_2} X_2$ such that $\ell_1(e_{1,h}) = \ell_2(e_{2,h})$ and $(X_{1,h}, X_{2,h}) \in \mathcal{B}$ for all $h = 1, \dots, n$, and vice versa. Note that $e_{1,h} \neq e_{1,k}$ and $e_{2,h} \neq e_{2,k}$ for all $h \neq k$ because in each transition the source configuration and the target configuration differ by one event, which is the executed event (see Definition 4).

Thus $\mathcal{C}_1 \sim_{\mathrm{HPB}} \mathcal{C}_2$ follows by proving that $\mathcal{B}' = \{(X_1, X_2, \{(e_{1,h}, e_{2,h}) \mid h \in H\}) \mid (X_1, X_2) \in \mathcal{B} \wedge X_{i,|H|} \xrightarrow{\ell_i(e_{i,|H|})}_{\mathcal{C}_i} X_{i,|H|-1} \dots X_{i,1} \xrightarrow{\ell_i(e_{i,1})}_{\mathcal{C}_i} X_i$ for $i \in \{1,2\} \wedge \ell_1(e_{1,h}) = \ell_2(e_{2,h})$for all $h \in H \wedge (X_{1,h}, X_{2,h}) \in \mathcal{B}$ for all $h \in H \wedge X_{1,|H|} = \emptyset = X_{2,|H|}\}$ is a hereditary history-preserving bisimulation. Observing that $(\emptyset, \emptyset) \in \mathcal{B}$ implies $(\emptyset, \emptyset, \emptyset) \in \mathcal{B}'$, take $(X_1, X_2, \{(e_{1,h}, e_{2,h}) \mid h \in H\}) \in \mathcal{B}'$, so that $(X_1, X_2) \in \mathcal{B}$:

- If $X_1 \xrightarrow{a}_{\mathcal{C}_1} X_1'$ due to some a-labeled event e_1, then $X_2 \xrightarrow{a}_{\mathcal{C}_2} X_2'$ due to some a-labeled event e_2 such that $(X_1', X_2') \in \mathcal{B}$. Since $e_1 \notin X_1$ and $e_2 \notin X_2$, it holds that $(X_1', X_2', \{(e_{1,h}, e_{2,h}) \mid h \in H\} \cup \{(e_1, e_2)\}) \in \mathcal{B}'$.
  If we start from $X_2 \xrightarrow{a}_{\mathcal{C}_2} X_2'$, then we reason in the same way.
- If $X_1' \xrightarrow{a}_{\mathcal{C}_1} X_1$ due to some a-labeled event e_1, then $X_2' \xrightarrow{a}_{\mathcal{C}_2} X_2$ due to some a-labeled event e_2 such that $(X_1', X_2') \in \mathcal{B}$. Since $e_1 \notin X_1'$, $e_2 \notin X_2'$, and $\mathit{brm}(X_1) = \mathit{brm}(X_2)$, the latter transition can be selected in such a way to satisfy $\{(e_{1,h}, e_{2,h}) \mid h \in H\} \upharpoonright X_1' = \{(e_{1,h}, e_{2,h}) \mid h \in H\} \setminus \{(e_1, e_2)\}$, hence $(X_1', X_2', \{(e_{1,h}, e_{2,h}) \mid h \in H\} \setminus \{(e_1, e_2)\}) \in \mathcal{B}'$.
  If we start from $X_2' \xrightarrow{a}_{\mathcal{C}_2} X_2$, then we reason in the same way.
- As for the clause related to the labeling- and causality-preserving bijection f, we can proceed as in the corresponding part of the proof of Theorem 33. The reason is that there we perform checks only on forward behaviors, which can be done also in this proof, and when checking backward behaviors the preservation of causality is guaranteed. ∎