# Noninterference Analysis of Stochastically Timed Reversible Systems

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**Abstract.** Noninterference theory aims at ensuring the absence of covert channels among different security levels. As far as the verification of information-flow properties via equivalence checking is concerned, in nondeterministic and probabilistic settings weak bisimilarity has turned out to be adequate only for standard systems, while branching bisimilarity has proven to be appropriate for reversible systems too. In this paper we investigate noninterference for stochastically timed systems represented in the interactive Markov chain model of Hermanns. After recasting a selection of noninterference properties via Markovian variants of weak and branching bisimilarities, we study their preservation and compositionality aspects, build their taxonomy, and compare it with the nondeterministic and probabilistic taxonomies. We show the adequacy of our proposal through some examples about a database management system.

## 1 Introduction

The notion of noninterference was introduced in [34] to reason about the way in which illegitimate information flows can occur in multi-level security systems due to covert channels from high-level agents to low-level ones. Since the first definition, conceived for deterministic systems, there have been several extensions to more expressive domains, such as nondeterministic systems, systems in which quantitative aspects like time and probability play a central role, and reversible systems; see, e.g., [26,2,47,37,64,59,8,5,3,41,25,24]. Likewise, different verification approaches have been proposed; see, e.g., [67,30,27,48,4].

Noninterference guarantees that low-level agents cannot infer from their observations what high-level ones are doing. Regardless of its specific definition, noninterference is closely tied to the notion of behavioral equivalence [32] because, given a multi-level security system, the idea is to compare the system behavior with high-level actions being prevented and the system behavior with the same actions being hidden. A natural framework in which to study system behavior is given by process algebra [49]. In this setting, weak bisimilarity has been employed in [26] to reason formally about covert channels and illegitimate information flows as well as to study a classification of noninterference properties for irreversible nondeterministic systems.

Noninterference analysis has been recently extended to reversible systems – featuring forward and backward computations – in the nondeterministic setting [25] and in the probabilistic one [24]. Reversibility has started to gain attention in computing since it has been shown that it may achieve lower levels of energy consumption [43,9]. Its applications range from biochemical reaction modeling [56,57] and parallel discrete-event simulation [53,61] to robotics [46], wireless communications [62], fault-tolerant systems [21,65,44,63], program debugging [29,45], and distributed algorithms [66,15].

As shown in [25,24], noninterference properties based on weak bisimilarity are not adequate in a reversible context because they fail to detect information flows emerging when backward computations are triggered. A more appropriate semantics turns out to be branching bisimilarity [33] because it coincides with weak back-and-forth bisimilarity [22]. The latter behavioral equivalence requires systems to be able to mimic each other's behavior stepwise not only when performing actions in the standard forward direction, but also when undoing those actions in the backward direction. Formally, weak back-and-forth bisimilarity is defined over computation paths instead of states thus preserving not only causality but also history, as backward moves are constrained to take place along the same path followed in the forward direction even in the presence of concurrency.

In this paper we extend the approach of [25,24] to a stochastically timed setting, so as to address noninterference properties in a framework featuring nondeterminism, time, and reversibility. To accomplish this we move to a model combining nondeterminism and stochastic time given by the interactive Markov chain model of [38], in which transitions are divided into action transitions, each labeled with an action, and rate transition, each labeled with a positive real number called rate that expresses an exponentially distributed delay. The reason for choosing this model in which time passing is orthogonal to action execution, instead of a model in which action execution and time passing are integrated [35,39,40,19,58,14,12,10] (see [13] for encodings between integrated-time and orthogonal-time calculi), is that the former naturally supports the definition of behavioral equivalences abstracting from unobservable actions [38] – which are necessary for noninterference analysis – whereas this is not the case in the latter [11], which was employed in [3,41] for stochastic variants of some noninterference properties.

Following [38] we build a process calculus featuring action prefix separated from rate prefix. As for behavioral equivalences, we adopt the weak Markovian bisimilarity of [38] and introduce a novel Markovian branching bisimilarity. By using these two equivalences we recast the noninterference properties of [26,28] for irreversible systems and the noninterference properties of [25] for reversible systems, respectively, to study their preservation and compositionality aspects as well as to provide a taxonomy similar to those in [26,25,24]. Reversibility comes into play by extending one of the results of [22] to the interactive Markov chain model; we show that a Markovian variant of weak back-and-forth bisimilarity coincides with our Markovian branching bisimilarity.

This paper is organized as follows. In Section 2 we recall the interactive Markov chain model of [38] along with various definitions of strong and weak bisimilarities for it and a process calculus interpreted on it. In Section 3 we recast in our stochastically timed framework a selection of noninterference properties taken from [26,28,25]. In Section 4 we study their preservation and compositionality characteristics as well as their taxonomy, which in Section 5 we relate to the nondeterministic taxonomy of [25] and the probabilistic one of [24]. In Section 6 we establish a connection with reversibility by introducing a weak Markovian back-and-forth bisimilarity and proving that it coincides with Markovian branching bisimilarity. In Section 7 we present examples of obfuscation and permission mechanisms in database management systems to show the adequacy of our approach to information flows in reversible systems featuring nondeterminism and stochastic time. Finally, in Section 8 we provide some concluding remarks.

## 2 Background Definitions and Results

In this section we recall the interactive Markov chain model of [38] (Section 2.1) along with its strong and weak Markovian bisimilarities and define a novel Markovian branching bisimilarity (Section 2.2). Then we introduce a Markovian process language inspired by [38] (Section 2.3) through which we will express bisimulation-based information-flow security properties accounting for nondeterminism and stochastic time.

### 2.1 Markovian Labeled Transition Systems

To represent the behavior of a process featuring nondeterminism and stochastic time, we use a Markovian labeled transition system. This is a variant of a labeled transition system [42] where, according to the interactive Markov chain model of [38], transitions are labeled with actions or positive real numbers called rates expressing exponentially distributed delays. We assume that the action set $\mathcal{A}_\tau$ contains a set $\mathcal{A}$ of observable actions and a single action $\tau \notin \mathcal{A}$ representing unobservable actions.

**Definition 1.** *A* Markovian labeled transition system (MLTS) *is a triple* $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ *where* $\mathcal{S}$ *is an at most countable set of states,* $\mathcal{A}_\tau = \mathcal{A} \cup \{\tau\}$ *is a countable set of actions, and* $\longrightarrow \, = \, \longrightarrow_{\rm a} \cup \longrightarrow_{\rm r}$ *is the transition relation, with* $\longrightarrow_{\rm a} \subseteq \mathcal{S} \times \mathcal{A}_\tau \times \mathcal{S}$ *being the action transition relation whilst* $\longrightarrow_{\rm r} \subseteq \mathcal{S} \times \mathbb{R}_{>0} \times \mathcal{S}$ *being the rate transition relation.* ■

An action transition $(s, a, s')$ is written $s \xrightarrow{a}_{\rm a} s'$ while a rate transition $(s, \lambda, s')$ is written $s \xrightarrow{\lambda}_{\rm r} s'$, where $s$ is the source state and $s'$ is the target state. We say that $s'$ is reachable from $s$, written $s' \in \mathit{reach}(s)$, iff $s' = s$ or there exists a sequence of finitely many transitions such that the target state of each of them coincides with the source state of the subsequent one, with the source of the first one being $s$ and the target of the last one being $s'$.

The label of a rate transition is the inverse of the average duration of the corresponding exponentially distributed delay, which enjoys the *memoryless property*: the residual duration after the execution starts is still exponentially distributed with the same rate. If the outgoing rate transitions of state $s$ are $s \xrightarrow{\lambda_i}_{\rm r} s_i$ for $1 \leq i \leq n$, then the *race policy* applies. This means that the average sojourn time in $s$ is given by the minimum of the $n$ exponentially distributed delays – which is exponentially distributed with rate $\sum_{1 \le i \le n} \lambda_i$ – and the execution probability of transition $j$ is given by $\lambda_j / \sum_{1 \le i \le n} \lambda_i$. As for the interplay between action transitions and rate transitions, like in [38] we assume *maximal progress*, i.e., $\tau$-transitions take precedence over rate transitions.

### 2.2 Bisimulation Equivalences

Bisimilarity [52,49] identifies processes that are able to mimic each other's behavior stepwise, i.e., having the same branching structure. In the interactive Markov chain model, this extends to stochastic behavior [38]. Let $rate(s, C) = \sum_{s \xrightarrow{\lambda}_{\mathrm{r}} s', s' \in C} \lambda$ be the cumulative rate with which state $s$ reaches a state in $C$. Due to maximal progress, cumulative rates are compared only in states with no outgoing $\tau$-transitions, denoted $\stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$.

**Definition 2.** *Let $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ be an MLTS. We say that $s_1, s_2 \in \mathcal{S}$ are* strongly Markovian bisimilar*, written $s_1 \sim_{\mathrm{m}} s_2$, iff $(s_1, s_2) \in \mathcal{B}$ for some strong Markovian bisimulation $\mathcal{B}$. An equivalence relation $\mathcal{B}$ over $\mathcal{S}$ is a* strong Markovian bisimulation *iff, whenever $(s_1, s_2) \in \mathcal{B}$, then:*

- *For each $s_1 \xrightarrow{a}_{\mathrm{a}} s'_1$ there exists $s_2 \xrightarrow{a}_{\mathrm{a}} s'_2$ such that $(s'_1, s'_2) \in \mathcal{B}$.*
- *If $s_1 \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$ then $rate(s_1, C) = rate(s_2, C)$ for all equivalence classes $C$ in the quotient set $\mathcal{S}/\mathcal{B}$.* ■

Weak bisimilarity [49] is additionally capable of abstracting from unobservable actions. Let $s \stackrel{\tau^*}{\Longrightarrow}_{\mathrm{a}} s'$ mean that $s' \in reach(s)$ and, when $s' \neq s$, there exists a finite sequence of transitions from $s$ to $s'$ each of which is labeled with $\tau$. Moreover let $\stackrel{\hat{a}}{\Longrightarrow}_{\mathrm{a}}$ stand for $\stackrel{\tau^*}{\Longrightarrow}_{\mathrm{a}}$ if $a = \tau$ or $\stackrel{\tau^*}{\Longrightarrow}_{\mathrm{a}} \xrightarrow{a}_{\mathrm{a}} \stackrel{\tau^*}{\Longrightarrow}_{\mathrm{a}}$ if $a \neq \tau$. The Markovian adaptation below is taken from [38].

**Definition 3.** *Let $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ be an MLTS. We say that $s_1, s_2 \in \mathcal{S}$ are* weakly Markovian bisimilar*, written $s_1 \approx_{\mathrm{mw}} s_2$, iff $(s_1, s_2) \in \mathcal{B}$ for some weak Markovian bisimulation $\mathcal{B}$. An equivalence relation $\mathcal{B}$ over $\mathcal{S}$ is a* weak Markovian bisimulation *iff, whenever $(s_1, s_2) \in \mathcal{B}$, then:*

- *For each $s_1 \xrightarrow{a}_{\mathrm{a}} s'_1$ there exists $s_2 \stackrel{\hat{a}}{\Longrightarrow}_{\mathrm{a}} s'_2$ such that $(s'_1, s'_2) \in \mathcal{B}$.*
- *If $s_1 \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$ then there exists $s_2 \stackrel{\tau^*}{\Longrightarrow}_{\mathrm{a}} \bar{s}_2$ such that $\bar{s}_2 \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$, $(s_1, \bar{s}_2) \in \mathcal{B}$, and $rate(s_1, C) = rate(\bar{s}_2, C)$ for all equivalence classes $C \in \mathcal{S}/\mathcal{B}$.* ■

Branching bisimilarity [33] is finer than weak bisimilarity as it preserves the branching structure of processes even when abstracting from $\tau$-actions – see condition $(s_1, \bar{s}_2) \in \mathcal{B}$ in the action transitions matching of the definition below. We adapt it to the Markovian setting as follows.

**Definition 4.** *Let $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ be an MLTS. We say that $s_1, s_2 \in \mathcal{S}$ are* Markovian branching bisimilar*, written $s_1 \approx_{\mathrm{mb}} s_2$, iff $(s_1, s_2) \in \mathcal{B}$ for some Markovian branching bisimulation $\mathcal{B}$. An equivalence relation $\mathcal{B}$ over $\mathcal{S}$ is a* Markovian branching bisimulation *iff, whenever $(s_1, s_2) \in \mathcal{B}$, then:*

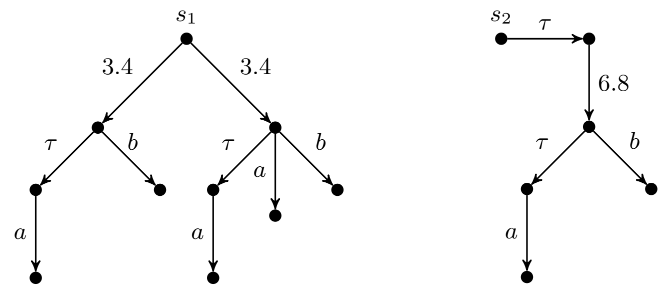


**Fig. 1.** States $s_1$ and $s_2$ are related by $\approx_{\mathrm{mw}}$ but distinguished by $\approx_{\mathrm{mb}}$

– *For each* $s_1 \xrightarrow{a}_{\mathrm{a}} s'_1$*:*
  - *either* $a = \tau$ *and* $(s'_1, s_2) \in \mathcal{B}$*;*
  - *or there exists* $s_2 \stackrel{\tau^*}{\Longrightarrow}_{\mathrm{a}} \bar{s}_2 \xrightarrow{a}_{\mathrm{a}} s'_2$ *such that* $(s_1, \bar{s}_2) \in \mathcal{B}$ *and* $(s'_1, s'_2) \in \mathcal{B}$*.*

– *If* $s_1 \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$ *then there exists* $s_2 \stackrel{\tau^*}{\Longrightarrow}_{\mathrm{a}} \bar{s}_2$ *such that* $\bar{s}_2 \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$*,* $(s_1, \bar{s}_2) \in \mathcal{B}$*, and* $\mathit{rate}(s_1, C) = \mathit{rate}(\bar{s}_2, C)$ *for all equivalence classes* $C \in \mathcal{S}/\mathcal{B}$*.* ■

In [38] it is argued that the weak bisimilarity of Definition 3 is already very close to branching bisimilarity, because maximal progress forces a check given by condition $(s_1, \bar{s}_2) \in \mathcal{B}$ on the branching structure of the considered processes. We show that our novel Definition 4, which sticks to the original one of [33], is more discriminating. Consider Figure 1, where every MLTS is depicted as a directed graph in which vertices represent states and action- or rate-labeled edges represent transitions. The initial states $s_1$ and $s_2$ of the two MLTSs are weakly Markovian bisimilar but not Markovian branching bisimilar. On the one hand, each of the two states reachable from $s_1$ with rate 3.4 and the state reachable from $s_2$ with rate 6.8 after a $\tau$-transition are all weakly Markovian bisimilar and hence the cumulative rate to reach them is the same from both initial states. On the other hand, the two states reachable from $s_1$ are not Markovian branching bisimilar, because if the one on the right performs $a$ then the one on the left cannot respond by performing $\tau$ followed by $a$ because the state reached after $\tau$ no longer enables $b$. Thus, with respect to Markovian branching bisimilarity, $s_1$ reaches with rate 3.4 two different equivalence classes, while $s_2$ reaches with rate 6.8 only one of them.

### 2.3 A Markovian Process Calculus with High and Low Actions

We now introduce a Markovian process calculus to formalize the security properties of interest. To address two security levels, we partition the set $\mathcal{A}$ of observable actions into $\mathcal{A}_{\mathcal{H}} \cup \mathcal{A}_{\mathcal{L}}$, with $\mathcal{A}_{\mathcal{H}} \cap \mathcal{A}_{\mathcal{L}} = \emptyset$, where $\mathcal{A}_{\mathcal{H}}$ is the set of high-level actions, ranged over by $h$, and $\mathcal{A}_{\mathcal{L}}$ is the set of low-level actions, ranged over by $l$. Note that $\tau \notin \mathcal{A}_{\mathcal{H}} \cup \mathcal{A}_{\mathcal{L}}$.

| | | |
|---|---|---|
| *Prefix* | $a \,.\, P \xrightarrow{a}_{\rm a} P$ | |
| *Choice* | $\dfrac{P_1 \xrightarrow{a}_{\rm a} P'_1}{P_1 + P_2 \xrightarrow{a}_{\rm a} P'_1}$ | $\dfrac{P_2 \xrightarrow{a}_{\rm a} P'_2}{P_1 + P_2 \xrightarrow{a}_{\rm a} P'_2}$ |
| *Parallel* | $\dfrac{P_1 \xrightarrow{a}_{\rm a} P'_1 \quad a \notin L}{P_1 \parallel_L P_2 \xrightarrow{a}_{\rm a} P'_1 \parallel_L P_2}$ | $\dfrac{P_2 \xrightarrow{a}_{\rm a} P'_2 \quad a \notin L}{P_1 \parallel_L P_2 \xrightarrow{a}_{\rm a} P_1 \parallel_L P'_2}$ |
| *Synch* | $\dfrac{P_1 \xrightarrow{a}_{\rm a} P'_1 \quad P_2 \xrightarrow{a}_{\rm a} P'_2 \quad a \in L}{P_1 \parallel_L P_2 \xrightarrow{a}_{\rm a} P'_1 \parallel_L P'_2}$ | |
| *Restriction* | $\dfrac{P \xrightarrow{a}_{\rm a} P' \quad a \notin L}{P \setminus L \xrightarrow{a}_{\rm a} P' \setminus L}$ | |
| *Hiding* | $\dfrac{P \xrightarrow{a}_{\rm a} P' \quad a \in L}{P \,/\, L \xrightarrow{\tau}_{\rm a} P' \,/\, L}$ | $\dfrac{P \xrightarrow{a}_{\rm a} P' \quad a \notin L}{P \,/\, L \xrightarrow{a}_{\rm a} P' \,/\, L}$ |

**Table 1.** Operational semantic rules for action transitions

The set $\mathbb{P}$ of process terms is obtained by considering typical operators from CCS [49] and CSP [18] together with rate prefix from [38]. In addition to prefix, choice, and parallel composition – which is taken from CSP so as not to hide synchronizations among high-level actions by turning them into $\tau$ as would happen with the CCS parallel composition – we include restriction and hiding as they are necessary to formalize noninterference properties. The syntax for $\mathbb{P}$ is:

$$P \;::=\; \underline{0} \mid a \,.\, P \mid (\lambda) \,.\, P \mid P + P \mid P \parallel_L P \mid P \setminus L \mid P \,/\, L$$

where:

- $\underline{0}$ is the terminated process.
- $a \,.\, \_$, for $a \in \mathcal{A}_\tau$, is the action prefix operator describing a process that can initially perform action $a$.
- $(\lambda) \,.\, \_$, for $\lambda \in \mathbb{R}_{>0}$, is the rate prefix operator describing a process that can initially let an exponentially distributed delay pass with average duration $1/\lambda$.
- $\_ + \_$ is the alternative composition operator expressing a choice between two processes, which is nondeterministic in case of actions, probabilistic in case of rates according to the race policy, or subject to maximal progress otherwise.
- $\_ \parallel_L \_$, for $L \subseteq \mathcal{A}$, is the parallel composition operator allowing two processes to proceed independently on any action not in $L$ as well as on rates thanks to the memoryless property of exponential distributions [38] and forcing them to synchronize on every action in $L$.
- $\_ \setminus L$, for $L \subseteq \mathcal{A}$, is the restriction operator, which prevents the execution of all actions belonging to $L$.
- $\_ \,/\, L$, for $L \subseteq \mathcal{A}$, is the hiding operator, which turns all the executed actions belonging to $L$ into the unobservable action $\tau$.

| | | |
|---|---|---|
| *RatePrefix* | $(\lambda)\,.\,P \xrightarrow{\lambda}_{\rm r} P$ | |
| *RateChoice* | $\dfrac{P_1 \xrightarrow{\lambda}_{\rm r} P'_1}{P_1 + P_2 \xrightarrow{\lambda}_{\rm r} P'_1}$ | $\dfrac{P_2 \xrightarrow{\lambda}_{\rm r} P'_2}{P_1 + P_2 \xrightarrow{\lambda}_{\rm r} P'_2}$ |
| *RateParallel* | $\dfrac{P_1 \xrightarrow{\lambda}_{\rm r} P'_1}{P_1 \parallel_L P_2 \xrightarrow{\lambda}_{\rm r} P'_1 \parallel_L P_2}$ | $\dfrac{P_2 \xrightarrow{\lambda}_{\rm r} P'_2}{P_1 \parallel_L P_2 \xrightarrow{\lambda}_{\rm r} P_1 \parallel_L P'_2}$ |
| *RateRestriction* | $\dfrac{P \xrightarrow{\lambda}_{\rm r} P'}{P \setminus L \xrightarrow{\lambda}_{\rm r} P' \setminus L}$ | |
| *RateHiding* | $\dfrac{P \xrightarrow{\lambda}_{\rm r} P'}{P \,/\, L \xrightarrow{\lambda}_{\rm r} P' \,/\, L}$ | |

**Table 2.** Operational semantic rules for rate transitions

The operational semantic rules for the process language are shown in Tables 1 and 2 for action and rate transitions respectively. Together they produce the MLTS $(\mathbb{P}, \mathcal{A}_\tau, \longrightarrow)$ where $\longrightarrow \; = \; \longrightarrow_{\rm a} \cup \longrightarrow_{\rm r}$, to which the bisimulation equivalences defined in Section 2.2 are applicable. While $\longrightarrow_{\rm a} \subseteq \mathbb{P} \times \mathcal{A}_\tau \times \mathbb{P}$ is a relation, $\longrightarrow_{\rm r} \subseteq \mathbb{P} \times \mathbb{R}_{>0} \times \mathbb{P}$ is deemed to be a multirelation [38]; e.g., from $(\lambda_1)\,.\,P + (\lambda_2)\,.\,P$ there must be two rate transitions to $P$ even when $\lambda_1 = \lambda_2$ otherwise the average sojourn time in the source process would be altered.

## 3 Markovian Information-Flow Security Properties

In this section we recast the definitions of noninteference properties of [26,28,25] – *Nondeterministic Non-Interference* (NNI) and *Non-Deducibility on Composition* (NDC) – by taking as behavioral equivalence the weak or branching bisimilarity of Section 2.2. The intuition behind noninterference in a two-level security system is that, if a group of agents at the high level performs some actions, the effect of those actions should not be seen by any agent at the low level. To formalize this, the restriction and hiding operators play a central role.

**Definition 5.** *Let* $P \in \mathbb{P}$ *and* $\approx \, \in \{\approx_{\rm mw}, \approx_{\rm mb}\}$*:*

– $P \in \mathrm{BSNNI}_{\approx} \iff P \setminus \mathcal{A}_{\mathcal{H}} \approx P \,/\, \mathcal{A}_{\mathcal{H}}$.
– $P \in \mathrm{BNDC}_{\approx} \iff$ *for all* $Q \in \mathbb{P}$ *such that each of its prefixes belongs to* $\mathcal{A}_{\mathcal{H}}$ *and for all* $L \subseteq \mathcal{A}_{\mathcal{H}}$, $P \setminus \mathcal{A}_{\mathcal{H}} \approx ((P \parallel_L Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$.
– $P \in \mathrm{SBSNNI}_{\approx} \iff$ *for all* $P' \in reach(P)$, $P' \in \mathrm{BSNNI}_{\approx}$.
– $P \in \mathrm{P\_BNDC}_{\approx} \iff$ *for all* $P' \in reach(P)$, $P' \in \mathrm{BNDC}_{\approx}$.
– $P \in \mathrm{SBNDC}_{\approx} \iff$ *for all* $P', P'' \in reach(P)$ *such that* $P' \xrightarrow{h}_{\rm a} P''$, $P' \setminus \mathcal{A}_{\mathcal{H}} \approx P'' \setminus \mathcal{A}_{\mathcal{H}}$. ■

*Bisimulation-based Strong Nondeterministic Non-Interference* (BSNNI) has been one of the first and most intuitive proposals. Basically, it is satisfied by any process $P$ that behaves the same when its high-level actions are prevented (as modeled by $P \setminus \mathcal{A_H}$) or when they are considered as hidden, unobservable actions (as modeled by $P \,/\, \mathcal{A_H}$). The equivalence between these two low-level views of $P$ states that a low-level agent cannot deduce the high-level behavior of the system. For instance, in our Markovian setting, a low-level agent that observes the execution of $l$ in $P = l \,.\, (2 \cdot \lambda) \,.\, \underline{0} + l \,.\, ((\lambda) \,.\, h \,.\, l_1 \,.\, \underline{0} + (\lambda) \,.\, h \,.\, l_2 \,.\, \underline{0}) + l \,.\, ((\lambda) \,.\, l_1 \,.\, \underline{0} + (\lambda) \,.\, l_2 \,.\, \underline{0})$ cannot infer anything about the execution of $h$. Indeed, after the execution of $l$, what the low-level agent observes is either a terminal state, reached with rate $2 \cdot \lambda$, or the execution of either $l_1$ or $l_2$, both with rate $\lambda$. Formally, $P \setminus \{h\} \approx P \,/\, \{h\}$ because $l \,.\, (2 \cdot \lambda) \,.\, \underline{0} + l \,.\, ((\lambda) \,.\, \underline{0} + (\lambda) \,.\, \underline{0}) + l \,.\, ((\lambda) \,.\, l_1 \,.\, \underline{0} + (\lambda) \,.\, l_2 \,.\, \underline{0}) \approx l \,.\, (2 \cdot \lambda) \,.\, \underline{0} + l \,.\, ((\lambda) \,.\, \tau \,.\, l_1 \,.\, \underline{0} + (\lambda) \,.\, \tau \,.\, l_2 \,.\, \underline{0}) + l \,.\, ((\lambda) \,.\, l_1 \,.\, \underline{0} + (\lambda) \,.\, l_2 \,.\, \underline{0})$, hence $P$ is $\mathrm{BSNNI}_{\approx}$.

$\mathrm{BSNNI}_{\approx}$ is not powerful enough to capture covert channels that derive from the behavior of a high-level agent interacting with the system. For instance, $l \,.\, (2 \cdot \lambda) \,.\, \underline{0} + l \,.\, ((\lambda) \,.\, h_1 \,.\, l_1 \,.\, \underline{0} + (\lambda) \,.\, h_2 \,.\, l_2 \,.\, \underline{0}) + l \,.\, ((\lambda) \,.\, l_1 \,.\, \underline{0} + (\lambda) \,.\, l_2 \,.\, \underline{0})$ is $\mathrm{BSNNI}_{\approx}$ for the same reason discussed above. However, a high-level agent could decide to enable only $h_1$, thus yielding the low-level view of the system $l \,.\, (2 \cdot \lambda) \,.\, \underline{0} + l \,.\, ((\lambda) \,.\, \tau \,.\, l_1 \,.\, \underline{0} + (\lambda) \,.\, \underline{0}) + l \,.\, ((\lambda) \,.\, l_1 \,.\, \underline{0} + (\lambda) \,.\, l_2 \,.\, \underline{0})$, which is clearly distinguishable from $l \,.\, (2 \cdot \lambda) \,.\, \underline{0} + l \,.\, ((\lambda) \,.\, \underline{0} + (\lambda) \,.\, \underline{0}) + l \,.\, ((\lambda) \,.\, l_1 \,.\, \underline{0} + (\lambda) \,.\, l_2 \,.\, \underline{0})$, as in the former there is a case in which the low-level agent can observe $l_1$ but not $l_2$ after the execution of $l$. To avoid such a limitation, the most obvious solution consists of checking explicitly the interaction on any action set $L \subseteq \mathcal{A_H}$ between the system and every possible high-level agent $Q$. The resulting property is the *Bisimulation-based Non-Deducibility on Composition* (BNDC), which features a universal quantification over $Q$ containing only high-level actions.

Note that in this Markovian setting the high-level agent $Q$ cannot exhibit any rate prefix by definition, otherwise no process would satisfy the BNDC property. To see why, consider the trivially safe process $l \,.\, \underline{0}$ and the high-level agent $(\lambda) \,.\, h \,.\, \underline{0}$. The processes $(l \,.\, \underline{0}) \setminus \mathcal{A_H}$ and $((l \,.\, \underline{0} \,\|_L (\lambda) \,.\, h \,.\, \underline{0}) \,/\, L) \setminus \mathcal{A_H}$ are not equivalent, regardless of the specific $L \subseteq \mathcal{A_H}$, because the former can only perform the low-level action $l$ while the latter can also let time pass before or after the execution of $l$.

To overcome the verification problems related to the quantification over $Q$, several properties stronger than BNDC have been proposed. They all express some persistency conditions, stating that the security checks have to be extended to all the processes reachable from a secure one. Three of the most representative ones among such properties are the variant of BSNNI that requires every reachable process to satisfy BSNNI itself, called *Strong* BSNNI (SBSNNI), the variant of BNDC that requires every reachable process to satisfy BNDC itself, called *Persistent* BNDC (P_BNDC), and *Strong* BNDC (SBNDC), which requires the low-level view of every reachable process to be the same before and after the execution of any high-level action, meaning that the execution of high-level actions must be completely transparent to low-level agents. In the nondeterministic and probabilistic settings, P_BNDC and SBSNNI have been proven to coincide in the case of both weak bisimilarity and branching bisimilarity [28,25,24].

## 4 Characteristics of Markovian Security Properties

In this section we investigate preservation and compositionality characteristics of the noninterference properties introduced in the previous section (Section 4.1) as well as the inclusion relationships between the ones based on $\approx_{\mathrm{mw}}$ and the ones based on $\approx_{\mathrm{mb}}$ (Section 4.2).

### 4.1 Preservation and Compositionality

All the Markovian noninterference properties of Definition 5 turn out to be preserved by the bisimilarity employed in their definition. This means that if a process $P_1$ is secure under any of such properties, then every other equivalent process $P_2$ is secure too according to the same property. This is very useful for automated property verification, as it allows us to work with the process with the smallest state space among the equivalent ones.

The preservation result of Theorem 1 immediately follows from Lemma 1 below, which ensures that $\approx_{\mathrm{mw}}$ and $\approx_{\mathrm{mb}}$ are congruences with respect to all the operators occurring in the aforementioned noninterference properties. Congruence with respect to action and rate prefixes is also addressed as it will be exploited in the proof of the compositionality result of Theorem 2. Some of the following congruence properties for $\approx_{\mathrm{mw}}$ are already known from [38].

**Lemma 1.** *Let $P_1, P_2 \in \mathbb{P}$ and $\approx \in \{\approx_{\mathrm{mw}}, \approx_{\mathrm{mb}}\}$. If $P_1 \approx P_2$ then:*

1. $a \,.\, P_1 \approx a \,.\, P_2$ *for all* $a \in \mathcal{A}_\tau$.
2. $(\lambda) \,.\, P_1 \approx (\lambda) \,.\, P_2$ *for all* $\lambda \in \mathbb{R}_{>0}$.
3. $P_1 \,\|_L\, P \approx P_2 \,\|_L\, P$ *and* $P \,\|_L\, P_1 \approx P \,\|_L\, P_2$ *for all* $L \subseteq \mathcal{A}$ *and* $P \in \mathbb{P}$.
4. $P_1 \setminus L \approx P_2 \setminus L$ *for all* $L \subseteq \mathcal{A}$.
5. $P_1 \,/\, L \approx P_2 \,/\, L$ *for all* $L \subseteq \mathcal{A}$. ■

**Theorem 1.** *Let $P_1, P_2 \in \mathbb{P}$, $\approx \in \{\approx_{\mathrm{mw}}, \approx_{\mathrm{mb}}\}$, and $\mathcal{P} \in \{\mathrm{BSNNI}_\approx, \mathrm{BNDC}_\approx, \mathrm{SBSNNI}_\approx, \mathrm{P\_BNDC}_\approx, \mathrm{SBNDC}_\approx\}$. If $P_1 \approx P_2$ then $P_1 \in \mathcal{P} \Longleftrightarrow P_2 \in \mathcal{P}$.* ■

As far as modular verification is concerned, like in the nondeterministic and probabilistic settings [26,25,24] only the local properties $\mathrm{SBSNNI}_\approx$, $\mathrm{P\_BNDC}_\approx$, and $\mathrm{SBNDC}_\approx$ are compositional, i.e., are preserved by some operators of the calculus in certain circumstances. Moreover, similar to [25,24], compositionality with respect to parallel composition is limited, for $\mathrm{SBSNNI}_{\approx_{\mathrm{mb}}}$ and $\mathrm{P\_BNDC}_{\approx_{\mathrm{mb}}}$, to the case in which synchronizations can take place only among low-level actions, i.e., $L \subseteq \mathcal{A}_{\mathcal{L}}$. A limitation to low-level actions applies to action prefix and hiding as well, whilst this is not the case for restriction. Another analogy with the nondeterministic and probabilistic settings [26,25,24] is that none of the considered noninterference properties is compositional with respect to alternative composition. As an example, let us examine processes $P_1 = l \,.\, \underline{0}$ and $P_2 = h \,.\, \underline{0}$. Both processes are $\mathrm{BSNNI}_\approx$, as $(l \,.\, \underline{0}) \setminus \{h\} \approx (l \,.\, \underline{0}) \,/\, \{h\}$ and $(h \,.\, \underline{0}) \setminus \{h\} \approx (h \,.\, \underline{0}) \,/\, \{h\}$, but $P_1 + P_2 \notin \mathrm{BSNNI}_\approx$, because $(l \,.\, \underline{0} + h \,.\, \underline{0}) \setminus \{h\} \approx l \,.\, \underline{0} \not\approx l \,.\, \underline{0} + \tau \,.\, \underline{0} \approx (l \,.\, \underline{0} + h \,.\, \underline{0}) \,/\, \{h\}$. It is easy to check that $P_1 + P_2 \notin \mathcal{P}$ also for $\mathcal{P} \in \{\mathrm{BNDC}_\approx, \mathrm{SBSNNI}_\approx, \mathrm{SBNDC}_\approx\}$.

**Theorem 2.** *Let* $P, P_1, P_2 \in \mathbb{P}$, $\approx \in \{\approx_{\rm mw}, \approx_{\rm mb}\}$, $\mathcal{P} \in \{\text{SBSNNI}_\approx, \text{P\_BNDC}_\approx, \text{SBNDC}_\approx\}$. *Then:*

1. $P \in \mathcal{P} \Longrightarrow a \,.\, P \in \mathcal{P}$ *for all* $a \in \mathcal{A}_\mathcal{L} \cup \{\tau\}$.
2. $P \in \mathcal{P} \Longrightarrow (\lambda) \,.\, P \in \mathcal{P}$ *for all* $\lambda \in \mathbb{R}_{>0}$.
3. $P_1, P_2 \in \mathcal{P} \Longrightarrow P_1 \parallel_L P_2 \in \mathcal{P}$ *for* $L \subseteq \mathcal{A}_\mathcal{L}$ *if* $\mathcal{P} \in \{\text{SBSNNI}_{\approx_{\rm mb}}, \text{P\_BNDC}_{\approx_{\rm mb}}\}$ *or for* $L \subseteq \mathcal{A}$ *if* $\mathcal{P} \in \{\text{SBSNNI}_{\approx_{\rm mw}}, \text{P\_BNDC}_{\approx_{\rm mw}}, \text{SBNDC}_{\approx_{\rm mw}}, \text{SBNDC}_{\approx_{\rm mb}}\}$.
4. $P \in \mathcal{P} \Longrightarrow P \setminus L \in \mathcal{P}$ *for all* $L \subseteq \mathcal{A}$.
5. $P \in \mathcal{P} \Longrightarrow P \,/\, L \in \mathcal{P}$ *for all* $L \subseteq \mathcal{A}_\mathcal{L}$. ■

### 4.2 Taxonomy of Security Properties

First of all, similar to the nondeterministic and probabilistic settings [26,25,24] the properties in Definition 5 turn out to be increasingly finer. This result holds for both those based on $\approx_{\rm mw}$ and those based on $\approx_{\rm mb}$.

**Theorem 3.** *Let* $\approx \in \{\approx_{\rm mw}, \approx_{\rm mb}\}$. *Then:*

$$\text{SBNDC}_\approx \subsetneq \text{SBSNNI}_\approx = \text{P\_BNDC}_\approx \subsetneq \text{BNDC}_\approx \subsetneq \text{BSNNI}_\approx$$ ■

Secondly, we observe that all the $\approx_{\rm mb}$-based noninterference properties imply the corresponding $\approx_{\rm mw}$-based ones, due to the fact that $\approx_{\rm mb}$ is finer than $\approx_{\rm mw}$.

**Theorem 4.** *The following inclusions hold:*

1. $\text{BSNNI}_{\approx_{\rm mb}} \subsetneq \text{BSNNI}_{\approx_{\rm mw}}$.
2. $\text{BNDC}_{\approx_{\rm mb}} \subsetneq \text{BNDC}_{\approx_{\rm mw}}$.
3. $\text{SBSNNI}_{\approx_{\rm mb}} \subsetneq \text{SBSNNI}_{\approx_{\rm mw}}$.
4. $\text{P\_BNDC}_{\approx_{\rm mb}} \subsetneq \text{P\_BNDC}_{\approx_{\rm mw}}$.
5. $\text{SBNDC}_{\approx_{\rm mb}} \subsetneq \text{SBNDC}_{\approx_{\rm mw}}$. ■

All the inclusions above are strict by virtue of the following result; for an example of $P_1$ and $P_2$ below, see Figure 1.

**Theorem 5.** *Let* $P_1, P_2 \in \mathbb{P}$ *be such that* $P_1 \approx_{\rm mw} P_2$ *but* $P_1 \not\approx_{\rm mb} P_2$. *If no high-level actions occur in* $P_1$ *and* $P_2$, *then* $Q \in \{P_1 + h \,.\, P_2, P_2 + h \,.\, P_1\}$ *is such that:*

1. $Q \in \text{BSNNI}_{\approx_{\rm mw}}$ *but* $Q \notin \text{BSNNI}_{\approx_{\rm mb}}$.
2. $Q \in \text{BNDC}_{\approx_{\rm mw}}$ *but* $Q \notin \text{BNDC}_{\approx_{\rm mb}}$.
3. $Q \in \text{SBSNNI}_{\approx_{\rm mw}}$ *but* $Q \notin \text{SBSNNI}_{\approx_{\rm mb}}$.
4. $Q \in \text{P\_BNDC}_{\approx_{\rm mw}}$ *but* $Q \notin \text{P\_BNDC}_{\approx_{\rm mb}}$.
5. $Q \in \text{SBNDC}_{\approx_{\rm mw}}$ *but* $Q \notin \text{SBNDC}_{\approx_{\rm mb}}$. ■

The diagram in Figure 2 summarizes the inclusions among the various noninterference properties based on the results in Theorems 3 and 4, where $\mathcal{P} \to \mathcal{Q}$ means that $\mathcal{P}$ is strictly included in $\mathcal{Q}$. These inclusions follow the same pattern as the nondeterministic and probabilistic settings [25,24].

The arrows missing in the diagram, witnessing incomparability, are justified by the following counterexamples:

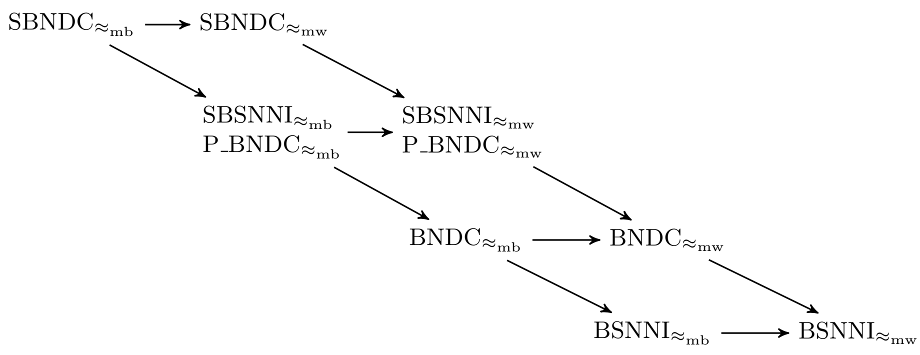


**Fig. 2.** Taxonomy of security properties based on Markovian bisimilarities

- $\text{SBNDC}_{\approx_{\rm mw}}$ vs. $\text{SBSNNI}_{\approx_{\rm mb}}$. The process $\tau \,.\, l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} + h \,.\, l \,.\, \underline{0}$ is $\text{BSNNI}_{\approx_{\rm mb}}$ as $(\tau \,.\, l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} + h \,.\, l \,.\, \underline{0}) \setminus \{h\} \approx_{\rm mb} \tau \,.\, l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} \approx_{\rm mb} \tau \,.\, l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} + \tau \,.\, l \,.\, \underline{0} \approx_{\rm mb} (\tau \,.\, l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} + h \,.\, l \,.\, \underline{0}) \,/\, \{h\}$. It is also $\text{SBSNNI}_{\approx_{\rm mb}}$ because every reachable process does not enable further high-level actions. However, it is not $\text{SBNDC}_{\approx_{\rm mw}}$ because after executing the high-level action $h$ it can perform a single $l$-action, while the original process with the restriction on high-level actions can go along a path where it performs two $l$-actions. On the other hand, the process $Q$ mentioned in Theorem 5 is $\text{SBNDC}_{\approx_{\rm mw}}$ but neither $\text{BSNNI}_{\approx_{\rm mb}}$ nor $\text{SBSNNI}_{\approx_{\rm mb}}$.
- $\text{SBSNNI}_{\approx_{\rm mw}}$ vs. $\text{BNDC}_{\approx_{\rm mb}}$. The process $l \,.\, h \,.\, l \,.\, \underline{0} + l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0}$ is $\text{BSNNI}_{\approx_{\rm mb}}$ as $(l \,.\, h \,.\, l \,.\, \underline{0} + l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0}) \setminus \{h\} \approx_{\rm mb} l \,.\, \underline{0} + l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} \approx_{\rm mb} l \,.\, \tau \,.\, l \,.\, \underline{0} + l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} \approx_{\rm mb} (l \,.\, h \,.\, l \,.\, \underline{0} + l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0}) \,/\, \{h\}$. The same process is $\text{BNDC}_{\approx_{\rm mb}}$ too as it includes only one high-level action, hence the only possible high-level strategy coincides with the check conducted by $\text{BSNNI}_{\approx_{\rm mb}}$. However, it is not $\text{SBSNNI}_{\approx_{\rm mw}}$ because of the reachable process $h \,.\, l \,.\, \underline{0}$, which is not $\text{BSNNI}_{\approx_{\rm mw}}$. On the other hand, the process $Q$ mentioned in Theorem 5 is $\text{SBSNNI}_{\approx_{\rm mw}}$ but not $\text{BSNNI}_{\approx_{\rm mb}}$ and, therefore, not even $\text{BNDC}_{\approx_{\rm mb}}$.
- $\text{BNDC}_{\approx_{\rm mw}}$ vs. $\text{BSNNI}_{\approx_{\rm mb}}$. The process $l \,.\, (2{\cdot}\lambda) \,.\, \underline{0} + l \,.\, ((\lambda) \,.\, h_1 \,.\, l_1 \,.\, \underline{0} + (\lambda) \,.\, h_1 \,.\, l_1 \,.\, \underline{0} + l \,.\, ((\lambda) \,.\, l_1 \,.\, \underline{0} + (\lambda) \,.\, l_2 \,.\, \underline{0}))$ is $\text{BSNNI}_{\approx_{\rm mb}}$ but not $\text{BNDC}_{\approx_{\rm mw}}$ as discussed in Section 3. In contrast, the process $Q$ mentioned in Theorem 5 is both $\text{BSNNI}_{\approx_{\rm mw}}$ and $\text{BNDC}_{\approx_{\rm mw}}$, but not $\text{BSNNI}_{\approx_{\rm mb}}$.

Like in the nondeterministic and probabilistic settings [25,24], the strongest property based on weak Markovian bisimilarity ($\text{SBNDC}_{\approx_{\rm mw}}$) and the weakest property based on Markovian branching bisimilarity ($\text{BSNNI}_{\approx_{\rm mb}}$) are incomparable too. The former is a very restrictive property because it requires a local check every time a high-level action is performed, while the latter requires a check only on the initial state. On the other hand, as shown in Theorem 5, it is very easy to construct processes that are secure under properties based on $\approx_{\rm mw}$ but not on $\approx_{\rm mb}$, due to the minimal number of high-level actions in $Q$.

# 5 Relating Nondeterministic, Probabilistic, and Markovian Taxonomies

Let us compare our Markovian taxonomy with the nondeterministic one of [25]. In the following, we assume that $\approx_{\rm w}$ denotes the weak nondeterministic bisimilarity of [49] and $\approx_{\rm b}$ denotes the nondeterministic branching bisimilarity of [33]. These can also be derived from the corresponding definitions in Section 2.2 by ignoring the clause involving the *rate* function. Since we are abstracting from time, given a process $P \in \mathbb{P}$ we can obtain its nondeterministic variant, denoted by $nd(P)$, by replacing every occurrence of $(\lambda) \,.\, P'$ with $\tau \,.\, P'$. However, to respect maximal progress, first we have to eliminate every subprocess starting with a rate prefix that is alternative to a subprocess starting with a $\tau$-prefix. To accomplish this transformation syntactically, we focus on the set $\mathbb{P}_{\rm seq}$ of sequential processes, i.e., without parallel composition; this is not too restrictive because, in the absence of recursion, parallel composition can be eliminated by repeatedly applying a Markovian variant of the expansion law [38].

The next proposition states that if two sequential processes are equivalent according to any of the weak bisimilarities in Section 2.2, then their nondeterministic variants are equivalent according to the corresponding nondeterministic weak bisimilarity. The inverse does not hold; e.g., processes $P_1 = (1) \,.\, a \,.\, \underline{0}$ and $P_2 = (2) \,.\, a \,.\, \underline{0}$ are such that $P_1 \not\approx_{\rm mw} P_2$ and $P_1 \not\approx_{\rm mb} P_2$, but their nondeterministic counterparts coincide as both of them are equal to $\tau \,.\, a \,.\, \underline{0}$.

**Proposition 1.** *Let $P_1, P_2 \in \mathbb{P}_{\rm seq}$. Then:*

- $P_1 \approx_{\rm mw} P_2 \Longrightarrow nd(P_1) \approx_{\rm w} nd(P_2)$.
- $P_1 \approx_{\rm mb} P_2 \Longrightarrow nd(P_1) \approx_{\rm b} nd(P_2)$. ∎

An immediate consequence is that if a sequential process is secure under any of the Markovian noninterference properties of Section 3, then its nondeterministic variant is secure under the corresponding nondeterministic property. The taxonomy of Figure 2 thus extends to the left the one in [25], as each of the properties of Section 3 is finer than its nondeterministic counterpart.

**Corollary 1.** *Let $\mathcal{P}_{\rm mk} \in \{\mathrm{BSNNI}_{\approx_{\rm mk}}, \mathrm{BNDC}_{\approx_{\rm mk}}, \mathrm{SBSNNI}_{\approx_{\rm mk}}, \mathrm{P\_BNDC}_{\approx_{\rm mk}}, \mathrm{SBNDC}_{\approx_{\rm mk}}\}$ and $\mathcal{P}_{\rm nd} \in \{\mathrm{BSNNI}_{\approx_{\rm nd}}, \mathrm{BNDC}_{\approx_{\rm nd}}, \mathrm{SBSNNI}_{\approx_{\rm nd}}, \mathrm{P\_BNDC}_{\approx_{\rm nd}}, \mathrm{SBNDC}_{\approx_{\rm nd}}\}$ for $\approx_{\rm mk} \in \{\approx_{\rm mw}, \approx_{\rm mb}\}$ and $\approx_{\rm nd} \in \{\approx_{\rm w}, \approx_{\rm b}\}$, where $\mathcal{P}_{\rm nd}$ is meant to be the nondeterministic variant of $\mathcal{P}_{\rm mk}$. Then $P \in \mathcal{P}_{\rm mk} \Longrightarrow nd(P) \in \mathcal{P}_{\rm nd}$ for all $P \in \mathbb{P}_{\rm seq}$.* ∎

We now compare our Markovian taxonomy with the probabilistic one of [24], which relies on the weak probabilistic bisimilarity $\approx_{\rm pw}$ of [54] and the probabilistic branching bisimilarity $\approx_{\rm pb}$ of [6], also derivable from the corresponding definitions in Section 2.2 by replacing the clause involving cumulative rates with a clause involving cumulative probabilities. We focus on the set $\mathbb{P}_{\rm alt,seq}$ of processes in which action prefixes and rate prefixes alternate – to comply with the strictly alternating model of [36] adopted for probabilistic processes – that are sequential – as rate transitions, as opposed to probabilistic ones, do not synchronize. Since we are abstracting from time, given a process $P \in \mathbb{P}_{\rm alt,seq}$ we can obtain its probabilistic variant, denoted by $pr(P)$, by replacing every occurrence of $\sum_{i \in I} (\lambda_i) \, . \, P_i$ with $\bigoplus_{i \in I} [p_i] pr(P_i)$ where $\bigoplus$ is the probabilistic choice operator and $p_i = \lambda_i / \sum_{j \in I} \lambda_j$. It is worth noting that over $\mathbb{P}_{\rm alt,seq}$ the weak bisimilarities $\approx_{\rm mw}$ and $\approx_{\rm mb}$ boil down to the strong bisimilarity $\sim_{\rm m}$ of Definition 2. This is due to the strict alternation between action prefixes and rate prefixes and the fact that the two weak bisimilarities do not abstract from rate transitions ($\approx_{\rm pw}$ and $\approx_{\rm pb}$ can instead abstract from probabilistic transitions).

The next proposition states that if two sequential alternating processes are equivalent according to any of the weak bisimilarities in Section 2.2, then their probabilistic variants are equivalent according to the corresponding probabilistic weak bisimilarity. The inverse does not hold; e.g., the probabilistic counterparts of the two inequivalent processes $(1) \, . \, a \, . \, \underline{0}$ and $(2) \, . \, a \, . \, \underline{0}$ coincide as both of them are equal to $[1] a \, . \, \underline{0}$.

**Proposition 2.** *Let $P_1, P_2 \in \mathbb{P}_{\rm alt,seq}$. Then:*

- $P_1 \approx_{\rm mw} P_2 \Longrightarrow pr(P_1) \approx_{\rm pw} pr(P_2)$.
- $P_1 \approx_{\rm mb} P_2 \Longrightarrow pr(P_1) \approx_{\rm pb} pr(P_2)$. ■

An immediate consequence is that if a sequential alternating process is secure under any of the Markovian noninterference properties of Section 3, then its probabilistic variant is secure under the corresponding probabilistic property. The taxonomy of Figure 2 thus extends to the left also the one in [24], as each of the properties of Section 3 is finer than its probabilistic counterpart.

**Corollary 2.** *Let $\mathcal{P}_{\rm mk} \in \{{\rm BSNNI}_{\approx_{\rm mk}}, {\rm BNDC}_{\approx_{\rm mk}}, {\rm SBSNNI}_{\approx_{\rm mk}}, {\rm P\_BNDC}_{\approx_{\rm mk}}, {\rm SBNDC}_{\approx_{\rm mk}}\}$ and $\mathcal{P}_{\rm pr} \in \{{\rm BSNNI}_{\approx_{\rm pr}}, {\rm BNDC}_{\approx_{\rm pr}}, {\rm SBSNNI}_{\approx_{\rm pr}}, {\rm P\_BNDC}_{\approx_{\rm pr}}, {\rm SBNDC}_{\approx_{\rm pr}}\}$ for $\approx_{\rm mk} \in \{\approx_{\rm mw}, \approx_{\rm mb}\}$ and $\approx_{\rm pr} \in \{\approx_{\rm pw}, \approx_{\rm pb}\}$, where $\mathcal{P}_{\rm pr}$ is meant to be the probabilistic variant of $\mathcal{P}_{\rm mk}$. Then $P \in \mathcal{P}_{\rm mk} \Longrightarrow pr(P) \in \mathcal{P}_{\rm pr}$ for all $P \in \mathbb{P}_{\rm alt,seq}$.* ■

## 6 Reversibility via Weak Markovian Back-and-Forth Bisimilarity

In [22] it was shown that, for nondeterministic processes, weak back-and-forth bisimilarity coincides with branching bisimilarity. We now extend that result so that Markovian branching bisimilarity can be employed in the noninterference analysis of reversible processes featuring nondeterminism and stochastic time.

An MLTS $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ represents a reversible process if each of its transitions is seen as bidirectional. When going backward, it is of paramount importance to respect causality, i.e., the last performed transition must be the first one to be undone. Following [22] we set up an equivalence that enforces not only causality but also history preservation. This means that, when going backward, a process can only move along the path representing the history that brought the process to the current state even in the presence of concurrency. To accomplish this, the equivalence has to be defined over computations, not over states, and the notion of transition has to be suitably revised. We start by adapting the notation of the nondeterministic setting of [22] to our nondeterministic and stochastically timed setting. We use $\ell$ for a label in $\mathcal{A}_\tau \cup \mathbb{R}_{>0}$.

**Definition 6.** *A sequence* $\xi = (s_0, \ell_1, s_1)(s_1, \ell_2, s_2) \dots (s_{n-1}, \ell_n, s_n) \in \ \longrightarrow^*$ *is a* path *of length* $n$ *from state* $s_0$*. We let* $\mathit{first}(\xi) = s_0$ *and* $\mathit{last}(\xi) = s_n$*; the empty path is indicated with* $\varepsilon$*. We denote by* $\mathit{path}(s)$ *the set of paths from* $s$*.* ■

**Definition 7.** *A pair* $\rho = (s, \xi)$ *is called a* run *from state* $s$ *iff* $\xi \in \mathit{path}(s)$*, in which case we let* $\mathit{path}(\rho) = \xi$*,* $\mathit{first}(\rho) = \mathit{first}(\xi) = s$*,* $\mathit{last}(\rho) = \mathit{last}(\xi)$*, with* $\mathit{first}(\rho) = \mathit{last}(\rho) = s$ *when* $\xi = \varepsilon$*. We denote by* $\mathit{run}(s)$ *the set of runs from state* $s$*. Given* $\rho = (s, \xi) \in \mathit{run}(s)$ *and* $\rho' = (s', \xi') \in \mathit{run}(s')$*, their composition* $\rho\rho' = (s, \xi\xi') \in \mathit{run}(s)$ *is defined iff* $\mathit{last}(\rho) = \mathit{first}(\rho') = s'$*. We write* $\rho \xrightarrow{\ell} \rho'$ *iff there exists* $\rho'' = (\bar{s}, (\bar{s}, \ell, s'))$ *with* $\bar{s} = \mathit{last}(\rho)$ *such that* $\rho' = \rho\rho''$*; note that* $\mathit{first}(\rho) = \mathit{first}(\rho')$*. Moreover rate is lifted in the expected way.* ■

In the considered MLTS we work with the set $\mathcal{U}$ of runs in lieu of $\mathcal{S}$. Following [22], given a run $\rho$, we distinguish between *outgoing* and *incoming* action transitions of $\rho$ during the weak bisimulation game. Like in [16], this does not apply to rate transitions, in the sense that the cumulative rates of incoming rate transitions are not compared. If this were not the case, states like $(\lambda_1) \,.\, (\underline{0} \backslash \emptyset) + (\lambda_2) \,.\, (\underline{0} \,/\, \emptyset)$ and $(\lambda_1 + \lambda_2) \,.\, \underline{0}$ – which are indistinguishable in the forward direction – would be told apart because the incoming cumulative rate from the class formed by those two states is $\lambda_1$, $\lambda_2$, or $\lambda_1 + \lambda_2$ depending on whether $\underline{0} \setminus \emptyset$, $\underline{0} \,/\, \emptyset$, or $\underline{0}$ is considered. When comparing the cumulative rates of outgoing transitions, we slightly deviate from the corresponding clause in Definition 4 to set up a more symmetric clause inspired by an alternative characterization of $\approx_{\mathrm{mw}}$ in [38] that is helpful to prove the forthcoming Lemma 2.

**Definition 8.** *Let* $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ *be an MLTS. We say that* $s_1, s_2 \in \mathcal{S}$ *are* weakly Markovian back-and-forth bisimilar*, written* $s_1 \approx_{\mathrm{mbf}} s_2$*, iff* $((s_1, \varepsilon), (s_2, \varepsilon)) \in \mathcal{B}$ *for some weak Markovian back-and-forth bisimulation* $\mathcal{B}$*. An equivalence relation* $\mathcal{B}$ *over* $\mathcal{U}$ *is a* weak Markovian back-and-forth bisimulation *iff, whenever* $(\rho_1, \rho_2) \in \mathcal{B}$*, then:*

- *For each* $\rho_1 \xrightarrow{a}_{\mathrm{a}} \rho'_1$ *there exists* $\rho_2 \xRightarrow{\hat{a}}_{\mathrm{a}} \rho'_2$ *such that* $(\rho'_1, \rho'_2) \in \mathcal{B}$*.*
- *For each* $\rho'_1 \xrightarrow{a}_{\mathrm{a}} \rho_1$ *there exists* $\rho'_2 \xRightarrow{\hat{a}}_{\mathrm{a}} \rho_2$ *such that* $(\rho'_1, \rho'_2) \in \mathcal{B}$*.*
- *For each* $\rho_1 \xRightarrow{\tau^*}_{\mathrm{a}} \rho'_1$ *with* $\rho'_1 \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$ *there exists* $\rho_2 \xRightarrow{\tau^*}_{\mathrm{a}} \rho'_2$ *with* $\rho'_2 \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$ *such that* $(\rho'_1, \rho'_2) \in \mathcal{B}$ *and* $\mathit{rate}(\rho'_1, C) = \mathit{rate}(\rho'_2, C)$ *for all equivalence classes* $C \in \mathcal{U}/\mathcal{B}$*.*
- *For each* $\rho'_1 \xrightarrow{\lambda_1}_{\mathrm{r}} \rho_1$ *with* $\rho'_1 \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$ *there exists* $\rho'_2 \xRightarrow{\tau^*}_{\mathrm{a}} \bar{\rho}'_2 \xrightarrow{\lambda_2}_{\mathrm{r}} \bar{\rho}_2 \xRightarrow{\tau^*}_{\mathrm{a}} \rho_2$ *with* $\bar{\rho}'_2 \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$ *such that* $(\rho_1, \bar{\rho}_2) \in \mathcal{B}$*,* $(\rho'_1, \bar{\rho}'_2) \in \mathcal{B}$*, and* $(\rho'_1, \rho'_2) \in \mathcal{B}$*.* ■

We show that weak Markovian back-and-forth bisimilarity over runs coincides with $\approx_{\rm mb}$, the forward-only Markovian branching bisimilarity over states. We proceed by adopting the proof strategy followed in [22] to show that their weak back-and-forth bisimilarity over runs coincides with the forward-only branching bisimilarity over states of [33]. Therefore we start by proving that $\approx_{\rm mbf}$ satisfies the *cross property*. This means that, whenever two runs of two $\approx_{\rm mbf}$-equivalent states can perform a sequence of finitely many $\tau$-transitions such that each of the two target runs is $\approx_{\rm mbf}$-equivalent to the source run of the other sequence, then the two target runs are $\approx_{\rm mbf}$-equivalent to each other as well.

**Lemma 2.** *Let $s_1, s_2 \in \mathcal{S}$ with $s_1 \approx_{\rm mbf} s_2$. For all $\rho'_1, \rho''_1 \in run(s_1)$ such that $\rho'_1 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho''_1$ and for all $\rho'_2, \rho''_2 \in run(s_2)$ such that $\rho'_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho''_2$, if $\rho'_1 \approx_{\rm mbf} \rho''_2$ and $\rho''_1 \approx_{\rm mbf} \rho'_2$ then $\rho''_1 \approx_{\rm mbf} \rho''_2$.* ■

**Theorem 6.** *Let $s_1, s_2 \in \mathcal{S}$. Then $s_1 \approx_{\rm mbf} s_2 \iff s_1 \approx_{\rm mb} s_2$.* ■

Therefore the properties $\text{BSNNI}_{\approx_{\rm mb}}$, $\text{BNDC}_{\approx_{\rm mb}}$, $\text{SBSNNI}_{\approx_{\rm mb}}$, $\text{P\_BNDC}_{\approx_{\rm mb}}$, and $\text{SBNDC}_{\approx_{\rm mb}}$ do not change if $\approx_{\rm mb}$ is replaced by $\approx_{\rm mbf}$. This allows us to study noninterference properties for reversible systems featuring nondeterminism and stochastic time by using $\approx_{\rm mb}$ in a standard Markovian process calculus like the one of Section 2.3, without having to resort to external memories [20], communication keys [55], or executed action decorations [17].

## 7 Use Case: DBMS Obfuscation and Permission Mechanisms

In [25] we have modeled the authentication mechanism of a database management system (DBMS) in which the database can be used to feed a machine learning (ML) module for training purposes, where reversible transactions are supported [23]. Due to privacy issues, DBMS users are not allowed to know which data are actually chosen to train the ML module [7]. Hence, for analysis purposes, the interactions between users and the DBMS are considered to be low level, while the interactions between the DBMS and the ML module are considered to be high level. The aim of the noninterference analysis is thus to check whether users can infer the utilization of their data in the ML dataset. In this section we present two novel examples for that scenario, which show the nature of the interferences emerging in a stochastically timed setting and the greater expressive power of branching bisimulation semantics in this setting.

Let $l_{\rm w}$ be a low-level action expressing the execution of a write transaction and $l_{\rm ow}$ be an analogous action that includes also the additional application of an obfuscation mechanism over written data for privacy purposes [1]. We assume that only obfuscated data can feed the ML module. Given the high-level actions $h$ and $h'$ denoting interactions between the DBMS and the ML module, consider the following process:

$$\begin{aligned} DBMS \;=\; & h \,.\, \tau \,.\, (l_{\rm w} \,.\, \underline{0} + l_{\rm ow} \,.\, h' \,.\, \underline{0}) \;+ \\ & \tau \,.\, (\tau \,.\, (l_{\rm w} \,.\, \underline{0} + l_{\rm ow} \,.\, \underline{0}) + l_{\rm w} \,.\, \underline{0}) \end{aligned}$$

The subprocess guarded by the high-level action $h$ represents the behavior of the DBMS whenever the ML module is activated through the $h$-based interaction. After an internal activity, the DBMS offers a choice between the two available transaction mechanisms, by assuming that only in the second case the transaction data will feed the ML module (through the $h'$-based interaction). The alternative subprocess guarded by a $\tau$-action describes the behavior of the DBMS whenever the ML module is not involved. Note that this subprocess replicates the behavior above to simulate the presence of the ML module and, thus, makes it transparent from the viewpoint of users. In addition, the subprocess immediately enables also action $l_{\mathrm{w}}$ for efficiency reasons and because, in any case, the transaction data will not feed the ML module.

Since the two low views $\tau \,.\, (l_{\mathrm{w}} \,.\, \underline{0} + l_{\mathrm{ow}} \,.\, \tau \,.\, \underline{0})$ and $\tau \,.\, (l_{\mathrm{w}} \,.\, \underline{0} + l_{\mathrm{ow}} \,.\, \underline{0}) + l_{\mathrm{w}} \,.\, \underline{0}$ are both weakly bisimilar and branching bisimilar, we immediately derive that all the noninterference properties of the nondeterministic taxonomy are satisfied. In particular, note that $\mathit{DBMS} \setminus \{h, h'\}$ and $\mathit{DBMS} \,/\, \{h, h'\}$ enable weakly/branching bisimilar behaviors by virtue of the observation above. However, if we add to the model the time spent by the DBMS in the internal activity before the choice about the possible obfuscation, we obtain:

$$\begin{aligned} \mathit{DBMS}_{\mathrm{stoch\_timed}} \;=\;& h \,.\, (\lambda_1) \,.\, (l_{\mathrm{w}} \,.\, \underline{0} + l_{\mathrm{ow}} \,.\, h' \,.\, \underline{0}) + \\ & \tau \,.\, ((\lambda_2) \,.\, (l_{\mathrm{w}} \,.\, \underline{0} + l_{\mathrm{ow}} \,.\, \underline{0}) + l_{\mathrm{w}} \,.\, \underline{0}) \end{aligned}$$

where the rates $\lambda_1$ and $\lambda_2$ govern the delays discussed above for the ML module being involved or not respectively (note that $\mathit{DBMS}$ is the nondeterministic version of $\mathit{DBMS}_{\mathrm{stoch\_timed}}$). In this enriched process, the equivalence between the two low views $(\lambda_1) \,.\, (l_{\mathrm{w}} \,.\, \underline{0} + l_{\mathrm{ow}} \,.\, \tau \,.\, \underline{0})$ and $(\lambda_2) \,.\, (l_{\mathrm{w}} \,.\, \underline{0} + l_{\mathrm{ow}} \,.\, \underline{0}) + l_{\mathrm{w}} \,.\, \underline{0}$ does not hold for the Markovian versions of the two bisimilarities. This means that no noninterference property of the Markovian taxonomy is satisfied. Note that this negative result holds also in the case $\lambda_1 = \lambda_2$, because only in the second subprocess it is possible to observe action $l_{\mathrm{w}}$ with no delay.

Let us consider a more sophisticated variant of the system above, including an explicit permission mechanism involving users. Let $l_{\mathrm{no\_auth}}$ be a low-level action expressing that users do not authorize the DBMS to feed the ML module with the data of their transaction, $l_{\mathrm{no\_auth\_o}}$ be a low-level action expressing that users do not authorize the obfuscation of the data of their transaction, and $l_{\mathrm{commit}}$ be a low-level action expressing the execution of the transaction. Then in the following process:

$$\begin{aligned} \mathit{DBMS}' = h \,.\, & (l_{\mathrm{no\_auth}} \,.\, l_{\mathrm{commit}} \,.\, \underline{0} + \tau \,.\, (l_{\mathrm{no\_auth\_o}} \,.\, l_{\mathrm{commit}} \,.\, \underline{0} + \tau \,.\, l_{\mathrm{commit}} \,.\, h' \,.\, \underline{0})) + \\ & \tau \,.\, ((l_{\mathrm{no\_auth}} \,.\, l_{\mathrm{commit}} \,.\, \underline{0} + \tau \,.\, (l_{\mathrm{no\_auth\_o}} \,.\, l_{\mathrm{commit}} \,.\, \underline{0} + \tau \,.\, l_{\mathrm{commit}} \,.\, \underline{0})) + \\ & \quad \tau \,.\, l_{\mathrm{commit}} \,.\, \underline{0}) \end{aligned}$$

the subprocess guarded by the high-level action $h$ – call it $P$ – expresses the behavior of the system whenever the ML module is active. In particular, in such a case, once that no authorization has been forbidden, the committed data are transferred to the training set (through the $h'$-based interaction). Now, consider the alternative subprocess guarded by a $\tau$-action and modeling the absence of the ML module – call it $Q$. This subprocess simulates the same behavior as $P$ in the absence of the ML module and, in addition, enables the branch $\tau \,.\, l_{\mathrm{commit}} \,.\, \underline{0}$ expressing the immediate execution of the transaction, which does not require any authorization because the ML module is not active. The two subprocesses $P \,/\, \{h'\}$ and $Q$ are weakly bisimilar but not branching bisimilar. In fact, $P \,/\, \{h'\}$ cannot respond to the $\tau$-action of $Q$ leading to $l_{\mathrm{commit}} \,.\, \underline{0}$ in a way that complies with the branching bisimulation semantics.

From the back-and-forth perspective, consider executing the run $\tau \,.\, l_{\mathrm{commit}} \,.\, \underline{0}$ of $Q$ and the run $\tau \,.\, \tau \,.\, l_{\mathrm{commit}} \,.\, \tau \,.\, \underline{0}$ of $P \,/\, \{h'\}$. By undoing the actions of the $Q$-run it is not possible to go back to a state enabling action $l_{\mathrm{no\_auth\_o}}$ before enabling action $l_{\mathrm{no\_auth}}$. Instead, this is possible by undoing the other run. This is enough to distinguish $P \,/\, \{h'\}$ and $Q$ in the setting of reversible transactions. Therefore, by following the same observations as the previous example, it turns out that the weak-bisimilarity-based noninterference properties are satisfied, while those based on branching bisimilarity are not. Finally, if we add the same rate $\lambda$ just before the execution of any action $l_{\mathrm{commit}}$ – thus yielding $DBMS'_{\mathrm{stoch\_timed}}$ – the same considerations continue to hold, thereby confirming the greater expressive power of branching bisimulation semantics even in the Markovian setting.

## 8 Conclusions

In this paper we have extended to a stochastically timed setting our previous preservation, compositionality, and classification results about a selection of noninterference properties for (irreversible and) reversible systems developed in a nondeterministic setting [25] and in a probabilistic one [24]. The two behavioral equivalences for those noninterference properties are the weak Markovian bisimilarity of [38] and a newly defined Markovian branching bisimilarity. Both equivalences are designed to comply with the assumption of maximal progress.

Since we have shown that Markovian branching bisimilarity coincides with a Markovian variant of the weak back-and-forth bisimilarity of [22], noninterference properties based on this equivalence can be applied to reversible Markovian systems. This extends the analogous result in [25] for nondeterministic systems as well as the one in [24] for systems featuring nondeterminism and probabilities.

Regarding future extensions, we are working on incorporating recursion into the process language under consideration, which will enable us to model systems that may not terminate. This requires identifying an adequate Markovian variant of the up-to technique for weak and branching bisimilarities [60,31], to be used in the proof of some results where we can now proceed by induction on the depth of the tree-like MLTS underlying the considered process term.

Another direction to pursue is the comparison of our work with those, based on an integrated-time Markovian model, of [3], addressing stochastic variants of BSNNI and SBNDC, and [41], which studies a stochastic variant of P_BNDC.

Finally, we would like to develop a taxonomy for deterministically timed systems, in which action execution is separated from time passing according to the model of [50,51] governed by time determinism and time additivity.

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## A Proofs of Results

**Proof of Lemma 1.** We first prove the five results for the $\approx_{\rm pw}$-based properties. The congruence of $\approx_{\rm mw}$ with respect to action prefix, rate prefix, parallel composition, and hiding has already been proven in [38], so we focus only on restriction. Let $\mathcal{B}$ be a weak Markovian bisimulation witnessing $P_1 \approx_{\rm mw} P_2$. The equivalence relation $\mathcal{B}' = \mathcal{I}_{\mathbb{P}} \cup \{(Q_1 \setminus L, Q_2 \setminus L) \mid (Q_1, Q_2) \in \mathcal{B}\}$ is a weak Markovian bisimulation too. Given $(Q_1 \setminus L, Q_2 \setminus L) \in \mathcal{B}'$ with $(Q_1, Q_2) \in \mathcal{B}$, there are two cases for action transitions based on the operational semantic rules in Table 1:

- If $Q_1 \setminus L \xrightarrow{\tau}_{\rm a} Q'_1 \setminus L$ with $Q_1 \xrightarrow{\tau}_{\rm a} Q'_1$, then there exists $Q_2 \xRightarrow{\tau^*}_{\rm a} Q'_2$ such that $(Q'_1, Q'_2) \in \mathcal{B}$. Since the restriction operator does not apply to $\tau$, we have that $Q_2 \setminus L \xRightarrow{\tau^*}_{\rm a} Q'_2 \setminus L$ with $(Q'_1 \setminus L, Q'_2 \setminus L) \in \mathcal{B}'$.
- If $Q_1 \setminus L \xrightarrow{a}_{\rm a} Q'_1 \setminus L$ with $Q_1 \xrightarrow{a}_{\rm a} Q'_1$ and $a \notin L \cup \{\tau\}$, then there exists $Q_2 \xRightarrow{\tau^*}_{\rm a} \xrightarrow{a}_{\rm a} \xRightarrow{\tau^*}_{\rm a} Q'_2$ such that $(Q'_1, Q'_2) \in \mathcal{B}$. Since the restriction operator does not apply to $\tau$ and $a \notin L$, we have that $Q_2 \setminus L \xRightarrow{\tau^*}_{\rm a} \xrightarrow{a}_{\rm a} \xRightarrow{\tau^*}_{\rm a} Q'_2 \setminus L$ with $(Q'_1 \setminus L, Q'_2 \setminus L) \in \mathcal{B}'$.

As for rates, to avoid trivial cases consider an equivalence class $C' = C \setminus L = \{Q \setminus L \mid Q \in C\}$ for some $C \in \mathbb{P}/\mathcal{B}$. Suppose that $Q_1 \setminus L \;\;\not\xrightarrow{\tau}_{\rm a}$ so that $Q_1 \not\xrightarrow{\tau}_{\rm a}$ too and hence from $(Q_1, Q_2) \in \mathcal{B}$ it follows that there exists $Q_2 \xRightarrow{\tau^*}_{\rm a} \bar{Q}_2$ such that $\bar{Q}_2 \;\;\not\xrightarrow{\tau}_{\rm a}$, $(Q_1, \bar{Q}_2) \in \mathcal{B}$, and $rate(Q_1, C) = rate(\bar{Q}_2, C)$. Since the restriction operator does not apply to $\tau$ and rate transitions, we have that $Q_2 \setminus L \xRightarrow{\tau^*}_{\rm a} \bar{Q}_2 \setminus L$ with $\bar{Q}_2 \setminus L \not\xrightarrow{\tau}_{\rm a}$, $(Q_1 \setminus L, \bar{Q}_2 \setminus L) \in \mathcal{B}'$, and $rate(Q_1 \setminus L, C') = rate(Q_1, C) = rate(\bar{Q}_2, C) = rate(\bar{Q}_2 \setminus L, C')$.

We then prove the five results for the $\approx_{\rm mb}$-based properties. Let $\mathcal{B}$ be a Markovian branching bisimulation witnessing $P_1 \approx_{\rm mb} P_2$:

1. The equivalence relation $\mathcal{B}' = (\mathcal{B} \cup \{(a \,.\, Q_1, a \,.\, Q_2) \mid (Q_1, Q_2) \in \mathcal{B}\})^+$ is a Markovian branching bisimulation too. The result immediately follows from the fact that, given $(a \,.\, Q_1, a \,.\, Q_2) \in \mathcal{B}'$ with $(Q_1, Q_2) \in \mathcal{B}$, $a \,.\, Q_1 \xrightarrow{a}_{\rm a} Q_1$ is matched by $a \,.\, Q_2 \xRightarrow{\tau^*}_{\rm a} a \,.\, Q_2 \xrightarrow{a}_{\rm a} Q_2$ with $(a \,.\, Q_1, a \,.\, Q_2) \in \mathcal{B}'$ and $(Q_1, Q_2) \in \mathcal{B}'$ as well as, in the case $a \neq \tau$, $a \,.\, Q_1 \;\not\xrightarrow{\tau}_{\rm a}$ with $a \,.\, Q_2 \xRightarrow{\tau^*}_{\rm a} a \,.\, Q_2 \;\not\xrightarrow{\tau}_{\rm a}$ and $rate(a \,.\, Q_1, C') = rate(a \,.\, Q_2, C') = 0$ for all $C' \in \mathbb{P}/\mathcal{B}'$.
2. The equivalence relation $\mathcal{B}' = (\mathcal{B} \cup \{((\lambda) \,.\, Q_1, (\lambda) \,.\, Q_2) \mid (Q_1, Q_2) \in \mathcal{B}\})^+$ is a Markovian branching bisimulation too. The result immediately follows from the fact that, given $((\lambda) \,.\, Q_1, (\lambda) \,.\, Q_2) \in \mathcal{B}'$ with $(Q_1, Q_2) \in \mathcal{B}$, both processes can only perform a $\lambda$-transition. Precisely, $(\lambda) \,.\, Q_1 \;\;\not\xrightarrow{\tau}_{\rm a}$ with $(\lambda) \,.\, Q_2 \xRightarrow{\tau^*}_{\rm a} (\lambda) \,.\, Q_2 \;\;\not\xrightarrow{\tau}_{\rm a}$ and $rate((\lambda) \,.\, Q_1, \bar{C}) = rate((\lambda) \,.\, Q_2, \bar{C}) = \lambda$ for $\bar{C} = [Q_1]_{\mathcal{B}'}$ while $rate((\lambda) \,.\, Q_1, C') = rate((\lambda) \,.\, Q_2, C') = 0$ for any other $C' \in \mathbb{P}/\mathcal{B}'$.
3. The equivalence relation $\mathcal{B}' = \mathcal{I}_{\mathbb{P}} \cup \{(Q_1 \parallel_L Q, Q_2 \parallel_L Q) \mid (Q_1, Q_2) \in \mathcal{B} \wedge Q \in \mathbb{P}\}$ and its variant $\mathcal{B}''$ in which $Q$ occurs to the left of parallel composition

in each pair are Markovian branching bisimulations too. Let us focus on $\mathcal{B}'$. Given $(Q_1 \parallel_L Q, Q_2 \parallel_L Q) \in \mathcal{B}'$ with $(Q_1, Q_2) \in \mathcal{B}$, there are three cases for action transitions based on the operational semantic rules in Table 1:

- If $Q_1 \parallel_L Q \xrightarrow{a}_{\rm a} Q_1' \parallel_L Q$ with $Q_1 \xrightarrow{a}_{\rm a} Q_1'$ and $a \notin L$, then either $a = \tau$ and $(Q_1', Q_2) \in \mathcal{B}$, or there exists $Q_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \xrightarrow{a}_{\rm a} Q_2'$ such that $(Q_1, \bar{Q}_2) \in \mathcal{B}$ and $(Q_1', Q_2') \in \mathcal{B}$. Since synchronization does not apply to $\tau$ and $a \notin L$, in the former subcase $Q_2 \parallel_L Q$ is allowed to stay idle with $(Q_1' \parallel_L Q, Q_2 \parallel_L Q) \in \mathcal{B}'$, while in the latter subcase $Q_2 \parallel_L Q \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \parallel_L Q \xrightarrow{a}_{\rm a} Q_2' \parallel_L Q$ with $(Q_1 \parallel_L Q, \bar{Q}_2 \parallel_L Q) \in \mathcal{B}'$ and $(Q_1' \parallel_L Q, Q_2' \parallel_L Q) \in \mathcal{B}'$.
- The case $Q_1 \parallel_L Q \xrightarrow{a}_{\rm a} Q_1 \parallel_L Q'$ with $Q \xrightarrow{a}_{\rm a} Q'$ and $a \notin L$ is trivial.
- If $Q_1 \parallel_L Q \xrightarrow{a}_{\rm a} Q_1' \parallel_L Q'$ with $Q_1 \xrightarrow{a}_{\rm a} Q_1'$, $Q \xrightarrow{a}_{\rm a} Q'$, and $a \in L$, then there exists $Q_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \xrightarrow{a}_{\rm a} Q_2'$ such that $(Q_1, \bar{Q}_2) \in \mathcal{B}$ and $(Q_1', Q_2') \in \mathcal{B}$. Since synchronization does not apply to $\tau$ and $a \in L$, we have that $Q_2 \parallel_L Q \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \parallel_L Q \xrightarrow{a}_{\rm a} Q_2' \parallel_L Q'$ with $(Q_1 \parallel_L Q, \bar{Q}_2 \parallel_L Q) \in \mathcal{B}'$ and $(Q_1' \parallel_L Q', Q_2' \parallel_L Q') \in \mathcal{B}'$.

As for rates, to avoid trivial cases consider an equivalence class $C' = C \parallel_L Q' = \{R \parallel_L Q' \mid R \in C\}$ for some $C \in \mathbb{P}/\mathcal{B}$. Suppose that $Q_1 \parallel_L Q \ \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ so that $Q_1 \ \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ and $Q \ \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ too and hence from $(Q_1, Q_2) \in \mathcal{B}$ it follows that there exists $Q_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2$ such that $\bar{Q}_2 \ \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, $(Q_1, \bar{Q}_2) \in \mathcal{B}$, and $\mathit{rate}(Q_1, C) = \mathit{rate}(\bar{Q}_2, C)$. Since synchronization does not apply to $\tau$ and rate transitions, we have that $Q_2 \parallel_L Q \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \parallel_L Q$ with $\bar{Q}_2 \parallel_L Q \ \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, $(Q_1 \parallel_L Q, \bar{Q}_2 \parallel_L Q) \in \mathcal{B}'$, and $\mathit{rate}(Q_1 \parallel_L Q, C') = \mathit{rate}(Q_1, C) = \mathit{rate}(\bar{Q}_2, C) = \mathit{rate}(\bar{Q}_2 \parallel_L Q, C')$ if $Q = Q'$, $\mathit{rate}(Q_1 \parallel_L Q, C') = \mathit{rate}(Q, \{Q'\}) = \mathit{rate}(\bar{Q}_2 \parallel_L Q, C')$ if $Q_1, \bar{Q}_2 \in C$, $\mathit{rate}(Q_1 \parallel_L Q, C') = 0 = \mathit{rate}(\bar{Q}_2 \parallel_L Q, C')$ otherwise.

4. The equivalence relation $\mathcal{B}' = \mathcal{I}_{\mathbb{P}} \cup \{(Q_1 \setminus L, Q_2 \setminus L) \mid (Q_1, Q_2) \in \mathcal{B}\}$ is a Markovian branching bisimulation too. Given $(Q_1 \setminus L, Q_2 \setminus L) \in \mathcal{B}'$ with $(Q_1, Q_2) \in \mathcal{B}$, there are two cases for action transitions based on the operational semantic rules in Table 1:

   - If $Q_1 \setminus L \xrightarrow{\tau}_{\rm a} Q_1' \setminus L$ with $Q_1 \xrightarrow{\tau}_{\rm a} Q_1'$, then either $(Q_1', Q_2) \in \mathcal{B}$, or there exists $Q_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \xrightarrow{\tau}_{\rm a} Q_2'$ such that $(Q_1, \bar{Q}_2) \in \mathcal{B}$ and $(Q_1', Q_2') \in \mathcal{B}$. Since the restriction operator does not apply to $\tau$, in the former subcase $Q_2 \setminus L$ is allowed to stay idle with $(Q_1' \setminus L, Q_2 \setminus L) \in \mathcal{B}'$, while in the latter subcase $Q_2 \setminus L \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \setminus L \xrightarrow{\tau}_{\rm a} Q_2' \setminus L$ with $(Q_1 \setminus L, \bar{Q}_2 \setminus L) \in \mathcal{B}'$ and $(Q_1' \setminus L, Q_2' \setminus L) \in \mathcal{B}'$.
   - If $Q_1 \setminus L \xrightarrow{a}_{\rm a} Q_1' \setminus L$ with $Q_1 \xrightarrow{a}_{\rm a} Q_1'$ and $a \notin L \cup \{\tau\}$, then there exists $Q_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \xrightarrow{a}_{\rm a} Q_2'$ such that $(Q_1, \bar{Q}_2) \in \mathcal{B}$ and $(Q_1', Q_2') \in \mathcal{B}$. Since the restriction operator does not apply to $\tau$ and $a \notin L$, we have that $Q_2 \setminus L \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \setminus L \xrightarrow{a}_{\rm a} Q_2' \setminus L$ with $(Q_1 \setminus L, \bar{Q}_2 \setminus L) \in \mathcal{B}'$ and $(Q_1' \setminus L, Q_2' \setminus L) \in \mathcal{B}'$.

   As for rates, we reason like in the proof of the corresponding result for $\approx_{\rm mw}$.

5. The equivalence relation $\mathcal{B}' = \mathcal{I}_{\mathbb{P}} \cup \{(Q_1 \,/\, L, Q_2 \,/\, L) \mid (Q_1, Q_2) \in \mathcal{B}\}$ is a Markovian branching bisimulation too. Given $(Q_1 \,/\, L, Q_2 \,/\, L) \in \mathcal{B}'$ with $(Q_1, Q_2) \in \mathcal{B}$, there are two cases for action transitions based on the operational semantic rules in Table 1:
   - If $Q_1 \,/\, L \xrightarrow{\tau}_{\rm a} Q'_1 \,/\, L$ with $Q_1 \xrightarrow{\tau}_{\rm a} Q'_1$, then either $(Q'_1, Q_2) \in \mathcal{B}$, or there exists $Q_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \xrightarrow{\tau}_{\rm a} Q'_2$ such that $(Q_1, \bar{Q}_2) \in \mathcal{B}$ and $(Q'_1, Q'_2) \in \mathcal{B}$. Since the hiding operator does not apply to $\tau$, in the former subcase $Q_2 \,/\, L$ is allowed to stay idle with $(Q'_1 \,/\, L, Q_2 \,/\, L) \in \mathcal{B}'$, while in the latter subcase $Q_2 \,/\, L \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \,/\, L \xrightarrow{\tau}_{\rm a} Q'_2 \,/\, L$ with $(Q_1 \,/\, L, \bar{Q}_2 \,/\, L) \in \mathcal{B}'$ and $(Q'_1 \,/\, L, Q'_2 \,/\, L) \in \mathcal{B}'$.
   - If $Q_1 \,/\, L \xrightarrow{a}_{\rm a} Q'_1 \,/\, L$ with $Q_1 \xrightarrow{b}_{\rm a} Q'_1$ and $b \in L \wedge a = \tau$ or $b \notin L \cup \{\tau\} \wedge a = b$, then there exists $Q_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \xrightarrow{b}_{\rm a} Q'_2$ such that $(Q_1, \bar{Q}_2) \in \mathcal{B}$ and $(Q'_1, Q'_2) \in \mathcal{B}$. Since the hiding operator does not apply to $\tau$, we have that $Q_2 \,/\, L \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \,/\, L \xrightarrow{a}_{\rm a} Q'_2 \,/\, L$ with $(Q_1 \,/\, L, \bar{Q}_2 \,/\, L) \in \mathcal{B}'$ and $(Q'_1 \,/\, L, Q'_2 \,/\, L) \in \mathcal{B}'$.

   As for rates, to avoid trivial cases consider an equivalence class $C' = C \,/\, L = \{Q \,/\, L \mid Q \in C\}$ for some $C \in \mathbb{P}/\mathcal{B}$. Suppose that $Q_1 \,/\, L \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ so that $Q_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ too and hence from $(Q_1, Q_2) \in \mathcal{B}$ it follows that there exists $Q_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2$ such that $\bar{Q}_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, $(Q_1, \bar{Q}_2) \in \mathcal{B}$, and $\mathit{rate}(Q_1, C) = \mathit{rate}(\bar{Q}_2, C)$. Since the hiding operator does not apply to $\tau$ and rate transitions, we have that $Q_2 \,/\, L \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_2 \,/\, L$ with $\bar{Q}_2 \,/\, L \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, $(Q_1 \,/\, L, \bar{Q}_2 \,/\, L) \in \mathcal{B}'$, and $\mathit{rate}(Q_1 \,/\, L, C') = \mathit{rate}(Q_1, C) = \mathit{rate}(\bar{Q}_2, C) = \mathit{rate}(\bar{Q}_2 \,/\, L, C')$. ■

**Proof of Theorem 1.** A straightforward consequence of the definition of the various properties, i.e., Definition 5, and Lemma 1. ■

**Lemma 3.** *Let $P_1, P_2, P \in \mathbb{P}$ and $\approx \in \{\approx_{\rm mw}, \approx_{\rm mb}\}$. Then:*

1. *If $P_1, P_2 \in \mathrm{SBSNNI}_{\approx}$ and $L \subseteq \mathcal{A} \setminus \{\tau\}$ for $\approx_{\rm mw}$ or $L \subseteq \mathcal{A}_{\mathcal{L}}$ for $\approx_{\rm mb}$, then $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A}_{\mathcal{H}} \approx (R_1 \,\|_L\, R_2) \,/\, \mathcal{A}_{\mathcal{H}}$ for all $Q_1, R_1 \in \mathit{reach}(P_1)$ and $Q_2, R_2 \in \mathit{reach}(P_2)$ such that $Q_1 \,\|_L\, Q_2, R_1 \,\|_L\, R_2 \in \mathit{reach}(P_1 \,\|_L\, P_2)$, $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \approx R_1 \,/\, \mathcal{A}_{\mathcal{H}}$, and $Q_2 \setminus \mathcal{A}_{\mathcal{H}} \approx R_2 \,/\, \mathcal{A}_{\mathcal{H}}$.*
2. *If $P \in \mathrm{SBSNNI}_{\approx}$ and $L \subseteq \mathcal{A} \setminus \{\tau\}$, then $(Q \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L \approx (R \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}$ for all $Q, R \in \mathit{reach}(P)$ such that $Q \,/\, \mathcal{A}_{\mathcal{H}} \approx R \setminus \mathcal{A}_{\mathcal{H}}$.*
3. *If $P_1, P_2 \in \mathrm{SBNDC}_{\approx}$ and $L \subseteq \mathcal{A} \setminus \{\tau\}$, then $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A}_{\mathcal{H}} \approx (R_1 \,\|_L\, R_2) \setminus \mathcal{A}_{\mathcal{H}}$ for all $Q_1, R_1 \in \mathit{reach}(P_1)$ and $Q_2, R_2 \in \mathit{reach}(P_2)$ such that $Q_1 \,\|_L\, Q_2, R_1 \,\|_L\, R_2 \in \mathit{reach}(P_1 \,\|_L\, P_2)$, $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \approx R_1 \setminus \mathcal{A}_{\mathcal{H}}$ and $Q_2 \setminus \mathcal{A}_{\mathcal{H}} \approx R_2 \setminus \mathcal{A}_{\mathcal{H}}$.*

*Proof. We first prove the three results for the $\approx_{\rm mw}$-based properties. Let $\mathcal{B}$ be an equivalence relation containing all the pairs of processes that have to be shown to be $\approx_{\rm mw}$-equivalent according to the considered result:*

1. *Starting from $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A}_{\mathcal{H}}$ and $(R_1 \,\|_L\, R_2) \,/\, \mathcal{A}_{\mathcal{H}}$ related by $\mathcal{B}$, so that $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} R_1 \,/\, \mathcal{A}_{\mathcal{H}}$ and $Q_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} R_2 \,/\, \mathcal{A}_{\mathcal{H}}$, there are thirteen cases for action transitions based on the operational semantic rules in Table 1. In the first five cases, it is $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A}_{\mathcal{H}}$ to move first:*

- *If* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}$ *with* $Q_1 \xrightarrow{l}_{\rm a} Q'_1$ *and* $l \notin L$*, then* $Q_1 \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} Q'_1 \setminus \mathcal{A_H}$ *as* $l \notin \mathcal{A_H}$*. From* $Q_1 \setminus \mathcal{A_H} \approx_{\rm mw} R_1 \,/\, \mathcal{A_H}$ *it follows that there exists* $R_1 \,/\, \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \xrightarrow{l}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} R'_1 \,/\, \mathcal{A_H}$ *such that* $Q'_1 \setminus \mathcal{A_H} \approx_{\rm mw} R'_1 \,/\, \mathcal{A_H}$*. Since synchronization does not apply to* $\tau$ *and* $l \notin L$*, we have that* $(Q_1 \parallel_L Q_2) \,/\, \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \xrightarrow{l}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}$ *with* $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}, (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}) \in \mathcal{B}$*.*
- *If* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} (Q_1 \parallel_L Q'_2) \setminus \mathcal{A_H}$ *with* $Q_2 \xrightarrow{l}_{\rm a} Q'_2$ *and* $l \notin L$*, then the proof is similar to the one of the previous case.*
- *If* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A_H}$ *with* $Q_i \xrightarrow{l}_{\rm a} Q'_i$ *for* $i \in \{1, 2\}$ *and* $l \in L$*, then* $Q_i \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} Q'_i \setminus \mathcal{A_H}$ *as* $l \notin \mathcal{A_H}$*. From* $Q_i \setminus \mathcal{A_H} \approx_{\rm mw} R_i \,/\, \mathcal{A_H}$ *it follows that there exists* $R_i \,/\, \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \xrightarrow{l}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} R'_i \,/\, \mathcal{A_H}$ *such that* $Q'_i \setminus \mathcal{A_H} \approx_{\rm mw} R'_i \,/\, \mathcal{A_H}$*. Since synchronization does not apply to* $\tau$ *and* $l \in L$*, we have that* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \xrightarrow{l}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (R'_1 \parallel_L R'_2) \,/\, \mathcal{A_H}$ *with* $((Q'_1 \parallel_L Q'_2) \setminus \mathcal{A_H}, (R'_1 \parallel_L R'_2) \,/\, \mathcal{A_H}) \in \mathcal{B}$*.*
- *If* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}$ *with* $Q_1 \xrightarrow{\tau}_{\rm a} Q'_1$*, then* $Q_1 \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} Q'_1 \setminus \mathcal{A_H}$ *as* $\tau \notin \mathcal{A_H}$*. From* $Q_1 \setminus \mathcal{A_H} \approx_{\rm mw} R_1 \,/\, \mathcal{A_H}$ *it follows that there exists* $R_1 \,/\, \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} R'_1 \,/\, \mathcal{A_H}$ *such that* $Q'_1 \setminus \mathcal{A_H} \approx_{\rm mw} R'_1 \,/\, \mathcal{A_H}$*. Since synchronization does not apply to* $\tau$*, we have that* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}$ *with* $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}, (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}) \in \mathcal{B}$*.*
- *If* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (Q_1 \parallel_L Q'_2) \setminus \mathcal{A_H}$ *with* $Q_2 \xrightarrow{\tau}_{\rm a} Q'_2$*, then the proof is similar to the one of the previous case.*

*In the other eight cases, instead, it is* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H}$ *to move first:*

- *If* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}$ *with* $R_1 \xrightarrow{l}_{\rm a} R'_1$ *and* $l \notin L$*, then* $R_1 \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} R'_1 \,/\, \mathcal{A_H}$ *as* $l \notin \mathcal{A_H}$*. From* $R_1 \,/\, \mathcal{A_H} \approx_{\rm mw} Q_1 \setminus \mathcal{A_H}$ *it follows that there exists* $Q_1 \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \xrightarrow{l}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} Q'_1 \setminus \mathcal{A_H}$ *such that* $R'_1 \,/\, \mathcal{A_H} \approx_{\rm mw} Q'_1 \setminus \mathcal{A_H}$*. Since synchronization does not apply to* $\tau$ *and* $l \notin L$*, we have that* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \xrightarrow{l}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}$ *with* $((R'_1 \parallel_L R_2) / \mathcal{A_H}, (Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}) \in \mathcal{B}$*.*
- *If* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} (R_1 \parallel_L R'_2) \,/\, \mathcal{A_H}$ *with* $R_2 \xrightarrow{l}_{\rm a} R'_2$ *and* $l \notin L$*, then the proof is similar to the one of the previous case.*
- *If* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} (R'_1 \parallel_L R'_2) \,/\, \mathcal{A_H}$ *with* $R_i \xrightarrow{l}_{\rm a} R'_i$ *for* $i \in \{1, 2\}$ *and* $l \in L$*, then* $R_i \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} R'_i \,/\, \mathcal{A_H}$ *as* $l \notin \mathcal{A_H}$*. From* $R_i \,/\, \mathcal{A_H} \approx_{\rm mw} Q_i \setminus \mathcal{A_H}$ *it follows that there exists* $Q_i \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \xrightarrow{l}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} Q'_i \setminus \mathcal{A_H}$ *such that* $R'_i \,/\, \mathcal{A_H} \approx_{\rm mw} Q'_i \setminus \mathcal{A_H}$*. Since synchronization does not apply to* $\tau$ *and* $l \in L$*, we have that* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \xrightarrow{l}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A_H}$ *with* $((R'_1 \parallel_L R'_2) / \mathcal{A_H}, (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A_H}) \in \mathcal{B}$*.*
- *If* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}$ *with* $R_1 \xrightarrow{\tau}_{\rm a} R'_1$*, then* $R_1 \,/ \mathcal{A_H} \xrightarrow{\tau}_{\rm a} R'_1 \,/ \mathcal{A_H}$ *as* $\tau \notin \mathcal{A_H}$*. From* $R_1 \,/\, \mathcal{A_H} \approx_{\rm mw} Q_1 \setminus \mathcal{A_H}$ *it follows that there exists* $Q_1 \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} Q'_1 \setminus \mathcal{A_H}$ *such that* $R'_1 \,/\, \mathcal{A_H} \approx_{\rm mw} Q'_1 \setminus \mathcal{A_H}$*.*

*Since synchronization does not apply to* $\tau$*, we have that* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \overset{\tau^*}{\Longrightarrow}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $((R'_1 \parallel_L R_2)/\mathcal{A}_{\mathcal{H}}, (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$.

- *If* $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} (R_1 \parallel_L R'_2) \,/\, \mathcal{A}_{\mathcal{H}}$ *with* $R_2 \xrightarrow{\tau}_{\rm a} R'_2$*, then the proof is similar to the one of the previous case.*
- *If* $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A}_{\mathcal{H}}$ *with* $R_1 \xrightarrow{h}_{\rm a} R'_1$ *and* $h \notin L$*, then* $R_1 \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} R'_1 \,/\, \mathcal{A}_{\mathcal{H}}$ *as* $h \in \mathcal{A}_{\mathcal{H}}$*. The rest of the proof is like the one of the fourth case.*
- *If* $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} (R_1 \parallel_L R'_2) \,/\, \mathcal{A}_{\mathcal{H}}$ *with* $R_2 \xrightarrow{h}_{\rm a} R'_2$ *and* $h \notin L$*, then the proof is similar to the one of the previous case.*
- *If* $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} (R'_1 \parallel_L R'_2) \,/\, \mathcal{A}_{\mathcal{H}}$ *with* $R_i \xrightarrow{h}_{\rm a} R'_i$ *for* $i \in \{1, 2\}$ *and* $h \in L$*, then* $R_i \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} R'_i \,/\, \mathcal{A}_{\mathcal{H}}$ *as* $h \in \mathcal{A}_{\mathcal{H}}$*. From* $R_i \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} Q_i \setminus \mathcal{A}_{\mathcal{H}}$ *it follows that there exists* $Q_i \setminus \mathcal{A}_{\mathcal{H}} \overset{\tau^*}{\Longrightarrow}_{\rm a} Q'_i \setminus \mathcal{A}_{\mathcal{H}}$ *such that* $R'_i \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} Q'_i \setminus \mathcal{A}_{\mathcal{H}}$*. Since synchronization does not apply to* $\tau$ *and* $h \in L$*, we have that* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \overset{\tau^*}{\Longrightarrow}_{\rm a} (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $((R'_1 \parallel_L R'_2) \,/\, \mathcal{A}_{\mathcal{H}}, (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$.

*As for rates, to avoid trivial cases consider an equivalence class* $C \in \mathbb{P}/\mathcal{B}$ *that involves processes reachable from* $P_1 \parallel_L P_2$*, specifically* $C = \{(S_{1,i} \parallel_L S_{2,i}) \setminus \mathcal{A}_{\mathcal{H}}, (S_{1,j} \parallel_L S_{2,j}) \,/\, \mathcal{A}_{\mathcal{H}} \mid S_{k,h} \in \mathit{reach}(P_k) \wedge S_{1,h} \parallel_L S_{2,h} \in \mathit{reach}(P_1 \parallel_L P_2) \wedge S_{k,i} \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} S_{k,j} \,/\, \mathcal{A}_{\mathcal{H}}\}$*. Suppose that* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ *so that* $Q_k \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ *too and hence from* $Q_k \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} R_k \,/\, \mathcal{A}_{\mathcal{H}}$ *it follows that there exists* $R_k \,/\, \mathcal{A}_{\mathcal{H}} \overset{\tau^*}{\Longrightarrow}_{\rm a} \bar{R}_k \,/\, \mathcal{A}_{\mathcal{H}}$ *such that* $\bar{R}_k \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, $Q_k \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \bar{R}_k \,/\, \mathcal{A}_{\mathcal{H}}$*, and* $\mathit{rate}(Q_k \setminus \mathcal{A}_{\mathcal{H}}, C') = \mathit{rate}(\bar{R}_k \,/\, \mathcal{A}_{\mathcal{H}}, C')$ *for all* $C' \in \mathbb{P}/\approx_{\rm mw}$*. Since synchronization does not apply to* $\tau$*, we have that* $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_{\mathcal{H}} \overset{\tau^*}{\Longrightarrow}_{\rm a} (\bar{R}_1 \parallel_L \bar{R}_2) \,/\, \mathcal{A}_{\mathcal{H}}$ *with* $(\bar{R}_1 \parallel_L \bar{R}_2) \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ *and* $((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}, (\bar{R}_1 \parallel_L \bar{R}_2) \,/\, \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$*. Since the restriction and hiding operators do not apply to rate transitions, we have that:*

$$\begin{aligned} \mathit{rate}((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}, C) &= \mathit{rate}((Q_1 \setminus \mathcal{A}_{\mathcal{H}}) \parallel_L (Q_2 \setminus \mathcal{A}_{\mathcal{H}}), C) \\ \mathit{rate}((\bar{R}_1 \parallel_L \bar{R}_2) \,/\, \mathcal{A}_{\mathcal{H}}, C) &= \mathit{rate}((\bar{R}_1 \,/\, \mathcal{A}_{\mathcal{H}}) \parallel_L (\bar{R}_2 \,/\, \mathcal{A}_{\mathcal{H}}), C) \end{aligned}$$

*Based on which subprocess moves so that the overall process reaches* $C$ *(which we assume to be reachable in one move to avoid trivial cases in which cumulative rates are zero), we have that:*

$$\begin{aligned} \mathit{rate}((Q_1 \setminus \mathcal{A}_{\mathcal{H}}) \parallel_L (Q_2 \setminus \mathcal{A}_{\mathcal{H}}), C) &= \mathit{rate}(Q_1 \setminus \mathcal{A}_{\mathcal{H}}, C_1) \\ \mathit{rate}((\bar{R}_1 \,/\, \mathcal{A}_{\mathcal{H}}) \parallel_L (\bar{R}_2 \,/\, \mathcal{A}_{\mathcal{H}}), C) &= \mathit{rate}(\bar{R}_1 \,/\, \mathcal{A}_{\mathcal{H}}, C_1) \end{aligned}$$

*or:*

$$\begin{aligned} \mathit{rate}((Q_1 \setminus \mathcal{A}_{\mathcal{H}}) \parallel_L (Q_2 \setminus \mathcal{A}_{\mathcal{H}}), C) &= \mathit{rate}(Q_2 \setminus \mathcal{A}_{\mathcal{H}}, C_2) \\ \mathit{rate}((\bar{R}_1 \,/\, \mathcal{A}_{\mathcal{H}}) \parallel_L (\bar{R}_2 \,/\, \mathcal{A}_{\mathcal{H}}), C) &= \mathit{rate}(\bar{R}_2 \,/\, \mathcal{A}_{\mathcal{H}}, C_2) \end{aligned}$$

*where:*

$$\begin{aligned} C_1 = {} & \{S_{1,h} \setminus \mathcal{A}_{\mathcal{H}} \mid (S_{1,h} \parallel_L S_{2,h}) \setminus \mathcal{A}_{\mathcal{H}} \in C\} \cup \\ & \{S_{1,h} \,/\, \mathcal{A}_{\mathcal{H}} \mid (S_{1,h} \parallel_L S_{2,h}) \,/\, \mathcal{A}_{\mathcal{H}} \in C\} \\ C_2 = {} & \{S_{2,h} \setminus \mathcal{A}_{\mathcal{H}} \mid (S_{1,h} \parallel_L S_{2,h}) \setminus \mathcal{A}_{\mathcal{H}} \in C\} \cup \\ & \{S_{2,h} \,/\, \mathcal{A}_{\mathcal{H}} \mid (S_{1,h} \parallel_L S_{2,h}) \,/\, \mathcal{A}_{\mathcal{H}} \in C\} \end{aligned}$$

*Since* $Q_k \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \bar{R}_k \,/\, \mathcal{A}_{\mathcal{H}}$ *and* $C_k$ *is the union of some* $\approx_{\rm mw}$*-equivalence classes for* $k \in \{1, 2\}$*, we have that:*

$$rate(Q_1 \setminus \mathcal{A}_\mathcal{H}, C_1) = rate(\bar{R}_1 \,/\, \mathcal{A}_\mathcal{H}, C_1)$$
$$rate(Q_2 \setminus \mathcal{A}_\mathcal{H}, C_2) = rate(\bar{R}_2 \,/\, \mathcal{A}_\mathcal{H}, C_2)$$

*If we start from* $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \stackrel{\tau}{\not\longrightarrow}_\mathrm{a}$*, then the proof is similar.*

2. *Starting from* $(Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L$ *and* $(R \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ *related by* $\mathcal{B}$*, so that* $Q \,/\, \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} R \setminus \mathcal{A}_\mathcal{H}$*, there are six cases for action transitions based on the operational semantic rules in Table 1. In the first three cases, it is* $(Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L$ *to move first:*

   – *If* $(Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L \xrightarrow{l}_\mathrm{a} (Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L$ *with* $Q \xrightarrow{l}_\mathrm{a} Q'$ *and* $l \notin L$*, then* $Q \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_\mathrm{a} Q' \,/\, \mathcal{A}_\mathcal{H}$ *as* $l \notin \mathcal{A}_\mathcal{H}$*. From* $Q \,/\, \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} R \setminus \mathcal{A}_\mathcal{H}$ *it follows that there exists* $Q \setminus \mathcal{A}_\mathcal{H} \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} \xrightarrow{l}_\mathrm{a} \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} R' \setminus \mathcal{A}_\mathcal{H}$ *such that* $Q' \,/\, \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} R' \setminus \mathcal{A}_\mathcal{H}$*. Since the restriction and hiding operators do not apply to* $\tau$ *and* $l$*, we have that* $(R \setminus L) \,/\, \mathcal{A}_\mathcal{H} \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} \xrightarrow{l}_\mathrm{a} \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} (R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ *with* $((Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L, (R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$*.*
   – *If* $(Q/\mathcal{A}_\mathcal{H}) \setminus L \xrightarrow{\tau}_\mathrm{a} (Q'/\mathcal{A}_\mathcal{H}) \setminus L$ *with* $Q \xrightarrow{\tau}_\mathrm{a} Q'$*, then* $Q/\mathcal{A}_\mathcal{H} \xrightarrow{\tau}_\mathrm{a} Q'/\mathcal{A}_\mathcal{H}$ *as* $\tau \notin \mathcal{A}_\mathcal{H}$*. From* $Q \,/\, \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} R \setminus \mathcal{A}_\mathcal{H}$ *it follows that there exists* $R \setminus \mathcal{A}_\mathcal{H} \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} R' \setminus \mathcal{A}_\mathcal{H}$ *such that* $Q' \,/\, \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} R' \setminus \mathcal{A}_\mathcal{H}$*. Since the restriction and hiding operators do not apply to* $\tau$*, we have that* $(R \setminus L) \,/\, \mathcal{A}_\mathcal{H} \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} (R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ *with* $((Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L, (R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$*.*
   – *If* $(Q/\mathcal{A}_\mathcal{H}) \setminus L \xrightarrow{\tau}_\mathrm{a} (Q'/\mathcal{A}_\mathcal{H}) \setminus L$ *with* $Q \xrightarrow{h}_\mathrm{a} Q'$*, then* $Q/\mathcal{A}_\mathcal{H} \xrightarrow{\tau}_\mathrm{a} Q'/\mathcal{A}_\mathcal{H}$ *as* $h \in \mathcal{A}_\mathcal{H}$*. The rest of the proof is like the one of the previous case.*

   *In the other three cases, instead, it is* $(R \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ *to move first:*

   – *If* $(R \setminus L) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_\mathrm{a} (R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ *with* $R \xrightarrow{l}_\mathrm{a} R'$ *and* $l \notin L$*, then* $R \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_\mathrm{a} R' \setminus \mathcal{A}_\mathcal{H}$ *as* $l \notin \mathcal{A}_\mathcal{H}$*. From* $R \setminus \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} Q \,/\, \mathcal{A}_\mathcal{H}$ *it follows that there exists* $Q \,/\, \mathcal{A}_\mathcal{H} \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} \xrightarrow{l}_\mathrm{a} \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} Q' \,/\, \mathcal{A}_\mathcal{H}$ *such that* $R' \setminus \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} Q' \,/\, \mathcal{A}_\mathcal{H}$*. Since the restriction operator does not apply to* $\tau$ *and* $l$*, we have that* $(Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} \xrightarrow{l}_\mathrm{a} \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} (Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L$ *with* $((R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}, (Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L) \in \mathcal{B}$*.*
   – *If* $(R \setminus L) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_\mathrm{a} (R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ *with* $R \xrightarrow{\tau}_\mathrm{a} R'$*, then* $R \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_\mathrm{a} R' \setminus \mathcal{A}_\mathcal{H}$ *as* $\tau \notin \mathcal{A}_\mathcal{H}$*. From* $R \setminus \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} Q \,/\, \mathcal{A}_\mathcal{H}$ *it follows that there exists* $Q \,/\, \mathcal{A}_\mathcal{H} \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} Q' \,/\, \mathcal{A}_\mathcal{H}$ *such that* $R' \setminus \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} Q' \,/\, \mathcal{A}_\mathcal{H}$*. Since the restriction operator does not apply to* $\tau$*, we have that* $(Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} (Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L$ *with* $((R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}, (Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L) \in \mathcal{B}$*.*
   – *If* $(R \setminus L) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_\mathrm{a} (R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ *with* $R \xrightarrow{h}_\mathrm{a} R'$ *and* $h \notin L$*, then* $R \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_\mathrm{a} R' \,/\, \mathcal{A}_\mathcal{H}$ *as* $h \in \mathcal{A}_\mathcal{H}$ *(note that* $R \setminus \mathcal{A}_\mathcal{H}$ *cannot perform* $h$*). From* $R \,/\, \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} R \setminus \mathcal{A}_\mathcal{H}$ *– as* $P \in \mathrm{SBSNNI}_{\approx_\mathrm{mw}}$ *and* $R \in reach(P)$ *– and* $R \setminus \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} Q \,/\, \mathcal{A}_\mathcal{H}$ *it follows that there exists* $Q \,/\, \mathcal{A}_\mathcal{H} \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} Q' \,/\, \mathcal{A}_\mathcal{H}$ *such that* $R' \,/\, \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} Q' \,/\, \mathcal{A}_\mathcal{H}$ *and hence* $R' \setminus \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} Q' \,/\, \mathcal{A}_\mathcal{H}$ *– as* $R' \,/\, \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} R' \setminus \mathcal{A}_\mathcal{H}$ *due to* $P \in \mathrm{SBSNNI}_{\approx_\mathrm{mw}}$ *and* $R' \in reach(P)$*. Since the restriction operator does not apply to* $\tau$*, we have that* $(Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L \stackrel{\tau^*}{\Longrightarrow}_\mathrm{a} (Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L$ *with* $((R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}, (Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L) \in \mathcal{B}$*.*

*As for rates, to avoid trivial cases consider an equivalence class* $C \in \mathbb{P}/\mathcal{B}$ *that involves processes reachable from* $P$*, specifically* $C = \{(S_i \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L, (S_j \setminus L) \,/\, \mathcal{A}_{\mathcal{H}} \mid S_h \in reach(P) \wedge S_i \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} S_j \,/\, \mathcal{A}_{\mathcal{H}}\}$*. Suppose that* $(Q \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ *so that* $Q \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ *too and hence from* $Q \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} R \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} R \,/\, \mathcal{A}_{\mathcal{H}}$ *– as* $P \in \text{SBSNNI}_{\approx_{\rm mw}}$ *and* $R \in reach(P)$ *– it follows that there exists* $R \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{R} \,/\, \mathcal{A}_{\mathcal{H}}$ *such that* $\bar{R} \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$*,* $Q \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \bar{R} \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \bar{R} \setminus \mathcal{A}_{\mathcal{H}}$*, and* $rate(Q \,/\, \mathcal{A}_{\mathcal{H}}, C) = rate(\bar{R} \setminus \mathcal{A}_{\mathcal{H}}, C)$ *for all* $C \in \mathbb{P}/\approx_{\rm mw}$*. Since the restriction and hiding operators do not apply to* $\tau$*, we have that* $(R \setminus L) \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (\bar{R} \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}$ *with* $(\bar{R} \setminus L) \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ *– as* $\bar{R} \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ *– and* $((Q \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L, (\bar{R} \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$*. Since the restriction and hiding operators do not apply to rate transitions, we have that:*

$$rate((Q \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L, C) = rate(Q \setminus \mathcal{A}_{\mathcal{H}}, \bar{C})$$
$$rate((\bar{R} \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}, C) = rate(\bar{R} \,/\, \mathcal{A}_{\mathcal{H}}, \bar{C})$$

*where:*

$$\bar{C} = \{S_i \setminus \mathcal{A}_{\mathcal{H}} \mid (S_i \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L \in C\} \cup \{S_j \,/\, \mathcal{A}_{\mathcal{H}} \mid (S_j \setminus L) \,/\, \mathcal{A}_{\mathcal{H}} \in C\}$$

*Since* $Q \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \bar{R} \,/\, \mathcal{A}_{\mathcal{H}}$ *and* $\bar{C}$ *is the union of some* $\approx_{\rm mw}$*-equivalence classes, we have that:*

$$rate(Q \setminus \mathcal{A}_{\mathcal{H}}, \bar{C}) = rate(\bar{R} \,/\, \mathcal{A}_{\mathcal{H}}, \bar{C})$$

*If we start from* $(R \setminus L) \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$*, then the proof is similar.*

3. *Starting from* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}$ *and* $(R_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}}$ *related by* $\mathcal{B}$*, so that* $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} R_1 \setminus \mathcal{A}_{\mathcal{H}}$ *and* $Q_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} R_2 \setminus \mathcal{A}_{\mathcal{H}}$*, there are five cases for action transitions based on the operational semantic rules in Table 1:*

   – *If* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\longrightarrow}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $Q_1 \stackrel{l}{\longrightarrow}_{\rm a} Q'_1$ *and* $l \notin L$*, then* $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\longrightarrow}_{\rm a} Q'_1 \setminus \mathcal{A}_{\mathcal{H}}$ *as* $l \notin \mathcal{A}_{\mathcal{H}}$*. From* $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} R_1 \setminus \mathcal{A}_{\mathcal{H}}$ *it follows that there exists* $R_1 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \stackrel{l}{\longrightarrow}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} R'_1 \setminus \mathcal{A}_{\mathcal{H}}$ *such that* $Q'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} R'_1 \setminus \mathcal{A}_{\mathcal{H}}$*. Since synchronization does not apply to* $\tau$ *and* $l \notin L$*, we have that* $(R_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \stackrel{l}{\longrightarrow}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (R'_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}, (R'_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$*.*
   – *If* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\longrightarrow}_{\rm a} (Q_1 \parallel_L Q'_2) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $Q_2 \stackrel{l}{\longrightarrow}_{\rm a} Q'_2$ *and* $l \notin L$*, then the proof is similar to the one of the previous case.*
   – *If* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\longrightarrow}_{\rm a} (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $Q_i \stackrel{l}{\longrightarrow}_{\rm a} Q'_i$ *for* $i \in \{1, 2\}$ *and* $l \in L$*, then* $Q_i \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\longrightarrow}_{\rm a} Q'_i \setminus \mathcal{A}_{\mathcal{H}}$ *as* $l \notin \mathcal{A}_{\mathcal{H}}$*. From* $Q_i \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} R_i \setminus \mathcal{A}_{\mathcal{H}}$ *it follows that there exists* $R_i \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \stackrel{l}{\longrightarrow}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} R'_i \setminus \mathcal{A}_{\mathcal{H}}$ *such that* $Q'_i \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} R'_i \setminus \mathcal{A}_{\mathcal{H}}$*. Since synchronization does not apply to* $\tau$ *and* $l \in L$*, we have that* $(R_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \stackrel{l}{\longrightarrow}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (R'_1 \parallel_L R'_2) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $((Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_{\mathcal{H}}, (R'_1 \parallel_L R'_2) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$*.*
   – *If* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $Q_1 \stackrel{\tau}{\longrightarrow}_{\rm a} Q'_1$*, then* $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow}_{\rm a} Q'_1 \setminus \mathcal{A}_{\mathcal{H}}$ *as* $\tau \notin \mathcal{A}_{\mathcal{H}}$*. From* $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} R_1 \setminus \mathcal{A}_{\mathcal{H}}$ *it follows that there exists* $R_1 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} R'_1 \setminus \mathcal{A}_{\mathcal{H}}$ *such that* $Q'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} R'_1 \setminus \mathcal{A}_{\mathcal{H}}$*. Since synchronization does not apply to* $\tau$*, we have that* $(R_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (R'_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}, (R'_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$*.*

– *If* $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (Q_1 \,\|_L\, Q'_2) \setminus \mathcal{A_H}$ *with* $Q_2 \xrightarrow{\tau}_{\rm a} Q'_2$*, then the proof is similar to the one of the previous case.*

*As for rates, to avoid trivial cases consider an equivalence class* $C \in \mathbb{P}/\mathcal{B}$ *that involves processes reachable from* $P_1 \,\|_L\, P_2$*, specifically* $C = \{(S_{1,i} \,\|_L\, S_{2,i}) \setminus \mathcal{A_H} \mid S_{k,h} \in \mathit{reach}(P_k) \wedge S_{1,h} \,\|_L\, S_{2,h} \in \mathit{reach}(P_1 \,\|_L\, P_2) \wedge S_{k,i} \setminus \mathcal{A_H} \approx_{\rm mw} S_{k,j} \setminus \mathcal{A_H}\}$*. Suppose that* $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H} \;\overset{\tau}{\not\longrightarrow}_{\rm a}$ *so that* $Q_k \setminus \mathcal{A_H} \;\overset{\tau}{\not\longrightarrow}_{\rm a}$ *too and hence from* $Q_k \setminus \mathcal{A_H} \approx_{\rm mw} R_k \setminus \mathcal{A_H}$ *it follows that there exists* $R_k \setminus \mathcal{A_H} \overset{\tau^*}{\Longrightarrow}_{\rm a} \bar{R}_k \setminus \mathcal{A_H}$ *such that* $\bar{R}_k \setminus \mathcal{A_H} \overset{\tau}{\not\longrightarrow}_{\rm a}$*,* $Q_k \setminus \mathcal{A_H} \approx_{\rm mw} \bar{R}_k \setminus \mathcal{A_H}$*, and* $\mathit{rate}(Q_k \setminus \mathcal{A_H}, C') = \mathit{rate}(\bar{R}_k \setminus \mathcal{A_H}, C')$ *for all* $C' \in \mathbb{P}/\approx_{\rm mw}$*. Since synchronization does not apply to* $\tau$*, we have that* $(R_1 \,\|_L\, R_2) \setminus \mathcal{A_H} \overset{\tau^*}{\Longrightarrow}_{\rm a} (\bar{R}_1 \,\|_L\, \bar{R}_2) \setminus \mathcal{A_H}$ *with* $(\bar{R}_1 \,\|_L\, \bar{R}_2) \setminus \mathcal{A_H} \overset{\tau}{\not\longrightarrow}_{\rm a}$ *and* $((Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}, (\bar{R}_1 \,\|_L\, \bar{R}_2) \setminus \mathcal{A_H}) \in \mathcal{B}$*. Since the restriction operator does not apply to rate transitions, we have that:*

$$\mathit{rate}((Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}, C) = \mathit{rate}((Q_1 \setminus \mathcal{A_H}) \,\|_L (Q_2 \setminus \mathcal{A_H}), C)$$
$$\mathit{rate}((\bar{R}_1 \,\|_L\, \bar{R}_2) \setminus \mathcal{A_H}, C) = \mathit{rate}((\bar{R}_1 \setminus \mathcal{A_H}) \,\|_L (\bar{R}_2 \setminus \mathcal{A_H}), C)$$

*Based on which subprocess moves so that the overall process reaches* $C$ *(which we assume to be reachable in one move to avoid trivial cases in which cumulative rates are zero), we have that:*

$$\mathit{rate}((Q_1 \setminus \mathcal{A_H}) \,\|_L (Q_2 \setminus \mathcal{A_H}), C) = \mathit{rate}(Q_1 \setminus \mathcal{A_H}, C_1)$$
$$\mathit{rate}((\bar{R}_1 \setminus \mathcal{A_H}) \,\|_L (\bar{R}_2 \setminus \mathcal{A_H}), C) = \mathit{rate}(\bar{R}_1 \setminus \mathcal{A_H}, C_1)$$

*or:*

$$\mathit{rate}((Q_1 \setminus \mathcal{A_H}) \,\|_L (Q_2 \setminus \mathcal{A_H}), C) = \mathit{rate}(Q_2 \setminus \mathcal{A_H}, C_2)$$
$$\mathit{rate}((\bar{R}_1 \setminus \mathcal{A_H}) \,\|_L (\bar{R}_2 \setminus \mathcal{A_H}), C) = \mathit{rate}(\bar{R}_2 \setminus \mathcal{A_H}, C_2)$$

*where:*

$$C_1 = \{S_{1,h} \setminus \mathcal{A_H} \mid (S_{1,h} \,\|_L\, S_{2,h}) \setminus \mathcal{A_H} \in C\}$$
$$C_2 = \{S_{2,h} \setminus \mathcal{A_H} \mid (S_{1,h} \,\|_L\, S_{2,h}) \setminus \mathcal{A_H} \in C\}$$

*Since* $Q_k \setminus \mathcal{A_H} \approx_{\rm mw} \bar{R}_k \setminus \mathcal{A_H}$ *and* $C_k$ *is the union of some* $\approx_{\rm mw}$*-equivalence classes for* $k \in \{1, 2\}$*, we have that:*

$$\mathit{rate}(Q_1 \setminus \mathcal{A_H}, C_1) = \mathit{rate}(\bar{R}_1 \setminus \mathcal{A_H}, C_1)$$
$$\mathit{rate}(Q_2 \setminus \mathcal{A_H}, C_2) = \mathit{rate}(\bar{R}_2 \setminus \mathcal{A_H}, C_2)$$

*If we start from* $(R_1 \,\|_L\, R_2) \setminus \mathcal{A_H} \overset{\tau}{\not\longrightarrow}_{\rm a}$*, then the proof is similar.*

*We then prove the three results for the* $\approx_{\rm mb}$*-based properties. Let* $\mathcal{B}$ *be an equivalence relation containing all the pairs of processes that have to be shown to be* $\approx_{\rm mb}$*-equivalent according to the considered result:*

1. *Starting from* $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}$ *and* $(R_1 \,\|_L\, R_2) \,/\, \mathcal{A_H}$ *related by* $\mathcal{B}$*, so that* $Q_1 \setminus \mathcal{A_H} \approx_{\rm mb} R_1 \,/\, \mathcal{A_H}$ *and* $Q_2 \setminus \mathcal{A_H} \approx_{\rm mb} R_2 \,/\, \mathcal{A_H}$*, there are twelve cases for action transitions based on the operational semantic rules in Table 1. In the first five cases, it is* $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}$ *to move first:*

   – *If* $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} (Q'_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}$ *with* $Q_1 \xrightarrow{l}_{\rm a} Q'_1$ *and* $l \notin L$*, then* $Q_1 \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} Q'_1 \setminus \mathcal{A_H}$ *as* $l \notin \mathcal{A_H}$*. From* $Q_1 \setminus \mathcal{A_H} \approx_{\rm mb} R_1 \,/\, \mathcal{A_H}$ *it follows that there exists* $R_1 \,/\, \mathcal{A_H} \overset{\tau^*}{\Longrightarrow}_{\rm a} \bar{R}_1 \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} R'_1 \,/\, \mathcal{A_H}$ *such that* $Q_1 \setminus \mathcal{A_H} \approx_{\rm mb} \bar{R}_1 \,/\, \mathcal{A_H}$ *and* $Q'_1 \setminus \mathcal{A_H} \approx_{\rm mb} R'_1 \,/\, \mathcal{A_H}$*. Since synchronization does not apply to* $\tau$ *and* $l \notin L$*, we have that* $(R_1 \,\|_L\, R_2) \,/\, \mathcal{A_H} \overset{\tau^*}{\Longrightarrow}_{\rm a}$

$(\bar{R}_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}$ *with* $((Q_1 \parallel_L Q_2) \setminus \mathcal{A_H}, (\bar{R}_1 \parallel_L R_2) \,/\, \mathcal{A_H}) \in \mathcal{B}$ *and* $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}, (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}) \in \mathcal{B}$.

- *If* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} (Q_1 \parallel_L Q'_2) \setminus \mathcal{A_H}$ *with* $Q_2 \xrightarrow{l}_{\rm a} Q'_2$ *and* $l \notin L$*, then the proof is similar to the one of the previous case.*
- *If* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A_H}$ *with* $Q_i \xrightarrow{l}_{\rm a} Q'_i$ *for* $i \in \{1, 2\}$ *and* $l \in L$*, then* $Q_i \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} Q'_i \setminus \mathcal{A_H}$ *as* $l \notin \mathcal{A_H}$*. From* $Q_i \setminus \mathcal{A_H} \approx_{\rm mb} R_i \,/\, \mathcal{A_H}$ *it follows that there exists* $R_i \,/\, \mathcal{A_H} \xRightarrow{\tau^*}_{\rm a} \bar{R}_i \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} R'_i \,/\, \mathcal{A_H}$ *such that* $Q_i \setminus \mathcal{A_H} \approx_{\rm mb} \bar{R}_i \,/\, \mathcal{A_H}$ *and* $Q'_i \setminus \mathcal{A_H} \approx_{\rm mb} R'_i \,/\, \mathcal{A_H}$*. Since synchronization does not apply to* $\tau$ *and* $l \in L$*, we have that* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xRightarrow{\tau^*}_{\rm a} (\bar{R}_1 \parallel_L \bar{R}_2) \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} (R'_1 \parallel_L R'_2) \,/\, \mathcal{A_H}$ *with* $((Q_1 \parallel_L Q_2) \setminus \mathcal{A_H}, (\bar{R}_1 \parallel_L \bar{R}_2) \,/\, \mathcal{A_H}) \in \mathcal{B}$ *and* $((Q'_1 \parallel_L Q'_2) \setminus \mathcal{A_H}, (R'_1 \parallel_L R'_2) \,/\, \mathcal{A_H}) \in \mathcal{B}$.
- *If* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}$ *with* $Q_1 \xrightarrow{\tau}_{\rm a} Q'_1$*, then* $Q_1 \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} Q'_1 \setminus \mathcal{A_H}$ *as* $\tau \notin \mathcal{A_H}$*. From* $Q_1 \setminus \mathcal{A_H} \approx_{\rm mb} R_1 \,/\, \mathcal{A_H}$ *it follows that either* $Q'_1 \setminus \mathcal{A_H} \approx_{\rm mb} R_1 \,/\, \mathcal{A_H}$*, or there exists* $R_1 \,/\, \mathcal{A_H} \xRightarrow{\tau^*}_{\rm a} \bar{R}_1 \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} R'_1 \,/\, \mathcal{A_H}$ *such that* $Q_1 \setminus \mathcal{A_H} \approx_{\rm mb} \bar{R}_1 \,/\, \mathcal{A_H}$ *and* $Q'_1 \setminus \mathcal{A_H} \approx_{\rm mb} R'_1 \,/\, \mathcal{A_H}$*. Since synchronization does not apply to* $\tau$*, in the former subcase* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H}$ *is allowed to stay idle with* $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}, (R_1 \parallel_L R_2) \,/\, \mathcal{A_H}) \in \mathcal{B}$*, while in the latter subcase* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xRightarrow{\tau^*}_{\rm a} (\bar{R}_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}$ *with* $((Q_1 \parallel_L Q_2) \setminus \mathcal{A_H}, (\bar{R}_1 \parallel_L R_2) \,/\, \mathcal{A_H}) \in \mathcal{B}$ *and* $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}, (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}) \in \mathcal{B}$.
- *If* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (Q_1 \parallel_L Q'_2) \setminus \mathcal{A_H}$ *with* $Q_2 \xrightarrow{\tau}_{\rm a} Q'_2$*, then the proof is similar to the one of the previous case.*

*In the other seven cases, instead, it is* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H}$ *to move first:*

- *If* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}$ *with* $R_1 \xrightarrow{l}_{\rm a} R'_1$ *and* $l \notin L$*, then* $R_1 \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} R'_1 \,/\, \mathcal{A_H}$ *as* $l \notin \mathcal{A_H}$*. From* $R_1 \,/\, \mathcal{A_H} \approx_{\rm mb} Q_1 \setminus \mathcal{A_H}$ *it follows that there exists* $Q_1 \setminus \mathcal{A_H} \xRightarrow{\tau^*}_{\rm a} \bar{Q}_1 \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} Q'_1 \setminus \mathcal{A_H}$ *such that* $R_1 \,/\, \mathcal{A_H} \approx_{\rm mb} \bar{Q}_1 \setminus \mathcal{A_H}$ *and* $R'_1 \,/\, \mathcal{A_H} \approx_{\rm mb} Q'_1 \setminus \mathcal{A_H}$*. Since synchronization does not apply to* $\tau$ *and* $l \notin L$*, we have that* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xRightarrow{\tau^*}_{\rm a} (\bar{Q}_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}$ *with* $((R_1 \parallel_L R_2)/\mathcal{A_H}, (\bar{Q}_1 \parallel_L Q_2) \setminus \mathcal{A_H}) \in \mathcal{B}$ *and* $((R'_1 \parallel_L R_2)/\mathcal{A_H}, (Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}) \in \mathcal{B}$.
- *If* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} (R_1 \parallel_L R'_2) \,/\, \mathcal{A_H}$ *with* $R_2 \xrightarrow{l}_{\rm a} R'_2$ *and* $l \notin L$*, then the proof is similar to the one of the previous case.*
- *If* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} (R'_1 \parallel_L R'_2) \,/\, \mathcal{A_H}$ *with* $R_i \xrightarrow{l}_{\rm a} R'_i$ *for* $i \in \{1, 2\}$ *and* $l \in L$*, then* $R_i \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} R'_i \,/\, \mathcal{A_H}$ *as* $l \notin \mathcal{A_H}$*. From* $R_i \,/\, \mathcal{A_H} \approx_{\rm mb} Q_i \setminus \mathcal{A_H}$ *it follows that there exists* $Q_i \setminus \mathcal{A_H} \xRightarrow{\tau^*}_{\rm a} \bar{Q}_i \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} Q'_i \setminus \mathcal{A_H}$ *such that* $R_i \,/\, \mathcal{A_H} \approx_{\rm mb} \bar{Q}_i \setminus \mathcal{A_H}$ *and* $R'_i \,/\, \mathcal{A_H} \approx_{\rm mb} Q'_i \setminus \mathcal{A_H}$*. Since synchronization does not apply to* $\tau$ *and* $l \in L$*, we have that* $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xRightarrow{\tau^*}_{\rm a} (\bar{Q}_1 \parallel_L \bar{Q}_2) \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A_H}$ *with* $((R_1 \parallel_L R_2)/\mathcal{A_H}, (\bar{Q}_1 \parallel_L \bar{Q}_2) \setminus \mathcal{A_H}) \in \mathcal{B}$ *and* $((R'_1 \parallel_L R'_2) \,/\, \mathcal{A_H}, (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A_H}) \in \mathcal{B}$.
- *If* $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}$ *with* $R_1 \xrightarrow{\tau}_{\rm a} R'_1$*, then* $R_1 \,/ \mathcal{A_H} \xrightarrow{\tau}_{\rm a} R'_1 \,/ \mathcal{A_H}$ *as* $\tau \notin \mathcal{A_H}$*. From* $R_1 \,/\, \mathcal{A_H} \approx_{\rm mb} Q_1 \setminus \mathcal{A_H}$ *it follows that*

*either* $R'_1 \,/\, \mathcal{A_H} \approx_{\rm mb} Q_1 \setminus \mathcal{A_H}$, *or there exists* $Q_1 \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q}_1 \setminus \mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} Q'_1 \setminus \mathcal{A_H}$ *such that* $R_1 \,/\, \mathcal{A_H} \approx_{\rm mb} \bar{Q}_1 \setminus \mathcal{A_H}$ *and* $R'_1 \,/\, \mathcal{A_H} \approx_{\rm mb} Q'_1 \setminus \mathcal{A_H}$. *Since synchronization does not apply to* $\tau$, *in the former subcase* $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}$ *is allowed to stay idle with* $((R'_1 \,\|_L\, R_2)/\mathcal{A_H}, (Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}) \in \mathcal{B}$, *while in the latter subcase* $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (\bar{Q}_1 \,\|_L\, Q_2) \setminus \mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} (Q'_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}$ *with* $((R_1 \,\|_L\, R_2)/\mathcal{A_H}, (\bar{Q}_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}) \in \mathcal{B}$ *and* $((R'_1 \,\|_L\, R_2)/\mathcal{A_H}, (Q'_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}) \in \mathcal{B}$.

- *If* $(R_1 \,\|_L\, R_2) \,/\, \mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} (R_1 \,\|_L\, R'_2) \,/\, \mathcal{A_H}$ *with* $R_2 \stackrel{\tau}{\longrightarrow}_{\rm a} R'_2$, *then the proof is similar to the one of the previous case.*
- *If* $(R_1 \,\|_L\, R_2) \,/\, \mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} (R'_1 \,\|_L\, R_2) \,/\, \mathcal{A_H}$ *with* $R_1 \stackrel{h}{\longrightarrow}_{\rm a} R'_1$ *and* $h \notin L$, *then* $R_1 \,/\, \mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} R'_1 \,/\, \mathcal{A_H}$ *as* $h \in \mathcal{A_H}$. *The rest of the proof is like the one of the fourth case.*
- *If* $(R_1 \,\|_L\, R_2) \,/\, \mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} (R_1 \,\|_L\, R'_2) \,/\, \mathcal{A_H}$ *with* $R_2 \stackrel{h}{\longrightarrow}_{\rm a} R'_2$ *and* $h \notin L$, *then the proof is similar to the one of the previous case.*

*As for rates, we reason like in the proof of the corresponding result for* $\approx_{\rm mw}$.

2. *Starting from* $(Q \,/\, \mathcal{A_H}) \setminus L$ *and* $(R \setminus L) \,/\, \mathcal{A_H}$ *related by* $\mathcal{B}$, *so that* $Q \,/\, \mathcal{A_H} \approx_{\rm mb} R \setminus \mathcal{A_H}$, *there are six cases for action transitions based on the operational semantic rules in Table 1. In the first three cases, it is* $(Q \,/\, \mathcal{A_H}) \setminus L$ *to move first:*

- *If* $(Q \,/\, \mathcal{A_H}) \setminus L \stackrel{l}{\longrightarrow}_{\rm a} (Q' \,/\, \mathcal{A_H}) \setminus L$ *with* $Q \stackrel{l}{\longrightarrow}_{\rm a} Q'$ *and* $l \notin L$, *then* $Q \,/\, \mathcal{A_H} \stackrel{l}{\longrightarrow}_{\rm a} Q' \,/\, \mathcal{A_H}$ *as* $l \notin \mathcal{A_H}$. *From* $Q \,/\, \mathcal{A_H} \approx_{\rm mb} R \setminus \mathcal{A_H}$ *it follows that there exists* $R \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{R} \setminus \mathcal{A_H} \stackrel{l}{\longrightarrow}_{\rm a} R' \setminus \mathcal{A_H}$ *such that* $Q \,/\, \mathcal{A_H} \approx_{\rm mb} \bar{R} \setminus \mathcal{A_H}$ *and* $Q' \,/\, \mathcal{A_H} \approx_{\rm mb} R' \setminus \mathcal{A_H}$. *Since the restriction and hiding operators do not apply to* $\tau$ *and* $l$, *we have that* $(R \setminus L) \,/\, \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (\bar{R} \setminus L) \,/\, \mathcal{A_H} \stackrel{l}{\longrightarrow}_{\rm a} (R' \setminus L) \,/\, \mathcal{A_H}$ *with* $((Q \,/\, \mathcal{A_H}) \setminus L, (\bar{R} \setminus L) \,/\, \mathcal{A_H}) \in \mathcal{B}$ *and* $((Q' \,/\, \mathcal{A_H}) \setminus L, (R' \setminus L) \,/\, \mathcal{A_H}) \in \mathcal{B}$.
- *If* $(Q/\mathcal{A_H}) \setminus L \stackrel{\tau}{\longrightarrow}_{\rm a} (Q'/\mathcal{A_H}) \setminus L$ *with* $Q \stackrel{\tau}{\longrightarrow}_{\rm a} Q'$, *then* $Q/\mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} Q'/\mathcal{A_H}$ *as* $\tau \notin \mathcal{A_H}$. *From* $Q \,/\, \mathcal{A_H} \approx_{\rm mb} R \setminus \mathcal{A_H}$ *it follows that either* $Q' \,/\, \mathcal{A_H} \approx_{\rm mb} R \setminus \mathcal{A_H}$, *or there exists* $R \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{R} \setminus \mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} R' \setminus \mathcal{A_H}$ *such that* $Q \,/\, \mathcal{A_H} \approx_{\rm mb} \bar{R} \setminus \mathcal{A_H}$ *and* $Q' \,/\, \mathcal{A_H} \approx_{\rm mb} R' \setminus \mathcal{A_H}$. *Since the restriction and hiding operators do not apply to* $\tau$, *in the former subcase* $(R \setminus L) \,/\, \mathcal{A_H}$ *is allowed to stay idle with* $((Q' \,/\, \mathcal{A_H}) \setminus L, (R \setminus L) \,/\, \mathcal{A_H}) \in \mathcal{B}$, *while in the latter subcase* $(R \setminus L) \,/\, \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (\bar{R} \setminus L) \,/\, \mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} (R' \setminus L) \,/\, \mathcal{A_H}$ *with* $((Q \,/\, \mathcal{A_H}) \setminus L, (\bar{R} \setminus L) \,/\, \mathcal{A_H}) \in \mathcal{B}$ *and* $((Q' \,/\, \mathcal{A_H}) \setminus L, (R' \setminus L) \,/\, \mathcal{A_H}) \in \mathcal{B}$.
- *If* $(Q/\mathcal{A_H}) \setminus L \stackrel{\tau}{\longrightarrow}_{\rm a} (Q'/\mathcal{A_H}) \setminus L$ *with* $Q \stackrel{h}{\longrightarrow}_{\rm a} Q'$, *then* $Q/\mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} Q'/\mathcal{A_H}$ *as* $h \in \mathcal{A_H}$. *The rest of the proof is like the one of the previous case.*

*In the other three cases, instead, it is* $(R \setminus L) \,/\, \mathcal{A_H}$ *to move first:*

- *If* $(R \setminus L) \,/\, \mathcal{A_H} \stackrel{l}{\longrightarrow}_{\rm a} (R' \setminus L) \,/\, \mathcal{A_H}$ *with* $R \stackrel{l}{\longrightarrow}_{\rm a} R'$ *and* $l \notin L$, *then* $R \setminus \mathcal{A_H} \stackrel{l}{\longrightarrow}_{\rm a} R' \setminus \mathcal{A_H}$ *as* $l \notin \mathcal{A_H}$. *From* $R \setminus \mathcal{A_H} \approx_{\rm mb} Q \,/\, \mathcal{A_H}$ *it follows that there exists* $Q \,/\, \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q} \,/\, \mathcal{A_H} \stackrel{l}{\longrightarrow}_{\rm a} Q' \,/\, \mathcal{A_H}$ *such that* $R \setminus \mathcal{A_H} \approx_{\rm mb} \bar{Q} \,/\, \mathcal{A_H}$ *and* $R' \setminus \mathcal{A_H} \approx_{\rm mb} Q' \,/\, \mathcal{A_H}$. *Since the restriction operator does not apply* *to $\tau$ and $l$, we have that $(Q \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (\bar{Q} \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L \stackrel{l}{\longrightarrow}_{\rm a} (Q' \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L$ with $((R \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}, (\bar{Q} \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L) \in \mathcal{B}$ and $((R' \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}, (Q' \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L) \in \mathcal{B}$.*

- *If $(R \setminus L) \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow}_{\rm a} (R' \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}$ with $R \stackrel{\tau}{\longrightarrow}_{\rm a} R'$, then $R \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow}_{\rm a} R' \setminus \mathcal{A}_{\mathcal{H}}$ as $\tau \notin \mathcal{A}_{\mathcal{H}}$. From $R \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} Q \,/\, \mathcal{A}_{\mathcal{H}}$ it follows that either $R' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} Q \,/\, \mathcal{A}_{\mathcal{H}}$, or there exists $Q \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q} \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow}_{\rm a} Q' \,/\, \mathcal{A}_{\mathcal{H}}$ such that $R \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} \bar{Q} \,/\, \mathcal{A}_{\mathcal{H}}$ and $R' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} Q' \,/\, \mathcal{A}_{\mathcal{H}}$. Since the restriction operator does not apply to $\tau$, in the former subcase $(Q \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L$ is allowed to stay idle with $((R' \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}, (Q \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L) \in \mathcal{B}$, while in the latter subcase $(Q \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (\bar{Q} \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L \stackrel{\tau}{\longrightarrow}_{\rm a} (Q' \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L$ with $((R \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}, (\bar{Q} \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L) \in \mathcal{B}$ and $((R' \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}, (Q' \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L) \in \mathcal{B}$.*
- *If $(R \setminus L) \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow}_{\rm a} (R' \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}$ with $R \stackrel{h}{\longrightarrow}_{\rm a} R'$ and $h \notin L$, then $R \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow}_{\rm a} R' \,/\, \mathcal{A}_{\mathcal{H}}$ as $h \in \mathcal{A}_{\mathcal{H}}$ (note that $R \setminus \mathcal{A}_{\mathcal{H}}$ cannot perform $h$). From $R \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} R \setminus \mathcal{A}_{\mathcal{H}}$ – as $P \in \mathrm{SBSNNI}_{\approx_{\rm mb}}$ and $R \in reach(P)$ – and $R \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} Q \,/\, \mathcal{A}_{\mathcal{H}}$ it follows that either $R' \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} Q \,/\, \mathcal{A}_{\mathcal{H}}$ and hence $R' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} Q \,/\, \mathcal{A}_{\mathcal{H}}$ – as $R' \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} R' \setminus \mathcal{A}_{\mathcal{H}}$ due to $P \in \mathrm{SBSNNI}_{\approx_{\rm mb}}$ and $R' \in reach(P)$ – or there exists $Q \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{Q} \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow}_{\rm a} Q' \,/\, \mathcal{A}_{\mathcal{H}}$ such that $R \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} \bar{Q} \,/\, \mathcal{A}_{\mathcal{H}}$ and $R' \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} Q' \,/\, \mathcal{A}_{\mathcal{H}}$ and hence $R \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} \bar{Q} \,/\, \mathcal{A}_{\mathcal{H}}$ and $R' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} Q' \,/\, \mathcal{A}_{\mathcal{H}}$. Since the restriction operator does not apply to $\tau$, in the former subcase $(Q \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L$ is allowed to stay idle with $((R' \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}, (Q \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L) \in \mathcal{B}$, while in the latter subcase $(Q \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (\bar{Q} \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L \stackrel{\tau}{\longrightarrow}_{\rm a} (Q' \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L$ with $((R \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}, (\bar{Q} \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L) \in \mathcal{B}$ and $((R' \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}, (Q' \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L) \in \mathcal{B}$.*

*As for rates, we reason like in the proof of the corresponding result for $\approx_{\rm mw}$.*

3. *Starting from $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A}_{\mathcal{H}}$ and $(R_1 \,\|_L\, R_2) \setminus \mathcal{A}_{\mathcal{H}}$ related by $\mathcal{B}$, so that $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} R_1 \setminus \mathcal{A}_{\mathcal{H}}$ and $Q_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} R_2 \setminus \mathcal{A}_{\mathcal{H}}$, there are five cases for action transitions based on the operational semantic rules in Table 1:*
   - *If $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\longrightarrow}_{\rm a} (Q'_1 \,\|_L\, Q_2) \setminus \mathcal{A}_{\mathcal{H}}$ with $Q_1 \stackrel{l}{\longrightarrow}_{\rm a} Q'_1$ and $l \notin L$, then $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\longrightarrow}_{\rm a} Q'_1 \setminus \mathcal{A}_{\mathcal{H}}$ as $l \notin \mathcal{A}_{\mathcal{H}}$. From $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} R_1 \setminus \mathcal{A}_{\mathcal{H}}$ it follows that there exists $R_1 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{R}_1 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\longrightarrow}_{\rm a} R'_1 \setminus \mathcal{A}_{\mathcal{H}}$ such that $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} \bar{R}_1 \setminus \mathcal{A}_{\mathcal{H}}$ and $Q'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} R'_1 \setminus \mathcal{A}_{\mathcal{H}}$. Since synchronization does not apply to $\tau$ and $l \notin L$, we have that $(R_1 \,\|_L\, R_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (\bar{R}_1 \,\|_L\, R_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\longrightarrow}_{\rm a} (R'_1 \,\|_L\, R_2) \setminus \mathcal{A}_{\mathcal{H}}$ with $((Q_1 \,\|_L\, Q_2) \setminus \mathcal{A}_{\mathcal{H}}, (\bar{R}_1 \,\|_L\, R_2) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ and $((Q'_1 \,\|_L\, Q_2) \setminus \mathcal{A}_{\mathcal{H}}, (R'_1 \,\|_L\, R_2) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$.*
   - *If $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\longrightarrow}_{\rm a} (Q_1 \,\|_L\, Q'_2) \setminus \mathcal{A}_{\mathcal{H}}$ with $Q_2 \stackrel{l}{\longrightarrow}_{\rm a} Q'_2$ and $l \notin L$, then the proof is similar to the one of the previous case.*
   - *If $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\longrightarrow}_{\rm a} (Q'_1 \,\|_L\, Q'_2) \setminus \mathcal{A}_{\mathcal{H}}$ with $Q_i \stackrel{l}{\longrightarrow}_{\rm a} Q'_i$ for $i \in \{1, 2\}$ and $l \in L$, then $Q_i \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\longrightarrow}_{\rm a} Q'_i \setminus \mathcal{A}_{\mathcal{H}}$ as $l \notin \mathcal{A}_{\mathcal{H}}$. From $Q_i \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} R_i \setminus \mathcal{A}_{\mathcal{H}}$ it follows that there exists $R_i \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{R}_i \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\longrightarrow}_{\rm a} R'_i \setminus \mathcal{A}_{\mathcal{H}}$ such that $Q_i \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} \bar{R}_i \setminus \mathcal{A}_{\mathcal{H}}$ and $Q'_i \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mb} R'_i \setminus \mathcal{A}_{\mathcal{H}}$. Since synchronization does not apply to $\tau$ and $l \in L$, we have that $(R_1 \,\|_L\, R_2) \setminus$*

$\mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (\bar{R}_1 \,\|_L\, \bar{R}_2) \setminus \mathcal{A_H} \stackrel{l}{\longrightarrow}_{\rm a} (R'_1 \,\|_L\, R'_2) \setminus \mathcal{A_H}$ *with* $((Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}, (\bar{R}_1 \,\|_L\, \bar{R}_2) \setminus \mathcal{A_H}) \in \mathcal{B}$ *and* $((Q'_1 \,\|_L\, Q'_2) \setminus \mathcal{A_H}, (R'_1 \,\|_L\, R'_2) \setminus \mathcal{A_H}) \in \mathcal{B}$.

– *If* $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} (Q'_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}$ *with* $Q_1 \stackrel{\tau}{\longrightarrow}_{\rm a} Q'_1$*, then* $Q_1 \setminus \mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} Q'_1 \setminus \mathcal{A_H}$ *as* $\tau \notin \mathcal{A_H}$*. From* $Q_1 \setminus \mathcal{A_H} \approx_{\rm mb} R_1 \setminus \mathcal{A_H}$ *it follows that either* $Q'_1 \setminus \mathcal{A_H} \approx_{\rm mb} R_1 \setminus \mathcal{A_H}$*, or there exists* $R_1 \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{R}_1 \setminus \mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} R'_1 \setminus \mathcal{A_H}$ *such that* $Q_1 \setminus \mathcal{A_H} \approx_{\rm mb} \bar{R}_1 \setminus \mathcal{A_H}$ *and* $Q'_1 \setminus \mathcal{A_H} \approx_{\rm mb} R'_1 \setminus \mathcal{A_H}$*. Since synchronization does not apply to* $\tau$*, in the former subcase* $(R_1 \,\|_L\, R_2) \setminus \mathcal{A_H}$ *is allowed to stay idle with* $((Q'_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}, (R_1 \,\|_L\, R_2) \setminus \mathcal{A_H}) \in \mathcal{B}$*, while in the latter subcase* $(R_1 \,\|_L\, R_2) \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} (\bar{R}_1 \,\|_L\, R_2) \setminus \mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} (R'_1 \,\|_L\, R_2) \setminus \mathcal{A_H}$ *with* $((Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}, (\bar{R}_1 \,\|_L\, R_2) \setminus \mathcal{A_H}) \in \mathcal{B}$ *and* $((Q'_1 \,\|_L\, Q_2) \setminus \mathcal{A_H}, (R'_1 \,\|_L\, R_2) \setminus \mathcal{A_H}) \in \mathcal{B}$.

– *If* $(Q_1 \,\|_L\, Q_2) \setminus \mathcal{A_H} \stackrel{\tau}{\longrightarrow}_{\rm a} (Q_1 \,\|_L\, Q'_2) \setminus \mathcal{A_H}$ *with* $Q_2 \stackrel{\tau}{\longrightarrow}_{\rm a} Q'_2$*, then the proof is similar to the one of the previous case.*

*As for rates, we reason like in the proof of the corresponding result for* $\approx_{\rm mw}$*.* ■

**Proof of Theorem 2.** We first prove the five results for SBSNNI$_\approx$, from which it will follow that they hold for P_BNDC$_\approx$ too by virtue of the forthcoming Theorem 3:

1. Given an arbitrary $P \in$ SBSNNI$_\approx$ and an arbitrary $a \in \mathcal{A_L} \cup \{\tau\}$, from $P \setminus \mathcal{A_H} \approx P \,/\, \mathcal{A_H}$ we derive that $a \,.\, (P \setminus \mathcal{A_H}) \approx a \,.\, (P \,/\, \mathcal{A_H})$ because $\approx$ is a congruence with respect to action prefix by virtue of Lemma 1(1), from which it follows that $(a \,.\, P) \setminus \mathcal{A_H} \approx (a \,.\, P) \,/\, \mathcal{A_H}$, i.e., $a \,.\, P \in$ BSNNI$_\approx$, because $a \notin \mathcal{A_H}$. To conclude the proof, it suffices to observe that all the processes reachable from $a \,.\, P$ after performing $a$ are processes reachable from $P$, which are known to be BSNNI$_\approx$.
2. Given an arbitrary $P \in$ SBSNNI$_\approx$ and an arbitrary $\lambda \in \mathbb{R}_{>0}$, from $P \setminus \mathcal{A_H} \approx P \,/\, \mathcal{A_H}$ we derive that $(\lambda) \,.\, (P \setminus \mathcal{A_H}) \approx (\lambda) \,.\, (P \,/\, \mathcal{A_H})$ because $\approx$ is a congruence with respect to rate prefix by virtue of Lemma 1(2), from which it follows that $((\lambda) \,.\, P) \setminus \mathcal{A_H} \approx ((\lambda) \,.\, P) \,/\, \mathcal{A_H}$, i.e., $(\lambda) \,.\, P \in$ BSNNI$_\approx$, because the restriction and hiding operators do not apply to rates. To conclude the proof, it suffices to observe that all the processes reachable from $(\lambda) \,.\, P$ after a delay governed by $\lambda$ has elapsed are processes reachable from $P$, which are known to be BSNNI$_\approx$.
3. Given two arbitrary $P_1, P_2 \in \mathbb{P}$ such that $P_1, P_2 \in$ SBSNNI$_\approx$ and an arbitrary $L \subseteq \mathcal{A_L}$, the result follows from Lemma 3(1) by taking $Q_1$ identical to $R_1$ and $Q_2$ identical to $R_2$.
4. Given an arbitrary $P \in$ SBSNNI$_\approx$ and an arbitrary $L \subseteq \mathcal{A} \setminus \{\tau\}$, the result follows from Lemma 3(2) by taking $Q$ identical to $R$ – which will be denoted by $P'$ – because:
   – $(P' \setminus L) \setminus \mathcal{A_H} \approx (P' \setminus \mathcal{A_H}) \setminus L$ as the order in which restriction sets are considered is unimportant.
   – $(P' \setminus \mathcal{A_H}) \setminus L \approx (P' \,/\, \mathcal{A_H}) \setminus L$ because $P' \setminus \mathcal{A_H} \approx P' \,/\, \mathcal{A_H}$ – as $P \in$ SBSNNI$_\approx$ and $P' \in \mathit{reach}(P)$ – and $\approx$ is a congruence with respect to the restriction operator due to Lemma 1(4).

   - $(P' \,/\, \mathcal{A}_{\mathcal{H}}) \setminus L \approx (P' \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}$ as shown in Lemma 3(2).
   - From the transitivity of $\approx$ we obtain that $(P' \setminus L) \setminus \mathcal{A}_{\mathcal{H}} \approx (P' \setminus L) \,/\, \mathcal{A}_{\mathcal{H}}$.
5. Given an arbitrary $P \in \mathrm{SBSNNI}_{\approx}$ and an arbitrary $L \subseteq \mathcal{A}_{\mathcal{L}}$, for every $P' \in$ *reach*$(P)$
it holds that $P' \setminus \mathcal{A}_{\mathcal{H}} \approx P' \,/\, \mathcal{A}_{\mathcal{H}}$, from which we derive that $(P' \setminus \mathcal{A}_{\mathcal{H}}) \,/\, L \approx (P'/\mathcal{A}_{\mathcal{H}}) \,/\, L$ because $\approx$ is a congruence with respect to the hiding operator due to Lemma 1(5). Since $L \cap \mathcal{A}_{\mathcal{H}} = \emptyset$, we have that $(P' \setminus \mathcal{A}_{\mathcal{H}}) \,/\, L$ is isomorphic to $(P' \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ and $(P' \,/\, \mathcal{A}_{\mathcal{H}}) \,/\, L$ is isomorphic to $(P' \,/\, L) \,/\, \mathcal{A}_{\mathcal{H}}$, hence $(P' \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \approx (P' \,/\, L) \,/\, \mathcal{A}_{\mathcal{H}}$, i.e., $P' \,/\, L$ is $\mathrm{BSNNI}_{\approx}$.

We then prove the five results for $\mathrm{SBNDC}_{\approx}$:

1. Given an arbitrary $P \in \mathrm{SBNDC}_{\approx}$ and an arbitrary $a \in \mathcal{A}_{\mathcal{L}} \cup \{\tau\}$, it trivially holds that $a \,.\, P \in \mathrm{SBNDC}_{\approx}$ because $a$ is not high and all the processes reachable from $a \,.\, P$ after performing $a$ are processes reachable from $P$, which is known to be $\mathrm{SBNDC}_{\approx}$.
2. Given an arbitrary $P \in \mathrm{SBNDC}_{\approx}$ and an arbitrary $\lambda \in \mathbb{R}_{>0}$, it trivially holds that $(\lambda) \,.\, P \in \mathrm{SBNDC}_{\approx}$ because all the processes reachable from $(\lambda) \,.\, P$ after a delay governed by $\lambda$ has elapsed are processes reachable from $P$, which is known to be $\mathrm{SBNDC}_{\approx}$.
3. Given two arbitrary $P_1, P_2 \in \mathbb{P}_{\mathrm{mk}}$ such that $P_1, P_2 \in \mathrm{SBNDC}_{\approx}$ and an arbitrary $L \subseteq \mathcal{A} \setminus \{\tau\}$, the result follows from Lemma 3(3) as can be seen by observing that whenever $P'_1 \parallel_L P'_2 \xrightarrow{h}_{\mathrm{a}} P''_1 \parallel_L P''_2$ for $P'_1 \parallel_L P'_2 \in$ *reach*$(P_1 \parallel_L P_2)$:
   - If $P'_1 \xrightarrow{h}_{\mathrm{a}} P''_1$, $P''_2 = P'_2$ (hence $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx P''_2 \setminus \mathcal{A}_{\mathcal{H}}$), and $h \notin L$, then from $P_1 \in \mathrm{SBNDC}_{\approx}$ it follows that $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx P''_1 \setminus \mathcal{A}_{\mathcal{H}}$, which in turn entails that $(P'_1 \parallel_L P'_2) \setminus \mathcal{A}_{\mathcal{H}} \approx (P''_1 \parallel_L P''_2) \setminus \mathcal{A}_{\mathcal{H}}$ because $\approx$ is a congruence with respect to the parallel composition operator due to Lemma 1(3) and restriction distributes over parallel composition.
   - If $P'_2 \xrightarrow{h}_{\mathrm{a}} P''_2$, $P''_1 = P'_1$, and $h \notin L$, then we reason like in the previous case.
   - If $P'_1 \xrightarrow{h}_{\mathrm{a}} P''_1$, $P'_2 \xrightarrow{h}_{\mathrm{a}} P''_2$, and $h \in L$, then from $P_1, P_2 \in \mathrm{SBNDC}_{\approx}$ it follows that $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx P''_1 \setminus \mathcal{A}_{\mathcal{H}}$ and $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx P''_2 \setminus \mathcal{A}_{\mathcal{H}}$, which in turn entail that $(P'_1 \parallel_L P'_2) \setminus \mathcal{A}_{\mathcal{H}} \approx (P''_1 \parallel_L P''_2) \setminus \mathcal{A}_{\mathcal{H}}$ because $\approx$ is a congruence with respect to the parallel composition operator due to Lemma 1(3) and restriction distributes over parallel composition.
4. Given an arbitrary $P \in \mathrm{SBNDC}_{\approx}$ and an arbitrary $L \subseteq \mathcal{A} \setminus \{\tau\}$, for every $P' \in$ *reach*$(P)$ and for every $P''$ such that $P' \xrightarrow{h}_{\mathrm{a}} P''$ it holds that $P' \setminus \mathcal{A}_{\mathcal{H}} \approx P'' \setminus \mathcal{A}_{\mathcal{H}}$, from which we derive that $(P' \setminus \mathcal{A}_{\mathcal{H}}) \setminus L \approx (P'' \setminus \mathcal{A}_{\mathcal{H}}) \setminus L$ because $\approx$ is a congruence with respect to the restriction operator due to Lemma 1(4). Since $(P' \setminus \mathcal{A}_{\mathcal{H}}) \setminus L$ is isomorphic to $(P' \setminus L) \setminus \mathcal{A}_{\mathcal{H}}$ and $(P'' \setminus \mathcal{A}_{\mathcal{H}}) \setminus L$ is isomorphic to $(P'' \setminus L) \setminus \mathcal{A}_{\mathcal{H}}$, we have that $(P' \setminus L) \setminus \mathcal{A}_{\mathcal{H}} \approx (P'' \setminus L) \setminus \mathcal{A}_{\mathcal{H}}$.
5. Given an arbitrary $P \in \mathrm{SBNDC}_{\approx}$ and an arbitrary $L \subseteq \mathcal{A}_{\mathcal{L}}$, for every $P' \in$ *reach*$(P)$ and for every $P''$ such that $P' \xrightarrow{h}_{\mathrm{a}} P''$ it holds that $P' \setminus \mathcal{A}_{\mathcal{H}} \approx P'' \setminus \mathcal{A}_{\mathcal{H}}$, from which we derive that $(P' \setminus \mathcal{A}_{\mathcal{H}}) \,/\, L \approx (P'' \setminus \mathcal{A}_{\mathcal{H}}) \,/\, L$ because $\approx$ is a congruence with respect to the hiding operator due to Lemma 1(5).

Since $L \cap \mathcal{A}_{\mathcal{H}} = \emptyset$, we have that $(P' \setminus \mathcal{A}_{\mathcal{H}}) \,/\, L$ is isomorphic to $(P' \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ and $(P'' \setminus \mathcal{A}_{\mathcal{H}}) \,/\, L$ is isomorphic to $(P'' \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$, hence $(P' \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \approx (P'' \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$. ■

**Lemma 4.** *Let $P, P_1, P_2 \in \mathbb{P}$ and $\approx \in \{\approx_{\rm mw}, \approx_{\rm mb}\}$. Then:*

1. *If $P \in \text{SBNDC}_{\approx}$, $P' \in reach(P)$, and $P' \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} P'' \,/\, \mathcal{A}_{\mathcal{H}}$, then $P' \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \hat{P}'' \setminus \mathcal{A}_{\mathcal{H}}$ with $P'' \setminus \mathcal{A}_{\mathcal{H}} \approx \hat{P}'' \setminus \mathcal{A}_{\mathcal{H}}$.*
2. *If $P_1, P_2 \in \text{SBNDC}_{\approx} \cap \mathbb{P}_{\rm mk,nhc}$ and $P_1 \setminus \mathcal{A}_{\mathcal{H}} \approx P_2 \setminus \mathcal{A}_{\mathcal{H}}$, then $P_1 \,/\, \mathcal{A}_{\mathcal{H}} \approx P_2 \,/\, \mathcal{A}_{\mathcal{H}}$.*
3. *If $P_2 \in \text{SBSNNI}_{\approx}$ and $L \subseteq \mathcal{A}_{\mathcal{H}}$, then $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx ((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ for all $Q \in \mathbb{P}$ having only prefixes in $\mathcal{A}_{\mathcal{H}}$ and for all $P'_1 \in reach(P_1)$ and $P'_2 \in reach(P_2)$ such that $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$.*

*Proof.* *We first prove the three results for the $\approx_{\rm mw}$-based properties:*

1. *We proceed by induction on the number $n \in \mathbb{N}$ of $\tau$-transitions along $P' \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} P'' \,/\, \mathcal{A}_{\mathcal{H}}$:*
   - *If $n = 0$ then $P' \,/\, \mathcal{A}_{\mathcal{H}}$ stays idle and $P'' \,/\, \mathcal{A}_{\mathcal{H}}$ is $P' \,/\, \mathcal{A}_{\mathcal{H}}$. Likewise, $P' \setminus \mathcal{A}_{\mathcal{H}}$ can stay idle, i.e., $P' \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} P' \setminus \mathcal{A}_{\mathcal{H}}$, with $P' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P' \setminus \mathcal{A}_{\mathcal{H}}$ as $\approx_{\rm mw}$ is reflexive.*
   - *Let $n > 0$ and $P'_0/\mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow}_{\rm a} P'_1/\mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow}_{\rm a} \dots \stackrel{\tau}{\longrightarrow}_{\rm a} P'_{n-1}/\mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow}_{\rm a} P'_n/\mathcal{A}_{\mathcal{H}}$ where $P'_0$ is $P'$ and $P'_n$ is $P''$. From the induction hypothesis it follows that $P' \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$ with $P'_{n-1} \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$. As far as the $n$-th $\tau$-transition $P'_{n-1} \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow} P'_n \,/\, \mathcal{A}_{\mathcal{H}}$ is concerned, there are two cases depending on whether it is originated from $P'_{n-1} \stackrel{\tau}{\longrightarrow}_{\rm a} P'_n$ or $P'_{n-1} \stackrel{h}{\longrightarrow}_{\rm a} P'_n$:*
     - *If $P'_{n-1} \stackrel{\tau}{\longrightarrow}_{\rm a} P'_n$ then $P'_{n-1} \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow}_{\rm a} P'_n \setminus \mathcal{A}_{\mathcal{H}}$. Since $P'_{n-1} \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$, there exists $\hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \hat{P}'_n \setminus \mathcal{A}_{\mathcal{H}}$ such that $P'_n \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \hat{P}'_n \setminus \mathcal{A}_{\mathcal{H}}$. Therefore $P' \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \hat{P}'_n \setminus \mathcal{A}_{\mathcal{H}}$ with $P'' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \hat{P}'_n \setminus \mathcal{A}_{\mathcal{H}}$.*
     - *If $P'_{n-1} \stackrel{h}{\longrightarrow}_{\rm a} P'_n$ then from $P \in \text{SBNDC}_{\approx_{\rm mw}}$ it follows that $P'_{n-1} \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P'_n \setminus \mathcal{A}_{\mathcal{H}}$. Since $P'_{n-1} \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$ and $\approx_{\rm mw}$ is symmetric and transitive, we obtain $P'_n \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$. Therefore $P' \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$ with $P'' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$.*
2. *Let $\mathcal{B}$ be an equivalence relation containing all the pairs of processes that have to be shown to be $\approx_{\rm mw}$-equivalent according to the considered result. Starting from $(P_1 \,/\, \mathcal{A}_{\mathcal{H}}, P_2 \,/\, \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$, so that $P_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P_2 \setminus \mathcal{A}_{\mathcal{H}}$, there are three cases for action transitions based on the operational semantic rules in Table 1:*
   - *If $P_1 \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\longrightarrow}_{\rm a} P'_1 \,/\, \mathcal{A}_{\mathcal{H}}$ with $P_1 \stackrel{h}{\longrightarrow}_{\rm a} P'_1$, then $P_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ as $h \in \mathcal{A}_{\mathcal{H}}$ and $P_1 \in \text{SBNDC}_{\approx_{\rm mw}}$. Since $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P_2 \setminus \mathcal{A}_{\mathcal{H}}$,*

*as* $P_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P_2 \setminus \mathcal{A}_{\mathcal{H}}$ *and* $\approx_{\rm mw}$ *is symmetric and transitive, with* $P'_1, P_2 \in \text{SBNDC}_{\approx_{\rm mw}}$*, we have that* $P_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *is allowed to stay idle with* $(P'_1 \,/\, \mathcal{A}_{\mathcal{H}}, P_2 \,/\, \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$.

- *If* $P_1 \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} P'_1 \,/\, \mathcal{A}_{\mathcal{H}}$ *with* $P_1 \xrightarrow{l}_{\rm a} P'_1$*, then* $P_1 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ *as* $l \notin \mathcal{A}_{\mathcal{H}}$*. From* $P_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P_2 \setminus \mathcal{A}_{\mathcal{H}}$ *it follows that there exists* $P_2 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\hat{l}}{\Longrightarrow}_{\rm a} P'_2 \setminus \mathcal{A}_{\mathcal{H}}$ *such that* $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P'_2 \setminus \mathcal{A}_{\mathcal{H}}$*. Thus* $P_2 \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\hat{l}}{\Longrightarrow}_{\rm a} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *as* $l, \tau \notin \mathcal{A}_{\mathcal{H}}$*. Since* $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P'_2 \setminus \mathcal{A}_{\mathcal{H}}$ *with* $P'_1, P'_2 \in \text{SBNDC}_{\approx_{\rm mw}}$*, we have that* $(P'_1 \,/\, \mathcal{A}_{\mathcal{H}}, P'_2 \,/\, \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$.
- *If* $P_1 \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} P'_1 \,/\, \mathcal{A}_{\mathcal{H}}$ *with* $P_1 \xrightarrow{\tau}_{\rm a} P'_1$*, then the proof is like the one of the previous case.*

*As for rates, suppose that* $P_1 \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ *so that* $P_1 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ *too and hence from* $P_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P_2 \setminus \mathcal{A}_{\mathcal{H}}$ *it follows that there exists* $P_2 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{P}_2 \setminus \mathcal{A}_{\mathcal{H}}$ *such that* $\bar{P}_2 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, $P_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \bar{P}_2 \setminus \mathcal{A}_{\mathcal{H}}$*, and* $rate(P_1 \setminus \mathcal{A}_{\mathcal{H}}, C) = rate(\bar{P}_2 \setminus \mathcal{A}_{\mathcal{H}}, C)$ *for all* $C \in \mathbb{P}/\mathcal{B}$*. Since the hiding and restriction operators do not apply to* $\tau$ *and rate transitions, it follows that* $P_2 \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{P}_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *with* $\bar{P}_2 \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ *(if* $\bar{P}_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *could perform* $\tau$ *due to* $\bar{P}_2 \xrightarrow{h}_{\rm a} \bar{P}'_2$*, then* $\bar{P}_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} \bar{P}'_2 \setminus \mathcal{A}_{\mathcal{H}}$ *as* $\bar{P}_2 \in \text{SBNDC}_{\approx_{\rm mw}}$*, hence it would just be a matter of going ahead until one not enabling* $\tau$ *is encountered, which certainly happens because the considered processes belong to* $\mathbb{P}_{\rm mk,nhc}$*),* $(P_1 \,/\, \mathcal{A}_{\mathcal{H}}, \bar{P}_2 \,/\, \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$*, and* $rate(P_1/\mathcal{A}_{\mathcal{H}}, C) = rate(P_1 \setminus \mathcal{A}_{\mathcal{H}}, C) = rate(\bar{P}_2 \setminus \mathcal{A}_{\mathcal{H}}, C) = rate(\bar{P}_2/\mathcal{A}_{\mathcal{H}}, C)$ *for all* $C \in \mathbb{P}/\mathcal{B}$.

3. *Let* $\mathcal{B}$ *be an equivalence relation containing all the pairs of processes that have to be shown to be* $\approx_{\rm mw}$*-equivalent according to the considered result. Starting from* $P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ *and* $((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ *related by* $\mathcal{B}$*, so that* $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$*, there are six cases for action transitions based on the operational semantic rules in Table 1. In the first two cases, it is* $P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ *to move first:*

   - *Let* $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} P''_1 \setminus \mathcal{A}_{\mathcal{H}}$*. We observe that from* $P'_2 \in reach(P_2)$ *and* $P_2 \in \text{SBSNNI}_{\approx_{\rm mw}}$ *it follows that* $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$*, so that* $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P'_2 \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P'_2 \setminus \mathcal{A}_{\mathcal{H}}$*, i.e.,* $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P'_2 \setminus \mathcal{A}_{\mathcal{H}}$*, as* $\approx_{\rm mw}$ *is symmetric and transitive. As a consequence, since* $l \neq \tau$ *there exists* $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\Longrightarrow}_{\rm a} P''_2 \setminus \mathcal{A}_{\mathcal{H}}$ *such that* $P''_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P''_2 \setminus \mathcal{A}_{\mathcal{H}}$*. Thus* $((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\Longrightarrow}_{\rm a} ((P''_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $(P''_1 \setminus \mathcal{A}_{\mathcal{H}}, ((P''_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ *because* $P''_1 \in reach(P_1)$*,* $P''_2 \in reach(P_2)$*, and* $P''_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P''_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *as* $P_2 \in \text{SBSNNI}_{\approx_{\rm mw}}$.
   - *Let* $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} P''_1 \setminus \mathcal{A}_{\mathcal{H}}$*. The proof is like the one of the previous case with* $\stackrel{\tau^*}{\Longrightarrow}_{\rm a}$ *used in place of* $\stackrel{l}{\Longrightarrow}_{\rm a}$.

   *In the other four cases, instead, it is* $((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ *to move first:*

   - *Let* $((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} ((P''_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $P'_2 \xrightarrow{l}_{\rm a} P''_2$ *so that* $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} P''_2 \setminus \mathcal{A}_{\mathcal{H}}$ *as* $l \notin \mathcal{A}_{\mathcal{H}}$*. We observe that from* $P'_2 \in reach(P_2)$ *and* $P_2 \in \text{SBSNNI}_{\approx_{\rm mw}}$ *it follows that* $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$*, so that* $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P'_2 \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P'_1 \setminus \mathcal{A}_{\mathcal{H}}$*, i.e.,* $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm mw} P'_1 \setminus \mathcal{A}_{\mathcal{H}}$*, as*

$\approx_{\mathrm{mw}}$ *is symmetric and transitive. As a consequence, since* $l \neq \tau$ *there exists* $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\Longrightarrow}_{\mathrm{a}} P''_1 \setminus \mathcal{A}_{\mathcal{H}}$ *such that* $P''_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} P''_1 \setminus \mathcal{A}_{\mathcal{H}}$. *Thus* $(((P''_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}, P''_1 \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ *because* $P''_1 \in \mathit{reach}(P_1)$, $P''_2 \in \mathit{reach}(P_2)$, *and* $P''_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} P''_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *as* $P_2 \in \mathrm{SBSNNI}_{\approx_{\mathrm{mw}}}$.

- *Let* $((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} ((P''_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $P'_2 \xrightarrow{\tau}_{\mathrm{a}} P''_2$ *so that* $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} P''_2 \setminus \mathcal{A}_{\mathcal{H}}$ *as* $\tau \notin \mathcal{A}_{\mathcal{H}}$. *The proof is like the one of the previous case with* $\stackrel{\tau^*}{\Longrightarrow}_{\mathrm{a}}$ *used in place of* $\stackrel{l}{\Longrightarrow}_{\mathrm{a}}$.
- *If* $((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} ((P'_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $Q \xrightarrow{\tau}_{\mathrm{a}} Q'$, *then trivially* $(((P'_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}, P'_1 \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ *as* $P'_2 \approx_{\mathrm{mw}} P'_2$ *and hence* $P'_2 \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *by Lemma 1(5).*
- *Let* $((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} ((P''_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $P'_2 \xrightarrow{h}_{\mathrm{a}} P''_2$ – *so that* $P'_2 \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} P''_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *as* $h \in \mathcal{A}_{\mathcal{H}}$ – *and* $Q \xrightarrow{h}_{\mathrm{a}} Q'$ *for* $h \in L$. *We observe that from* $P'_2, P''_2 \in \mathit{reach}(P_2)$ *and* $P_2 \in \mathrm{SBSNNI}_{\approx_{\mathrm{mw}}}$ *it follows that* $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *and* $P''_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} P''_2 \,/\, \mathcal{A}_{\mathcal{H}}$, *so that* $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\mathrm{a}} P''_2 \setminus \mathcal{A}_{\mathcal{H}}$ *as* $P'_2 \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} P''_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *and* $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} P'_2 \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} P'_1 \setminus \mathcal{A}_{\mathcal{H}}$, *i.e.,* $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} P'_1 \setminus \mathcal{A}_{\mathcal{H}}$, *as* $\approx_{\mathrm{mw}}$ *is symmetric and transitive. As a consequence there exists* $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\mathrm{a}} P''_1 \setminus \mathcal{A}_{\mathcal{H}}$ *such that* $P''_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} P''_1 \setminus \mathcal{A}_{\mathcal{H}}$. *Thus* $(((P''_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}, P''_1 \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ *because* $P''_1 \in \mathit{reach}(P_1)$, $P''_2 \in \mathit{reach}(P_2)$, *and* $P''_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} P''_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *as* $P_2 \in \mathrm{SBSNNI}_{\approx_{\mathrm{mw}}}$.

*As for rates, to avoid trivial cases consider an equivalence class* $C \in \mathbb{P}/\mathcal{B}$ *that involves processes reachable from* $P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ *and* $((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$, *specifically* $C = \{R_{1,i} \setminus \mathcal{A}_{\mathcal{H}}, ((R_{2,j} \parallel_L S_j) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \mid S_j \in \mathbb{P}$ *having only prefixes in* $\mathcal{A}_{\mathcal{H}} \wedge R_{k,h} \in \mathit{reach}(P_k) \wedge R_{1,i} \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} R_{2,j} \,/\, \mathcal{A}_{\mathcal{H}}\}$. *If* $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$ *then from* $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *it follows that there exists* $P'_2 \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\mathrm{a}} \bar{P}'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *such that* $\bar{P}'_2 \,/\, \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$, $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} \bar{P}'_2 \,/\, \mathcal{A}_{\mathcal{H}}$, *and* $\mathit{rate}(P'_1 \setminus \mathcal{A}_{\mathcal{H}}, C') = \mathit{rate}(\bar{P}'_2 \,/\, \mathcal{A}_{\mathcal{H}}, C')$ *for all* $C' \in \mathbb{P}/\approx_{\mathrm{mw}}$. *Since synchronization as well as the restriction and hiding operators do not apply to* $\tau$, *we have that* $((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau^*}{\Longrightarrow}_{\mathrm{a}} ((\bar{P}'_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ *with* $((\bar{P}'_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$ *and* $(P'_1 \setminus \mathcal{A}_{\mathcal{H}}, ((\bar{P}'_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$. *Since the restriction and hiding operators do not apply to rate transitions and* $Q$ *cannot perform any rate transition, we have that:*

$$\mathit{rate}(P'_1 \setminus \mathcal{A}_{\mathcal{H}}, C) = \mathit{rate}(P'_1 \setminus \mathcal{A}_{\mathcal{H}}, \bar{C})$$
$$\mathit{rate}(((\bar{P}'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}, C) = \mathit{rate}(\bar{P}'_2 \,/\, \mathcal{A}_{\mathcal{H}}, \bar{C})$$

*where:*

$$\bar{C} = \{R_{1,i} \setminus \mathcal{A}_{\mathcal{H}} \in C\} \cup \{R_{2,j} \,/\, \mathcal{A}_{\mathcal{H}} \mid ((R_{2,j} \parallel_L S_j) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \in C\}$$

*Since* $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mw}} \bar{P}'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ *and* $\bar{C}$ *is the union of some* $\approx_{\mathrm{mw}}$*-equivalence classes, we have that:*

$$\mathit{rate}(P'_1 \setminus \mathcal{A}_{\mathcal{H}}, \bar{C}) = \mathit{rate}(\bar{P}'_2 \,/\, \mathcal{A}_{\mathcal{H}}, \bar{C})$$

*If we start from* $((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$, *then the proof is similar.*

*We then prove the three results for the $\approx_{\mathrm{mb}}$-based properties:*

1. *We proceed by induction on the number $n \in \mathbb{N}$ of $\tau$-transitions along $P' \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau^*}_{\mathrm{a}} P'' \,/\, \mathcal{A}_{\mathcal{H}}$:*
   - *If $n = 0$ then the proof is like the one of the corresponding result for $\approx_{\mathrm{mw}}$.*
   - *Let $n > 0$ and $P'_0/\mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} P'_1/\mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} \dots \xrightarrow{\tau}_{\mathrm{a}} P'_{n-1}/\mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} P'_n/\mathcal{A}_{\mathcal{H}}$ where $P'_0$ is $P'$ and $P'_n$ is $P''$. From the induction hypothesis it follows that $P' \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau^*}_{\mathrm{a}} \hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$ with $P'_{n-1} \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} \hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$. The rest of the proof is like the one of the corresponding result for $\approx_{\mathrm{mw}}$ with the following difference:*
     - *If $P'_{n-1} \xrightarrow{\tau}_{\mathrm{a}} P'_n$ then $P'_{n-1} \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} P'_n \setminus \mathcal{A}_{\mathcal{H}}$. Since $P'_{n-1} \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} \hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$:*
       - *either $P'_n \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} \hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$, in which case $\hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$ stays idle and hence $P' \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau^*}_{\mathrm{a}} \hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$ with $P'' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} \hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$;*
       - *or there exists $\hat{P}'_{n-1} \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau^*}_{\mathrm{a}} \bar{P}_{n-1} \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} \hat{P}'_n \setminus \mathcal{A}_{\mathcal{H}}$ such that $P'_{n-1} \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} \bar{P}_{n-1} \setminus \mathcal{A}_{\mathcal{H}}$ and $P'_n \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} \hat{P}'_n \setminus \mathcal{A}_{\mathcal{H}}$, hence $P' \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau^*}_{\mathrm{a}} \hat{P}'_n \setminus \mathcal{A}_{\mathcal{H}}$ with $P'' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} \hat{P}'_n \setminus \mathcal{A}_{\mathcal{H}}$.*
2. *Let $\mathcal{B}$ be an equivalence relation containing all the pairs of processes that have to be shown to be $\approx_{\mathrm{mb}}$-equivalent according to the considered result. Starting from $(P_1 \,/\, \mathcal{A}_{\mathcal{H}}, P_2 \,/\, \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$, so that $P_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} P_2 \setminus \mathcal{A}_{\mathcal{H}}$, there are three cases for action transitions based on the operational semantic rules in Table 1:*
   - *If $P_1 \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} P'_1 \,/\, \mathcal{A}_{\mathcal{H}}$ with $P_1 \xrightarrow{h}_{\mathrm{a}} P'_1$, then the proof is like the one of the corresponding result for $\approx_{\mathrm{mw}}$.*
   - *If $P_1 \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\mathrm{a}} P'_1 \,/\, \mathcal{A}_{\mathcal{H}}$ with $P_1 \xrightarrow{l}_{\mathrm{a}} P'_1$, then $P_1 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\mathrm{a}} P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ as $l \notin \mathcal{A}_{\mathcal{H}}$. From $P_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} P_2 \setminus \mathcal{A}_{\mathcal{H}}$ it follows that there exists $P_2 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau^*}_{\mathrm{a}} \bar{P}_2 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\mathrm{a}} P'_2 \setminus \mathcal{A}_{\mathcal{H}}$ such that $P_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} \bar{P}_2 \setminus \mathcal{A}_{\mathcal{H}}$ and $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} P'_2 \setminus \mathcal{A}_{\mathcal{H}}$. Thus $P_2 \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau^*}_{\mathrm{a}} \bar{P}_2 \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\mathrm{a}} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ as $l, \tau \notin \mathcal{A}_{\mathcal{H}}$. Since $P_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} \bar{P}_2 \setminus \mathcal{A}_{\mathcal{H}}$ with $P_1, \bar{P}_2 \in \mathrm{SBNDC}_{\approx_{\mathrm{mb}}}$ and $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} P'_2 \setminus \mathcal{A}_{\mathcal{H}}$ with $P'_1, P'_2 \in \mathrm{SBNDC}_{\approx_{\mathrm{mb}}}$, we have that $(P_1 \,/\, \mathcal{A}_{\mathcal{H}}, \bar{P}_2 \,/\, \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ and $(P'_1 \,/\, \mathcal{A}_{\mathcal{H}}, P'_2 \,/\, \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$.*
   - *If $P_1 \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} P'_1 \,/\, \mathcal{A}_{\mathcal{H}}$ with $P_1 \xrightarrow{\tau}_{\mathrm{a}} P'_1$, then $P_1 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ as $\tau \notin \mathcal{A}_{\mathcal{H}}$. There are two subcases:*
     - *If $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} P_2 \setminus \mathcal{A}_{\mathcal{H}}$ then $P_2 \setminus \mathcal{A}_{\mathcal{H}}$ is allowed to stay idle with $(P'_1 \,/\, \mathcal{A}_{\mathcal{H}}, P_2 \,/\, \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ because $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{mb}} P_2 \setminus \mathcal{A}_{\mathcal{H}}$ and $P'_1, P_2 \in \mathrm{SBNDC}_{\approx_{\mathrm{mb}}}$.*
     - *If $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \not\approx_{\mathrm{mb}} P_2 \setminus \mathcal{A}_{\mathcal{H}}$ then the proof is like the one of the previous case with $\xrightarrow{\tau}_{\mathrm{a}}$ used in place of $\xrightarrow{l}_{\mathrm{a}}$.*

   *As for rates, we reason like in the proof of the corresponding result for $\approx_{\mathrm{mw}}$.*

*3. Let $\mathcal{B}$ be an equivalence relation containing all the pairs of processes that have to be shown to be $\approx_{\rm mb}$-equivalent according to the considered result. Starting from $P'_1 \setminus \mathcal{A_H}$ and $((P'_2 \parallel_L Q) / L) \setminus \mathcal{A_H}$ related by $\mathcal{B}$, so that $P'_1 \setminus \mathcal{A_H} \approx_{\rm mb} P'_2 / \mathcal{A_H}$, there are six cases for action transitions based on the operational semantic rules in Table 1. In the first two cases, it is $P'_1 \setminus \mathcal{A_H}$ to move first:*

- *Let $P'_1 \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} P''_1 \setminus \mathcal{A_H}$. We observe that from $P'_2 \in reach(P_2)$ and $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm mb}}$ it follows that $P'_2 \setminus \mathcal{A_H} \approx_{\rm mb} P'_2 / \mathcal{A_H}$, so that $P'_1 \setminus \mathcal{A_H} \approx_{\rm mb} P'_2 / \mathcal{A_H} \approx_{\rm mb} P'_2 \setminus \mathcal{A_H}$, i.e., $P'_1 \setminus \mathcal{A_H} \approx_{\rm mb} P'_2 \setminus \mathcal{A_H}$, as $\approx_{\rm mb}$ is symmetric and transitive. As a consequence, since $l \neq \tau$ there exists $P'_2 \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{P}'_2 \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} P''_2 \setminus \mathcal{A_H}$ such that $P'_1 \setminus \mathcal{A_H} \approx_{\rm mb} \bar{P}'_2 \setminus \mathcal{A_H}$ and $P''_1 \setminus \mathcal{A_H} \approx_{\rm mb} P''_2 \setminus \mathcal{A_H}$. Thus $((P'_2 \parallel_L Q) / L) \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} ((\bar{P}'_2 \parallel_L Q) / L) \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} ((P''_2 \parallel_L Q) / L) \setminus \mathcal{A_H}$ with $(P'_1 \setminus \mathcal{A_H}, ((\bar{P}'_2 \parallel_L Q) / L) \setminus \mathcal{A_H}) \in \mathcal{B}$ – because $P'_1 \in reach(P_1)$, $\bar{P}'_2 \in reach(P_2)$, and $P'_1 \setminus \mathcal{A_H} \approx_{\rm mb} \bar{P}'_2 / \mathcal{A_H}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm mb}}$ – and $(P''_1 \setminus \mathcal{A_H}, ((P''_2 \parallel_L Q) / L) \setminus \mathcal{A_H}) \in \mathcal{B}$ – because $P''_1 \in reach(P_1)$, $P''_2 \in reach(P_2)$, and $P''_1 \setminus \mathcal{A_H} \approx_{\rm mb} P''_2 / \mathcal{A_H}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm mb}}$.*
- *If $P'_1 \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} P''_1 \setminus \mathcal{A_H}$ there are two subcases:*
  - *If $P''_1 \setminus \mathcal{A_H} \approx_{\rm mb} P'_2 / \mathcal{A_H}$ then $(P'_2 \parallel_L Q) / L) \setminus \mathcal{A_H}$ is allowed to stay idle with $(P''_1 \setminus \mathcal{A_H}, ((P'_2 \parallel_L Q) / L) \setminus \mathcal{A_H}) \in \mathcal{B}$ because $P''_1 \in reach(P_1)$ and $P'_2 \in reach(P_2)$.*
  - *If $P''_1 \setminus \mathcal{A_H} \not\approx_{\rm mb} P'_2 / \mathcal{A_H}$ then the proof is like the one of the previous case with $\xrightarrow{\tau}_{\rm a}$ used in place of $\xrightarrow{l}_{\rm a}$.*

*In the other four cases, instead, it is $((P'_2 \parallel_L Q) / L) \setminus \mathcal{A_H}$ to move first:*

- *Let $((P'_2 \parallel_L Q) / L) \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} ((P''_2 \parallel_L Q) / L) \setminus \mathcal{A_H}$ with $P'_2 \xrightarrow{l}_{\rm a} P''_2$ so that $P'_2 \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} P''_2 \setminus \mathcal{A_H}$ as $l \notin \mathcal{A_H}$. We observe that from $P'_2 \in reach(P_2)$ and $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm mb}}$ it follows that $P'_2 \setminus \mathcal{A_H} \approx_{\rm mb} P'_2 / \mathcal{A_H}$, so that $P'_2 \setminus \mathcal{A_H} \approx_{\rm mb} P'_2 / \mathcal{A_H} \approx_{\rm mb} P'_1 \setminus \mathcal{A_H}$, i.e., $P'_2 \setminus \mathcal{A_H} \approx_{\rm mb} P'_1 \setminus \mathcal{A_H}$, as $\approx_{\rm mb}$ is symmetric and transitive. As a consequence, since $l \neq \tau$ there exists $P'_1 \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{P}'_1 \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} P''_1 \setminus \mathcal{A_H}$ such that $P'_2 \setminus \mathcal{A_H} \approx_{\rm mb} \bar{P}'_1 \setminus \mathcal{A_H}$ and $P''_2 \setminus \mathcal{A_H} \approx_{\rm mb} P''_1 \setminus \mathcal{A_H}$. Thus $(((P'_2 \parallel_L Q) / L) \setminus \mathcal{A_H}, \bar{P}'_1 \setminus \mathcal{A_H}) \in \mathcal{B}$ – because $\bar{P}'_1 \in reach(P_1)$, $P'_2 \in reach(P_2)$, and $\bar{P}'_1 \setminus \mathcal{A_H} \approx_{\rm mb} P'_2 / \mathcal{A_H}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm mb}}$ – and $(((P''_2 \parallel_L Q) / L) \setminus \mathcal{A_H}, P''_1 \setminus \mathcal{A_H}) \in \mathcal{B}$ – because $P''_1 \in reach(P_1)$, $P''_2 \in reach(P_2)$, and $P''_1 \setminus \mathcal{A_H} \approx_{\rm mb} P''_2 / \mathcal{A_H}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm mb}}$.*
- *If $((P'_2 \parallel_L Q) / L) \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} ((P''_2 \parallel_L Q) / L) \setminus \mathcal{A_H}$ with $P'_2 \xrightarrow{\tau}_{\rm a} P''_2$ so that $P'_2 \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} P''_2 \setminus \mathcal{A_H}$ as $\tau \notin \mathcal{A_H}$, there are two subcases:*
  - *If $P''_2 \setminus \mathcal{A_H} \approx_{\rm mb} P'_1 \setminus \mathcal{A_H}$ then $P'_1 \setminus \mathcal{A_H}$ is allowed to stay idle with $(((P''_2 \parallel_L Q) / L) \setminus \mathcal{A_H}, P'_1 \setminus \mathcal{A_H}) \in \mathcal{B}$ because $P'_1 \in reach(P_1)$, $P''_2 \in reach(P_2)$, and $P'_1 \setminus \mathcal{A_H} \approx_{\rm mb} P''_2 / \mathcal{A_H}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm mb}}$.*
  - *If $P''_2 \setminus \mathcal{A_H} \not\approx_{\rm mb} P'_1 \setminus \mathcal{A_H}$ then the proof is like the one of the previous case with $\xrightarrow{\tau}_{\rm a}$ used in place of $\xrightarrow{l}_{\rm a}$.*
- *If $((P'_2 \parallel_L Q) / L) \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} ((P'_2 \parallel_L Q') / L) \setminus \mathcal{A_H}$ with $Q \xrightarrow{\tau}_{\rm a} Q'$, then trivially $(((P'_2 \parallel_L Q') / L) \setminus \mathcal{A_H}, P'_1 \setminus \mathcal{A_H}) \in \mathcal{B}$ as $P'_2 \approx_{\rm mb} P'_2$ and hence $P'_2 / \mathcal{A_H} \approx_{\rm mb} P'_2 / \mathcal{A_H}$ by Lemma 1(5).*

– *Let* $((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} ((P''_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A_H}$ *with* $P'_2 \xrightarrow{h}_{\rm a} P''_2$ *– so that* $P'_2 \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} P''_2 \,/\, \mathcal{A_H}$ *as* $h \in \mathcal{A_H}$ *– and* $Q \xrightarrow{h}_{\rm a} Q'$ *for* $h \in L$*. We observe that from* $P'_2, P''_2 \in \mathit{reach}(P_2)$ *and* $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm mb}}$ *it follows that* $P'_2 \setminus \mathcal{A_H} \approx_{\rm mb} P'_2 \,/\, \mathcal{A_H}$ *and* $P''_2 \setminus \mathcal{A_H} \approx_{\rm mb} P''_2 \,/\, \mathcal{A_H}$*, so that* $P'_2 \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} P''_2 \setminus \mathcal{A_H}$ *and* $P'_2 \setminus \mathcal{A_H} \approx_{\rm mb} P'_2 \,/\, \mathcal{A_H} \approx_{\rm mb} P'_1 \setminus \mathcal{A_H}$*, i.e.,* $P'_2 \setminus \mathcal{A_H} \approx_{\rm mb} P'_1 \setminus \mathcal{A_H}$*, as* $\approx_{\rm mb}$ *is symmetric and transitive. There are two subcases:*
  - *If* $P''_2 \setminus \mathcal{A_H} \approx_{\rm mb} P'_1 \setminus \mathcal{A_H}$ *then* $P'_1 \setminus \mathcal{A_H}$ *is allowed to stay idle with* $(((P''_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A_H}, P'_1 \setminus \mathcal{A_H}) \in \mathcal{B}$ *because* $P'_1 \in \mathit{reach}(P_1)$*,* $P''_2 \in \mathit{reach}(P_2)$*, and* $P'_1 \setminus \mathcal{A_H} \approx_{\rm mb} P''_2 \,/\, \mathcal{A_H}$ *as* $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm mb}}$*.*
  - *If* $P''_2 \setminus \mathcal{A_H} \not\approx_{\rm mb} P'_1 \setminus \mathcal{A_H}$ *then there exists* $P'_1 \setminus \mathcal{A_H} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{P}'_1 \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} P''_1 \setminus \mathcal{A_H}$ *such that* $P'_2 \setminus \mathcal{A_H} \approx_{\rm mb} \bar{P}'_1 \setminus \mathcal{A_H}$ *and* $P''_2 \setminus \mathcal{A_H} \approx_{\rm mb} P''_1 \setminus \mathcal{A_H}$*. Thus* $(((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A_H}, \bar{P}'_1 \setminus \mathcal{A_H}) \in \mathcal{B}$ *– because* $\bar{P}'_1 \in \mathit{reach}(P_1)$*,* $P'_2 \in \mathit{reach}(P_2)$*, and* $\bar{P}'_1 \setminus \mathcal{A_H} \approx_{\rm mb} P'_2 \,/\, \mathcal{A_H}$ *as* $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm mb}}$ *– and* $(((P''_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A_H}, P''_1 \setminus \mathcal{A_H}) \in \mathcal{B}$ *– because* $P''_1 \in \mathit{reach}(P_1)$*,* $P''_2 \in \mathit{reach}(P_2)$*, and* $P''_1 \setminus \mathcal{A_H} \approx_{\rm mb} P''_2 \,/\, \mathcal{A_H}$ *as* $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm mb}}$*.*

*As for rates, we reason like in the proof of the corresponding result for* $\approx_{\rm mw}$*.*

■

**Proof of Theorem 3.** We first prove the results for the $\approx_{\rm mw}$-based properties. Let us examine each relationship separately:

– $\mathrm{SBNDC}_{\approx_{\rm mw}} \subseteq \mathrm{SBSNNI}_{\approx_{\rm mw}}$. We need to prove that for a given $P \in \mathbb{P}$, if $P \in$ SBNDC, it follows that for every $P'$ reachable from $P$, $P' \in \mathrm{BSNNI}_{\approx_{\rm mw}}$. Since the processes we are considering are not recursive we can treat them as trees, and hence we can proceed by induction on their depth. In this case we will proceed by induction on the depth of $P$:
  - If the depth of $P$ is 0 then $P$ has no outgoing transitions and it behaves as $\underline{0}$. This obviously entails that $P \setminus \mathcal{A_H} \approx_{\rm mw} P \,/\, \mathcal{A_H}$.
  - If the depth of $P$ is $n+1$ with $n \in \mathbb{N}$, then take any $P'$ of depth $n$ such that $P \xrightarrow{a}_{\rm a} P'$ or $P \xrightarrow{\lambda}_{\rm r} P'$. By hypothesis, $P, P' \in \mathrm{SBNDC}_{\approx_{\rm mw}}$ and by induction hypothesis $P' \in \mathrm{SBSNNI}_{\approx_{\rm mw}}$. Hence, we just need to prove that $P \setminus \mathcal{A_H} \approx_{\rm mw} P \,/\, \mathcal{A_H}$. There are three cases:
    - ∗ If $a \notin \mathcal{A_H}$ then both $P \setminus \mathcal{A_H}$ and $P \,/\, \mathcal{A_H}$ can execute $a$ and reach, respectively, $P' \setminus \mathcal{A_H}$ and $P' \,/\, \mathcal{A_H}$, which are weakly probabilistic bisimilar by induction hypothesis. Thus Definition 3 is respected.
    - ∗ If $a \in \mathcal{A_H}$ we have that $P \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} P' \,/\, \mathcal{A_H}$, with $P \xrightarrow{a}_{\rm a} P'$. By induction hypothesis we have that $P' \setminus \mathcal{A_H} \approx_{\rm mw} P' \,/\, \mathcal{A_H}$, and since $a \in \mathcal{A_H}$ and $P \in \mathrm{SBNDC}_{\approx_{\rm mw}}$ we have $P \setminus \mathcal{A_H} \approx_{\rm mw} P' \setminus \mathcal{A_H}$. By transitivity it follows that $P \setminus \mathcal{A_H} \approx_{\rm mw} P' \,/\, \mathcal{A_H}$ which, combined with $P \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} P' \,/\, \mathcal{A_H}$, determines the condition required by Definition 3.
    - ∗ If $P \xrightarrow{\lambda}_{\rm r} P'$ then both $P \setminus \mathcal{A_H}$ and $P \,/\, \mathcal{A_H}$ can perform the same transitions, i.e., $P \setminus \mathcal{A_H} \xrightarrow{\lambda}_{\rm r} P' \setminus \mathcal{A_H}$ and $P \,/\, \mathcal{A_H} \xrightarrow{\lambda}_{\rm r} P' \,/\, \mathcal{A_H}$, because the hiding and restriction operators do not apply to Markovian

transitions. The processes $P' \,/\, \mathcal{A}_{\mathcal{H}}$ and $P' \setminus \mathcal{A}_{\mathcal{H}}$ are weakly Markovian bisimilar because of the induction hypothesis.

– $\text{SBSNNI}_{\approx_{\text{mw}}} = \text{P\_BNDC}_{\approx_{\text{mw}}}$. $\text{SBSNNI}_{\approx_{\text{w}}} \subseteq \text{P\_BNDC}_{\approx_{\text{mw}}}$ follows from Lemma 4(3) by taking $P'_1$ identical to $P'_2$ and both reachable from $P \in \text{SBSNNI}_{\approx_{\text{mw}}}$.
On the other hand, if $P \in \text{P\_BNDC}_{\approx_{\text{mw}}}$ then $P' \in \text{BNDC}_{\approx_{\text{mw}}}$ for every $P' \in \mathit{reach}(P)$. Since $\text{BNDC}_{\approx_{\text{mw}}} \subseteq \text{BSNNI}_{\approx_{\text{mw}}}$ as will be shown in the last case of the proof of this part of the theorem, $P' \in \text{BSNNI}_{\approx_{\text{mw}}}$ for every $P' \in \mathit{reach}(P)$, i.e., $P \in \text{SBSNNI}_{\approx_{\text{mw}}}$.
– $\text{SBSNNI}_{\approx_{\text{mw}}} \subseteq \text{BNDC}_{\approx_{\text{mw}}}$. If $P \in \text{SBSNNI}_{\approx_{\text{mw}}} = \text{P\_BNDC}_{\approx_{\text{mw}}}$ then it immediately follows that $P \in \text{BNDC}_{\approx_{\text{mw}}}$.
– $\text{BNDC}_{\approx_{\text{mw}}} \subseteq \text{BSNNI}_{\approx_{\text{mw}}}$. If $P \in \text{BNDC}_{\approx_{\text{mw}}}$, i.e., $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{mw}} (P \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ for all $Q \in \mathbb{P}$ such that each of its prefixes belongs to $\mathcal{A}_{\mathcal{H}}$ and for all $L \subseteq \mathcal{A}_{\mathcal{H}}$, then we can consider in particular $\hat{Q}$ capable of stepwise mimicking the high-level behavior of $P$, in the sense that $\hat{Q}$ is able to synchronize with all the high-level actions executed by $P$ and its reachable processes, along with $\hat{L} = \mathcal{A}_{\mathcal{H}}$. As a consequence $(P \,\|_{\hat{L}}\, \hat{Q}) \,/\, \hat{L}) \setminus \mathcal{A}_{\mathcal{H}}$ is isomorphic to $P \,/\, \mathcal{A}_{\mathcal{H}}$, hence $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{mw}} (P \,\|_{\hat{L}}\, \hat{Q}) \,/\, \hat{L}) \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{mw}} P \,/\, \mathcal{A}_{\mathcal{H}}$, i.e., $P \in \text{BSNNI}_{\approx_{\text{mw}}}$, as $\approx_{\text{mw}}$ is transitive.

We then prove the results for the $\approx_{\text{mb}}$-based properties. Let us examine each relationship separately:

– $\text{SBNDC}_{\approx_{\text{mb}}} \subseteq \text{SBSNNI}_{\approx_{\text{mb}}}$. We need to prove that for a given $P \in \mathbb{P}$, if $P \in$ SBNDC, it follows that for every $P'$ reachable from $P$, $P' \in \text{BSNNI}_{\approx_{\text{mb}}}$. Since the processes we are considering are not recursive we can treat them as trees, and hence we can proceed by induction on their depth. In this case we will proceed by induction on the depth of $P$:
  - If the depth of $P$ is 0 then $P$ has no outgoing transitions and it behaves as $\underline{0}$. This obviously entails that $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{mb}} P \,/\, \mathcal{A}_{\mathcal{H}}$.
  - If the depth of $P$ is $n+1$ with $n \in \mathbb{N}$, then take any $P'$ of depth $n$ such that $P \xrightarrow{a}_{\text{a}} P'$. By hypothesis, $P, P' \in \text{SBNDC}_{\approx_{\text{mb}}}$ and by induction hypothesis $P' \in \text{SBSNNI}_{\approx_{\text{mb}}}$. Hence, we just need to prove that $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{mb}} P \,/\, \mathcal{A}_{\mathcal{H}}$. There are three cases:
    * If $a \notin \mathcal{A}_{\mathcal{H}}$ then both $P \setminus \mathcal{A}_{\mathcal{H}}$ and $P \,/\, \mathcal{A}_{\mathcal{H}}$ can execute $a$ and reach, respectively, $P' \setminus \mathcal{A}_{\mathcal{H}}$ and $P' \,/\, \mathcal{A}_{\mathcal{H}}$, which are Markovian branching bisimilar by induction hypothesis. Thus Definition 4 is respected.
    * If $a \in \mathcal{A}_{\mathcal{H}}$ we have that $P \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\text{a}} P' \,/\, \mathcal{A}_{\mathcal{H}}$, with $P \xrightarrow{a}_{\text{a}} P'$. By induction hypothesis we have that $P' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{mb}} P' \,/\, \mathcal{A}_{\mathcal{H}}$, and since $a \in \mathcal{A}_{\mathcal{H}}$ and $P \in \text{SBNDC}_{\approx_{\text{mb}}}$ we have $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{mb}} P' \setminus \mathcal{A}_{\mathcal{H}}$. By transitivity it follows that $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{mb}} P' \,/\, \mathcal{A}_{\mathcal{H}}$ which, combined with $P \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\text{a}} P' \,/\, \mathcal{A}_{\mathcal{H}}$, determines the condition required by Definition 4.
    * If $P \xrightarrow{\lambda}_{\text{r}} P'$ then both $P \setminus \mathcal{A}_{\mathcal{H}}$ and $P \,/\, \mathcal{A}_{\mathcal{H}}$ can perform the same transitions, i.e., $P \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\lambda}_{\text{r}} P' \setminus \mathcal{A}_{\mathcal{H}}$ and $P \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\lambda}_{\text{r}} P' \,/\, \mathcal{A}_{\mathcal{H}}$, because the hiding and restriction operators do not apply to Markovian

transitions. The processes $P' \,/\, \mathcal{A}_\mathcal{H}$ and $P' \backslash \mathcal{A}_\mathcal{H}$ are weakly Markovian bisimilar because of the induction hypothesis.

– $\mathrm{SBSNNI}_{\approx_\mathrm{mb}} = \mathrm{P\_BNDC}_{\approx_\mathrm{mb}}$. The proof is like the one of the corresponding result for $\approx_\mathrm{mw}$.
– $\mathrm{SBSNNI}_{\approx_\mathrm{mb}} \subseteq \mathrm{BNDC}_{\approx_\mathrm{mb}}$. The proof is like the one of the corresponding result for $\approx_\mathrm{mw}$.
– $\mathrm{BNDC}_{\approx_\mathrm{mb}} \subseteq \mathrm{BSNNI}_{\approx_\mathrm{mb}}$. The proof is like the one of the corresponding result for $\approx_\mathrm{mw}$. ■

**Proof of Theorem 5** Let $Q$ be $P_1 + h \,.\, P_2$ (the proof is similar for $Q$ equal to $P_2 + h \,.\, P_1$) and observe that no high-level actions occur in every process reachable from $Q$ except $Q$ itself:

1. Since the only high-level action occurring in $Q$ is $h$, in the proof of $Q \in \mathrm{BSNNI}_{\approx_\mathrm{mw}}$ the only interesting case is the transition $Q \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_\mathrm{a} P_2 \,/\, \mathcal{A}_\mathcal{H}$, to which $Q \backslash \mathcal{A}_\mathcal{H}$ responds by staying idle because $P_2 \,/\, \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} P_2 \approx_\mathrm{mw} P_1 \approx_\mathrm{mw} Q \setminus \mathcal{A}_\mathcal{H}$, i.e., $P_2 \,/\, \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} Q \setminus \mathcal{A}_\mathcal{H}$ as $\approx_\mathrm{mw}$ is symmetric and transitive.
On the other hand, $Q \notin \mathrm{BSNNI}_{\approx_\mathrm{mb}}$ because $P_2 \not\approx_\mathrm{mb} P_1$ in the same situation as before.
2. Since $Q \in \mathrm{BSNNI}_{\approx_\mathrm{mw}}$ by the previous result and no high-level actions occur in every process reachable from $Q$ other than $Q$, it holds that $Q \in \mathrm{SBSNNI}_{\approx_\mathrm{mw}}$ and hence $Q \in \mathrm{BNDC}_{\approx_\mathrm{mw}}$ by virtue of Theorem 3.
On the other hand, from $Q \notin \mathrm{BSNNI}_{\approx_\mathrm{mb}}$ by the previous result it follows that $Q \notin \mathrm{BNDC}_{\approx_\mathrm{mb}}$ by virtue of Theorem 3.
3. We already know from the proof of the previous result that $Q \in \mathrm{SBSNNI}_{\approx_\mathrm{mw}}$. On the other hand, from $Q \notin \mathrm{BSNNI}_{\approx_\mathrm{mb}}$ by the first result it follows that $Q \notin \mathrm{SBSNNI}_{\approx_\mathrm{mb}}$ by virtue of Theorem 3.
4. An immediate consequence of $\mathrm{P\_BNDC}_{\approx_\mathrm{mw}} = \mathrm{SBSNNI}_{\approx_\mathrm{mw}}$ and $\mathrm{P\_BNDC}_{\approx_\mathrm{mb}} = \mathrm{SBSNNI}_{\approx_\mathrm{mb}}$ as established by Theorem 3.
5. Since the only high-level action occurring in $Q$ is $h$, in the proof of $Q \in \mathrm{SBNDC}_{\approx_\mathrm{mw}}$ the only interesting case is the transition $Q \xrightarrow{h}_\mathrm{a} P_2$, for which it holds that $Q \backslash \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} P_1 \approx_\mathrm{mw} P_2 \approx_\mathrm{mw} P_2 \backslash \mathcal{A}_\mathcal{H}$, i.e., $Q \backslash \mathcal{A}_\mathcal{H} \approx_\mathrm{mw} P_2 \backslash \mathcal{A}_\mathcal{H}$ as $\approx_\mathrm{mw}$ is transitive.
On the other hand, $Q \notin \mathrm{SBNDC}_{\approx_\mathrm{mb}}$ because $P_1 \not\approx_\mathrm{mb} P_2$ in the same situation as before. ■

**Proof of Proposition 1** We prove the two results separately:

– We need to prove that the symmetric relation $\mathcal{B} = \{(nd(P_1), nd(P_2)) \mid P_1 \approx_\mathrm{mw} P_2\}$ is a weak bisimulation. We start by observing that from $P_1 \approx_\mathrm{mw} P_2$ it follows that for each $P_1 \xrightarrow{a}_\mathrm{a} P'_1$ there exists $P_2 \stackrel{\hat{a}}{\Longrightarrow}_\mathrm{a} P'_2$ such that $P'_1 \approx_\mathrm{mw} P'_2$. Since $nd(P_1)$ and $nd(P_2)$ are obtained by eliminating every rate transition that is alternative to a $\tau$-transition and replacing each remaining rate transition with a $\tau$-transition, for each $nd(P_1) \xrightarrow{a}_\mathrm{a} nd(P'_1)$ there exists $nd(P_2) \stackrel{\hat{a}}{\Longrightarrow}_\mathrm{a} nd(P'_2)$ such that $(nd(P'_1), nd(P'_2)) \in \mathcal{B}$.

– We need to prove that the symmetric relation $\mathcal{B} = \{nd(P_1), nd(P_2)) \mid P_1 \approx_{\rm mb} P_2\}$ is a branching bisimulation. We start by observing that from $P_1 \approx_{\rm mb} P_2$ it follows that for each $P_1 \xrightarrow{a}_{\rm a} P'_1$ either $a = \tau$ and $P'_1 \approx_{\rm mb} P_2$, or there exists $P_2 \xRightarrow{\tau^*}_{\rm a} \bar{P}_2 \xrightarrow{a}_{\rm a} P'_2$ such that $P_1 \approx_{\rm mb} \bar{P}_2$ and $P'_1 \approx_{\rm mb} P'_2$. Since $nd(P_1)$ and $nd(P_2)$ are obtained by eliminating every rate transition that is alternative to a $\tau$-transition and replacing each remaining rate transition with a $\tau$-transition, for each $nd(P_1) \xrightarrow{a}_{\rm a} nd(P'_1)$ either $a = \tau$ and $(nd(P'_1), nd(P_2)) \in \mathcal{B}$, or there exists $nd(P_2) \xRightarrow{\tau^*}_{\rm a} nd(\bar{P}_2) \xrightarrow{a}_{\rm a} nd(P'_2)$ such that $(nd(P_1), nd(\bar{P}_2)) \in \mathcal{B}$ and $(nd(P'_1), nd(P'_2)) \in \mathcal{B}$. ■

**Proof of Corollary 1** The result directly follows from Proposition 1. ■

**Definition 9.** *A* probabilistic labeled transition system (PLTS) *is a triple* $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ *where* $\mathcal{S} = \mathcal{S}_{\rm n} \cup \mathcal{S}_{\rm p}$ *with* $\mathcal{S}_{\rm n} \cap \mathcal{S}_{\rm p} = \emptyset$ *is an at most countable set of states,* $\mathcal{A}_\tau = \mathcal{A} \cup \{\tau\}$ *is a countable set of actions, and* $\longrightarrow \, = \, \longrightarrow_{\rm a} \cup \longrightarrow_{\rm p}$ *is the transition relation, with* $\longrightarrow_{\rm a} \subseteq \mathcal{S}_{\rm n} \times \mathcal{A}_\tau \times \mathcal{S}_{\rm p}$ *being the action transition relation whilst* $\longrightarrow_{\rm p} \subseteq \mathcal{S}_{\rm p} \times \mathbb{R}_{]0,1]} \times \mathcal{S}_{\rm n}$ *being the probabilistic transition relation satisfying* $\sum_{(s,p,s') \in \longrightarrow_{\rm p}} p \in \{0, 1\}$ *for all* $s \in \mathcal{S}_{\rm p}$*. We further define function* *prob* *as follows:*

$$prob(s, s') \;=\; \begin{cases} p & \text{if } s \in \mathcal{S}_{\rm p} \wedge \sum_{s \xrightarrow{p'}_{\rm p} s'} p' = p > 0 \\ 1 & \text{if } s \in \mathcal{S}_{\rm n} \wedge s' = s \\ 0 & \text{otherwise} \end{cases}$$

*and denote by* $\Longrightarrow$ *a sequence of alternating* $\tau$*- and probabilistic transitions.* ■

**Definition 10.** *Let* $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ *be a PLTS. We say that* $s_1, s_2 \in \mathcal{S}$ *are* weakly probabilistic bisimilar*, written* $s_1 \approx_{\rm p} s_2$*, iff* $(s_1, s_2) \in \mathcal{B}$ *for some weak probabilistic bisimulation* $\mathcal{B}$*. An equivalence relation* $\mathcal{B}$ *over* $\mathcal{S}$ *is a* weak probabilistic bisimulation *iff, whenever* $(s_1, s_2) \in \mathcal{B}$*, then:*

– *For each* $s_1 \xrightarrow{a}_{\rm a} s'_1$ *there exists* $s_2 \xRightarrow{\hat{a}} s'_2$ *such that* $(s'_1, s'_2) \in \mathcal{B}$*.*
– $prob(s_1, C) = prob(s_2, C)$ *for all equivalence classes* $C \in \mathcal{S}/\mathcal{B}$*.* ■

**Definition 11.** *Let* $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ *be a PLTS. We say that* $s_1, s_2 \in \mathcal{S}$ *are* probabilistic branching bisimilar*, written* $s_1 \approx_{\rm pb} s_2$*, iff* $(s_1, s_2) \in \mathcal{B}$ *for some probabilistic branching bisimulation* $\mathcal{B}$*. An equivalence relation* $\mathcal{B}$ *over* $\mathcal{S}$ *is a* probabilistic branching bisimulation *iff, whenever* $(s_1, s_2) \in \mathcal{B}$*, then:*

– *For each* $s_1 \xrightarrow{a}_{\rm a} s'_1$*:*
  - *either* $a = \tau$ *and* $(s'_1, s_2) \in \mathcal{B}$*;*
  - *or there exists* $s_2 \Longrightarrow \bar{s}_2 \xrightarrow{a}_{\rm a} s'_2$ *such that* $(s_1, \bar{s}_2) \in \mathcal{B}$ *and* $(s'_1, s'_2) \in \mathcal{B}$*.*

– $prob(s_1, C) = prob(s_2, C)$ *for all equivalence classes* $C \in \mathcal{S}/\mathcal{B}$*.* ■

**Proof of Proposition 2** We prove the two results separately:

- We need to prove that the equivalence relation $\mathcal{B} = \{(pr(P_1), pr(P_2)) \mid P_1 \approx_{\rm mw} P_2\}$ is a weak probabilistic bisimulation.
  As for action transitions, we start by observing that from $P_1 \approx_{\rm mw} P_2$ it follows that for each $P_1 \xrightarrow{a}_{\rm a} P'_1$ there exists $P_2 \xrightarrow{a}_{\rm a} P'_2$ – due to the strict alternation – such that $P'_1 \approx_{\rm mw} P'_2$. Since $pr(P_1)$ and $pr(P_2)$ are obtained by replacing each rate transition with a probabilistic one, for each $pr(P_1) \xrightarrow{a}_{\rm a} pr(P'_1)$ there exists $pr(P_2) \xrightarrow{a}_{\rm a} pr(P'_2)$ such that $(pr(P'_1), pr(P'_2)) \in \mathcal{B}$.
  As for probabilities, for each $P \xrightarrow{\gamma}_{\rm r} P'$ there exists $pr(P) \xrightarrow{p}_{\rm p} pr(P')$ with $p = \gamma / \sum_{P \xrightarrow{\delta}_{\rm r} Q} \delta$. Due to the strict alternation, from $P_1 \approx_{\rm mw} P_2$ it follows that $\sum_{P_1 \xrightarrow{\lambda}_{\rm r} P'_1, P'_1 \in C} \lambda = \sum_{P_2 \xrightarrow{\mu}_{\rm r} P'_2, P'_2 \in C} \mu$ for each $C \in \mathbb{P}_{\rm mk} / \approx_{\rm mw}$ and hence $\sum_{P_1 \xrightarrow{\lambda}_{\rm r} P'_1} \lambda = \sum_{P_2 \xrightarrow{\mu}_{\rm r} P'_2} \mu$. Since every equivalence class $C' \in \mathbb{P}_{\rm pr} / \mathcal{B}$ is of the form $[pr(Q)]_{\mathcal{B}} = \{pr(Q') \mid Q \approx_{\rm mw} Q'\}$, we have that $\sum_{pr(P_1) \xrightarrow{p}_{\rm p} pr(P'_1), pr(P'_1) \in C'} p = \sum_{pr(P_2) \xrightarrow{q}_{\rm p} pr(P'_2), pr(P'_2) \in C'} q$ where every $p$ and every $q$ is obtained from the corresponding rate ratios respectively involving $\lambda$ and $\mu$.
- We need to prove that the equivalence relation $\mathcal{B} = \{(pr(P_1), pr(P_2)) \mid P_1 \approx_{\rm mb} P_2\}$ is a probabilistic branching bisimulation.
  As for action transitions, we start by observing that from $P_1 \approx_{\rm mb} P_2$ it follows that for each $P_1 \xrightarrow{a}_{\rm a} P'_1$ either $a = \tau$ and $P'_1 \approx_{\rm mb} P_2$, or there exists $P_2 \xRightarrow{\tau^*}_{\rm a} P_2 \xrightarrow{a}_{\rm a} P'_2$ – due to the strict alternation – such that $P'_1 \approx_{\rm mb} P'_2$. Since $pr(P_1)$ and $pr(P_2)$ are obtained by replacing each rate transition with a probabilistic one, for each $pr(P_1) \xrightarrow{a}_{\rm a} pr(P'_1)$ either $a = \tau$ and $(pr(P'_1), pr(P_2)) \in \mathcal{B}$, or there exists $pr(P_2) \xRightarrow{\tau^*}_{\rm a} pr(P_2) \xrightarrow{a}_{\rm a} pr(P'_2)$ such that $(pr(P'_1), pr(P'_2)) \in \mathcal{B}$.
  As for probabilities, we reason like in the proof of the corresponding result for $\approx_{\rm pw}$. ■

**Proof of Corollary 2** The result directly follows from Proposition 2. ■

**Proof of Lemma 2.** Given $s_1, s_2 \in \mathcal{S}$ with $s_1 \approx_{\rm mbf} s_2$, consider the transitive closure $\mathcal{B}^+$ of the reflexive and symmetric relation $\mathcal{B} = \approx_{\rm mbf} \cup \{(\rho''_1, \rho''_2), (\rho''_2, \rho''_1) \in (run(s_1) \times run(s_2)) \cup (run(s_2) \times run(s_1)) \mid \exists \rho'_1 \in run(s_1), \rho'_2 \in run(s_2).\, \rho'_1 \xRightarrow{\tau^*}_{\rm a} \rho''_1 \wedge \rho'_2 \xRightarrow{\tau^*}_{\rm a} \rho''_2 \wedge \rho'_1 \approx_{\rm mbf} \rho''_2 \wedge \rho''_1 \approx_{\rm mbf} \rho'_2\}$. The result will follow by proving that $\mathcal{B}^+$ is a weak Markovian back-and-forth bisimulation, because this implies that $\rho''_1 \approx_{\rm mbf} \rho''_2$ for every additional pair – i.e., $\mathcal{B}^+$ satisfies the cross property – as well as $\mathcal{B}^+ = \approx_{\rm mbf}$ – hence $\approx_{\rm mbf}$ satisfies the cross property too.
Let $(\rho''_1, \rho''_2) \in \mathcal{B} \setminus \approx_{\rm mbf}$ to avoid trivial cases. Then there exist $\rho'_1 \in run(s_1)$ and $\rho'_2 \in run(s_2)$ such that $\rho'_1 \xRightarrow{\tau^*}_{\rm a} \rho''_1$, $\rho'_2 \xRightarrow{\tau^*}_{\rm a} \rho''_2$, $\rho'_1 \approx_{\rm mbf} \rho''_2$, and $\rho''_1 \approx_{\rm mbf} \rho'_2$. There are two cases for action transitions:

- In the forward case, assume that $\rho''_1 \xrightarrow{a}_{\rm a} \rho'''_1$, from which we derive $\rho'_1 \xRightarrow{\tau^*}_{\rm a} \rho''_1 \xrightarrow{a}_{\rm a} \rho'''_1$. From $\rho'_1 \approx_{\rm mbf} \rho''_2$ it follows that there exists $\rho''_2 \xRightarrow{\tau^*}_{\rm a} \rho'''_2$ if $a = \tau$ or

$\rho''_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \stackrel{a}{\longrightarrow}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'''_2$ if $a \neq \tau$, such that $\rho'''_1 \approx_{\rm mbf} \rho'''_2$ and hence $(\rho'''_1, \rho'''_2) \in \mathcal{B}$.

When starting from $\rho''_2 \stackrel{a}{\longrightarrow}_{\rm a} \rho'''_2$, we exploit $\rho'_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho''_2$ and $\rho''_1 \approx_{\rm mbf} \rho'_2$ instead.

- In the backward case, assume that $\rho'''_1 \stackrel{a}{\longrightarrow}_{\rm a} \rho''_1$. From $\rho''_1 \approx_{\rm mbf} \rho'_2$ it follows that there exists $\rho'''_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'_2$ if $a = \tau$, so $\rho'''_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho''_2$, or $\rho'''_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'_2$ if $a \neq \tau$, so $\rho'''_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \stackrel{a}{\longrightarrow}_{\rm a} \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho''_2$, such that $\rho'''_1 \approx_{\rm mbf} \rho'''_2$ and hence $(\rho'''_1, \rho'''_2) \in \mathcal{B}$.

  When starting from $\rho'''_2 \stackrel{a}{\longrightarrow}_{\rm a} \rho''_2$, we exploit $\rho'_1 \approx_{\rm mbf} \rho''_2$ and $\rho'_1 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho''_1$ instead.

Likewise, there are two cases for rate transitions:

- In the forward case, assume that $\rho''_1 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'''_1$ with $\rho'''_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, from which we derive $\rho'_1 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'''_1$. From $\rho'_1 \approx_{\rm mbf} \rho''_2$ it follows that there exists $\rho''_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'''_2$ with $\rho'''_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ such that $\rho'''_1 \approx_{\rm mbf} \rho'''_2$ and $\mathit{rate}(\rho'''_1, C) = \mathit{rate}(\rho'''_2, C)$ for all $C \in \mathcal{U}/\approx_{\rm mbf}$. Since every equivalence class $C' \in \mathcal{U}/\mathcal{B}^+$ is the union of equivalence classes with respect to $\approx_{\rm mbf}$, it holds that $\mathit{rate}(\rho'''_1, C') = \mathit{rate}(\rho'''_2, C')$.

  When starting from $\rho''_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'''_2$ with $\rho'''_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, we exploit $\rho'_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho''_2$ and $\rho''_1 \approx_{\rm mbf} \rho'_2$ instead.
- In the backward case, assume that $\rho'''_1 \stackrel{\lambda_1}{\longrightarrow}_{\rm r} \rho''_1$ with $\rho'''_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$. From $\rho''_1 \approx_{\rm mbf} \rho'_2$ it follows that there exists $\rho'''_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{\rho}'''_2 \stackrel{\lambda_2}{\longrightarrow}_{\rm r} \bar{\rho}'_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'_2$ with $\bar{\rho}'''_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, so $\rho'''_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{\rho}'''_2 \stackrel{\lambda_2}{\longrightarrow}_{\rm r} \bar{\rho}'_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho''_2$ with $\bar{\rho}'''_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, such that $\rho''_1 \approx_{\rm mbf} \bar{\rho}'_2$, $\rho'''_1 \approx_{\rm mbf} \bar{\rho}'''_2$, and $\rho'''_1 \approx_{\rm mbf} \rho'''_2$, hence $(\rho''_1, \bar{\rho}'_2) \in \mathcal{B}$, $(\rho'''_1, \bar{\rho}'''_2) \in \mathcal{B}$, and $(\rho'''_1, \rho'''_2) \in \mathcal{B}$.

  When starting from $\rho'''_2 \stackrel{\lambda_2}{\longrightarrow}_{\rm r} \rho''_2$ with $\rho'''_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, we exploit $\rho'_1 \approx_{\rm mbf} \rho''_2$ and $\rho'_1 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho''_1$ instead. ■

**Proof of Theorem 6.** The proof is divided into two parts:

- Suppose that $s_1 \approx_{\rm mbf} s_2$ and let $\mathcal{B}$ be a weak Markovian back-and-forth bisimulation over $\mathcal{U}$ such that $((s_1, \varepsilon), (s_2, \varepsilon)) \in \mathcal{B}$. Assume that $\mathcal{B}$ only contains all the pairs of $\approx_{\rm mbf}$-equivalent runs from $s_1$ and $s_2$, so that Lemma 2 is applicable to $\mathcal{B}$. We show that $\mathcal{B}' = \{(\mathit{last}(\rho_1), \mathit{last}(\rho_2)) \mid (\rho_1, \rho_2) \in \mathcal{B}\}$ is a Markovian branching bisimulation over the states in $\mathcal{S}$ reachable from $s_1$ and $s_2$, from which $s_1 \approx_{\rm mb} s_2$ will follow. Note that $\mathcal{B}'$ is an equivalence relation because so is $\mathcal{B}$.

  Given $(\mathit{last}(\rho_1), \mathit{last}(\rho_2)) \in \mathcal{B}'$, by definition of $\mathcal{B}'$ we have that $(\rho_1, \rho_2) \in \mathcal{B}$. Let $r_k = \mathit{last}(\rho_k)$ for $k \in \{1, 2\}$, so that $(r_1, r_2) \in \mathcal{B}'$. Suppose that $r_1 \stackrel{a}{\longrightarrow}_{\rm a} r'_1$, i.e., $\rho_1 \stackrel{a}{\longrightarrow}_{\rm a} \rho'_1$ where $\mathit{last}(\rho'_1) = r'_1$. There are two cases:

  - If $a = \tau$ then from $(\rho_1, \rho_2) \in \mathcal{B}$ it follows that there exists $\rho_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'_2$ such that $(\rho'_1, \rho'_2) \in \mathcal{B}$. This means that we have a sequence of $n \geq 0$

transitions of the form $\rho_{2,i} \xrightarrow{\tau}_{\rm a} \rho_{2,i+1}$ for all $0 \leq i \leq n-1$ where $\rho_{2,0}$ is $\rho_2$ while $\rho_{2,n}$ is $\rho'_2$ so that $(\rho'_1, \rho_{2,n}) \in \mathcal{B}$ as $(\rho'_1, \rho'_2) \in \mathcal{B}$.
If $n = 0$ then we are done because $\rho'_2$ is $\rho_2$ and hence $(\rho'_1, \rho_2) \in \mathcal{B}$ as $(\rho'_1, \rho'_2) \in \mathcal{B}$ – thus $(r'_1, r_2) \in \mathcal{B}'$ – otherwise from $\rho_{2,n}$ we go back to $\rho_{2,n-1}$ via $\rho_{2,n-1} \xrightarrow{\tau}_{\rm a} \rho_{2,n}$. Recalling that $(\rho'_1, \rho_{2,n}) \in \mathcal{B}$, if $\rho'_1$ can respond by staying idle, so that $(\rho'_1, \rho_{2,n-1}) \in \mathcal{B}$, and $n = 1$, then we are done because $\rho_{2,n-1}$ is $\rho_2$ and hence $(\rho'_1, \rho_2) \in \mathcal{B}$ as $(\rho'_1, \rho_{2,n-1}) \in \mathcal{B}$ – thus $(r'_1, r_2) \in \mathcal{B}'$ – otherwise we go further back to $\rho_{2,n-2}$ via $\rho_{2,n-2} \xrightarrow{\tau}_{\rm a} \rho_{2,n-1}$. If $\rho'_1$ can respond by staying idle, so that $(\rho'_1, \rho_{2,n-2}) \in \mathcal{B}$, and $n = 2$, then we are done because $\rho_{2,n-2}$ is $\rho_2$ and hence $(\rho'_1, \rho_2) \in \mathcal{B}$ as $(\rho'_1, \rho_{2,n-2}) \in \mathcal{B}$ – thus $(r'_1, r_2) \in \mathcal{B}'$ – otherwise we keep going backward.
By repeating this procedure, since $(\rho'_1, \rho_{2,n}) \in \mathcal{B}$ either we get to $(\rho'_1, \rho_{2,n-n}) \in \mathcal{B}$ and we are done because this implies that $(\rho'_1, \rho_2) \in \mathcal{B}$ – thus $(r'_1, r_2) \in \mathcal{B}'$ – or for some $0 < m \leq n$ such that $(\rho'_1, \rho_{2,m}) \in \mathcal{B}$ the incoming transition $\rho_{2,m-1} \xrightarrow{\tau}_{\rm a} \rho_{2,m}$ is matched by $\bar{\rho}_1 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho_1 \xrightarrow{\tau}_{\rm a} \rho'_1$ with $(\bar{\rho}_1, \rho_{2,m-1}) \in \mathcal{B}$. In the latter case, since $\bar{\rho}_1 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho_1$, $\rho_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho_{2,m-1}$, $(\bar{\rho}_1, \rho_{2,m-1}) \in \mathcal{B}$, and $(\rho_1, \rho_2) \in \mathcal{B}$, from Lemma 2 we derive that $(\rho_1, \rho_{2,m-1}) \in \mathcal{B}$. Consequently $\rho_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho_{2,m-1} \xrightarrow{\tau}_{\rm a} \rho_{2,m}$ with $(\rho_1, \rho_{2,m-1}) \in \mathcal{B}$ and $(\rho'_1, \rho_{2,m}) \in \mathcal{B}$, thus $r_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \mathit{last}(\rho_{2,m-1}) \xrightarrow{\tau}_{\rm a} \mathit{last}(\rho_{2,m})$ with $(r_1, \mathit{last}(\rho_{2,m-1})) \in \mathcal{B}'$ and $(r'_1, \mathit{last}(\rho_{2,m})) \in \mathcal{B}'$.

- If $a \neq \tau$ then from $(\rho_1, \rho_2) \in \mathcal{B}$ it follows that there exists $\rho_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{\rho}_2 \xrightarrow{a}_{\rm a} \bar{\rho}'_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'_2$ such that $(\rho'_1, \rho'_2) \in \mathcal{B}$.
From $(\rho'_1, \rho'_2) \in \mathcal{B}$ and $\bar{\rho}'_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'_2$ it follows that there exists $\bar{\rho}'_1 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'_1$ such that $(\bar{\rho}'_1, \bar{\rho}'_2) \in \mathcal{B}$. Since $\rho_1 \xrightarrow{a}_{\rm a} \rho'_1$ and hence the last transition in $\rho'_1$ is labeled with $a$, we derive that $\bar{\rho}'_1$ is $\rho'_1$ and hence $(\rho'_1, \bar{\rho}'_2) \in \mathcal{B}$.
From $(\rho'_1, \bar{\rho}'_2) \in \mathcal{B}$ and $\bar{\rho}_2 \xrightarrow{a}_{\rm a} \bar{\rho}'_2$ it follows that there exists $\bar{\rho}_1 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho_1 \xrightarrow{a}_{\rm a} \rho'_1$ such that $(\bar{\rho}_1, \bar{\rho}_2) \in \mathcal{B}$.
Since $\bar{\rho}_1 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho_1$, $\rho_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{\rho}_2$, $(\bar{\rho}_1, \bar{\rho}_2) \in \mathcal{B}$, and $(\rho_1, \rho_2) \in \mathcal{B}$, from Lemma 2 we derive that $(\rho_1, \bar{\rho}_2) \in \mathcal{B}$.
Consequently $\rho_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{\rho}_2 \xrightarrow{a}_{\rm a} \bar{\rho}'_2$ with $(\rho_1, \bar{\rho}_2) \in \mathcal{B}$ and $(\rho'_1, \bar{\rho}'_2) \in \mathcal{B}$, thus $r_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \mathit{last}(\bar{\rho}_2) \xrightarrow{a}_{\rm a} \mathit{last}(\bar{\rho}'_2)$ with $(r_1, \mathit{last}(\bar{\rho}_2)) \in \mathcal{B}'$ and $(r'_1, \mathit{last}(\bar{\rho}'_2)) \in \mathcal{B}'$.

As for rates, given $\rho \in \mathit{run}(s_1) \cup \mathit{run}(s_2)$, the equivalence class $C'_\rho$ with respect to $\mathcal{B}'$ is of the form $[\mathit{last}(\rho)]_{\mathcal{B}'} = \{\mathit{last}(\rho') \mid (\mathit{last}(\rho), \mathit{last}(\rho')) \in \mathcal{B}'\} = \mathit{last}(\{\rho' \mid (\rho, \rho') \in \mathcal{B}\}) = \mathit{last}([\rho]_{\mathcal{B}})$, i.e., $C'_\rho = \mathit{last}(C_\rho)$ for some equivalence class $C_\rho$ with respect to $\mathcal{B}$, provided that function *last* is lifted from runs to sets of runs.
Suppose that $r_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ so that $\rho_1 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho_1$ with $\rho_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$. From $(\rho_1, \rho_2) \in \mathcal{B}$ it follows that there exists $\rho_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'_2$ with $\rho'_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ such that $(\rho_1, \rho'_2) \in \mathcal{B}$ and $\mathit{rate}(\rho_1, C) = \mathit{rate}(\rho'_2, C)$ for all $C \in \mathcal{U}/\mathcal{B}$. Thus there exists $r_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} r'_2$

with $r'_2 = \mathit{last}(\rho'_2)$ and $r'_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ such that $(r_1, r'_2) \in \mathcal{B}'$ and $\mathit{rate}(r_1, C'_\rho) = \mathit{rate}(\rho_1, C_\rho) = \mathit{rate}(\rho'_2, C_\rho) = \mathit{rate}(r'_2, C'_\rho)$ for all equivalence classes $C'_\rho$ with respect to $\mathcal{B}'$ such that $C'_\rho = \mathit{last}(C_\rho)$ for some equivalence class $C_\rho$ with respect to $\mathcal{B}$.

– Suppose that $s_1 \approx_{\rm mb} s_2$ and let $\mathcal{B}$ be a Markovian branching bisimulation over $\mathcal{S}$ such that $(s_1, s_2) \in \mathcal{B}$. Assume that $\mathcal{B}$ only contains all the pairs of $\approx_{\rm mb}$-equivalent states reachable from $s_1$ and $s_2$. We show that the reflexive and transitive closure $\mathcal{B}'^*$ of $\mathcal{B}' = \{(\rho_1, \rho_2), (\rho_2, \rho_1) \in (\mathit{run}(s_1) \times \mathit{run}(s_2)) \cup (\mathit{run}(s_2) \times \mathit{run}(s_1)) \mid (\mathit{last}(\rho_1), \mathit{last}(\rho_2)) \in \mathcal{B}\}$ is a weak Markovian back-and-forth bisimulation over the runs in $\mathcal{U}$ from $s_1$ and $s_2$, from which $(s_1, \varepsilon) \approx_{\rm mbf} (s_2, \varepsilon)$, i.e., $s_1 \approx_{\rm mbf} s_2$, will follow.
Given $(\rho_1, \rho_2) \in \mathcal{B}'$, by definition of $\mathcal{B}'$ we have that $(\mathit{last}(\rho_1), \mathit{last}(\rho_2)) \in \mathcal{B}$. Let $r_k = \mathit{last}(\rho_k)$ for $k \in \{1, 2\}$, so that $(r_1, r_2) \in \mathcal{B}$. There are two cases for action transitions:
  - If $\rho_1 \xrightarrow{a}_{\rm a} \rho'_1$, i.e., $r_1 \xrightarrow{a}_{\rm a} r'_1$ where $r'_1 = \mathit{last}(\rho'_1)$, then either $a = \tau$ and $(r'_1, r'_2) \in \mathcal{B}$ where $r'_2 = r_2$, or there exists $r_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{r}_2 \xrightarrow{a}_{\rm a} r'_2$ such that $(r_1, \bar{r}_2) \in \mathcal{B}$ and $(r'_1, r'_2) \in \mathcal{B}$. In both cases $\rho_2 \stackrel{\hat{a}}{\Longrightarrow}_{\rm a} \rho'_2$ where $\mathit{last}(\rho'_2) = r'_2$, so that $(\rho'_1, \rho'_2) \in \mathcal{B}'$.
  - If $\rho'_1 \xrightarrow{a}_{\rm a} \rho_1$, i.e., $r'_1 \xrightarrow{a}_{\rm a} r_1$ where $r'_1 = \mathit{last}(\rho'_1)$, there are two subcases:
    - ∗ If $\rho'_1$ is $(s_1, \varepsilon)$, i.e., $r'_1 \xrightarrow{a}_{\rm a} r_1$ is $s_1 \xrightarrow{a}_{\rm a} r_1$ and $\mathit{last}(\rho'_1) = s_1$, then from $(s_1, s_2) \in \mathcal{B}$ it follows that either $a = \tau$ and $(r_1, r_2) \in \mathcal{B}$ where $r_2 = s_2$, or there exists $s_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{r}_2 \xrightarrow{a}_{\rm a} r_2$ such that $(s_1, \bar{r}_2) \in \mathcal{B}$ and $(r_1, r_2) \in \mathcal{B}$. In both cases $\rho'_2 \stackrel{\hat{a}}{\Longrightarrow}_{\rm a} \rho_2$ where $\mathit{last}(\rho'_2) = s_2$, so that $(\rho'_1, \rho'_2) \in \mathcal{B}'$.
    - ∗ If $\rho'_1$ is not $(s_1, \varepsilon)$ then from $(s_1, s_2) \in \mathcal{B}$ it follows that $s_1$ reaches $r'_1$ with a sequence of moves that are $\mathcal{B}$-compatible with those with which $s_2$ reaches some $r'_2$ such that $(r'_1, r'_2) \in \mathcal{B}$ as $\mathcal{B}$ only contains all the states reachable from $s_1$ and $s_2$. Therefore either $a = \tau$ and $(r_1, r'_2) \in \mathcal{B}$ where $r'_2 = r_2$, or there exists $r'_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{r}_2 \xrightarrow{a}_{\rm a} r_2$ such that $(r'_1, \bar{r}_2) \in \mathcal{B}$ and $(r_1, r_2) \in \mathcal{B}$. In both cases $\rho'_2 \stackrel{\hat{a}}{\Longrightarrow}_{\rm a} \rho_2$ where $\mathit{last}(\rho'_2) = r'_2$, so that $(\rho'_1, \rho'_2) \in \mathcal{B}'$.

Likewise, there are two cases for rate transitions:
  - Given $\rho \in \mathit{run}(s_1) \cup \mathit{run}(s_2)$, the equivalence class $C'_\rho$ with respect to $\mathcal{B}'^*$ is of the form $[\rho]_{\mathcal{B}'^*} = \{\rho' \in \mathit{run}(s_1) \cup \mathit{run}(s_2) \mid \mathit{last}(\rho') \in [\mathit{last}(\rho)]_{\mathcal{B}}\}$, i.e., $C'_\rho$ corresponds to some equivalence class $C_\rho$ with respect to $\mathcal{B}$. Suppose that $\rho_1 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'_1$ with $\rho'_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ so that $r_1 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} r'_1$ with $r'_1 = \mathit{last}(\rho'_1) \stackrel{\tau}{\not\longrightarrow}_{\rm a}$. From $(r_1, r_2) \in \mathcal{B}$ it follows that there exists $r_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \bar{r}_2$ such that $(r'_1, \bar{r}_2) \in \mathcal{B}$ and, since $r'_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, there exists $\bar{r}_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} r'_2$ with $r'_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ such that $(r'_1, r'_2) \in \mathcal{B}$ and $\mathit{rate}(r'_1, C) = \mathit{rate}(r'_2, C)$ for all $C \in \mathcal{S}/\mathcal{B}$. Thus there exists $\rho_2 \stackrel{\tau^*}{\Longrightarrow}_{\rm a} \rho'_2$ with $\mathit{last}(\rho'_2) = r'_2$ and $\rho'_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ such that $(\rho'_1, \rho'_2) \in \mathcal{B}$ and $\mathit{rate}(\rho'_1, C'_\rho) = \mathit{rate}(\mathit{last}(\rho'_1), C_\rho) = \mathit{rate}(\mathit{last}(\rho'_2), C_\rho) = \mathit{rate}(\rho'_2, C'_\rho)$ for all equivalence classes $C'_\rho$ with respect to $\mathcal{B}'^*$.

- If $\rho'_1 \xrightarrow{\lambda_1}_{\rm r} \rho_1$ with $\rho'_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, i.e., $r'_1 \xrightarrow{\lambda_1}_{\rm r} r_1$ where $r'_1 = \mathit{last}(\rho'_1) \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, there are two subcases:
  - ∗ If $\rho'_1$ is $(s_1, \varepsilon)$, i.e., $r'_1 \xrightarrow{\lambda_1}_{\rm r} r_1$ is $s_1 \xrightarrow{\lambda_1}_{\rm r} r_1$ and $\mathit{last}(\rho'_1) = s_1$, then from $(s_1, s_2) \in \mathcal{B}$ and $s_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ it follows that there exists $s_2 \xrightarrow{\tau^*}_{\rm a} \bar{r}'_2$ with $\bar{r}'_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ and $r_2 \in \mathit{reach}(\bar{r}'_2)$ such that $(r'_1, \bar{r}'_2) \in \mathcal{B}$, which in turn implies that there exists $\bar{r}'_2 \xrightarrow{\lambda_2}_{\rm r} \bar{r}_2$ such that $(r_1, \bar{r}_2) \in \mathcal{B}$, hence $(r_2, \bar{r}_2) \in \mathcal{B}$ as $\approx_{\rm mb}$ is symmetric and transitive. If $r_2$ and $\bar{r}_2$ coincide then we are done because $\rho'_2 \xrightarrow{\tau^*}_{\rm a} \bar{\rho}'_2 \xrightarrow{\lambda_2}_{\rm r} \rho_2 \xrightarrow{\tau^*}_{\rm a} \rho_2$, where $\mathit{last}(\rho'_2) = s_2$ and $\mathit{last}(\bar{\rho}'_2) = \bar{r}'_2$, and $(\rho'_1, \bar{\rho}'_2) \in \mathcal{B}'$ and $(\rho'_1, \rho'_2) \in \mathcal{B}'$. Otherwise, from $r_2 \in \mathit{reach}(\bar{r}_2)$ and $(r_2, \bar{r}_2) \in \mathcal{B}$ it follows that there must exists $\bar{r}_2 \xrightarrow{\tau^*}_{\rm a} r_2$ and hence we are done because $\rho'_2 \xrightarrow{\tau^*}_{\rm a} \bar{\rho}'_2 \xrightarrow{\lambda_2}_{\rm r} \bar{\rho}_2 \xrightarrow{\tau^*}_{\rm a} \rho_2$, where $\mathit{last}(\bar{\rho}_2) = \bar{r}_2$, and $(\rho_1, \bar{\rho}_2) \in \mathcal{B}'$, $(\rho'_1, \bar{\rho}'_2) \in \mathcal{B}'$, and $(\rho'_1, \rho'_2) \in \mathcal{B}'$.
  - ∗ If $\rho'_1$ is not $(s_1, \varepsilon)$ then from $(s_1, s_2) \in \mathcal{B}$ it follows that $s_1$ reaches $r'_1$ with a sequence of moves that are $\mathcal{B}$-compatible with those with which $s_2$ reaches some $r'_2$ such that $(r'_1, r'_2) \in \mathcal{B}$ as $\mathcal{B}$ only contains all the states reachable from $s_1$ and $s_2$. From $(r'_1, r'_2) \in \mathcal{B}$ and $r'_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ it follows that there exists $r'_2 \xrightarrow{\tau^*}_{\rm a} \bar{r}'_2$ with $\bar{r}'_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ and $r_2 \in \mathit{reach}(\bar{r}'_2)$ such that $(r'_1, \bar{r}'_2) \in \mathcal{B}$, at which points the proof continues like the one of the previous subcase. ■