Revisiting Trace and Testing Equivalences for Nondeterministic and Probabilistic Processes ⋆

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Abstract. One of the most studied extensions of testing theory to nondeterministic and probabilistic processes yields unrealistic probabilities estimations that give rise to two anomalies. First, probabilistic testing equivalence does not imply probabilistic trace equivalence. Second, probabilistic testing equivalence differentiates processes that perform the same sequence of actions with the same probability but make internal choices in different moments and thus, when applied to processes without probabilities, does not coincide with classical testing equivalence. In this paper, new versions of probabilistic trace and testing equivalences are presented for nondeterministic and probabilistic processes that resolve the two anomalies. Instead of focussing only on suprema and infima of the set of success probabilities of resolutions of interaction systems, our testing equivalence matches all the resolutions on the basis of the success probabilities of their identically labeled computations. A simple spectrum is provided to relate the new relations with existing ones. It is also shown that, with our approach, the standard probabilistic testing equivalences for generative and reactive probabilistic processes can be retrieved.

1 Introduction

The testing theory for concurrent processes [5] is based on the idea that two processes are equivalent if and only if they cannot be told apart when interacting with their environment, which is represented by arbitrary processes with distinguished successful actions or states, often called observers. In case purely nondeterministic processes are considered, this approach has been very successful and the induced relations enjoy a number of interesting properties. Testing equivalence has been used in many contexts and it is a good compromise between abstraction and inspection capabilities; it distinguishes processes that have different behaviors with respect to deadlock, but abstracts from the exact moment in which a process performs internal (unobservable) choices.

When probabilities enter the game, the possible choices in defining the set of observers, in deciding how to resolve nondeterminism, or in assembling the results of the observations are many more and can give rise to significantly different behavioral relations.

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One of the most studied variants of testing equivalence for nondeterministic and probabilistic processes [17, 10, 14, 6] considers the probability of performing computations along which the same tests are passed. Due to the possible presence of equally labeled transitions departing from the same state, there is not necessarily a single probability value with which a nondeterministic and probabilistic process passes a test. Given two states s_1 and s_2 and the initial state o of an observer, the above mentioned probabilistic testing equivalence computes the probability of performing a successful computation from (s_1, o) and (s_2, o) in every resolution of the interaction systems, then it compares the suprema ($\bigsqcup$) and the infima ($\bigsqcap$) of these values over all possible resolutions of the interaction systems. This equivalence, which we denote by $\sim_{\mathrm{PTe},\sqcup\sqcap}$, enjoys nice properties and possesses logical and equational characterizations but, if contrasted with classical testing for purely nondeterministic processes [5], suffers from two anomalies.

The first anomaly is that $\sim_{\mathrm{PTe},\sqcup\sqcap}$ is not always included in one of the most well-studied probabilistic versions of trace equivalence, namely $\sim_{\mathrm{PTr,dis}}$ of [13]. Actually, the inclusion depends on the class of schedulers used for deriving resolutions of interaction systems. It holds if randomized schedulers are admitted [14], while it does not hold when only deterministic schedulers are considered like in [17, 10, 6]. This anomaly could be solved by (i) considering a coarser probabilistic trace equivalence $\sim_{\mathrm{PTr,new}}$ that compares the execution probabilities of single traces rather than trace distributions and (ii) replacing $\sim_{\mathrm{PTe},\sqcup\sqcap}$ with a finer probabilistic testing equivalence $\sim_{\mathrm{PTe,new}}$, which does not focus only on the highest and the lowest probability of passing a test but matches the maximal resolutions of the interaction systems according to their success probability. Unfortunately, $\sim_{\mathrm{PTe,new}}$ does not overcome the other anomaly.

The second anomaly of $\sim_{\mathrm{PTe},\sqcup\sqcap}$ (which also affects the testing equivalence of [14]) is that, when used to test purely nondeterministic processes, it does not preserve classical testing equivalence. In fact, given two fully nondeterministic processes that are testing equivalent according to [5], they may be told apart by $\sim_{\mathrm{PTe},\sqcup\sqcap}$ because observers with probabilistic choices make the latter equivalence sensitive to the moment of occurrence of internal choices, thus yielding unrealistic probability estimations. This problem has been recently tackled in [8] for a significantly different probabilistic model by relying on a label massaging that avoids over-/under-estimations of success probabilities in a parallel context.

In this paper, by using $\sim_{\mathrm{PTe,new}}$ as a stepping stone, we propose a new probabilistic testing equivalence, $\sim_{\mathrm{PTe,tbt}}$, that solves both anomalies by matching success probabilities in a trace-by-trace fashion rather than on entire resolutions. With respect to [8], the interesting feature of our definition is that it does not require any model modification. We also show that the standard notions of testing equivalences for generative probabilistic processes and reactive probabilistic processes can be redefined, by following the same approach taken for the general model, without altering their discriminating power. Finally, we relate $\sim_{\mathrm{PTe,tbt}}$ with some of the other probabilistic equivalences by showing that it is comprised between $\sim_{\mathrm{PTr,new}}$ and a novel probabilistic failure equivalence $\sim_{\mathrm{PF,new}}$, which in turn is comprised between $\sim_{\mathrm{PTe,tbt}}$ and $\sim_{\mathrm{PTe,new}}$.

2 Nondeterministic and Probabilistic Processes

Processes combining nondeterminism and probability are typically represented by means of extensions of labeled transitions systems (LTS), in which every action-labeled transition goes from a source state to a probability distribution over target states rather than to a single target state. The resulting processes are essentially Markov decision processes [7] and correspond to a number of slightly different probabilistic computational models including real nondeterminism, among which we mention concurrent Markov chains [16], alternating probabilistic models [9, 17], and probabilistic automata in the sense of [12].

Definition 1. *A nondeterministic and probabilistic labeled transition system, NPLTS for short, is a triple* $(S, A, \longrightarrow)$ *where* S *is an at most countable set of states,* A *is a countable set of transition-labeling actions, and* $\longrightarrow \subseteq S \times A \times \mathit{Distr}(S)$ *is a transition relation where* $\mathit{Distr}(S)$ *is the set of discrete probability distributions over* S. ■

Given a transition $s \xrightarrow{a} \mathcal{D}$, we say that $s' \in S$ is not reachable from s via that transition if $\mathcal{D}(s') = 0$, otherwise we say that it is reachable with probability $p = \mathcal{D}(s')$. The choice among all the transitions departing from s is external and nondeterministic, while the choice of the target state for a specific transition is internal and probabilistic. A NPLTS represents (i) a fully nondeterministic process when every transition leads to a distribution that concentrates all the probability mass into a single target state, (ii) a fully probabilistic process when every state has at most one outgoing transition, or (iii) a reactive probabilistic process [15] when no state has several transitions labeled with the same action.

A NPLTS can be depicted as a directed graph-like structure in which vertices represent states and action-labeled edges represent action-labeled transitions. Given a transition $s \xrightarrow{a} \mathcal{D}$, the corresponding a-labeled edge goes from the vertex representing state s to a set of vertices linked by a dashed line, each of which represents a state s' such that $\mathcal{D}(s') > 0$ and is labeled with $\mathcal{D}(s')$ – label omitted if $\mathcal{D}(s') = 1$. Figure 1 shows six NPLTS models, with the first two mixing nondeterminism and probability and the last four being fully probabilistic.

In this setting, a computation is a sequence of state-to-state steps (denoted by $\longrightarrow_{\rm s}$) derived from the state-to-distribution transitions of the NPLTS.

Definition 2. *Let* $\mathcal{L} = (S, A, \longrightarrow)$ *be a NPLTS,* $n \in \mathbb{N}$, $s_i \in S$ *for all* $i = 0, \dots, n$, *and* $a_i \in A$ *for all* $i = 1, \dots, n$. *We say that:*

$$c \equiv s_0 \xrightarrow{a_1}_{\rm s} s_1 \xrightarrow{a_2}_{\rm s} s_2 \dots s_{n-1} \xrightarrow{a_n}_{\rm s} s_n$$

is a computation of $\mathcal{L}$ *of length* n *going from* s_0 *to* s_n *iff for all* $i = 1, \dots, n$ *there exists a transition* $s_{i-1} \xrightarrow{a_i} \mathcal{D}_i$ *such that* $\mathcal{D}_i(s_i) > 0$, *with* $\mathcal{D}_i(s_i)$ *being the execution probability of step* $s_{i-1} \xrightarrow{a_i}_{\rm s} s_i$ *of* c *conditioned on the selection of transition* $s_{i-1} \xrightarrow{a_i} \mathcal{D}_i$ *of* $\mathcal{L}$ *at state* s_{i-1}. ■

In the following, given $s \in S$ we denote by $\mathcal{C}_{\rm fin}(s)$ the set of finite-length computations from s. Given $c \in \mathcal{C}_{\rm fin}(s)$, we say that c is maximal iff it cannot be further extended, i.e., it is not a proper prefix of any other element of $\mathcal{C}_{\rm fin}(s)$.

In order to define testing equivalence, we also need to introduce the notion of parallel composition of two NPLTS models, borrowed from [10].

Definition 3. *Let $\mathcal{L}_i = (S_i, A, \longrightarrow_i)$ be a NPLTS for $i = 1, 2$. The parallel composition of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the NPLTS $\mathcal{L}_1 \parallel \mathcal{L}_2 = (S_1 \times S_2, A, \longrightarrow)$ where $\longrightarrow \subseteq (S_1 \times S_2) \times A \times Distr(S_1 \times S_2)$ is such that $(s_1, s_2) \xrightarrow{a} \mathcal{D}$ iff $s_1 \xrightarrow{a}_1 \mathcal{D}_1$ and $s_2 \xrightarrow{a}_2 \mathcal{D}_2$ with $\mathcal{D}(s'_1, s'_2) = \mathcal{D}_1(s'_1) \cdot \mathcal{D}_2(s'_2)$ for each $(s'_1, s'_2) \in S_1 \times S_2$.* ■

3 Trace Equivalences for NPLTS Models

Trace equivalence for NPLTS models [13] examines the probability of performing computations labeled with the same action sequences for each possible way of solving nondeterminism. To formalize this for a NPLTS $\mathcal{L}$, given a state s of $\mathcal{L}$ we take the set of resolutions of s. Each of them is a tree-like structure whose branching points represent probabilistic choices. This is obtained by unfolding from s the graph structure underlying $\mathcal{L}$ and by selecting at each state a single transition of $\mathcal{L}$ – deterministic scheduler – or a convex combination of equally labeled transitions of $\mathcal{L}$ – randomized scheduler – among all the transitions possible from that state. Below, we introduce the notion of resolution arising from a deterministic scheduler as a fully probabilistic NPLTS. In this case, resolutions coincide with computations if $\mathcal{L}$ is fully nondeterministic.

Definition 4. *Let $\mathcal{L} = (S, A, \longrightarrow)$ be a NPLTS and $s \in S$. We say that a NPLTS $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}})$ is a resolution of s obtained via a deterministic scheduler iff there exists a state correspondence function $corr : Z \to S$ such that $s = corr(z_s)$, for some $z_s \in Z$, and for all $z \in Z$:*

- *If $z \xrightarrow{a}_{\mathcal{Z}} \mathcal{D}$, then $corr(z) \xrightarrow{a} \mathcal{D}'$ with $\mathcal{D}(z') = \mathcal{D}'(corr(z'))$ for all $z' \in Z$.*
- *If $z \xrightarrow{a_1}_{\mathcal{Z}} \mathcal{D}_1$ and $z \xrightarrow{a_2}_{\mathcal{Z}} \mathcal{D}_2$, then $a_1 = a_2$ and $\mathcal{D}_1 = \mathcal{D}_2$.* ■

Given a state s of a NPLTS $\mathcal{L}$, we denote by $Res(s)$ the set of resolutions of s obtained via deterministic schedulers. Since $\mathcal{Z} \in Res(s)$ is fully probabilistic, the probability $prob(c)$ of executing $c \in \mathcal{C}_{\text{fin}}(z_s)$ can be defined as the product of the (no longer conditional) execution probabilities of the individual steps of c, with $prob(c)$ being always equal to 1 if $\mathcal{L}$ is fully nondeterministic. This notion is lifted to $C \subseteq \mathcal{C}_{\text{fin}}(z_s)$ by letting $prob(C) = \sum_{c \in C} prob(c)$ whenever C is finite and none of its computations is a proper prefix of one of the others.

Given $\alpha \in A^*$, we then say that c is compatible with α iff the sequence of actions labeling the steps of c is equal to α. We denote by $\mathcal{CC}(z_s, \alpha)$ the set of computations in $\mathcal{C}_{\text{fin}}(z_s)$ that are compatible with α. Below we introduce a variant of the probabilistic trace distribution equivalence of [13] in which only deterministic schedulers are admitted.

Definition 5. *Let $(S, A, \longrightarrow)$ be a NPLTS. We say that $s_1, s_2 \in S$ are* probabilistic trace distribution equivalent*, written $s_1 \sim_{\text{PTr,dis}} s_2$, iff:*

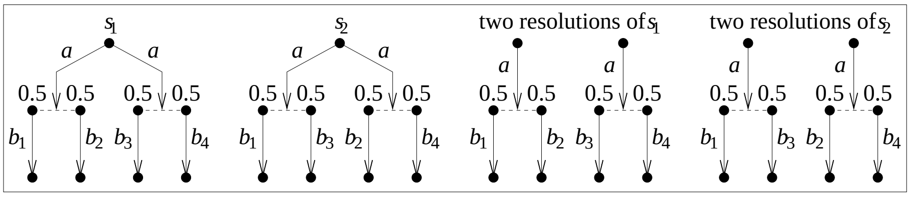


Fig. 1. Counterexample for probabilistic trace equivalences

- *For each* $\mathcal{Z}_1 \in Res(s_1)$ *there exists* $\mathcal{Z}_2 \in Res(s_2)$ *such that for all* $\alpha \in A^*$*:*
$$prob(\mathcal{CC}(z_{s_1}, \alpha)) \; = \; prob(\mathcal{CC}(z_{s_2}, \alpha))$$
- *For each* $\mathcal{Z}_2 \in Res(s_2)$ *there exists* $\mathcal{Z}_1 \in Res(s_1)$ *such that for all* $\alpha \in A^*$*:*
$$prob(\mathcal{CC}(z_{s_2}, \alpha)) \; = \; prob(\mathcal{CC}(z_{s_1}, \alpha))$$ ■

The relation $\sim_{\rm PTr,dis}$ is quite discriminating, because it compares trace distributions and hence imposes a constraint on the execution probability of *all* the traces of any pair of matching resolutions. This constraint can be relaxed by considering a *single* trace at a time, i.e., by anticipating the quantification over traces. In this way, differently labeled computations of a resolution are allowed to be matched by computations of different resolutions, which leads to the following coarser probabilistic trace equivalence that we will use later on.

Definition 6. *Let* $(S, A, \longrightarrow)$ *be a NPLTS. We say that* $s_1, s_2 \in S$ *are* probabilistic trace equivalent*, written* $s_1 \sim_{\rm PTr,new} s_2$*, iff for all traces* $\alpha \in A^*$*:*

- *For each* $\mathcal{Z}_1 \in Res(s_1)$ *there exists* $\mathcal{Z}_2 \in Res(s_2)$ *such that:*
$$prob(\mathcal{CC}(z_{s_1}, \alpha)) \; = \; prob(\mathcal{CC}(z_{s_2}, \alpha))$$
- *For each* $\mathcal{Z}_2 \in Res(s_2)$ *there exists* $\mathcal{Z}_1 \in Res(s_1)$ *such that:*
$$prob(\mathcal{CC}(z_{s_2}, \alpha)) \; = \; prob(\mathcal{CC}(z_{s_1}, \alpha))$$ ■

Theorem 1. *Let* $(S, A, \longrightarrow)$ *be a NPLTS and* $s_1, s_2 \in S$*. Then:*
$$s_1 \sim_{\rm PTr,dis} s_2 \; \Longrightarrow \; s_1 \sim_{\rm PTr,new} s_2$$ ■

The inclusion of $\sim_{\rm PTr,dis}$ in $\sim_{\rm PTr,new}$ is strict. Indeed, if we consider the two NPLTS models on the left-hand side of Fig. 1, when $b_i \neq b_j$ for $i \neq j$ we have that $s_1 \sim_{\rm PTr,new} s_2$ while $s_1 \not\sim_{\rm PTr,dis} s_2$. In fact, the sets of traces of the two resolutions of s_1 depicted in the figure are $\{\varepsilon, a, a\,b_1, a\,b_2\}$ and $\{\varepsilon, a, a\,b_3, a\,b_4\}$, respectively, while the sets of traces of the two resolutions of s_2 depicted in the figure are $\{\varepsilon, a, a\,b_1, a\,b_3\}$ and $\{\varepsilon, a, a\,b_2, a\,b_4\}$, respectively. As a consequence, neither of the two considered resolutions of s_1 (resp. s_2) can have the same trace distribution as one of the two considered resolutions of s_2 (resp. s_1).

Both probabilistic trace equivalences are totally compatible with classical trace equivalence $\sim_{\rm Tr}$ [2], i.e., two fully nondeterministic NPLTS models are related by $\sim_{\rm Tr}$ iff $\sim_{\rm PTr,dis}$ and $\sim_{\rm PTr,new}$ relate them.

Theorem 2. *Let* $(S, A, \longrightarrow)$ *be a fully nondeterministic NPLTS and* $s_1, s_2 \in S$*. Then:*
$$s_1 \sim_{\rm Tr} s_2 \; \Longleftrightarrow \; s_1 \sim_{\rm PTr,dis} s_2 \; \Longleftrightarrow \; s_1 \sim_{\rm PTr,new} s_2$$ ■

4 Testing Equivalences for NPLTS Models

Testing equivalence for NPLTS models [17, 10, 14, 6] considers the probability of performing computations along which the same tests are passed, where tests specify which actions of a process are permitted at each state and are formalized as NPLTS models equipped with a success state. For the sake of simplicity, we restrict ourselves to tests whose underlying graph structure is acyclic – i.e., only finite-length computations are considered – and finitely branching – i.e., only a choice among finitely many alternative actions is made available at each state.

Definition 7. *A nondeterministic and probabilistic test is an acyclic and finitely-branching NPLTS $\mathcal{T} = (O, A, \longrightarrow)$ where O contains a distinguished success state denoted by ω that has no outgoing transitions. We say that a computation of $\mathcal{T}$ is successful iff its last state is ω.* ■

Definition 8. *Let $\mathcal{L} = (S, A, \longrightarrow)$ be a NPLTS and $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ be a nondeterministic and probabilistic test. The interaction system of $\mathcal{L}$ and $\mathcal{T}$ is the acyclic and finitely-branching NPLTS $\mathcal{I}(\mathcal{L}, \mathcal{T}) = \mathcal{L} \parallel \mathcal{T}$ where:*

- *Every element $(s, o) \in S \times O$ is called a configuration and is said to be successful iff $o = \omega$.*
- *A computation of $\mathcal{I}(\mathcal{L}, \mathcal{T})$ is said to be successful iff its last configuration is successful. Given $s \in S$, $o \in O$, and $\mathcal{Z} \in Res(s, o)$, we denote by $\mathcal{SC}(z_{s,o})$ the set of successful computations from the state of $\mathcal{Z}$ corresponding to (s, o).* ■

Due to the possible presence of equally labeled transitions departing from the same state, there is not necessarily a single probability value with which a NPLTS passes a test. Thus, given two states s_1 and s_2 of the NPLTS and the initial state o of the test, we need to compute the probability of performing a successful computation from (s_1, o) and (s_2, o) in every resolution of the interaction system. Then, one option is comparing, for the two configurations, the suprema ($\bigsqcup$) and the infima ($\bigsqcap$) of these values over all resolutions of the interaction system.

Given a state s of a NPLTS $\mathcal{L}$ and the initial state o of a nondeterministic and probabilistic test $\mathcal{T}$, we denote by $Res_{\max}(s, o)$ the set of resolutions in $Res(s, o)$ that are maximal, i.e., that cannot be further extended in accordance with the graph structure of $\mathcal{I}(\mathcal{L}, \mathcal{T})$ and the constraints of Def. 4. In the following, we will consider only maximal resolutions because the non-maximal ones would lead to obtain always 0 as infimum being them unsuccessful.

Definition 9. *Let $(S, A, \longrightarrow)$ be a NPLTS. We say that $s_1, s_2 \in S$ are* probabilistic testing equivalent according to [17, 10, 6]*, written $s_1 \sim_{\mathrm{PTe},\sqcup\sqcap} s_2$, iff for all nondeterministic and probabilistic tests $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:*

$$\bigsqcup_{\mathcal{Z}_1 \in Res_{\max}(s_1, o)} prob(\mathcal{SC}(z_{s_1,o})) = \bigsqcup_{\mathcal{Z}_2 \in Res_{\max}(s_2, o)} prob(\mathcal{SC}(z_{s_2,o}))$$

$$\bigsqcap_{\mathcal{Z}_1 \in Res_{\max}(s_1, o)} prob(\mathcal{SC}(z_{s_1,o})) = \bigsqcap_{\mathcal{Z}_2 \in Res_{\max}(s_2, o)} prob(\mathcal{SC}(z_{s_2,o}))$$

■

Following the structure of classical testing equivalence $\sim_{\rm Te}$ for fully nondeterministic processes [5], the constraint on suprema is the may-part of $\sim_{\rm PTe,\sqcup\sqcap}$ while the constraint on infima is the must-part of $\sim_{\rm PTe,\sqcup\sqcap}$. The probabilistic testing equivalence of [14] is defined in a similar way, but resolves nondeterminism through randomized schedulers and makes use of countably many success actions. The relation $\sim_{\rm PTe,\sqcup\sqcap}$ possesses several properties investigated in [17, 10, 6, 14], but suffers from two anomalies when contrasting it with $\sim_{\rm Te}$.

The first anomaly of $\sim_{\rm PTe,\sqcup\sqcap}$ is that it is not always included in $\sim_{\rm PTr,dis}$ and $\sim_{\rm PTr,new}$. The inclusion depends on the class of schedulers that are considered for deriving resolutions of interaction systems. If randomized schedulers are admitted, then inclusion holds as shown in [14]. However, this is no longer the case when only deterministic schedulers are taken into account like in [17, 10, 6].

For instance, if we take the two NPLTS models on the left-hand side of Fig. 2(i), when $b \neq c$ it turns out that $s_1 \sim_{\rm PTe,\sqcup\sqcap} s_2$ while $s_1 \not\sim_{\rm PTr,dis} s_2$ and $s_1 \not\sim_{\rm PTr,new} s_2$. States s_1 and s_2 are not related by the two probabilistic trace equivalences because the maximal resolution of s_1 starting with the central a-transition is not matched by any of the two maximal resolutions of s_2 (note that we would have $s_1 \sim_{\rm PTr,dis} s_2$ if randomized schedulers were admitted). It holds that $s_1 \sim_{\rm PTe,\sqcup\sqcap} s_2$ because, for any test, the same maximal resolution of s_1 cannot give rise to a success probability not comprised between the success probabilities of the other two maximal resolutions of s_1, which basically coincide with the two maximal resolutions of s_2.

The inclusion problem can be overcome by considering $\sim_{\rm PTr,new}$ instead of $\sim_{\rm PTr,dis}$ and by replacing $\sim_{\rm PTe,\sqcup\sqcap}$ with the finer probabilistic testing equivalence below. This equivalence does not only focus on the highest and the lowest probability of passing a test but requires matching all maximal resolutions of the interaction system according to their success probabilities.

Definition 10. *Let $(S, A, \longrightarrow)$ be a NPLTS. We say that $s_1, s_2 \in S$ are* probabilistic testing equivalent*, written $s_1 \sim_{\rm PTe,new} s_2$, iff for all nondeterministic and probabilistic tests $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:*

- *For each $\mathcal{Z}_1 \in Res_{\max}(s_1, o)$ there exists $\mathcal{Z}_2 \in Res_{\max}(s_2, o)$ such that:* $$prob(\mathcal{SC}(z_{s_1,o})) = prob(\mathcal{SC}(z_{s_2,o}))$$
- *For each $\mathcal{Z}_2 \in Res_{\max}(s_2, o)$ there exists $\mathcal{Z}_1 \in Res_{\max}(s_1, o)$ such that:* $$prob(\mathcal{SC}(z_{s_2,o})) = prob(\mathcal{SC}(z_{s_1,o}))$$ ■

Theorem 3. *Let $(S, A, \longrightarrow)$ be a NPLTS and $s_1, s_2 \in S$. Then:*
$$s_1 \sim_{\rm PTe,new} s_2 \implies s_1 \sim_{\rm PTe,\sqcup\sqcap} s_2$$ ■

The inclusion of $\sim_{\rm PTe,new}$ in $\sim_{\rm PTe,\sqcup\sqcap}$ is strict. Indeed, if we consider again the two NPLTS models on the left-hand side of Fig. 2(i) together with the test next to them, it turns out that $s_1 \sim_{\rm PTe,\sqcup\sqcap} s_2$ while $s_1 \not\sim_{\rm PTe,new} s_2$. The considered test distinguishes s_1 from s_2 with respect to $\sim_{\rm PTe,new}$ because – looking at the interaction system on the right-hand side of Fig. 2(i) – the maximal resolution of (s_1, o) starting with the central a-transition gives rise to a success probability

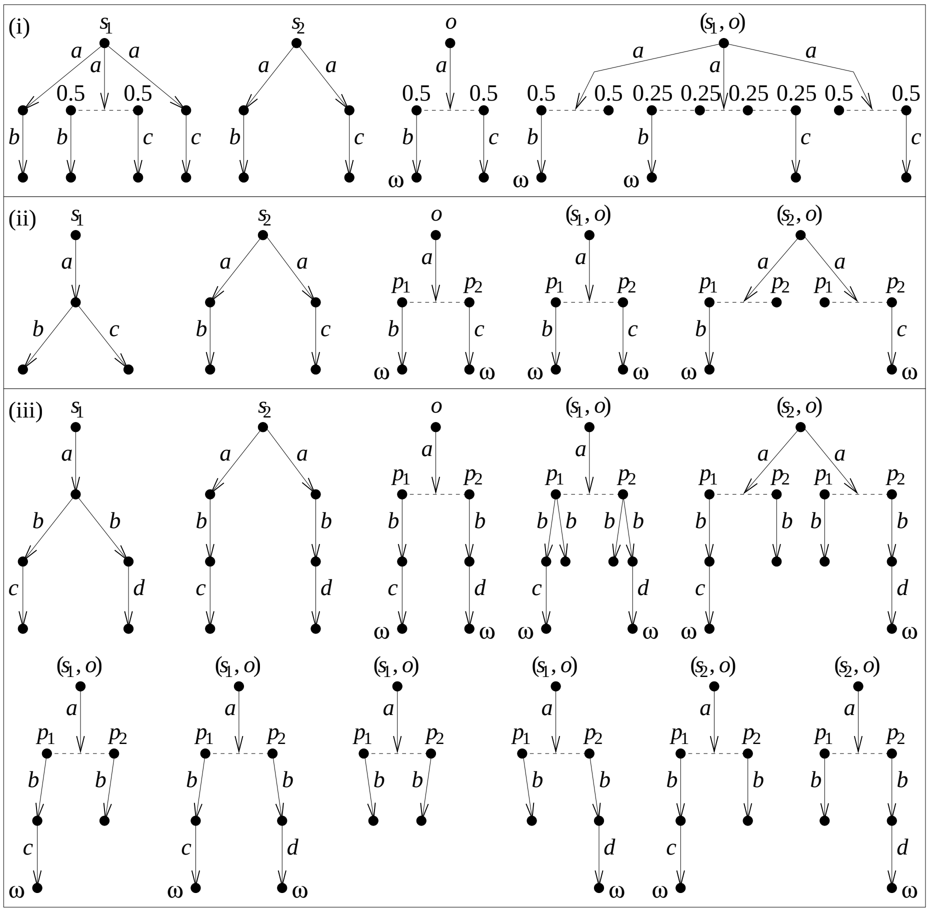


Fig. 2. Counterexamples for probabilistic testing and trace equivalences

equal to 0.25 that is not matched by any of the two maximal resolutions of (s_2, o). These resolutions – not shown in the figure – basically coincide with the maximal resolutions of (s_1, o) starting with the two outermost a-transitions, hence their success probabilities are respectively 0.5 and 0.

We now show that $\sim_{\rm PTe,new}$ does not suffer from the first anomaly because it is included in $\sim_{\rm PTr,new}$. As expected, the inclusion is strict. For instance, if we consider the two NPLTS models on the left-hand side of Fig. 2(ii) together with the test next to them, when $b \neq c$ it turns out that ($s_1 \sim_{\rm PTr,dis} s_2$ and) $s_1 \sim_{\rm PTr,new} s_2$ while $s_1 \not\sim_{\rm PTe,new} s_2$. In fact, the considered test distinguishes s_1 from s_2 with respect to $\sim_{\rm PTe,new}$ because – looking at the two interaction systems on the right-hand side of Fig. 2(ii) – the only maximal resolution of (s_1, o) gives rise to a success probability equal to 1 that is not matched by any of the two maximal resolutions of (s_2, o), whose success probabilities are p_1 and p_2.

Theorem 4. *Let* $(S, A, \longrightarrow)$ *be a NPLTS and* $s_1, s_2 \in S$. *Then:*

$$s_1 \sim_{\rm PTe,new} s_2 \implies s_1 \sim_{\rm PTr,new} s_2 \qquad \blacksquare$$

Unfortunately, $\sim_{\rm PTe,new}$ still does not avoid the second anomaly of $\sim_{\rm PTe,\sqcup\sqcap}$ (which affects [14] too) because it does not preserve $\sim_{\rm Te}$. In fact, there exist two fully nondeterministic processes that are testing equivalent according to [5], but are differentiated by $\sim_{\rm PTe,new}$ when probabilistic choices are present within tests. The reason is that such probabilistic choices make it possible to take copies of intermediate states of the processes under test, and thus to enhance the discriminating power of observers [1].

As an example, if we consider the two NPLTS models on the left-hand side of the upper part of Fig. 2(iii) together with the test next to them, when $c \neq d$ it turns out that $s_1 \sim_{\rm Te} s_2$ while $s_1 \not\sim_{\rm PTe,\sqcup\sqcap} s_2$ and $s_1 \not\sim_{\rm PTe,new} s_2$. Let us look at the two interaction systems on the right-hand side of the upper part of Fig. 2(iii), whose maximal resolutions are shown in the lower part of the same figure. The considered test distinguishes s_1 from s_2 with respect to $\sim_{\rm PTe,\sqcup\sqcap}$ because the supremum of the success probabilities of the four maximal resolutions of (s_1, o) is 1 – see the second maximal resolution of (s_1, o) – whereas the supremum of the success probabilities of the two maximal resolutions of (s_2, o) is equal to the maximum between p_1 and p_2. The considered test distinguishes s_1 from s_2 with respect to $\sim_{\rm PTe,new}$ because the third maximal resolution of (s_1, o) gives rise to a success probability equal to 0 that is not matched by any of the two maximal resolutions of (s_2, o), whose success probabilities are p_1 and p_2.

The second anomaly is essentially originated from an unrealistic estimation of success probabilities. For instance, if we consider again the four maximal resolutions of (s_1, o) in the lower part of Fig. 2(iii), we have that their success probabilities are p_1, 1, 0, and p_2, respectively. However, value 1 is clearly an overestimation of the success probability, in the same way as value 0 is an underestimation. These two values come from the fact that in each of the two corresponding maximal resolutions of (s_1, o) the deterministic scheduler selects a different b-transition in the two states of the probabilistic choice. The selection is instead consistent in the other two maximal resolutions of (s_1, o), which thus yield realistic estimations of the success probability.

The issue of realistic probability estimation has been recently addressed in [8]. Their models are significantly different from ours, with three kinds of transition (visible, invisible, and probabilistic) and each state having only one kind of outgoing transition. Equally labeled transitions departing from the same state are tagged to be kept distinct. Moreover, in presence of cycles, models are unfolded and the tagged transitions are further decorated with the unfolding stage. Since schedulers, while testing, might encounter several instances of a given state with tagged transitions, they must resolve nondeterminism consistently in all the instances at the same stage; choices at different stages are instead independent. Therefore, in Fig. 2(iii) the two pairs of b-transitions in the interaction system with initial configuration (s_1, o) would be identically tagged with b_l and b_r and the only allowed maximal resolutions of that interaction system would be the first one (choice of b_l) and the fourth one (choice of b_r).

5 Trace-by-Trace Redefinition of Testing Equivalence

In this section, we propose a solution to the problem of estimating success probabilities – and hence to the second anomaly – which is alternative to the solution in [8]. Our solution is not invasive at all, in the sense that it does not require any transition relabeling. In order to counterbalance the strong discriminating power deriving from the presence of probabilistic choices within tests, our basic idea is changing the definition of $\sim_{\mathrm{PTe,new}}$ by considering success probabilities in a *trace-by-trace fashion* rather than on entire resolutions.

In the following, given a state s of a NPLTS, a state o of a nondeterministic and probabilistic test, and $\alpha \in A^*$, we denote by $Res_{\max,\alpha}(s, o)$ the set of resolutions $\mathcal{Z} \in Res_{\max}(s, o)$ such that $\mathcal{CC}_{\max}(z_{s,o}, \alpha) \neq \emptyset$, where $\mathcal{CC}_{\max}(z_{s,o}, \alpha)$ is the set of computations in $\mathcal{CC}(z_{s,o}, \alpha)$ that are maximal. Moreover, for each such resolution $\mathcal{Z}$ we denote by $\mathcal{SCC}(z_{s,o}, \alpha)$ the set of computations in $\mathcal{SC}(z_{s,o})$ that are compatible with α.

Definition 11. *Let $(S, A, \longrightarrow)$ be a NPLTS. We say that $s_1, s_2 \in S$ are* trace-by-trace probabilistic testing equivalent*, written $s_1 \sim_{\mathrm{PTe,tbt}} s_2$, iff for all nondeterministic and probabilistic tests $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all traces $\alpha \in A^*$:*

- *For each $\mathcal{Z}_1 \in Res_{\max,\alpha}(s_1, o)$ there exists $\mathcal{Z}_2 \in Res_{\max,\alpha}(s_2, o)$ such that:*
  $$prob(\mathcal{SCC}(z_{s_1,o}, \alpha)) = prob(\mathcal{SCC}(z_{s_2,o}, \alpha))$$
- *For each $\mathcal{Z}_2 \in Res_{\max,\alpha}(s_2, o)$ there exists $\mathcal{Z}_1 \in Res_{\max,\alpha}(s_1, o)$ such that:*
  $$prob(\mathcal{SCC}(z_{s_2,o}, \alpha)) = prob(\mathcal{SCC}(z_{s_1,o}, \alpha))$$ ■

If we consider again the two NPLTS models on the left-hand side of the upper part of Fig. 2(iii), it turns out that $s_1 \sim_{\mathrm{PTe,tbt}} s_2$. As an example, let us examine the interaction with the test in the same figure, which originates maximal computations from (s_1, o) or (s_2, o) that are all labeled with traces $a\,b$, $a\,b\,c$, or $a\,b\,d$. It is easy to see that, for each of these traces, say α, the probability of performing a successful computation compatible with it in any of the four maximal resolutions of (s_1, o) having a maximal computation labeled with α is matched by the probability of performing a successful computation compatible with α in one of the two maximal resolutions of (s_2, o), and vice versa. For instance, the probability p_1 (resp. p_2) of performing a successful computation compatible with $a\,b\,c$ (resp. $a\,b\,d$) in the second maximal resolution of (s_1, o) is matched by the probability of performing a successful computation compatible with that trace in the first (resp. second) maximal resolution of (s_2, o). As another example, the probability 0 of performing a successful computation compatible with $a\,b$ in the third maximal resolution of (s_1, o) is matched by the probability of performing a successful computation compatible with that trace in any of the two maximal resolutions of (s_2, o).

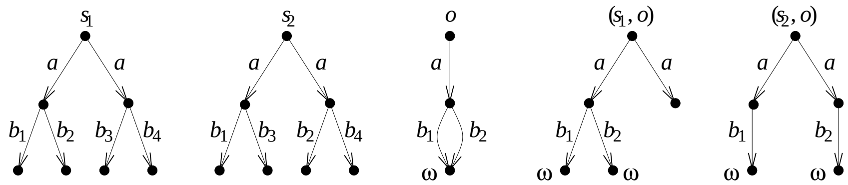
The previous example shows that $\sim_{\mathrm{PTe,tbt}}$ is included neither in $\sim_{\mathrm{PTe},\sqcup\sqcap}$ nor in $\sim_{\mathrm{PTe,new}}$. On the other hand, $\sim_{\mathrm{PTe},\sqcup\sqcap}$ is not included in $\sim_{\mathrm{PTe,tbt}}$ as witnessed by the two NPLTS models in Fig. 2(i), where the considered test distinguishes

s_1 from s_2 with respect to $\sim_{\rm PTe,tbt}$. In fact, the probability 0.25 of performing a successful computation compatible with $a\,b$ in the maximal resolution of (s_1, o) beginning with the central a-transition is not matched by the probability 0.5 of performing a successful computation compatible with $a\,b$ in the only maximal resolution of (s_2, o) that has a maximal computation labeled with $a\,b$. In contrast, $\sim_{\rm PTe,new}$ is (strictly) included in $\sim_{\rm PTe,tbt}$.

Theorem 5. *Let $(S, A, \longrightarrow)$ be a NPLTS and $s_1, s_2 \in S$. Then:*

$$s_1 \sim_{\rm PTe,new} s_2 \implies s_1 \sim_{\rm PTe,tbt} s_2 \qquad \blacksquare$$

Apart from the use of $prob(\mathcal{SCC}(z_{s,o}, \alpha))$ values instead of $prob(\mathcal{SC}(z_{s,o}))$ values, another major difference between $\sim_{\rm PTe,tbt}$ and $\sim_{\rm PTe,new}$ is the consideration of resolutions in $Res_{\max,\alpha}(s, o)$ rather than in $Res_{\max}(s, o)$. The reason is that it is not appropriate to match the (zero) success probability of unsuccessful maximal computations labeled with α with the (zero) success probability of computations labeled with α that are not maximal, as it may happen when considering $Res_{\max}(s, o)$. For example, let us take the two NPLTS models on the left-hand side of the following figure:



where $b_i \neq b_j$ for $i \neq j$. If we employed maximal resolutions not necessarily having maximal computations labeled with a, then the test in the figure would not be able to distinguish s_1 from s_2 with respect to $\sim_{\rm PTe,tbt}$. In fact, the success probability of the maximal resolution of (s_1, o) formed by the rightmost a-transition departing from (s_1, o) – which is 0 – would be inappropriately matched by the success probability of the a-prefix of the only maximal computation of both maximal resolutions of (s_2, o).

We now show that $\sim_{\rm PTe,tbt}$ does *not* suffer from the two anomalies discussed in Sect. 4. We start with the inclusion in $\sim_{\rm PTr,new}$, which is easily met. As expected, the inclusion is strict. For instance, if we consider again the two NPLTS models on the left-hand side of Fig. 2(ii) together with the test next to them, it turns out that ($s_1 \sim_{\rm PTr,dis} s_2$ and) $s_1 \sim_{\rm PTr,new} s_2$ while $s_1 \not\sim_{\rm PTe,tbt} s_2$. In fact, the considered test distinguishes s_1 from s_2 with respect to $\sim_{\rm PTe,tbt}$ because – looking at the two interaction systems on the right-hand side of Fig. 2(ii) – each of the two maximal resolutions of (s_2, o) has a maximal computation labeled with a while the only maximal resolution of (s_1, o) has not.

Theorem 6. *Let $(S, A, \longrightarrow)$ be a NPLTS and $s_1, s_2 \in S$. Then:*

$$s_1 \sim_{\rm PTe,tbt} s_2 \implies s_1 \sim_{\rm PTr,new} s_2 \qquad \blacksquare$$

With regard to the second anomaly, we show that $\sim_{\rm PTe,tbt}$ is totally compatible with $\sim_{\rm Te}$, in the sense that two fully nondeterministic NPLTS models are related by $\sim_{\rm Te}$ iff they are related by $\sim_{\rm PTe,tbt}$ regardless of the class of tests. Due to the anomaly, only partial compatibility results could be provided in [17] for $\sim_{\rm PTe,\sqcup\sqcap}$, because only tests without probabilistic choices (i.e., only fully nondeterministic tests) could be considered.

Theorem 7. *Let* $(S, A, \longrightarrow)$ *be a fully nondeterministic NPLTS and* $s_1, s_2 \in S$. *Then:*

$$s_1 \sim_{\rm Te} s_2 \iff s_1 \sim_{\rm PTe,tbt} s_2$$ ■

We conclude by showing that $\sim_{\rm PTe,tbt}$ is a congruence with respect to parallel composition of NPLTS models (see Def. 3).

Theorem 8. *Let* $\mathcal{L}_i = (S_i, A, \longrightarrow_i)$ *be a NPLTS and* $s_i \in S_i$ *for* $i = 0, 1, 2$ *and consider* $\mathcal{L}_1 \parallel \mathcal{L}_0$ *and* $\mathcal{L}_2 \parallel \mathcal{L}_0$. *Then:*

$$s_1 \sim_{\rm PTe,tbt} s_2 \implies (s_1, s_0) \sim_{\rm PTe,tbt} (s_2, s_0)$$ ■

6 Placing Trace-by-Trace Testing in the Spectrum

In this section, we show that $\sim_{\rm PTe,tbt}$ is strictly comprised between $\sim_{\rm PTr,new}$ (see Thm. 6) and a novel probabilistic failure equivalence $\sim_{\rm PF,new}$ that, in turn, is strictly comprised between $\sim_{\rm PTe,tbt}$ and $\sim_{\rm PTe,new}$.

In the following, we denote by $2^A_{\rm fin}$ the set of finite subsets of A and we call failure pair any element β of $A^* \times 2^A_{\rm fin}$, which is formed by a trace α and a finite action set F. Given a state s of a NPLTS $\mathcal{L}$, $\mathcal{Z} \in Res(s)$, and $c \in \mathcal{C}_{\rm fin}(z_s)$, we say that c is compatible with β iff $c \in \mathcal{CC}(z_s, \alpha)$ and the last state reached by c has no outgoing transitions in $\mathcal{L}$ labeled with an action in F. We denote by $\mathcal{FCC}(z_s, \beta)$ the set of computations in $\mathcal{C}_{\rm fin}(z_s)$ that are compatible with β.

Definition 12. *Let* $(S, A, \longrightarrow)$ *be a NPLTS. We say that* $s_1, s_2 \in S$ *are* probabilistic failure equivalent, *written* $s_1 \sim_{\rm PF,new} s_2$, *iff for all* $\beta \in A^* \times 2^A_{\rm fin}$:

- *For each* $\mathcal{Z}_1 \in Res(s_1)$ *there exists* $\mathcal{Z}_2 \in Res(s_2)$ *such that:*
$$prob(\mathcal{FCC}(z_{s_1}, \beta)) = prob(\mathcal{FCC}(z_{s_2}, \beta))$$
- *For each* $\mathcal{Z}_2 \in Res(s_2)$ *there exists* $\mathcal{Z}_1 \in Res(s_1)$ *such that:*
$$prob(\mathcal{FCC}(z_{s_2}, \beta)) = prob(\mathcal{FCC}(z_{s_1}, \beta))$$ ■

Theorem 9. *Let* $(S, A, \longrightarrow)$ *be a NPLTS and* $s_1, s_2 \in S$. *Then:*

$$s_1 \sim_{\rm PF,new} s_2 \implies s_1 \sim_{\rm PTe,tbt} s_2$$ ■

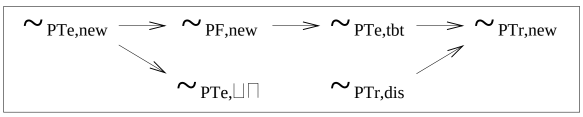


Fig. 3. Spectrum of the considered probabilistic equivalences for NPLTS models

The inclusion of $\sim_{\rm PF,new}$ in $\sim_{\rm PTe,tbt}$ is strict. For instance, if we consider again the two NPLTS models on the left-hand side of Fig. 1, when $b_i \neq b_j$ for $i \neq j$ it turns out that $s_1 \sim_{\rm PTe,tbt} s_2$ while $s_1 \not\sim_{\rm PF,new} s_2$. In fact, given the failure pair $\beta = (a, \{b_1, b_2\})$, the second maximal resolution of s_1 has probability 1 of performing a computation compatible with β, whilst each of the two maximal resolutions of s_2 has probability 0.5.

Theorem 10. *Let* $(S, A, \longrightarrow)$ *be a NPLTS and* $s_1, s_2 \in S$. *Then:*

$$s_1 \sim_{\rm PTe,new} s_2 \implies s_1 \sim_{\rm PF,new} s_2 \qquad \blacksquare$$

The inclusion of $\sim_{\rm PTe,new}$ in $\sim_{\rm PF,new}$ is strict as $\sim_{\rm PTe,new}$ suffers from the second anomaly. Indeed, if we consider again the two NPLTS models on the left-hand side of Fig. 2(iii), it holds that $s_1 \sim_{\rm PF,new} s_2$ while $s_1 \not\sim_{\rm PTe,new} s_2$.

We also note that $\sim_{\rm PTe,\sqcup\sqcap}$ is incomparable not only with $\sim_{\rm PTr,dis}$, $\sim_{\rm PTr,new}$, and $\sim_{\rm PTe,tbt}$, but with $\sim_{\rm PF,new}$ too. The NPLTS models on the left-hand side of Fig. 2(i) show that $\sim_{\rm PTe,\sqcup\sqcap}$ is not included in $\sim_{\rm PF,new}$ and the NPLTS models on the left-hand side of Fig. 2(iii) show that $\sim_{\rm PF,new}$ is not included in $\sim_{\rm PTe,\sqcup\sqcap}$. Similarly, $\sim_{\rm PTr,dis}$ is incomparable with $\sim_{\rm PTe,tbt}$, $\sim_{\rm PF,new}$, and $\sim_{\rm PTe,new}$. The NPLTS models on the left-hand side of Fig. 1 show that $\sim_{\rm PTe,tbt}$ is not included in $\sim_{\rm PTr,dis}$ and the NPLTS models on the left-hand side of Fig. 2(ii) show that $\sim_{\rm PTr,dis}$ is not included in $\sim_{\rm PTe,tbt}$. Moreover, the NPLTS models on the left-hand side of Fig. 2(ii) show that $\sim_{\rm PTr,dis}$ is included in neither $\sim_{\rm PF,new}$ nor $\sim_{\rm PTe,new}$. On the other hand, neither of $\sim_{\rm PF,new}$ and $\sim_{\rm PTe,new}$ is included in $\sim_{\rm PTr,dis}$ as can be seen by considering two NPLTS models both having four states $s_{i,a}$, $s_{i,b}$, $s_{i,c}$, and $s_{i,-}$ for $i = 1, 2$ such that: $s_{i,a} \xrightarrow{a} \mathcal{D}_{i,j}$ for $j = 1, 2, 3$ with $\mathcal{D}_{1,1}(s_{1,b}) = 0.6 = 1 - \mathcal{D}_{1,1}(s_{1,-})$, $\mathcal{D}_{1,2}(s_{1,b}) = 0.4 = 1 - \mathcal{D}_{1,2}(s_{1,c})$, $\mathcal{D}_{1,3}(s_{1,-}) = 0.6 = 1 - \mathcal{D}_{1,3}(s_{1,c})$, and $\mathcal{D}_{2,j}(s_{2,*}) = 1 - \mathcal{D}_{1,j}(s_{1,*})$ for $* = b, c, -$; $s_{i,b} \xrightarrow{b} \mathcal{D}_{i,b}$ with $\mathcal{D}_{i,b}(s_{i,-}) = 1$; and $s_{i,c} \xrightarrow{c} \mathcal{D}_{i,c}$ with $\mathcal{D}_{i,b}(s_{i,-}) = 1$.

The relationships among the examined equivalences are summarized in Fig. 3, where arrows mean more-discriminating-than. For the sake of completeness, we remark that $\sim_{\rm PTe,\sqcup\sqcap}$ has been characterized in [10, 6] through probabilistic simulation equivalence for the may-part and probabilistic failure simulation equivalence for the must-part. With regard to the variant of [14] based on randomized schedulers, the may-part coincides with the coarsest congruence contained in $\sim_{\rm PTr,dis}$ (again based on randomized schedulers) and the must-part coincides with the coarsest congruence contained in probabilistic failure distribution equivalence. The probabilistic testing equivalence of [8] has instead been characterized via probabilistic ready-trace equivalence. Its connection with $\sim_{\rm PTe,tbt}$ is hindered by the different underlying models and is still an open problem.

7 Trace-by-Trace Testing for GPLTS and RPLTS Models

Our variant of testing equivalence naturally fits to generative and reactive probabilistic processes [15]. In this section, we show that the two testing equivalences for those two classes of processes can be redefined in a uniform trace-by-trace fashion without altering their discriminating power.

Definition 13. *A probabilistic labeled transition system, PLTS for short, is a triple* $(S, A, \longrightarrow)$ *where* S *is an at most countable set of states,* A *is a countable set of transition-labeling actions, and* $\longrightarrow \subseteq S \times A \times \mathbb{R}_{(0,1]} \times S$ *is a transition relation satisfying one of the following two conditions:*

- $\sum \{\!| p \in \mathbb{R}_{(0,1]} \mid \exists a \in A. \exists s' \in S. s \xrightarrow{a,p} s' |\!\} \in \{0, 1\}$ *for all* $s \in S$ *(generative PLTS, or GPLTS for short).*
- $\sum \{\!| p \in \mathbb{R}_{(0,1]} \mid \exists s' \in S. s \xrightarrow{a,p} s' |\!\} \in \{0, 1\}$ *for all* $s \in S$ *and* $a \in A$ *(reactive PLTS, or RPLTS for short).* ■

A test consistent with a PLTS $\mathcal{L} = (S, A, \longrightarrow_{\mathcal{L}})$ is an acyclic and finitely-branching PLTS $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ equipped with a success state, which is generative (resp. reactive) if so is $\mathcal{L}$. Their interaction system is the acyclic and finitely-branching PLTS $\mathcal{I}(\mathcal{L}, \mathcal{T}) = (S \times O, A, \longrightarrow)$ whose transition relation $\longrightarrow \subseteq (S \times O) \times A \times \mathbb{R}_{(0,1]} \times (S \times O)$ is such that $(s, o) \xrightarrow{a,p} (s', o')$ iff $s \xrightarrow{a,p_1}_{\mathcal{L}} s'$ and $o \xrightarrow{a,p_2}_{\mathcal{T}} o'$ with p being equal to:

$$\begin{array}{ll} p_1 \cdot p_2 / \sum \{\!| q_1 \cdot q_2 \mid \exists b \in A, s'' \in S, o'' \in O. s \xrightarrow{b,q_1}_{\mathcal{L}} s'' \land o \xrightarrow{b,q_2}_{\mathcal{T}} o'' |\!\} & \text{if GPLTS} \\ p_1 \cdot p_2 & \text{if RPLTS} \end{array}$$

Given $s \in S$ and $o \in O$, we denote by $\mathcal{SC}(s, o)$ the set of successful computations of $\mathcal{I}(\mathcal{L}, \mathcal{T})$ with initial configuration (s, o) and by $\mathcal{SCC}(s, o, \alpha)$ the set of computations in $\mathcal{SC}(s, o)$ that are compatible with $\alpha \in A^*$. Moreover, we denote by $Tr_{\max}(s, o)$ the set of traces labeling the maximal computations from (s, o).

Definition 14. *Let* $(S, A, \longrightarrow)$ *be a GPLTS. We say that* $s_1, s_2 \in S$ *are* probabilistic testing equivalent according to [3, 4], *written* $s_1 \sim_{\mathrm{PTe,G}} s_2$*, iff for all generative probabilistic tests* $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ *with initial state* $o \in O$*:*

$$prob(\mathcal{SC}(s_1, o)) = prob(\mathcal{SC}(s_2, o))$$ ■

Definition 15. *Let* $(S, A, \longrightarrow)$ *be a RPLTS. We say that* $s_1, s_2 \in S$ *are* probabilistic testing equivalent according to [11], *written* $s_1 \sim_{\mathrm{PTe,R}} s_2$*, iff for all reactive probabilistic tests* $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ *with initial state* $o \in O$*:*

$$\bigsqcup_{\alpha \in Tr_{\max}(s_1, o)} prob(\mathcal{SCC}(s_1, o, \alpha)) = \bigsqcup_{\alpha \in Tr_{\max}(s_2, o)} prob(\mathcal{SCC}(s_2, o, \alpha))$$

$$\bigsqcap_{\alpha \in Tr_{\max}(s_1, o)} prob(\mathcal{SCC}(s_1, o, \alpha)) = \bigsqcap_{\alpha \in Tr_{\max}(s_2, o)} prob(\mathcal{SCC}(s_2, o, \alpha))$$ ■

Definition 16. *Let* $\mathcal{L} = (S, A, \longrightarrow)$ *be a PLTS. We say that* $s_1, s_2 \in S$ *are* trace-by-trace probabilistic testing equivalent*, written* $s_1 \sim_{\mathrm{PTe,tbt}} s_2$*, iff for all probabilistic tests* $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ *consistent with* $\mathcal{L}$ *with initial state* $o \in O$ *and for all traces* $\alpha \in A^*$*:*

$$prob(\mathcal{SCC}(s_1, o, \alpha)) = prob(\mathcal{SCC}(s_2, o, \alpha))$$ ■

Theorem 11. *Let* $(S, A, \longrightarrow)$ *be a GPLTS and* $s_1, s_2 \in S$*. Then:*

$$s_1 \sim_{\mathrm{PTe,G}} s_2 \iff s_1 \sim_{\mathrm{PTe,tbt}} s_2$$ ■

Theorem 12. *Let* $(S, A, \longrightarrow)$ *be a RPLTS and* $s_1, s_2 \in S$*. Then:*

$$s_1 \sim_{\mathrm{PTe,R}} s_2 \iff s_1 \sim_{\mathrm{PTe,tbt}} s_2$$ ■

8 Conclusion

In this paper, we have proposed solutions for avoiding two anomalies of probabilistic testing equivalence for NPLTS models by (i) matching all resolutions on the basis of their success probabilities rather than taking only maximal and minimal success probabilities and (ii) considering success probabilities in a trace-by-trace fashion rather than on entire resolutions. The trace-by-trace approach – which fits also testing equivalences for nondeterministic processes (Thm. 7), generative probabilistic processes (Thm. 11), and reactive probabilistic processes (Thm. 12) – thus annihilates the impact of the copying capability introduced by probabilistic observers. In the future, we would like to find equational and logical characterizations of $\sim_{\mathrm{PTe,tbt}}$. Moreover, we plan to investigate the whole spectrum of probabilistic behavioral equivalences for NPLTS models.

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Appendix: Proofs of Results

Proof of Thm. 1. Trivial. ■

Proof of Thm. 2. Let us recall from [2] that $s_1 \sim_{\mathrm{Tr}} s_2$ means that, for all $\alpha \in A^*$, there is a computation from s_1 labeled with α iff there is a computation from s_2 labeled with α. The result is a straightforward consequence of the fact that the considered NPLTS is fully nondeterministic and hence its resolutions coincide with its computations, so that the probability of performing a computation compatible with α within a resolution can only be either 0 or 1. ■

Proof of Thm. 3. Trivial. ■

Proof of Thm. 4. If $s_1 \sim_{\mathrm{PTe,new}} s_2$, then in particular for all nondeterministic and probabilistic tests $\mathcal{T}_\alpha = (O, A, \longrightarrow_{\mathcal{T}_\alpha})$ with initial state $o \in O$ having a single maximal computation that is labeled with $\alpha \in A^*$ and reaches success:

- For each $\mathcal{Z}_1 \in \mathit{Res}_{\max}(s_1, o)$ there exists $\mathcal{Z}_2 \in \mathit{Res}_{\max}(s_2, o)$ such that:
$$\mathit{prob}(\mathcal{SC}(z_{s_1,o})) \; = \; \mathit{prob}(\mathcal{SC}(z_{s_2,o}))$$
- For each $\mathcal{Z}_2 \in \mathit{Res}_{\max}(s_2, o)$ there exists $\mathcal{Z}_1 \in \mathit{Res}_{\max}(s_1, o)$ such that:
$$\mathit{prob}(\mathcal{SC}(z_{s_2,o})) \; = \; \mathit{prob}(\mathcal{SC}(z_{s_1,o}))$$

Since $\mathit{prob}(\mathcal{SC}^{\mathcal{Z}}(z_{s,o})) = \mathit{prob}(\mathcal{CC}^{\mathcal{Z}'}(z_s, \alpha))$ for all $s \in S$ due to the structure of $\mathcal{T}_\alpha$ – where $\mathcal{Z} \in \mathit{Res}_{\max}(s, o)$ and $\mathcal{Z}' \in \mathit{Res}(s)$ originates $\mathcal{Z}$ in the interaction with $\mathcal{T}_\alpha$ – we immediately derive that for all $\alpha \in A^*$:

- For each $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ such that:
$$\mathit{prob}(\mathcal{CC}(z_{s_1}, \alpha)) \; = \; \mathit{prob}(\mathcal{CC}(z_{s_2}, \alpha))$$
- For each $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ there exists $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ such that:
$$\mathit{prob}(\mathcal{CC}(z_{s_2}, \alpha)) \; = \; \mathit{prob}(\mathcal{CC}(z_{s_1}, \alpha))$$

This means that $s_1 \sim_{\mathrm{PTr,new}} s_2$. ■

Proof of Thm. 5. Let $\mathcal{T} = (O, A, \longrightarrow)$ be an arbitrary nondeterministic and probabilistic test with initial state $o \in O$. If $s_1 \sim_{\mathrm{PTe,new}} s_2$, then in particular for all variants $\mathcal{T}_\alpha = (O, A, \longrightarrow_{\mathcal{T}_\alpha})$ of $\mathcal{T}$ in which only the successful computations of $\mathcal{T}$ that are labeled with $\alpha \in A^*$ reach ω:

- For each $\mathcal{Z}_1 \in \mathit{Res}_{\max}(s_1, o)$ there exists $\mathcal{Z}_2 \in \mathit{Res}_{\max}(s_2, o)$ such that:
$$\mathit{prob}(\mathcal{SC}^{\mathcal{T}_\alpha}(z_{s_1,o})) \; = \; \mathit{prob}(\mathcal{SC}^{\mathcal{T}_\alpha}(z_{s_2,o}))$$
- For each $\mathcal{Z}_2 \in \mathit{Res}_{\max}(s_2, o)$ there exists $\mathcal{Z}_1 \in \mathit{Res}_{\max}(s_1, o)$ such that:
$$\mathit{prob}(\mathcal{SC}^{\mathcal{T}_\alpha}(z_{s_2,o})) \; = \; \mathit{prob}(\mathcal{SC}^{\mathcal{T}_\alpha}(z_{s_1,o}))$$

Since $\mathit{prob}(\mathcal{SC}^{\mathcal{T}_\alpha}(z_{s,o})) = \mathit{prob}(\mathcal{SCC}^{\mathcal{T}}(z_{s,o}, \alpha))$ for all $s \in S$ due to the structure of $\mathcal{T}_\alpha$, we immediately derive that for all $\alpha \in A^*$:

- For each $\mathcal{Z}_1 \in \mathit{Res}_{\max,\alpha}(s_1, o)$ there exists $\mathcal{Z}_2 \in \mathit{Res}_{\max,\alpha}(s_2, o)$ such that:
$$\mathit{prob}(\mathcal{SCC}^{\mathcal{T}}(z_{s_1,o}, \alpha)) \; = \; \mathit{prob}(\mathcal{SCC}^{\mathcal{T}}(z_{s_2,o}, \alpha))$$

– For each $\mathcal{Z}_2 \in Res_{\max,\alpha}(s_2, o)$ there exists $\mathcal{Z}_1 \in Res_{\max,\alpha}(s_1, o)$ such that:
$$prob(\mathcal{SCC}^{\mathcal{T}}(z_{s_2,o}, \alpha)) \ = \ prob(\mathcal{SCC}^{\mathcal{T}}(z_{s_1,o}, \alpha))$$

This means that $s_1 \sim_{\mathrm{PTe,tbt}} s_2$. ■

Proof of Thm. 6. If $s_1 \sim_{\mathrm{PTe,tbt}} s_2$, then in particular for all nondeterministic and probabilistic tests $\mathcal{T}_\alpha = (O, A, \longrightarrow_{\mathcal{T}_\alpha})$ with initial state $o \in O$ having a single maximal computation that is labeled with $\alpha \in A^*$ and reaches success:

– For each $\mathcal{Z}_1 \in Res_{\max,\alpha}(s_1, o)$ there exists $\mathcal{Z}_2 \in Res_{\max,\alpha}(s_2, o)$ such that:
$$prob(\mathcal{SCC}(z_{s_1,o}, \alpha)) \ = \ prob(\mathcal{SCC}(z_{s_2,o}, \alpha))$$
– For each $\mathcal{Z}_2 \in Res_{\max,\alpha}(s_2, o)$ there exists $\mathcal{Z}_1 \in Res_{\max,\alpha}(s_1, o)$ such that:
$$prob(\mathcal{SCC}(z_{s_2,o}, \alpha)) \ = \ prob(\mathcal{SCC}(z_{s_1,o}, \alpha))$$

Since $prob(\mathcal{SCC}^{\mathcal{Z}}(z_{s,o}, \alpha)) = prob(\mathcal{CC}^{\mathcal{Z}'}(z_s, \alpha))$ for all $s \in S$ due to the structure of $\mathcal{T}_\alpha$ – where $\mathcal{Z} \in Res_{\max,\alpha}(s, o)$ and $\mathcal{Z}' \in Res(s)$ originates $\mathcal{Z}$ in the interaction with $\mathcal{T}_\alpha$ – we immediately derive that for all $\alpha \in A^*$:

– For each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$ such that:
$$prob(\mathcal{CC}(z_{s_1}, \alpha)) \ = \ prob(\mathcal{CC}(z_{s_2}, \alpha))$$
– For each $\mathcal{Z}_2 \in Res(s_2)$ there exists $\mathcal{Z}_1 \in Res(s_1)$ such that:
$$prob(\mathcal{CC}(z_{s_2}, \alpha)) \ = \ prob(\mathcal{CC}(z_{s_1}, \alpha))$$

This means that $s_1 \sim_{\mathrm{PTr,new}} s_2$. ■

Proof of Thm. 7. First of all, we recall from [5] that $s_1 \sim_{\mathrm{Te}} s_2$ means that for all *fully nondeterministic* tests $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:

– There is a successful computation from (s_1, o) iff there is a successful computation from (s_2, o).
– All maximal computations from (s_1, o) are successful iff all maximal computations from (s_2, o) are successful.

The proof is divided into two parts:

– Let $s_1 \sim_{\mathrm{Te}} s_2$ and consider an arbitrary nondeterministic and probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$, an arbitrary trace $\alpha \in A^*$, and an arbitrary resolution $\mathcal{Z}_1 \in Res_{\max,\alpha}(s_1, o)$.
Suppose that $Res_{\max,\alpha}(s_2, o) = \emptyset$, i.e., suppose that for all $\mathcal{Z}_2 \in Res_{\max}(s_2, o)$ it holds that $\mathcal{CC}_{\max}(z_{s_2,o}, \alpha) = \emptyset$. Let $\mathcal{T}_\alpha = (O, A, \longrightarrow_{\mathcal{T}_\alpha})$ be a fully nondeterministic test obtained from $\mathcal{T}$ in which (i) only the maximal computations labeled with α reach ω and (ii) each transition $o' \xrightarrow{a}_{\mathcal{T}} \mathcal{D}$ such that set $R = \{o'' \in O \mid \mathcal{D}(o'') > 0\}$ has cardinality greater than 1 is transformed into $|R|$ transitions $o' \xrightarrow{a}_{\mathcal{T}_\alpha} \mathcal{D}_{o''}$, $o'' \in R$, where $\mathcal{D}_{o''}(o'') = 1$ and $\mathcal{D}_{o''}(o''') = 0$ for all $o''' \in O \setminus \{o''\}$. Observing that $\mathcal{T}_\alpha$ yields the same α-computations as $\mathcal{T}$ in the interaction systems, test $\mathcal{T}_\alpha$ would violate $s_1 \sim_{\mathrm{Te}} s_2$ because all

maximal computations from (s_1, o) are successful whilst there are no maximal computations from (s_2, o) that are successful. We have thus deduced that, whenever $s_1 \sim_{\mathrm{Te}} s_2$, then the existence of $\mathcal{Z}_1 \in \mathit{Res}_{\max,\alpha}(s_1, o)$ implies the existence of $\mathcal{Z}_2 \in \mathit{Res}_{\max,\alpha}(s_2, o)$.
Suppose now that for all $\mathcal{Z}_2 \in \mathit{Res}_{\max,\alpha}(s_2, o)$ it holds that:

$$\mathit{prob}(\mathcal{SCC}(z_{s_1,o}, \alpha)) \neq \mathit{prob}(\mathcal{SCC}(z_{s_2,o}, \alpha))$$

Observing that $\mathcal{T}$ must have a successful computation labeled with α because it cannot be $\mathit{prob}(\mathcal{SCC}(z_{s_1,o}, \alpha)) = 0 = \mathit{prob}(\mathcal{SCC}(z_{s_2,o}, \alpha))$, from $\mathcal{CC}_{\max}(z_{s_1,o}, \alpha) \neq \emptyset$ and $\mathcal{CC}_{\max}(z_{s_2,o}, \alpha) \neq \emptyset$ we derive $\mathit{prob}(\mathcal{SCC}(z_{s_1,o}, \alpha)) > 0$ and $\mathit{prob}(\mathcal{SCC}(z_{s_2,o}, \alpha)) > 0$. Denoting by $\mathcal{Z}'_1$ the element of $\mathit{Res}(s_1)$ that originates $\mathcal{Z}_1$, we would then have that for all $\mathcal{Z}'_2 \in \mathit{Res}(s_2)$ originating $\mathcal{Z}_2$:

$$\begin{aligned}\mathit{prob}(\mathcal{CC}(z_{s_1}, \alpha)) &= \mathit{prob}(\mathcal{SCC}(z_{s_1,o}, \alpha))/p \neq \\ &\neq \mathit{prob}(\mathcal{SCC}(z_{s_2,o}, \alpha))/p = \mathit{prob}(\mathcal{CC}(z_{s_2}, \alpha))\end{aligned}$$

where p is the probability of performing a successful computation labeled with α in the element $\mathcal{Z}$ of $\mathit{Res}(o)$ that originates $\mathcal{Z}_1$. Since the NPLTS under test is fully nondeterministic, $\mathcal{Z}'_1$ and $\mathcal{Z}'_2$ boil down to two computations both labeled with α and it holds that:

$$\mathit{prob}(\mathcal{CC}(z_{s_1}, \alpha)) = 1 = \mathit{prob}(\mathcal{CC}(z_{s_2}, \alpha))$$

which contradicts what established before.
In conclusion, whenever $s_1 \sim_{\mathrm{Te}} s_2$, then for all $\mathcal{Z}_1 \in \mathit{Res}_{\max,\alpha}(s_1, o)$ there exists $\mathcal{Z}_2 \in \mathit{Res}_{\max,\alpha}(s_2, o)$ such that:

$$\mathit{prob}(\mathcal{SCC}(z_{s_1,o}, \alpha)) = \mathit{prob}(\mathcal{SCC}(z_{s_2,o}, \alpha))$$

With a similar argument, we can prove that, whenever $s_1 \sim_{\mathrm{Te}} s_2$, then for all $\mathcal{Z}_2 \in \mathit{Res}_{\max,\alpha}(s_2, o)$ there exists $\mathcal{Z}_1 \in \mathit{Res}_{\max,\alpha}(s_1, o)$ such that:

$$\mathit{prob}(\mathcal{SCC}(z_{s_2,o}, \alpha)) = \mathit{prob}(\mathcal{SCC}(z_{s_1,o}, \alpha))$$

from which it follows that $s_1 \sim_{\mathrm{PTe,tbt}} s_2$.

– If $s_1 \sim_{\mathrm{PTe,tbt}} s_2$, then in particular for all fully nondeterministic tests $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:
  - For each $\mathcal{Z}_1 \in \mathit{Res}_{\max,\alpha}(s_1, o)$ there exists $\mathcal{Z}_2 \in \mathit{Res}_{\max,\alpha}(s_2, o)$ such that:
  $$\mathit{prob}(\mathcal{SCC}(z_{s_1,o}, \alpha)) = \mathit{prob}(\mathcal{SCC}(z_{s_2,o}, \alpha))$$
  - For each $\mathcal{Z}_2 \in \mathit{Res}_{\max,\alpha}(s_2, o)$ there exists $\mathcal{Z}_1 \in \mathit{Res}_{\max,\alpha}(s_1, o)$ such that:
  $$\mathit{prob}(\mathcal{SCC}(z_{s_2,o}, \alpha)) = \mathit{prob}(\mathcal{SCC}(z_{s_1,o}, \alpha))$$

  Since the NPLTS under test and the considered tests are all fully nondeterministic, the resulting interaction systems are fully nondeterministic too, and hence their resolutions coincide with their computations and each of the probability values above is either 0 or 1. As a consequence, the previous relationships among resolutions can be rephrased as follows:
  - For each maximal computation from (s_1, o) labeled with α there exists a maximal computation from (s_2, o) labeled with α such that the two computations are both successful or both unsuccessful.
  - For each maximal computation from (s_2, o) labeled with α there exists a maximal computation from (s_1, o) labeled with α such that the two computations are both successful or both unsuccessful.

From this, we immediately derive that $s_1 \sim_{\text{Te}} s_2$. ■

Proof of Thm. 8. Given an arbitrary nondeterministic and probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$, we observe that $\mathcal{L}_0 \parallel \mathcal{T}$ is still a nondeterministic and probabilistic test with initial state $(s_0, o) \in S_0 \times O$. If $s_1 \sim_{\text{PTe,tbt}} s_2$, then in particular for all $\alpha \in A^*$:

- For each $\mathcal{Z}_1 \in Res_{\max,\alpha}(s_1, (s_0, o))$ there exists $\mathcal{Z}_2 \in Res_{\max,\alpha}(s_2, (s_0, o))$ such that:
$$prob(\mathcal{SCC}(z_{s_1,(s_0,o)}, \alpha)) \; = \; prob(\mathcal{SCC}(z_{s_2,(s_0,o)}, \alpha))$$
- For each $\mathcal{Z}_2 \in Res_{\max,\alpha}(s_2, (s_0, o))$ there exists $\mathcal{Z}_1 \in Res_{\max,\alpha}(s_1, (s_0, o))$ such that:
$$prob(\mathcal{SCC}(z_{s_2,(s_0,o)}, \alpha)) \; = \; prob(\mathcal{SCC}(z_{s_1,(s_0,o)}, \alpha))$$

Since $Res(s, (s_0, o)) = Res((s, s_0), o)$ for all $s \in S_1 \cup S_2$ due to the associativity of the parallel composition operator of Def. 3, we immediately derive that for all $\alpha \in A^*$:

- For each $\mathcal{Z}_1 \in Res_{\max,\alpha}((s_1, s_0), o)$ there exists $\mathcal{Z}_2 \in Res_{\max,\alpha}((s_2, s_0), o)$ such that:
$$prob(\mathcal{SCC}(z_{(s_1,s_0),o}, \alpha)) \; = \; prob(\mathcal{SCC}(z_{(s_2,s_0),o}, \alpha))$$
- For each $\mathcal{Z}_2 \in Res_{\max,\alpha}((s_2, s_0), o)$ there exists $\mathcal{Z}_1 \in Res_{\max,\alpha}((s_1, s_0), o)$ such that:
$$prob(\mathcal{SCC}(z_{(s_2,s_0),o}, \alpha)) \; = \; prob(\mathcal{SCC}(z_{(s_1,s_0),o}, \alpha))$$

This means that $(s_1, s_0) \sim_{\text{PTe,tbt}} (s_2, s_0)$. ■

Proof of Thm. 9. Following the approach used for $\sim_{\text{Te}}$, we preliminarily introduce a further probabilistic equivalence based on the combination of two predicates named "after" and "must" in [5]. Let us call after-must pair any element γ of $A^* \times 2^A_{\text{fin}}$ formed by a trace α and a finite action set M. Given $s \in S$, $\mathcal{Z} \in Res(s)$, and $c \in \mathcal{C}_{\text{fin}}(z_s)$, we say that c is compatible with γ iff $c \in \mathcal{CC}(z_s, \alpha)$ and the last state reached by c has at least one outgoing transition (in the considered NPLTS) labeled with an action in M. We denote by $\mathcal{AMCC}(z_s, \gamma)$ the set of computations in $\mathcal{C}_{\text{fin}}(z_s)$ that are compatible with γ. We say that $s_1, s_2 \in S$ are *probabilistic after-must equivalent*, written $s_1 \sim_{\text{PAM}} s_2$, iff for all after-must pairs $\gamma \in A^* \times 2^A_{\text{fin}}$:

- For each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$ such that:
$$prob(\mathcal{AMCC}(z_{s_1}, \gamma)) \; = \; prob(\mathcal{AMCC}(z_{s_2}, \gamma))$$
- For each $\mathcal{Z}_2 \in Res(s_2)$ there exists $\mathcal{Z}_1 \in Res(s_1)$ such that:
$$prob(\mathcal{AMCC}(z_{s_2}, \gamma)) \; = \; prob(\mathcal{AMCC}(z_{s_1}, \gamma))$$

We show that $\sim_{\text{PAM}}$ is included in $\sim_{\text{PTe,tbt}}$. To this purpose, given $s_1, s_2 \in S$ such that $s_1 \sim_{\text{PAM}} s_2$, we consider an arbitrary nondeterministic and probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$, an arbitrary trace $\alpha \in A^*$, and an arbitrary resolution $\mathcal{Z}_1 \in Res_{\max,\alpha}(s_1, o)$.
Suppose that $Res_{\max,\alpha}(s_2, o) = \emptyset$, i.e., suppose that for all $\mathcal{Z}_2 \in Res_{\max}(s_2, o)$ it holds that $\mathcal{CC}_{\max}(z_{s_2,o}, \alpha) = \emptyset$. There are two cases:

– If for all $\mathcal{Z}_2 \in Res_{\max}(s_2, o)$ it further holds that $\mathcal{CC}(z_{s_2,o}, \alpha) = \emptyset$, then $\mathcal{CC}(s_2, \alpha) = \emptyset$ because the existence of $\mathcal{Z}_1 \in Res_{\max,\alpha}(s_1, o)$ witnesses the existence of a computation labeled with α in $\mathcal{T}$. Since $\mathcal{CC}_{\max}(z_{s_1,o}, \alpha) \neq \emptyset$, it must be $\alpha \neq \varepsilon$. Assuming $\alpha = \alpha' a$, in this case the after-must pair $\gamma = (\alpha', \{a\})$ would violate $s_1 \sim_{\text{PAM}} s_2$. In fact, denoting by $\mathcal{Z}'_1$ the element of $Res(s_1)$ that originates $\mathcal{Z}_1$, we would have that for all $\mathcal{Z}'_2 \in Res(s_2)$ originating $\mathcal{Z}_2$:
$$prob(\mathcal{AMCC}(z_{s_1}, \gamma)) \; > \; 0 \; = \; prob(\mathcal{AMCC}(z_{s_2}, \gamma))$$
– If there exists a resolution $\mathcal{Z}_2 \in Res_{\max}(s_2, o)$ such that $\mathcal{CC}(z_{s_2,o}, \alpha) \neq \emptyset$, from $\mathcal{CC}_{\max}(z_{s_2,o}, \alpha) = \emptyset$ it follows that the computations from $z_{s_2,o}$ labeled with α are all proper prefixes of maximal computations from $z_{s_2,o}$ itself. As a consequence, all the computations from o labeled with α are all proper prefixes of maximal computations from o itself. Therefore, none of the computations in $\mathcal{CC}_{\max}(z_{s_1,o}, \alpha)$ is successful. Indeed, each of them is interrupted because, after performing α, the first process under test is incapable of synchronizing with $\mathcal{T}$. Denoting by M the union of the sets of actions offered by each state of $\mathcal{T}$ reached after performing a computation labeled with α, in this case the after-must pair $\gamma = (\alpha, M)$ would violate $s_1 \sim_{\text{PAM}} s_2$. In fact, denoting by $\mathcal{Z}'_1$ the element of $Res(s_1)$ that originates $\mathcal{Z}_1$, we would have that for all $\mathcal{Z}'_2 \in Res(s_2)$ originating $\mathcal{Z}_2$:
$$prob(\mathcal{AMCC}(z_{s_1}, \gamma)) \; = \; 0 \; < \; prob(\mathcal{AMCC}(z_{s_2}, \gamma))$$

We have thus deduced that, whenever $s_1 \sim_{\text{PAM}} s_2$, then the existence of $\mathcal{Z}_1 \in Res_{\max,\alpha}(s_1, o)$ implies the existence of $\mathcal{Z}_2 \in Res_{\max,\alpha}(s_2, o)$.
Suppose now that for all $\mathcal{Z}_2 \in Res_{\max,\alpha}(s_2, o)$ it holds that:
$$prob(\mathcal{SCC}(z_{s_1,o}, \alpha)) \; \neq \; prob(\mathcal{SCC}(z_{s_2,o}, \alpha))$$
from which it follows that $\alpha \neq \varepsilon$. Assuming $\alpha = \alpha' a$, the after-must pair $\gamma = (\alpha', \{a\})$ would violate $s_1 \sim_{\text{PAM}} s_2$. In fact, observing that $\mathcal{T}$ must have a successful computation labeled with α because it cannot be $prob(\mathcal{SCC}(z_{s_1,o}, \alpha)) = 0 = prob(\mathcal{SCC}(z_{s_2,o}, \alpha))$, from $\mathcal{CC}_{\max}(z_{s_1,o}, \alpha) \neq \emptyset$ and $\mathcal{CC}_{\max}(z_{s_2,o}, \alpha) \neq \emptyset$ we derive $prob(\mathcal{SCC}(z_{s_1,o}, \alpha)) > 0$ and $prob(\mathcal{SCC}(z_{s_2,o}, \alpha)) > 0$. Denoting by $\mathcal{Z}'_1$ the element of $Res(s_1)$ that originates $\mathcal{Z}_1$, we would then have that for all $\mathcal{Z}'_2 \in Res(s_2)$ originating $\mathcal{Z}_2$:
$$\begin{aligned} prob(\mathcal{AMCC}(z_{s_1}, \gamma)) \; &= \; prob(\mathcal{SCC}(z_{s_1,o}, \alpha))/p \; \neq \\ &\neq \; prob(\mathcal{SCC}(z_{s_2,o}, \alpha))/p \; = \; prob(\mathcal{AMCC}(z_{s_2}, \gamma)) \end{aligned}$$
where p is the probability of performing a successful computation labeled with α in the element $\mathcal{Z}$ of $Res(o)$ that originates $\mathcal{Z}_1$.
In conclusion, whenever $s_1 \sim_{\text{PAM}} s_2$, then for all $\mathcal{Z}_1 \in Res_{\max,\alpha}(s_1, o)$ there exists $\mathcal{Z}_2 \in Res_{\max,\alpha}(s_2, o)$ such that:
$$prob(\mathcal{SCC}(z_{s_1,o}, \alpha)) \; = \; prob(\mathcal{SCC}(z_{s_2,o}, \alpha))$$
With a similar argument, we can prove that, whenever $s_1 \sim_{\text{PAM}} s_2$, then for all $\mathcal{Z}_2 \in Res_{\max,\alpha}(s_2, o)$ there exists $\mathcal{Z}_1 \in Res_{\max,\alpha}(s_1, o)$ such that:
$$prob(\mathcal{SCC}(z_{s_2,o}, \alpha)) \; = \; prob(\mathcal{SCC}(z_{s_1,o}, \alpha))$$
from which it follows that $s_1 \sim_{\text{PTe,tbt}} s_2$.
The inclusion of $\sim_{\text{PF,new}}$ in $\sim_{\text{PTe,tbt}}$ stems from the fact that $\sim_{\text{PF,new}}$ coincides with $\sim_{\text{PAM}}$, because for all $s \in S$, $\mathcal{Z} \in Res(s)$, $\alpha \in A^*$, and $F \in 2^A_{\text{fin}}$ it holds

that:

$$prob(\mathcal{FCC}(z_s, (\alpha, F))) \; = \; 1 - prob(\mathcal{AMCC}(z_s, (\alpha, F))) \qquad \blacksquare$$

Proof of Thm. 10. Let us preliminarily prove that $\sim_{\rm PTe,new}$ is included in $\sim_{\rm PAM}$ – with $\sim_{\rm PAM}$ being the probabilistic after-must equivalence introduced in the proof of Thm. 9 – from which the result will immediately follow because $\sim_{\rm PF,new}$ coincides with $\sim_{\rm PAM}$.
Given $s_1, s_2 \in S$ such that $s_1 \sim_{\rm PTe,new} s_2$, let us consider an arbitrary after-must pair $\gamma = (\alpha, M) \in A^* \times 2^A_{\rm fin}$. If $M = \emptyset$, then compatibility with γ means incapability of performing a computation labeled with α. In this case, $\sim_{\rm PAM}$ reduces to $\sim_{\rm PTr,new}$ and we already know that $\sim_{\rm PTe,new}$ is included in $\sim_{\rm PTr,new}$ (see Thm. 4).
If instead $M \neq \emptyset$, then we take a nondeterministic and probabilistic test $\mathcal{T}_\gamma = (O, A, \longrightarrow_{\mathcal{T}_\gamma})$ with initial state $o \in O$, which starts with a single computation labeled with α whose last state has $|M|$ outgoing transitions, each labeled with a distinct action in M and having ω as target state with probability 1. Due to the structure of $\mathcal{T}_\gamma$, for all $s \in S$ it holds that:

- For each $\mathcal{Z} \in Res(s)$ there exists $\mathcal{Z}' \in Res_{\max}(s, o)$ such that:
  $$prob(\mathcal{AMCC}(z_s, \gamma)) \; = \; prob(\mathcal{SC}(z_{s,o}))$$
- For each $\mathcal{Z} \in Res_{\max}(s, o)$ there exists $\mathcal{Z}' \in Res(s)$ such that:
  $$prob(\mathcal{SC}(z_{s,o})) \; = \; prob(\mathcal{AMCC}(z_s, \gamma))$$

As a consequence, from $s_1 \sim_{\rm PTe,new} s_2$ we derive that:

- For each $\mathcal{Z}_1 \in Res(s_1)$ there exist:
  - $\mathcal{Z}'_1 \in Res_{\max}(s_1, o)$ such that:
    $$prob(\mathcal{AMCC}(z_{s_1}, \gamma)) \; = \; prob(\mathcal{SC}(z_{s_1,o}))$$
  - $\mathcal{Z}'_2 \in Res_{\max}(s_2, o)$ such that:
    $$prob(\mathcal{SC}(z_{s_1,o})) \; = \; prob(\mathcal{SC}(z_{s_2,o}))$$
  - $\mathcal{Z}_2 \in Res(s_2)$ such that:
    $$prob(\mathcal{SC}(z_{s_2,o})) \; = \; prob(\mathcal{AMCC}(z_{s_2}, \gamma))$$
- For each $\mathcal{Z}_2 \in Res(s_2)$ there exist:
  - $\mathcal{Z}'_2 \in Res_{\max}(s_2, o)$ such that:
    $$prob(\mathcal{AMCC}(z_{s_2}, \gamma)) \; = \; prob(\mathcal{SC}(z_{s_2,o}))$$
  - $\mathcal{Z}'_1 \in Res_{\max}(s_1, o)$ such that:
    $$prob(\mathcal{SC}(z_{s_2,o})) \; = \; prob(\mathcal{SC}(z_{s_1,o}))$$
  - $\mathcal{Z}_1 \in Res(s_1)$ such that:
    $$prob(\mathcal{SC}(z_{s_1,o})) \; = \; prob(\mathcal{AMCC}(z_{s_1}, \gamma))$$

This means that $s_1 \sim_{\rm PAM} s_2$. $\blacksquare$

Proof of Thm. 11. The proof is divided into two parts:

– Suppose that $s_1 \sim_{\text{PTe,G}} s_2$. Given an arbitrary generative probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and an arbitrary trace $\alpha \in A^*$, consider a variant $\mathcal{T}_\alpha$ of $\mathcal{T}$ in which only the successful computations of $\mathcal{T}$ that are labeled with α reach ω. From $s_1 \sim_{\text{PTe,G}} s_2$, we derive that:

$$\begin{aligned} prob(\mathcal{SCC}^{\mathcal{T}}(s_1, o, \alpha)) &= prob(\mathcal{SCC}^{\mathcal{T}_\alpha}(s_1, o, \alpha)) = \\ &= prob(\mathcal{SC}^{\mathcal{T}_\alpha}(s_1, o)) = \\ &= prob(\mathcal{SC}^{\mathcal{T}_\alpha}(s_2, o)) = \\ &= prob(\mathcal{SCC}^{\mathcal{T}_\alpha}(s_2, o, \alpha)) = prob(\mathcal{SCC}^{\mathcal{T}}(s_2, o, \alpha)) \end{aligned}$$

As a consequence, $s_1 \sim_{\text{PTe,tbt}} s_2$.

– If $s_1 \sim_{\text{PTe,tbt}} s_2$, then for all generative probabilistic tests $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:

$$\begin{aligned} prob(\mathcal{SC}(s_1, o)) &= \sum_{\alpha \in A^*} prob(\mathcal{SCC}(s_1, o, \alpha)) = \\ &= \sum_{\alpha \in A^*} prob(\mathcal{SCC}(s_2, o, \alpha)) = prob(\mathcal{SC}(s_2, o)) \end{aligned}$$

which means that $s_1 \sim_{\text{PTe,G}} s_2$. ■

Proof of Thm. 12. The proof is divided into two parts:

– Suppose that $s_1 \sim_{\text{PTe,R}} s_2$. Given an arbitrary reactive probabilistic test $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and an arbitrary trace $\alpha \in A^*$, consider a variant $\mathcal{T}_\alpha$ of $\mathcal{T}$ in which only the successful computations of $\mathcal{T}$ that are labeled with α reach ω. From $s_1 \sim_{\text{PTe,R}} s_2$, we derive that:

$$\begin{aligned} prob(\mathcal{SCC}^{\mathcal{T}}(s_1, o, \alpha)) &= prob(\mathcal{SCC}^{\mathcal{T}_\alpha}(s_1, o, \alpha)) \\ &= \bigsqcup_{\alpha' \in Tr_{\max}(s_1, o)} prob(\mathcal{SCC}^{\mathcal{T}_\alpha}(s_1, o, \alpha')) \\ &= \bigsqcup_{\alpha' \in Tr_{\max}(s_2, o)} prob(\mathcal{SCC}^{\mathcal{T}_\alpha}(s_2, o, \alpha')) \\ &= prob(\mathcal{SCC}^{\mathcal{T}_\alpha}(s_2, o, \alpha)) \\ &= prob(\mathcal{SCC}^{\mathcal{T}}(s_2, o, \alpha)) \end{aligned}$$

As a consequence, $s_1 \sim_{\text{PTe,tbt}} s_2$.

– If $s_1 \sim_{\text{PTe,tbt}} s_2$, then for all reactive probabilistic tests $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:

$$\bigsqcup_{\alpha \in Tr_{\max}(s_1, o)} prob(\mathcal{SCC}(s_1, o, \alpha)) = \bigsqcup_{\alpha \in Tr_{\max}(s_2, o)} prob(\mathcal{SCC}(s_2, o, \alpha))$$

$$\bigsqcap_{\alpha \in Tr_{\max}(s_1, o)} prob(\mathcal{SCC}(s_1, o, \alpha)) = \bigsqcap_{\alpha \in Tr_{\max}(s_2, o)} prob(\mathcal{SCC}(s_2, o, \alpha))$$

which means that $s_1 \sim_{\text{PTe,R}} s_2$. ■