On the Weak Continuation of Reverse Bisimilarity vs. Forward Bisimilarity

Marco Bernardo[1,*], Andrea Esposito[1]

[1]Dipartimento di Scienze Pure e Applicate, Università di Urbino, Urbino, Italy

Abstract

We introduce a process calculus for nondeterministic systems that are reversible, i.e., capable of undoing their actions starting from the last performed one. The considered systems are sequential so as to be neutral with respect to interleaving semantics vs. truly concurrent semantics of parallel composition. As a natural continuation of previous work on strong bisimilarity in this reversible setting, we investigate compositionality properties and equational characterizations of weak variants of forward-reverse bisimilarity as well as of its two components, i.e., weak forward bisimilarity and weak reverse bisimilarity.

1. Introduction

Reversibility in computing started to gain attention since the seminal works of Landauer [1] and Bennett [2], where it was shown that reversible computations may achieve lower levels of heat dissipation. Nowadays *reversible computing* has many applications ranging from computational biochemistry and parallel discrete-event simulation to robotics, control theory, fault tolerant systems, and concurrent program debugging.

In a reversible system, two directions of computation can be observed: a *forward* one, coinciding with the normal way of computing, and a *backward* one, along which the effects of the forward one can be undone when needed in a *causally consistent* way, i.e., by returning to a past consistent state. The latter task is not easy to accomplish in a concurrent system, because the undo procedure necessarily starts from the last performed action and this may not be uniquely identifiable. The usually adopted strategy is that an action can be undone provided that all of its consequences, if any, have been undone beforehand [3].

In the process algebra literature, two approaches have been developed to reverse computations based on keeping track of past actions: the dynamic one of [3] and the static one of [4], later shown to be equivalent in terms of labeled transition systems isomorphism [5].

The former approach yields RCCS, a variant of CCS [6] that uses stack-based memories attached to processes so as to record all the actions executed by the processes themselves. A single transition relation is defined, while actions are divided into forward and backward resulting in forward and backward transitions. This approach is suitable when the operational

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*Corresponding author.
marco.bernardo@uniurb.it (M. Bernardo); a.esposito30@campus.uniurb.it (A. Esposito)
0000-0003-0267-6170 (M. Bernardo); 0009-0009-2259-902X (A. Esposito)

semantics is given in terms of reduction semantics, like in the case of very expressive calculi as well as programming languages.

In contrast, the latter approach proposes a general method, of which CCSK is a result, to reverse calculi, relying on the idea of retaining within the process syntax all executed actions, which are suitably decorated, and all dynamic operators, which are thus made static. A forward transition relation and a backward transition relation are separately defined, which are labeled with actions extended with communication keys so as to remember who synchronized with whom when going backward. This approach is very handy when it comes to deal with labeled transition systems and basic process calculi.

In [4] *forward-reverse bisimilarity* was introduced too. Unlike standard bisimilarity [7, 6], it is truly concurrent as it does not satisfy the expansion law of parallel composition into a choice among all possible action sequencings. The interleaving view can be restored in a reversible setting by employing *back-and-forth bisimilarity* [8]. This is defined on computation paths instead of states, thus preserving not only causality but also history as backward moves are constrained to take place along the path followed when going forward even in the presence of concurrency. In the latter setting, a single transition relation is considered, which is viewed as bidirectional, and in the bisimulation game the distinction between going forward or backward is made by matching outgoing or incoming transitions of the considered processes, respectively.

In [9] forward-reverse bisimilarity and its two components, i.e., forward bisimilarity and reverse bisimilarity, have been investigated in terms of compositionality properties and equational characterizations, both for nondeterministic processes and for Markovian processes. In order to remain neutral with respect to interleaving view vs. true concurrency, the study has been conducted over a sequential processes calculus, in which parallel composition is not admitted so that not even the communication keys of [4] are needed. Furthermore, a single transition relation viewed as bidirectional and the distinction between outgoing and incoming transitions in the bisimulation game have been adopted like in [8].

In this paper we extend the work done in [9] to *weak* variants of forward-reverse, forward, and reverse bisimilarities over nondeterministic reversible sequential processes, where by weak we mean that the considered equivalences abstract from unobservable actions, traditionally denoted by τ. As far as compositionality is concerned, compared to [9] we discover that an initiality condition is necessary not only for forward bisimilarity but also for forward-reverse bisimilarity, which additionally solves the congruence problem with respect to nondeterministic choice affecting all weak variants of bisimilarity [6, 10]. As for equational characterizations, we retrieve the τ-laws of weak bisimilarity [6] and branching bisimilarity [10] over standard forward-only processes in the case of forward bisimilarity and forward-reverse bisimilarity respectively, along with some variants of those laws in the case of reverse bisimilarity. Together with the results in [8, 11], this emphasizes once more the connection between forward-reverse bisimilarity and branching bisimilarity.

The paper is organized as follows. In Section 2 we recall syntax and semantics for the calculus of nondeterministic reversible sequential processes as well as the forward, reverse, and forward-reverse bisimilarities introduced in [9]. In Section 3 we define the weak variants of the three aforementioned bisimilarities. In Section 4 we study their compositionality properties. Finally, in Section 5 we provide sound and ground-complete equational characterizations for the three weak bisimilarities.

2. Background

2.1. Syntax of Nondeterministic Reversible Sequential Processes

Given a countable set A of actions – ranged over by a, b, c – including an unobservable action denoted by τ, the syntax of reversible sequential processes is as follows [9]:

$$P \ ::= \ \underline{0} \mid a \, . \, P \mid a^{\dagger} . \, P \mid P + P$$

where:

- $\underline{0}$ is the terminated process.
- $a \, . \, P$ is a process that can execute action a and whose continuation is P.
- $a^{\dagger} \, . \, P$ is a process that executed action a and whose continuation is in P.
- $P_1 + P_2$ expresses a nondeterministic choice between P_1 and P_2 as far as both of them have not executed any action yet.

We syntactically characterize through suitable predicates three classes of processes generated by the grammar above. Firstly, we have *initial* processes, i.e., processes in which all the actions are unexecuted:

$$\begin{array}{rcl} \mathit{initial}(\underline{0}) & & \\ \mathit{initial}(a \, . \, P) & \Longleftarrow & \mathit{initial}(P) \\ \mathit{initial}(P_1 + P_2) & \Longleftarrow & \mathit{initial}(P_1) \wedge \mathit{initial}(P_2) \end{array}$$

Secondly, we have *final* processes, i.e., processes in which all the actions along a single path have been executed:

$$\begin{array}{rcl} \mathit{final}(\underline{0}) & & \\ \mathit{final}(a^{\dagger} . \, P) & \Longleftarrow & \mathit{final}(P) \\ \mathit{final}(P_1 + P_2) & \Longleftarrow & (\mathit{final}(P_1) \wedge \mathit{initial}(P_2)) \vee (\mathit{initial}(P_1) \wedge \mathit{final}(P_2)) \end{array}$$

Multiple paths arise only in the presence of alternative compositions, i.e., nondeterministic choices. At each occurrence of $+$, only the subprocess chosen for execution can move, while the other one, although not selected, is kept as an initial subprocess within the overall process to support reversibility.

Thirdly, we have the processes *reachable* from an initial one, whose set we denote by $\mathbb{P}$:

$$\begin{array}{rcl} \mathit{reachable}(\underline{0}) & & \\ \mathit{reachable}(a \, . \, P) & \Longleftarrow & \mathit{initial}(P) \\ \mathit{reachable}(a^{\dagger} . \, P) & \Longleftarrow & \mathit{reachable}(P) \\ \mathit{reachable}(P_1 + P_2) & \Longleftarrow & (\mathit{reachable}(P_1) \wedge \mathit{initial}(P_2)) \vee (\mathit{initial}(P_1) \wedge \mathit{reachable}(P_2)) \end{array}$$

It is worth noting that:

- $\underline{0}$ is the only process that is both initial and final as well as reachable.
- Every initial or final process is reachable too.
- $\mathbb{P}$ also contains processes that are neither initial nor final, like e.g. $a^{\dagger} . \, b \, . \, \underline{0}$.
- The relative positions of already executed actions and actions to be executed matter; in particular, an action of the former kind can never follow one of the latter kind. For instance, $a^{\dagger} . \, b \, . \, \underline{0} \in \mathbb{P}$ whereas $b \, . \, a^{\dagger} . \, \underline{0} \notin \mathbb{P}$.

$(\textsc{Act}_{\rm f})\ \dfrac{initial(P)}{a\,.\,P \xrightarrow{a} a^{\dagger}\,.\,P}$	$(\textsc{Act}_{\rm p})\ \dfrac{P \xrightarrow{b} P'}{a^{\dagger}\,.\,P \xrightarrow{b} a^{\dagger}\,.\,P'}$
$(\textsc{Cho}_{\rm l})\ \dfrac{P_1 \xrightarrow{a} P_1' \quad initial(P_2)}{P_1 + P_2 \xrightarrow{a} P_1' + P_2}$	$(\textsc{Cho}_{\rm r})\ \dfrac{P_2 \xrightarrow{a} P_2' \quad initial(P_1)}{P_1 + P_2 \xrightarrow{a} P_1 + P_2'}$

Table 1
Operational semantic rules for reversible action prefix and nondeterministic choice

2.2. Operational Semantic Rules

According to the approach of [4], dynamic operators such as action prefix and alternative composition have to be made static by the semantics, so as to retain within the syntax all the information needed to enable reversibility. For the sake of minimality, unlike [4] we do not generate two distinct transition relations – a forward one $\longrightarrow$ and a backward one $\rightsquigarrow$ – but a single transition relation, which we implicitly regard as being symmetric like in [8] to enforce the *loop property*: each executed action can be undone and each undone action can be redone.

In our setting, a backward transition from P' to P ($P' \overset{a}{\rightsquigarrow} P$) is subsumed by the corresponding forward transition t from P to P' ($P \xrightarrow{a} P'$). As will become clear with the definition of bisimulation equivalences, like in [8] when going forward we view t as an *outgoing* transition of P, while when going backward we view t as an *incoming* transition of P'. The semantic rules for $\longrightarrow \subseteq \mathbb{P} \times A \times \mathbb{P}$ are defined in Table 1 and generate the labeled transition system $(\mathbb{P}, A, \longrightarrow)$ [9].

The first rule for action prefix ($\textsc{Act}_{\rm f}$ where f stands for forward) applies only if P is initial and retains the executed action in the target process of the generated forward transition by decorating the action itself with †. The second rule for action prefix ($\textsc{Act}_{\rm p}$ where p stands for propagation) propagates actions executed by inner initial subprocesses.

In both rules for alternative composition ($\textsc{Cho}_{\rm l}$ and $\textsc{Cho}_{\rm r}$ where l stands for left and r stands for right), the subprocess that has not been selected for execution is retained as an initial subprocess in the target process of the generated transition. When both subprocesses are initial, both rules for alternative composition are applicable, otherwise only one of them can be applied and in that case it is the non-initial subprocess that can move, because the other one has been discarded at the moment of the selection.

Every state corresponding to a non-final process has at least one outgoing transition, while every state corresponding to a non-initial process has exactly one incoming transition due to the decoration of executed actions. The labeled transition system underlying an initial process turns out to be a tree, whose branching points correspond to occurrences of $+$.

Example 2.1. The labeled transition systems generated by the rules in Table 1 for the two initial processes $a\,.\,\underline{0}$ and $a\,.\,\underline{0} + a\,.\,\underline{0}$ are depicted in Figure 1. As for the one on the right, we observe that, in the case of a standard process calculus, a single a-transition from $a\,.\,\underline{0} + a\,.\,\underline{0}$ to $\underline{0}$ would have been generated due to the absence of action decorations within processes. ■

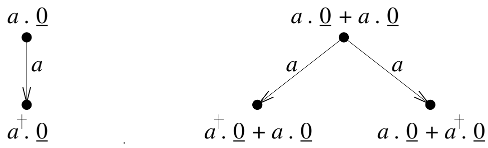

Figure 1: Labeled transition systems underlying $a\,.\,\underline{0}$ and $a\,.\,\underline{0} + a\,.\,\underline{0}$

2.3. Strong Forward, Reverse, and Forward-Reverse Bisimilarities

While forward bisimilarity considers only *outgoing* transitions [7, 6], reverse bisimilarity considers only *incoming* transitions. Forward-reverse bisimilarity [4] considers instead both outgoing transitions and incoming ones. Here are their *strong* versions studied in [9], where strong means not abstracting from τ-actions.

Definition 2.2. We say that $P_1, P_2 \in \mathbb{P}$ are *forward bisimilar*, written $P_1 \sim_{\mathrm{FB}} P_2$, iff $(P_1, P_2) \in \mathcal{B}$ for some forward bisimulation $\mathcal{B}$. A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a *forward bisimulation* iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:

- Whenever $P_1 \xrightarrow{a} P_1'$, then $P_2 \xrightarrow{a} P_2'$ with $(P_1', P_2') \in \mathcal{B}$. ■

Definition 2.3. We say that $P_1, P_2 \in \mathbb{P}$ are *reverse bisimilar*, written $P_1 \sim_{\mathrm{RB}} P_2$, iff $(P_1, P_2) \in \mathcal{B}$ for some reverse bisimulation $\mathcal{B}$. A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a *reverse bisimulation* iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:

- Whenever $P_1' \xrightarrow{a} P_1$, then $P_2' \xrightarrow{a} P_2$ with $(P_1', P_2') \in \mathcal{B}$. ■

Definition 2.4. We say that $P_1, P_2 \in \mathbb{P}$ are *forward-reverse bisimilar*, written $P_1 \sim_{\mathrm{FRB}} P_2$, iff $(P_1, P_2) \in \mathcal{B}$ for some forward-reverse bisimulation $\mathcal{B}$. A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a *forward-reverse bisimulation* iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:

- Whenever $P_1 \xrightarrow{a} P_1'$, then $P_2 \xrightarrow{a} P_2'$ with $(P_1', P_2') \in \mathcal{B}$.
- Whenever $P_1' \xrightarrow{a} P_1$, then $P_2' \xrightarrow{a} P_2$ with $(P_1', P_2') \in \mathcal{B}$. ■

$\sim_{\mathrm{FRB}} \subsetneq \sim_{\mathrm{FB}} \cap \sim_{\mathrm{RB}}$ with the inclusion being strict because, e.g., the two final processes $a^{\dagger}\,.\,\underline{0}$ and $a^{\dagger}\,.\,\underline{0} + c\,.\,\underline{0}$ are identified by $\sim_{\mathrm{FB}}$ (no outgoing transitions on both sides) and by $\sim_{\mathrm{RB}}$ (only an incoming a-transition on both sides), but distinguished by $\sim_{\mathrm{FRB}}$ as in the latter process action c is enabled again after undoing a (and hence there is an outgoing c-transition in addition to an outgoing a-transition). Moreover, $\sim_{\mathrm{FB}}$ and $\sim_{\mathrm{RB}}$ are incomparable because for instance:

$$a^{\dagger}\,.\,\underline{0} \sim_{\mathrm{FB}} \underline{0} \quad \text{but} \quad a^{\dagger}\,.\,\underline{0} \not\sim_{\mathrm{RB}} \underline{0}$$
$$a\,.\,\underline{0} \sim_{\mathrm{RB}} \underline{0} \quad \text{but} \quad a\,.\,\underline{0} \not\sim_{\mathrm{FB}} \underline{0}$$

Note that that $\sim_{\mathrm{FRB}} = \sim_{\mathrm{FB}}$ over initial processes, with $\sim_{\mathrm{RB}}$ strictly coarser, whilst $\sim_{\mathrm{FRB}} \neq \sim_{\mathrm{RB}}$ over final processes because, after going backward, previously discarded subprocesses come into play again in the forward direction.

Example 2.5. The two processes considered in Example 2.1 are identified by all the three equivalences. This is witnessed by any bisimulation that contains the pairs $(a\,.\,\underline{0}, a\,.\,\underline{0} + a\,.\,\underline{0})$, $(a^{\dagger}\,.\,\underline{0}, a^{\dagger}\,.\,\underline{0} + a\,.\,\underline{0})$, and $(a^{\dagger}\,.\,\underline{0}, a\,.\,\underline{0} + a^{\dagger}\,.\,\underline{0})$. ■

As observed in [9], it makes sense that $\sim_{\rm FB}$ identifies processes with a different past and that $\sim_{\rm RB}$ identifies processes with a different future, in particular with $\underline{0}$ that has neither past nor future. However, for $\sim_{\rm FB}$ this results in a compositionality violation with respect to alternative composition. As an example:

$$\begin{array}{rcl} a^{\dagger} . \, b \, . \, \underline{0} & \sim_{\rm FB} & b \, . \, \underline{0} \\ a^{\dagger} . \, b \, . \, \underline{0} + c \, . \, \underline{0} & \not\sim_{\rm FB} & b \, . \, \underline{0} + c \, . \, \underline{0} \end{array}$$

because in $a^{\dagger} . \, b \, . \, \underline{0} + c \, . \, \underline{0}$ action c is disabled due to the presence of the already executed action $a^{\dagger}$, while in $b \, . \, \underline{0} + c \, . \, \underline{0}$ action c is enabled as there are no past actions preventing it from occurring. Note that a similar phenomenon does not happen with $\sim_{\rm RB}$ as $a^{\dagger} . \, b \, . \, \underline{0} \not\sim_{\rm RB} b \, . \, \underline{0}$ due to the incoming a-transition of $a^{\dagger} . \, b \, . \, \underline{0}$.

This problem, which does not show up for $\sim_{\rm RB}$ and $\sim_{\rm FRB}$ because these two equivalences cannot identify an initial process with a non-initial one, leads to the following variant of $\sim_{\rm FB}$ that is sensitive to the presence of the past.

Definition 2.6. We say that $P_1, P_2 \in \mathbb{P}$ are *past-sensitive forward bisimilar*, written $P_1 \sim_{\rm FB:ps} P_2$, iff $(P_1, P_2) \in \mathcal{B}$ for some past-sensitive forward bisimulation $\mathcal{B}$. A relation $\mathcal{B}$ over $\mathbb{P}$ is a *past-sensitive forward bisimulation* iff it is a forward bisimulation such that *initial*$(P_1) \iff$ *initial*(P_2) for all $(P_1, P_2) \in \mathcal{B}$. ■

Now $\sim_{\rm FB:ps}$ is sensitive to the presence of the past:

$$a^{\dagger} . \, b \, . \, \underline{0} \not\sim_{\rm FB:ps} b \, . \, \underline{0}$$

but can still identify non-initial processes having a different past:

$$a_1^{\dagger} \, . \, P \sim_{\rm FB:ps} a_2^{\dagger} \, . \, P$$

It holds that $\sim_{\rm FRB} \subsetneq \sim_{\rm FB:ps} \cap \sim_{\rm RB}$, with $\sim_{\rm FRB} = \sim_{\rm FB:ps}$ over initial processes as well as $\sim_{\rm FB:ps}$ and $\sim_{\rm RB}$ being incomparable because, e.g., for $a_1 \neq a_2$:

$$\begin{array}{lcl} a_1^{\dagger} \, . \, P \sim_{\rm FB:ps} a_2^{\dagger} \, . \, P & \text{but} & a_1^{\dagger} \, . \, P \not\sim_{\rm RB} a_2^{\dagger} \, . \, P \\ a_1 \, . \, P \sim_{\rm RB} a_2 \, . \, P & \text{but} & a_1 \, . \, P \not\sim_{\rm FB:ps} a_2 \, . \, P \end{array}$$

In [9] it has been shown that all the considered bisimilarities are congruences with respect to action prefix, while only $\sim_{\rm FB:ps}$, $\sim_{\rm RB}$, and $\sim_{\rm FRB}$ are congruences with respect to alternative composition too, with $\sim_{\rm FB:ps}$ being the coarsest congruence with respect to $+$ contained in $\sim_{\rm FB}$. Sound and ground-complete equational characterizations have also been provided for the three equivalences that are congruences with respect to both operators.

3. Weak Bisimilarity and Reversibility

In this section we introduce *weak* variants of forward, reverse, and forward-reverse bisimilarities, i.e., variants capable of abstracting from τ-actions.

In the following definitions, $P \stackrel{\tau^*}{\Longrightarrow} P'$ means that $P' = P$ or there exists a nonempty sequence of finitely many τ-transitions such that the target of each of them coincides with the source of the subsequent one, with the source of the first one being P and the target of the last one being P'. Moreover, $\stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow}$ stands for an a-transition possibly preceded and followed by finitely many τ-transitions. We further let $\bar{A} = A \setminus \{\tau\}$.

Definition 3.1. We say that $P_1, P_2 \in \mathbb{P}$ are *weakly forward bisimilar*, written $P_1 \approx_{\mathrm{FB}} P_2$, iff $(P_1, P_2) \in \mathcal{B}$ for some weak forward bisimulation $\mathcal{B}$. A symmetric binary relation $\mathcal{B}$ over $\mathbb{P}$ is a *weak forward bisimulation* iff for all $(P_1, P_2) \in \mathcal{B}$:

- Whenever $P_1 \xrightarrow{\tau} P'_1$, then $P_2 \xRightarrow{\tau^*} P'_2$ and $(P'_1, P'_2) \in \mathcal{B}$.
- Whenever $P_1 \xrightarrow{a} P'_1$ for $a \in \bar{A}$, then $P_2 \xRightarrow{\tau^*} \xrightarrow{a} \xRightarrow{\tau^*} P'_2$ and $(P'_1, P'_2) \in \mathcal{B}$. ■

Definition 3.2. We say that $P_1, P_2 \in \mathbb{P}$ are *weakly reverse bisimilar*, written $P_1 \approx_{\mathrm{RB}} P_2$, iff $(P_1, P_2) \in \mathcal{B}$ for some weak reverse bisimulation $\mathcal{B}$. A symmetric binary relation $\mathcal{B}$ over $\mathbb{P}$ is a *weak reverse bisimulation* iff for all $(P_1, P_2) \in \mathcal{B}$:

- Whenever $P'_1 \xrightarrow{\tau} P_1$, then $P'_2 \xRightarrow{\tau^*} P_2$ and $(P'_1, P'_2) \in \mathcal{B}$.
- Whenever $P'_1 \xrightarrow{a} P_1$ for $a \in \bar{A}$, then $P'_2 \xRightarrow{\tau^*} \xrightarrow{a} \xRightarrow{\tau^*} P_2$ and $(P'_1, P'_2) \in \mathcal{B}$. ■

Definition 3.3. We say that $P_1, P_2 \in \mathbb{P}$ are *weakly forward-reverse bisimilar*, written $P_1 \approx_{\mathrm{FRB}} P_2$, iff $(P_1, P_2) \in \mathcal{B}$ for some weak forward-reverse bisimulation $\mathcal{B}$. A symmetric binary relation $\mathcal{B}$ over $\mathbb{P}$ is a *weak forward-reverse bisimulation* iff for all $(P_1, P_2) \in \mathcal{B}$:

- Whenever $P_1 \xrightarrow{\tau} P'_1$, then $P_2 \xRightarrow{\tau^*} P'_2$ and $(P'_1, P'_2) \in \mathcal{B}$.
- Whenever $P_1 \xrightarrow{a} P'_1$ for $a \in \bar{A}$, then $P_2 \xRightarrow{\tau^*} \xrightarrow{a} \xRightarrow{\tau^*} P'_2$ and $(P'_1, P'_2) \in \mathcal{B}$.
- Whenever $P'_1 \xrightarrow{\tau} P_1$, then $P'_2 \xRightarrow{\tau^*} P_2$ and $(P'_1, P'_2) \in \mathcal{B}$.
- Whenever $P'_1 \xrightarrow{a} P_1$ for $a \in \bar{A}$, then $P'_2 \xRightarrow{\tau^*} \xrightarrow{a} \xRightarrow{\tau^*} P_2$ and $(P'_1, P'_2) \in \mathcal{B}$. ■

Each of the three weak bisimilarities is strictly coarser than the corresponding strong one. Similar to the strong case, $\approx_{\mathrm{FRB}} \subsetneq \approx_{\mathrm{FB}} \cap \approx_{\mathrm{RB}}$ with $\approx_{\mathrm{FB}}$ and $\approx_{\mathrm{RB}}$ being incomparable. Unlike the strong case, $\approx_{\mathrm{FRB}} \neq \approx_{\mathrm{FB}}$ over initial processes. For instance, $\tau \,.\, a \,.\, \underline{0} + a \,.\, \underline{0} + b \,.\, \underline{0}$ and $\tau \,.\, a \,.\, \underline{0} + b \,.\, \underline{0}$ are identified by $\approx_{\mathrm{FB}}$ but told apart by $\approx_{\mathrm{FRB}}$: if the former performs a, the latter responds with τ followed by a and if it subsequently undoes a thus becoming $\tau^\dagger \,.\, a \,.\, \underline{0} + b \,.\, \underline{0}$ in which only a is enabled, the latter can only respond by undoing a thus becoming $\tau \,.\, a \,.\, \underline{0} + a \,.\, \underline{0} + b \,.\, \underline{0}$ in which both a and b are enabled. An analogous counterexample with non-initial τ-actions is given by $c \,.\, (\tau \,.\, a \,.\, \underline{0} + a \,.\, \underline{0} + b \,.\, \underline{0})$ and $c \,.\, (\tau \,.\, a \,.\, \underline{0} + b \,.\, \underline{0})$.

4. Congruence Properties

In this section we investigate the compositionality of the three weak bisimilarities with respect to the considered process operators. Firstly, we observe that $\approx_{\mathrm{FB}}$ suffers from the same problem with respect to alternative composition as $\sim_{\mathrm{FB}}$. Secondly, $\approx_{\mathrm{FB}}$ and $\approx_{\mathrm{FRB}}$ feature the same problem as weak bisimilarity for standard forward-only processes [6], i.e., for $\approx \in \{\approx_{\mathrm{FB}}, \approx_{\mathrm{FRB}}\}$ it holds that:

$$\begin{array}{rcl} \tau \,.\, a \,.\, \underline{0} & \approx & a \,.\, \underline{0} \\ \tau \,.\, a \,.\, \underline{0} + b \,.\, \underline{0} & \not\approx & a \,.\, \underline{0} + b \,.\, \underline{0} \end{array}$$

because if $\tau \,.\, a \,.\, \underline{0} + b \,.\, \underline{0}$ performs τ thereby evolving to $\tau^{\dagger} .\, a \,.\, \underline{0} + b \,.\, \underline{0}$ where only a is enabled in the forward direction, then $a \,.\, \underline{0} + b \,.\, \underline{0}$ can neither move nor idle in the attempt to evolve in such a way to match $\tau^{\dagger} .\, a \,.\, \underline{0} + b \,.\, \underline{0}$.

To solve both problems it is sufficient to redefine the two equivalences by making them sensitive to the presence of the past, exactly as in the strong case for forward bisimilarity. By so doing, $\tau \,.\, a \,.\, \underline{0}$ is no longer identified with $a \,.\, \underline{0}$: if the former performs τ thereby evolving to $\tau^{\dagger} .\, a \,.\, \underline{0}$ and the latter idles, then $\tau^{\dagger} .\, a \,.\, \underline{0}$ and $a \,.\, \underline{0}$ are told apart because they are not both initial or non-initial.

Definition 4.1. We say that $P_1, P_2 \in \mathbb{P}$ are *weakly past-sensitive forward bisimilar*, written $P_1 \approx_{\mathrm{FB:ps}} P_2$, iff $(P_1, P_2) \in \mathcal{B}$ for some weak past-sensitive forward bisimulation $\mathcal{B}$. A binary relation $\mathcal{B}$ over $\mathbb{P}$ is a *weak past-sensitive forward bisimulation* iff it is a weak forward bisimulation such that *initial*$(P_1) \Longleftrightarrow$ *initial*(P_2) for all $(P_1, P_2) \in \mathcal{B}$. ■

Definition 4.2. We say that $P_1, P_2 \in \mathbb{P}$ are *weakly past-sensitive forward-reverse bisimilar*, written $P_1 \approx_{\mathrm{FRB:ps}} P_2$, iff $(P_1, P_2) \in \mathcal{B}$ for some weak past-sensitive forward-reverse bisimulation $\mathcal{B}$. A binary relation $\mathcal{B}$ over $\mathbb{P}$ is a *weak past-sensitive forward-reverse bisimulation* iff it is a weak forward-reverse bisimulation such that *initial*$(P_1) \Longleftrightarrow$ *initial*(P_2) for all $(P_1, P_2) \in \mathcal{B}$. ■

Observing that $\sim_{\mathrm{FRB}} \subsetneq \approx_{\mathrm{FRB:ps}}$ as the former naturally satisfies the initiality condition, we show the following congruence results. When present, side conditions on subprocesses just ensure that the overall processes are reachable.

Theorem 4.3. Let $\approx \in \{\approx_{\mathrm{FB}}, \approx_{\mathrm{FB:ps}}, \approx_{\mathrm{RB}}, \approx_{\mathrm{FRB}}, \approx_{\mathrm{FRB:ps}}\}$, $\approx' \in \{\approx_{\mathrm{FB:ps}}, \approx_{\mathrm{RB}}, \approx_{\mathrm{FRB:ps}}\}$, and $P_1, P_2 \in \mathbb{P}$:

- If $P_1 \approx P_2$ then for all $a \in A$:
  - $a \,.\, P_1 \approx a \,.\, P_2$ provided that *initial*$(P_1) \wedge$ *initial*(P_2).
  - $a^{\dagger} .\, P_1 \approx a^{\dagger} .\, P_2$.
- If $P_1 \approx' P_2$ then for all $P \in \mathbb{P}$:
  - $P_1 + P \approx' P_2 + P$ and $P + P_1 \approx' P + P_2$ provided that *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.
- $\approx_{\mathrm{FB:ps}}$ is the coarsest congruence with respect to $+$ contained in $\approx_{\mathrm{FB}}$.
- $\approx_{\mathrm{FRB:ps}}$ is the coarsest congruence with respect to $+$ contained in $\approx_{\mathrm{FRB}}$. ■

Like in the non-past-sensitive case, $\approx_{\mathrm{FRB:ps}} \neq \approx_{\mathrm{FB:ps}}$ over initial processes, as shown by $\tau \,.\, a \,.\, \underline{0} + a \,.\, \underline{0}$ and $\tau \,.\, a \,.\, \underline{0}$: if the former performs a, the latter responds with τ followed by a and if it subsequently undoes a thus becoming the non-initial process $\tau^{\dagger} .\, a \,.\, \underline{0}$, the latter can only respond by undoing a thus becoming the initial process $\tau \,.\, a \,.\, \underline{0} + a \,.\, \underline{0}$. An analogous counterexample with non-initial τ-actions is given again by $c \,.\, (\tau \,.\, a \,.\, \underline{0} + a \,.\, \underline{0} + b \,.\, \underline{0})$ and $c \,.\, (\tau \,.\, a \,.\, \underline{0} + b \,.\, \underline{0})$.

It is worth noting that the aforementioned compositionality problems with respect to alternative composition may not be solved, in this reversible setting, by employing the construction of [6] for building a weak bisimulation congruence. If we introduced a variant $\approx'_{\rm FB}$ of $\approx_{\rm FB}$ such that, when considering two initial processes, a τ-transition on either side must be matched by a τ-transition on the other side – possibly preceded and followed by finitely many τ-transitions – with the two reached processes being related by $\approx_{\rm FB}$, then again $a^\dagger . \, b \, . \, \underline{0} \ \approx'_{\rm FB} \ b \, . \, \underline{0}$ but $a^\dagger . \, b \, . \, \underline{0} + c \, . \, \underline{0} \ \not\approx'_{\rm FB} \ b \, . \, \underline{0} + c \, . \, \underline{0}$ as explained in Section 2.3.

5. Equational Characterizations

In this section we investigate the equational characterizations of $\approx_{\rm FB:ps}$, $\approx_{\rm RB}$, and $\approx_{\rm FRB:ps}$ so as to highlight the fundamental laws of these behavioral equivalences. In the following, by deduction system we mean a set comprising the following axioms and inference rules over $\mathbb{P}$ – possibly enriched by a set $\mathcal{A}$ of additional axioms – corresponding to the fact that $\approx_{\rm FB:ps}$, $\approx_{\rm RB}$, and $\approx_{\rm FRB:ps}$ are equivalence relations as well as congruences with respect to action prefix and alternative composition as established by Theorem 4.3:

- Reflexivity, symmetry, transitivity: $P = P$, $\dfrac{P_1 = P_2}{P_2 = P_1}$, $\dfrac{P_1 = P_2 \quad P_2 = P_3}{P_1 = P_3}$.
- .-Substitutivity: $\dfrac{P_1 = P_2 \quad \mathit{initial}(P_1) \land \mathit{initial}(P_2)}{a \, . \, P_1 = a \, . \, P_2}$, $\dfrac{P_1 = P_2}{a^\dagger . \, P_1 = a^\dagger . \, P_2}$.
- $+$-Substitutivity: $\dfrac{P_1 = P_2 \quad \mathit{initial}(P) \lor (\mathit{initial}(P_1) \land \mathit{initial}(P_2))}{P_1 + P = P_2 + P \quad P + P_1 = P + P_2}$.

It is known from [9] that, for the three strong bisimilarities, alternative composition turns out to be associative and commutative and to admit $\underline{0}$ as neutral element, like in the case of bisimilarity over standard forward-only processes [12]. The same holds true for $\approx_{\rm FB:ps}$, $\approx_{\rm RB}$, and $\approx_{\rm FRB:ps}$ as they are strictly coarser than their strong counterparts. This is formalized by axioms $\mathcal{A}_1$ to $\mathcal{A}_3$ in Table 2.

Then, we have axioms specific to $\sim_{\rm FB:ps}$ [9], which are thus valid for $\approx_{\rm FB:ps}$ too. Axioms $\mathcal{A}_4$ and $\mathcal{A}_5$ together establish that the past can be neglected when moving only forward, but the presence of the past cannot be ignored. Axiom $\mathcal{A}_6$ states that a previously non-selected alternative can be discarded after starting moving only forward.

Likewise, we have axioms specific to $\sim_{\rm RB}$ [9], which are thus valid for $\approx_{\rm RB}$ too. Axiom $\mathcal{A}_7$ means that the future can be completely canceled when moving only backward. Axiom $\mathcal{A}_8$ states that a previously non-selected alternative can be discarded when moving only backward. Since there are no constraints on P, axiom $\mathcal{A}_8$ subsumes axiom $\mathcal{A}_3$.

Furthermore, the idempotency of alternative composition in the case of bisimilarity over standard forward-only processes, i.e., $P + P = P$ [12], changes as follows depending on the considered equivalence [9]:

- For $\sim_{\rm FB:ps}$, and hence $\approx_{\rm FB:ps}$ too, idempotency is explicitly formalized by axiom $\mathcal{A}_9$, which is disjoint from axiom $\mathcal{A}_6$ where P cannot be initial.

$(\mathcal{A}_1)$	$(P_1 + P_2) + P_3$	$=$	$P_1 + (P_2 + P_3)$	
$(\mathcal{A}_2)$	$P_1 + P_2$	$=$	$P_2 + P_1$	
$(\mathcal{A}_3)$	$P + \underline{0}$	$=$	P	
$(\mathcal{A}_4)$ $[\sim_{\rm FB:ps}]$	$a^{\dagger} . P$	$=$	P	if $\neg$*initial*(P)
$(\mathcal{A}_5)$ $[\sim_{\rm FB:ps}]$	$a_1^{\dagger} . P$	$=$	$a_2^{\dagger} . P$	if *initial*(P)
$(\mathcal{A}_6)$ $[\sim_{\rm FB:ps}]$	$P + Q$	$=$	P	if $\neg$*initial*(P), where *initial*(Q)
$(\mathcal{A}_7)$ $[\sim_{\rm RB}]$	$a . P$	$=$	P	where *initial*(P)
$(\mathcal{A}_8)$ $[\sim_{\rm RB}]$	$P + Q$	$=$	P	if *initial*(Q)
$(\mathcal{A}_9)$ $[\sim_{\rm FB:ps}]$	$P + P$	$=$	P	where *initial*(P)
$(\mathcal{A}_{10})$ $[\sim_{\rm FRB}]$	$P + Q$	$=$	P	if *initial*$(Q) \land$ *to_initial*$(P) = Q$
$(\mathcal{A}_1^{\tau})$ $[\approx_{\rm FB:ps}]$	$a . \tau . P$	$=$	$a . P$	where *initial*(P)
$(\mathcal{A}_2^{\tau})$ $[\approx_{\rm FB:ps}]$	$P + \tau . P$	$=$	$\tau . P$	where *initial*(P)
$(\mathcal{A}_3^{\tau})$ $[\approx_{\rm FB:ps}]$	$a . (P_1 + \tau . P_2) + a . P_2$	$=$	$a . (P_1 + \tau . P_2)$	where *initial*$(P_1) \land$ *initial*(P_2)
$(\mathcal{A}_4^{\tau})$ $[\approx_{\rm FB:ps}]$	$a^{\dagger} . \tau . P$	$=$	$a^{\dagger} . P$	where *initial*(P)
$(\mathcal{A}_5^{\tau})$ $[\approx_{\rm RB}]$	$\tau^{\dagger} . P$	$=$	P	
$(\mathcal{A}_6^{\tau})$ $[\approx_{\rm FRB:ps}]$	$a . (\tau . (P_1 + P_2) + P_1)$	$=$	$a . (P_1 + P_2)$	where *initial*$(P_1) \land$ *initial*(P_2)
$(\mathcal{A}_7^{\tau})$ $[\approx_{\rm FRB:ps}]$	$a^{\dagger} . (\tau . (P_1 + P_2) + P_1')$	$=$	$a^{\dagger} . (P_1' + P_2)$	if *to_initial*$(P_1') = P_1$, where *initial*$(P_1) \land$ *initial*(P_2)
$(\mathcal{A}_8^{\tau})$ $[\approx_{\rm FRB:ps}]$	$a^{\dagger} . (\tau^{\dagger} . (P_1' + P_2) + P_1)$	$=$	$a^{\dagger} . (P_1' + P_2)$	if *to_initial*$(P_1') = P_1$, where *initial*(P_1)

Table 2
Axioms characterizing $\approx_{\rm FB:ps}$, $\approx_{\rm RB}$, $\approx_{\rm FRB:ps}$

- For $\sim_{\rm RB}$, and hence $\approx_{\rm RB}$ either, an additional axiom is not needed as idempotency follows from axiom $\mathcal{A}_8$ by taking Q equal to P.

- For $\sim_{\rm FRB}$, and hence $\approx_{\rm FRB:ps}$ too, idempotency is formalized by axiom $\mathcal{A}_{10}$, where function *to_initial* brings a process back to its initial version by removing all action decorations:
$$\begin{array}{rcl} \textit{to\_initial}(\underline{0}) & = & \underline{0} \\ \textit{to\_initial}(a . P) & = & a . P \\ \textit{to\_initial}(a^{\dagger} . P) & = & a . \textit{to\_initial}(P) \\ \textit{to\_initial}(P_1 + P_2) & = & \textit{to\_initial}(P_1) + \textit{to\_initial}(P_2) \end{array}$$
This axiom appeared for the first time in [13] and subsumes axioms $\mathcal{A}_9$ and $\mathcal{A}_6$ for $\sim_{\rm FB:ps}$ and $\approx_{\rm FB:ps}$ as well as axiom $\mathcal{A}_8$ for $\sim_{\rm RB}$ and $\approx_{\rm RB}$.

Let us now focus on axioms specific to $\approx_{\rm FB:ps}$, $\approx_{\rm RB}$, and $\approx_{\rm FRB:ps}$, which are usually called τ-laws. Axioms $\mathcal{A}_1^{\tau}$ to $\mathcal{A}_3^{\tau}$ are valid for $\approx_{\rm FB:ps}$ and coincide with those for weak bisimulation congruence over standard forward-only processes [12]. A variant of $\mathcal{A}_1^{\tau}$ with a being decorated, i.e., axiom $\mathcal{A}_4^{\tau}$, is also valid for $\approx_{\rm FB:ps}$; note that $a^{\dagger} . \tau^{\dagger} . P = a^{\dagger} . P$ is valid too, but it follows from reflexity ($P = P$), axiom $\mathcal{A}_5$ or axiom $\mathcal{A}_4$ depending on whether P is initial or not ($\tau^{\dagger} . P = a^{\dagger} . P$), and axiom $\mathcal{A}_4$ applied to the lefthand side along with transitivity. As far as $\tau . P = P$ is concerned, which over standard forward-only processes is valid for weak bisimilarity but not for weak bisimulation congruence [12], its reverse counterpart holds for $\approx_{\rm RB}$, yielding axiom $\mathcal{A}_5^\tau$. Axioms $\mathcal{A}_6^\tau$, $\mathcal{A}_7^\tau$, $\mathcal{A}_8^\tau$ are valid for $\approx_{\rm FRB:ps}$ and are related to the only τ-law of branching bisimulation congruence [10].

In the following, we denote by $\vdash$ the deduction relation and we examine the three sets of additional axioms below:

- $\mathcal{A}^\tau_{\rm FB:ps} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_9, \mathcal{A}_1^\tau, \mathcal{A}_2^\tau, \mathcal{A}_3^\tau, \mathcal{A}_4^\tau\}$ for $\approx_{\rm FB:ps}$.
- $\mathcal{A}^\tau_{\rm RB} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_7, \mathcal{A}_8, \mathcal{A}_5^\tau\}$ for $\approx_{\rm RB}$.
- $\mathcal{A}^\tau_{\rm FRB:ps} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_{10}, \mathcal{A}_6^\tau, \mathcal{A}_7^\tau, \mathcal{A}_8^\tau\}$ for $\approx_{\rm FRB:ps}$.

After proving its soundness, we demonstrate the ground completeness of the equational characterization for each of the three considered weak bisimilarities by introducing as usual equivalence-specific normal forms to which every process is shown to be reducible, so that we then work with normal forms only. For each of the three weak bisimilarities, the normal form comes from the one of the corresponding strong bisimilarity in [9] and relies on the fact that alternative composition is associative and commutative, hence the binary $+$ can be generalized to the n-ary $\sum_{i \in I}$ for a finite nonempty index set I. The proofs of the ground completeness theorems will be by induction on the *size* of a process, which is inductively defined as follows:

$$\begin{array}{rcl} \textit{size}(\underline{0}) & = & 1 \\ \textit{size}(a \,.\, P) & = & 1 + \textit{size}(P) \\ \textit{size}(a^\dagger .\, P) & = & 1 + \textit{size}(P) \\ \textit{size}(P_1 + P_2) & = & \max(\textit{size}(P_1), \textit{size}(P_2)) \end{array}$$

We start with the soundness and ground completeness of $\mathcal{A}^\tau_{\rm FB:ps}$ with respect to $\approx_{\rm FB:ps}$. To this purpose, we introduce the following function that extracts the forward behavior from a process by eliminating executed actions and non-selected alternatives:

$$\begin{array}{rcll} \textit{to\_forward}(P) & = & P & \text{if } \textit{initial}(P) \\ \textit{to\_forward}(a^\dagger .\, P) & = & \textit{to\_forward}(P) & \\ \textit{to\_forward}(P_1 + P_2) & = & \textit{to\_forward}(P_1) & \text{if } \neg\textit{initial}(P_1) \land \textit{initial}(P_2) \\ \textit{to\_forward}(P_1 + P_2) & = & \textit{to\_forward}(P_2) & \text{if } \neg\textit{initial}(P_2) \land \textit{initial}(P_1) \end{array}$$

which yields an initial process and satisfies the following properties.

Proposition 5.1. Let $P, P', P'', Q \in \mathbb{P}$ and $a \in A$:

- *to_forward*(P) is initial, with *to_forward*$(P) = P$ when *initial*(P) while *to_forward*(P) $\sim_{\rm FB}$ P when $\neg$*initial*(P).
- $P \xrightarrow{a} P'$ iff *to_forward*$(P) \xrightarrow{a} P''$ with $P' \sim_{\rm FB:ps} P''$.
- If $P \approx_{\rm FB:ps} Q$, then *to_forward*$(P) \approx_{\rm FB:ps}$ *to_forward*(Q) when P and Q are initial or cannot execute τ-actions, else *to_forward*$(P) \approx_{\rm FB}$ *to_forward*(Q). ■

Theorem 5.2. Let $P_1, P_2 \in \mathbb{P}$. If $\mathcal{A}^\tau_{\rm FB:ps} \vdash P_1 = P_2$ then $P_1 \approx_{\rm FB:ps} P_2$. ■

Definition 5.3. We say that $P \in \mathbb{P}$ is in *forward normal form*, written *F-nf*, iff it is equal to one of the following:

- $\underline{0}$.
- $\sum_{i \in I} a_i \,.\, P_i$, where each P_i is initial and in F-nf.
- $a^{\dagger} .\, P'$, where P' is initial and in F-nf. ■

Lemma 5.4. For all $P \in \mathbb{P}$ there exists $Q \in \mathbb{P}$ in F-nf such that $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = Q$. ■

Following the approach adopted in [6] for weak bisimulation congruence over standard forward-only processes, for $\approx_{\mathrm{FB:ps}}$ we introduce a *saturated normal form* where, unlike [6], two distinct equivalent processes P' and P'' come into play instead of a single process due to the presence of action decorations within processes in our reversible setting. This leads to the so-called *saturation lemma*, which immediately follows the definition below and, unlike [6], features *to_forward*(P') in place of P' in the final part of its statement.

Definition 5.5. We say that $P \in \mathbb{P}$ is in *forward saturated normal form*, written *F-snf*, iff it is equal to one of the following:

- $\underline{0}$
- $\sum_{i \in I} a_i \,.\, P_i$, where each P_i is initial and in F-snf
- $a^{\dagger} .\, P'$, where P' is initial and in F-snf

and whenever $P \overset{\tau^*}{\Longrightarrow} \overset{a}{\longrightarrow} \overset{\tau^*}{\Longrightarrow} P'$, then $P \overset{a}{\longrightarrow} P''$ with $P' \approx_{\mathrm{FB:ps}} P''$. ■

Lemma 5.6. [saturation lemma] Let $P \in \mathbb{P}$ be initial. If $P \overset{\tau^*}{\Longrightarrow} \overset{a}{\longrightarrow} \overset{\tau^*}{\Longrightarrow} P'$ then $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + a \,.\, \textit{to\_forward}(P')$. ■

Lemma 5.7. For all $P \in \mathbb{P}$ in F-nf there exists $Q \in \mathbb{P}$ in F-snf such that $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = Q$. ■

Theorem 5.8. Let $P_1, P_2 \in \mathbb{P}$. If $P_1 \approx_{\mathrm{FB:ps}} P_2$ then $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P_1 = P_2$. ■

As for the soundness and ground completeness of $\mathcal{A}^{\tau}_{\mathrm{RB}}$ with respect to $\approx_{\mathrm{RB}}$, the latter does not require saturation as no choice occurs when going backward.

Theorem 5.9. Let $P_1, P_2 \in \mathbb{P}$. If $\mathcal{A}^{\tau}_{\mathrm{RB}} \vdash P_1 = P_2$ then $P_1 \approx_{\mathrm{RB}} P_2$. ■

Definition 5.10. We say that $P \in \mathbb{P}$ is in *reverse normal form*, written *R-nf*, iff it is equal to one of the following:

- $\underline{0}$.
- $a^{\dagger} .\, P'$, where P' is in R-nf. ■

Lemma 5.11. For all $P \in \mathbb{P}$ there exists $Q \in \mathbb{P}$ in R-nf such that $\mathcal{A}^{\tau}_{\mathrm{RB}} \vdash P = Q$. ■

Theorem 5.12. Let $P_1, P_2 \in \mathbb{P}$. If $P_1 \approx_{\mathrm{RB}} P_2$ then $\mathcal{A}^{\tau}_{\mathrm{RB}} \vdash P_1 = P_2$. ■

We conclude with the soundness and ground completeness of $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}}$ with respect to $\approx_{\mathrm{FRB:ps}}$.

Theorem 5.13. Let $P_1, P_2 \in \mathbb{P}$. If $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_1 = P_2$ then $P_1 \approx_{\mathrm{FRB:ps}} P_2$. ■

Definition 5.14. We say that $P \in \mathbb{P}$ is in *forward-reverse normal form*, written *FR-nf*, iff it is equal to one of the following:

- $\underline{0}$.
- $\sum_{i \in I} a_i \,.\, P_i$, where each P_i is initial and in FR-nf.
- $a^{\dagger} .\, P'$, where P' is in FR-nf.
- $a^{\dagger} .\, P' + \sum_{i \in I} a_i \,.\, P_i$, where P' is in FR-nf and each P_i is initial and in FR-nf. ■

Lemma 5.15. For all $P \in \mathbb{P}$ there exists $Q \in \mathbb{P}$ in FR-nf such that $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P = Q$. ■

Similar to branching bisimulation semantics over standard forward-only processes [14], saturation is unsound for $\approx_{\mathrm{FRB:ps}}$. In particular, a normal form based on saturation cannot be set up for $\approx_{\mathrm{FRB:ps}}$. First of all, the backward version of:

$$\text{whenever } P \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P', \text{ then } P \stackrel{a}{\longrightarrow} P'' \text{ with } P' \approx_{\mathrm{FRB:ps}} P''$$

which is:

$$\text{whenever } P' \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P, \text{ then } P'' \stackrel{a}{\longrightarrow} P \text{ with } P' \approx_{\mathrm{FRB:ps}} P''$$

can be satisfied only when P' and P'' coincide because P can have only one incoming transition. Secondly, not even the forward version of saturation works for $\approx_{\mathrm{FRB:ps}}$:

- Consider $P \triangleq \tau \,.\, (a \,.\, \tau \,.\, \underline{0} + b \,.\, \underline{0}) + a \,.\, \underline{0} + b \,.\, \underline{0}$ along with its two transitions:
  $$\begin{array}{llll} P & \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} & \tau^{\dagger} .\, (a^{\dagger} .\, \tau^{\dagger} .\, \underline{0} + b \,.\, \underline{0}) + a \,.\, \underline{0} + b \,.\, \underline{0} & \triangleq \; P' \\ P & \stackrel{a}{\longrightarrow} & \tau \,.\, (a \,.\, \tau \,.\, \underline{0} + b \,.\, \underline{0}) + a^{\dagger} .\, \underline{0} + b \,.\, \underline{0} & \triangleq \; P'' \end{array}$$
  Then $P' \not\approx_{\mathrm{FRB:ps}} P''$. Indeed, if P' undoes τ with P'' staying idle and then undoes a thus reaching the non-initial process $\tau^{\dagger} .\, (a \,.\, \tau \,.\, \underline{0} + b \,.\, \underline{0}) + a \,.\, \underline{0} + b \,.\, \underline{0}$, then P'' can only respond by undoing a thus reaching the initial process P.
- Consider $Q \triangleq \tau \,.\, a \,.\, (\tau \,.\, \underline{0} + b \,.\, \underline{0}) + a \,.\, \underline{0} + b \,.\, \underline{0}$ along with its two transitions:
  $$\begin{array}{llll} Q & \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} & \tau^{\dagger} .\, a^{\dagger} .\, (\tau^{\dagger} .\, \underline{0} + b \,.\, \underline{0}) + a \,.\, \underline{0} + b \,.\, \underline{0} & \triangleq \; Q' \\ Q & \stackrel{a}{\longrightarrow} & \tau \,.\, a \,.\, (\tau \,.\, \underline{0} + b \,.\, \underline{0}) + a^{\dagger} .\, \underline{0} + b \,.\, \underline{0} & \triangleq \; Q'' \end{array}$$
  Then $Q' \not\approx_{\mathrm{FRB:ps}} Q''$. Indeed, if Q' undoes τ thus reaching $\tau^{\dagger} .\, a^{\dagger} .\, (\tau \,.\, \underline{0} + b \,.\, \underline{0}) + a \,.\, \underline{0} + b \,.\, \underline{0}$ with Q'' staying idle, then in the forward direction the newly reached process can perform b whereas Q'' cannot.

To investigate the ground completeness of $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}}$ for $\approx_{\mathrm{FRB:ps}}$, first of all we develop an alternative characterization of $\approx_{\mathrm{FRB:ps}}$. This is inspired by the construction employed in [6] over forward-only processes to define weak bisimulation congruence on the basis of weak bisimulation equivalence. Consider for example $\tau \,.\, a \,.\, \underline{0}$ and $a \,.\, \underline{0}$, which are identified by $\approx_{\mathrm{FRB}}$

but told apart by $\approx_{\mathrm{FRB:ps}}$. The reason for distinguishing them is that if $\tau \,.\, a \,.\, \underline{0}$ performs τ thereby evolving to the non-initial process $\tau^{\dagger} .\, a \,.\, \underline{0}$, then the only way for $a \,.\, \underline{0}$ to respond is idling thus remaining in an initial process. In the weak bisimulation congruence setting of [6], this would be reformulated in terms of the fact that the latter process has no initial τ-transition and hence cannot match the initial τ-transition of the former process.

In our reversible setting, the construction of [6] needs to be adapted as follows. In the case of two initial processes, every transition of either process must be matched by an identically labeled transition of the other process, with the two reached non-initial processes being related by $\approx_{\mathrm{FRB}}$. In the case of two non-initial processes, in addition to requiring them to be $\approx_{\mathrm{FRB}}$-equivalent, we also have to make sure that their initial versions are equivalent in the sense above. For instance, the two non-initial processes $\tau^{\dagger} .\, a^{\dagger} .\, \underline{0}$ and $a^{\dagger} .\, \underline{0}$ are identified by $\approx_{\mathrm{FRB}}$, but $\mathit{to\_initial}(\tau^{\dagger} .\, a^{\dagger} .\, \underline{0}) = \tau \,.\, a \,.\, \underline{0} \not\approx_{\mathrm{FRB:ps}} a \,.\, \underline{0} = \mathit{to\_initial}(a^{\dagger} .\, \underline{0})$, hence $\tau^{\dagger} .\, a^{\dagger} .\, \underline{0} \not\approx_{\mathrm{FRB:ps}} a^{\dagger} .\, \underline{0}$ too. On the other hand, it is not enough to guarantee that the initial versions are equivalent, as for example $\mathit{to\_initial}(a^{\dagger} .\, b \,.\, \underline{0}) = a \,.\, b \,.\, \underline{0} = \mathit{to\_initial}(a^{\dagger} .\, b^{\dagger} .\, \underline{0})$ but $a^{\dagger} .\, b \,.\, \underline{0} \not\approx_{\mathrm{FRB}} a^{\dagger} .\, b^{\dagger} .\, \underline{0}$.

Definition 5.16. We say that $P_1, P_2 \in \mathbb{P}$ are *weakly forward-reverse bisimulation congruent*, written $P_1 \approx_{\mathrm{FRB:c}} P_2$, iff:

- either P_1 and P_2 are both initial and, for all $a \in A$, whenever $P_1 \xrightarrow{a} P'_1$, then $P_2 \xrightarrow{a} P'_2$ and $P'_1 \approx_{\mathrm{FRB}} P'_2$, and vice versa;
- or P_1 and P_2 are both non-initial, $P_1 \approx_{\mathrm{FRB}} P_2$, and $\mathit{to\_initial}(P_1) \approx_{\mathrm{FRB:c}} \mathit{to\_initial}(P_2)$. ■

Theorem 5.17. Let $P_1, P_2 \in \mathbb{P}$. Then $P_1 \approx_{\mathrm{FRB:c}} P_2$ iff $P_1 \approx_{\mathrm{FRB:ps}} P_2$. ■

Secondly, we recast in our reversible setting a preliminary result for the completeness of the axiomatization of branching bisimulation congruence provided in [15]. This yields two lemmas, where the former is about $\approx_{\mathrm{FRB}}$-equivalent initial processes that are then prefixed by an unexecuted action, while the latter has to do with $\approx_{\mathrm{FRB}}$-equivalent arbitrary processes that are then prefixed by an executed action. The proof of the former lemma and part of the latter lemma is inspired by the proof of the preliminary result in the aforementioned paper. Each lemma is followed by the corresponding ground completeness result of $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}}$ for $\approx_{\mathrm{FRB:ps}}$, in which the lemma itself can be employed thanks to the alternative characterization of $\approx_{\mathrm{FRB:ps}}$. The former completeness result thus deals with $\approx_{\mathrm{FRB:ps}}$-equivalent initial processes. The latter completeness result instead addresses $\approx_{\mathrm{FRB:ps}}$-equivalent non-initial processes, with the related lemma exploiting completeness over initial processes.

Lemma 5.18. Let $P_1, P_2 \in \mathbb{P}$ be initial and $a \in A$. If $P_1 \approx_{\mathrm{FRB}} P_2$ then $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash a \,.\, P_1 = a \,.\, P_2$. ■

Theorem 5.19. Let $P_1, P_2 \in \mathbb{P}$ be initial. If $P_1 \approx_{\mathrm{FRB:ps}} P_2$ then $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_1 = P_2$. ■

Lemma 5.20. Let $P_1, P_2 \in \mathbb{P}$ and $a \in A$. If $P_1 \approx_{\mathrm{FRB}} P_2$ then $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash a^{\dagger} .\, P_1 = a^{\dagger} .\, P_2$. ■

Theorem 5.21. Let $P_1, P_2 \in \mathbb{P}$ be not initial. If $P_1 \approx_{\mathrm{FRB:ps}} P_2$ then $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_1 = P_2$. ■

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A. Proofs of Results

Proof of Theorem 4.3.
Let $P_1, P_2 \in \mathbb{P}$:

- Let $P_1 \approx P_2$ and $a \in A$ and consider a $\approx$-bisimulation $\mathcal{B}$ containing the pair (P_1, P_2). Then:
$$\begin{array}{rcl} \mathcal{B}' & = & \mathcal{B} \cup \{(a \,.\, P'_1, a \,.\, P'_2) \mid \mathit{initial}(P'_1) \wedge \mathit{initial}(P'_2) \wedge (P'_1, P'_2) \in \mathcal{B}\} \\ & & \cup \, \{(a^\dagger .\, P'_1, a^\dagger .\, P'_2) \mid (P'_1, P'_2) \in \mathcal{B}\} \end{array}$$
is a $\approx$-bisimulation too because:
  - If $\approx$ considers moving forward, then both $a \,.\, P'_1$ and $a \,.\, P'_2$ with $\mathit{initial}(P'_1)$ and $\mathit{initial}(P'_2)$ turn out to have a single outgoing a-transition and these two a-transitions respectively reach $a^\dagger .\, P'_1$ and $a^\dagger .\, P'_2$, which form a pair of $\mathcal{B}'$. Note that whether $a = \tau$ or not is unimportant.
  - Moving backward is not allowed from $a \,.\, P'_1$ and $a \,.\, P'_2$ with $\mathit{initial}(P'_1)$ and $\mathit{initial}(P'_2)$ as they are both initial and hence have no incoming transitions.
  - $a^\dagger .\, P'_1$ and $a^\dagger .\, P'_2$ have $\approx$-matching outgoing/incoming transitions – depending on whether $\approx$ considers moving forward/backward – respectively determined by the two $\approx$-equivalent processes P'_1 and P'_2.
  In particular, if P'_1 and P'_2 are initial and $\approx$ considers moving backward, then $a^\dagger .\, P'_1$ and $a^\dagger .\, P'_2$ turn out to have a single incoming a-transition and these two a-transitions respectively depart from $a \,.\, P'_1$ and $a \,.\, P'_2$, which form a pair of $\mathcal{B}'$.

  Therefore $a \,.\, P_1 \approx a \,.\, P_2$, provided that $\mathit{initial}(P_1) \wedge \mathit{initial}(P_2)$, as well as $a^\dagger .\, P_1 \approx a^\dagger .\, P_2$.

- Let $P_1 \approx' P_2$ and $P \in \mathbb{P}$ and consider a $\approx'$-bisimulation $\mathcal{B}$ containing the pair (P_1, P_2):
  - Then:
  $$\begin{array}{rcl} \mathcal{B}' & = & \mathcal{B} \cup \{(P'_1 + P', P'_2 + P') \mid (P'_1, P'_2) \in \mathcal{B} \, \wedge \\ & & \qquad (\mathit{initial}(P') \vee (\mathit{initial}(P'_1) \wedge \mathit{initial}(P'_2)))\} \end{array}$$
  is a $\approx'$-bisimulation too because $P'_1 + P'$ and $P'_2 + P'$ have $\approx'$-matching outgoing/incoming transitions – depending on whether $\approx'$ considers moving forward/backward – determined by the two $\approx'$-equivalent processes P'_1 and P'_2 respectively when $\mathit{initial}(P')$ or by P' when $\mathit{initial}(P'_1) \wedge \mathit{initial}(P'_2)$.
  In the forward case, since from $(P'_1, P'_2) \in \mathcal{B}$ it follows that $\mathit{initial}(P'_1) \iff \mathit{initial}(P'_2)$, when $\mathit{initial}(P')$ all the initial actions of P' are enabled both in $P'_1 + P'$ and in $P'_2 + P'$ if $\mathit{initial}(P'_1) \wedge \mathit{initial}(P'_2)$ or in neither of them if $\neg\mathit{initial}(P'_1) \wedge \neg\mathit{initial}(P'_2)$.
  Therefore $P_1 + P \approx' P_2 + P$ provided that $\mathit{initial}(P) \vee (\mathit{initial}(P_1) \wedge \mathit{initial}(P_2))$.
  - The proof of $P + P_1 \approx' P + P_2$ is similar because the two operational semantic rules for alternative composition are symmetric.

- We have to prove that $P_1 \approx_{\mathrm{FB:ps}} P_2$ iff $P_1 + P \approx_{\mathrm{FB}} P_2 + P$ for all $P \in \mathbb{P}$ such that $\mathit{initial}(P) \vee (\mathit{initial}(P_1) \wedge \mathit{initial}(P_2))$.
If $P_1 \approx_{\mathrm{FB:ps}} P_2$ then $P_1 + P \approx_{\mathrm{FB:ps}} P_2 + P$ as we have proved before for all $P \in \mathbb{P}$

such that *initial*$(P) \vee (\textit{initial}(P_1) \wedge \textit{initial}(P_2))$, hence $P_1 + P \approx_{\rm FB} P_2 + P$ because $\approx_{\rm FB:ps} \subset \approx_{\rm FB}$.
As far as the reverse implication is concerned, we reason on the contrapositive. Suppose that $P_1 \not\approx_{\rm FB:ps} P_2$:

- If it is not the case that *initial*$(P_1) \Longleftrightarrow \textit{initial}(P_2)$, say $\neg\textit{initial}(P_1)$ and *initial*(P_2), then, even if P_1 and P_2 have matching outgoing transitions, it turns out that $P_1 + c\,.\,\underline{0} \not\approx_{\rm FB} P_2 + c\,.\,\underline{0}$, where $c \neq \tau$ is an action occurring neither in P_1 nor in P_2, because $P_2 + c\,.\,\underline{0}$ has an outgoing c-transition whilst $P_1 + c\,.\,\underline{0}$ has not (not even one that is preceded by finitely many τ-transitions). Note that *initial*$(c\,.\,\underline{0})$.
- If P_1 and P_2 are both initial or non-initial but have no matching outgoing transitions, then $P_1 + \underline{0}$ and $P_2 + \underline{0}$ have no matching outgoing transitions either, hence $P_1 + \underline{0} \not\approx_{\rm FB} P_2 + \underline{0}$. Note that *initial*$(\underline{0})$.

• The proof that $P_1 \approx_{\rm FRB:ps} P_2$ iff $P_1 + P \approx_{\rm FRB} P_2 + P$ for all $P \in \mathbb{P}$ such that *initial*$(P) \vee (\textit{initial}(P_1) \wedge \textit{initial}(P_2))$ is similar to the previous one. In particular, when reasoning on the contrapositive of the reverse implication, we have that:

- If $\neg\textit{initial}(P_1)$ and *initial*(P_2) then, even if P_1 and P_2 have matching outgoing and incoming transitions, it turns out that $P_1 + c\,.\,\underline{0} \not\approx_{\rm FRB} P_2 + c\,.\,\underline{0}$, where $c \neq \tau$ is an action occurring neither in P_1 nor in P_2.
- If P_1 and P_2 have no matching outgoing or incoming transitions, then $P_1 + \underline{0}$ and $P_2 + \underline{0}$ have no matching outgoing or incoming transitions either, hence $P_1 + \underline{0} \not\approx_{\rm FRB} P_2 + \underline{0}$. ■

Proof of Proposition 5.1.
The first property is a straightforward consequence of the definition of *to_forward* and the fact that $\sim_{\rm FB}$ considers only the forward behavior of processes. Note that *to_forward*$(P) \sim_{\rm FB:ps} P$ cannot hold when P is not initial because *to_forward*(P) is initial.
As for the second property, by construction *to_forward*(P) is obtained from P by removing all decorated (executed) actions as well as all non-selected alternatives, which are all the parts of P from which an outgoing transition cannot be generated. As a consequence $P \xrightarrow{a} P'$ iff *to_forward*$(P) \xrightarrow{a} P''$ with $P' \sim_{\rm FB:ps} P''$. Due to the first property, P' does not coincide with P'' when P is not initial, because in that case P' contains decorated actions along with possible non-selected alternatives that cannot be present in P''. However $P' \sim_{\rm FB:ps} P''$ (instead of $P' \sim_{\rm FB} P''$ only) because both P' and P'' are not initial.
As for the third property, we distinguish two cases:

• If P and Q are initial, then *to_forward*$(P) = P \approx_{\rm FB:ps} Q =$ *to_forward*(Q).

• If P and Q are not initial, then *to_forward*$(P) \neq P$ and *to_forward*$(Q) \neq Q$.
Suppose that *to_forward*$(P) \xrightarrow{a} P'$. Then, due to the second property, $P \xrightarrow{a} P''$ with $P' \sim_{\rm FB:ps} P''$ and hence $P' \approx_{\rm FB:ps} P''$ because $\sim_{\rm FB:ps}$ is contained in $\approx_{\rm FB:ps}$. There are two subcases:

– If $a \neq \tau$, from $P \approx_{\mathrm{FB:ps}} Q$ it follows that $Q \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} Q''$ with $P'' \approx_{\mathrm{FB:ps}} Q''$. By repeatedly applying the second property we get *to_forward*$(Q) \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} Q'$ with $Q' \approx_{\mathrm{FB:ps}} Q''$ (as neither Q' nor Q'' is initial). The result stems from $P' \approx_{\mathrm{FB:ps}} P'' \approx_{\mathrm{FB:ps}} Q'' \approx_{\mathrm{FB:ps}} Q'$ by exploiting the fact that $\approx_{\mathrm{FB:ps}}$ is symmetric and transitive.

– If $a = \tau$, from $P \approx_{\mathrm{FB:ps}} Q$ it follows that $Q \stackrel{\tau^*}{\Longrightarrow} Q''$ with $P'' \approx_{\mathrm{FB:ps}} Q''$. By repeatedly applying the second property we get *to_forward*$(Q) \stackrel{\tau^*}{\Longrightarrow} Q'$ with $Q' \approx_{\mathrm{FB}} Q''$ (instead of $Q' \approx_{\mathrm{FB:ps}} Q''$) as Q'' is not initial while Q' may be initial (this is the case when no τ is performed by *to_forward*(Q)). Since $\approx_{\mathrm{FB:ps}}$ is contained in $\approx_{\mathrm{FB}}$, the result stems from $P' \approx_{\mathrm{FB}} P'' \approx_{\mathrm{FB}} Q'' \approx_{\mathrm{FB}} Q'$ by exploiting the fact that $\approx_{\mathrm{FB}}$ is symmetric and transitive. ■

Proof of Theorem 5.2.
A straightforward consequence of the axioms and inference rules behind $\vdash$ together with the fact that $\approx_{\mathrm{FB:ps}}$ is an equivalence relation and a congruence (Theorem 4.3) and the fact that the lefthand side process of each additional axiom in $\mathcal{A}^{\tau}_{\mathrm{FB:ps}}$ is $\approx_{\mathrm{FB:ps}}$-equivalent to the righthand side process of the same axiom. ■

Proof of Lemma 5.4.
Similar to the proof of [9, Lemma 1] (which uses axioms $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{A}_6$) because, in the considered normal form, τ-actions do not play a role different from the one of visible actions; in particular, unexecuted τ-actions are not abstracted away unless they are inside non-selected alternatives. ■

Proof of Lemma 5.6.
Suppose that P is in F-nf. Should this not be the case, thanks to Lemma 5.4 we could find Q in F-nf such that $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = Q$, hence proving the result for Q would entail the validity of the result for P by substitutivity. In particular:

- If $P \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'$, then $Q \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} Q'$ with $P' \approx_{\mathrm{FB:ps}} Q'$ due to $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = Q$ implying $P \approx_{\mathrm{FB:ps}} Q$ by soundness (Theorem 5.2) and the fact that Q cannot idle when $a = \tau$ because P and Q are both initial.
- $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash$ *to_forward*$(P') =$ *to_forward*(Q') because $\underline{0}$ summands possibly occurring in *to_forward*(P') can be eliminated via $\mathcal{A}_3$ and Q is a F-nf for P so that Q' cannot abstract from unexecuted τ-actions unless they are inside non-selected alternatives (which by the way can occur neither in *to_forward*(P') nor in Q' and hence *to_forward*(Q')).

We thus proceed by induction on the syntactical structure of the initial process P in F-nf such that $P \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'$ (note that P cannot be $\underline{0}$), where in the following the finite index set I can be empty in which case the corresponding summation is meant to disappear:

- If P is $\sum_{i \in I} a_i \,.\, P_i + a \,.\, \bar{P}$ and P' is $\sum_{i \in I} a_i \,.\, P_i + a^{\dagger} .\, \bar{P}$ – i.e., no τ-transitions precede and follow the a-transition in $P \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'$ – where we note that $\bar{P}$ is in F-nf and initial because so is P, then $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + a \,.\, \bar{P}$ by $\mathcal{A}_9$ applied to $a \,.\, \bar{P}$ inside P and substitutivity, with $\bar{P} = \mathit{to\_forward}(P')$.

- If P is $\sum_{i \in I} a_i \,.\, P_i + a \,.\, Q$ and $\sum_{i \in I} a_i \,.\, P_i + a^{\dagger} .\, Q \stackrel{\tau^*}{\Longrightarrow} \stackrel{\tau}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'$ – i.e., no τ-transitions precede but at least one τ-transition follows the a-transition in $P \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'$ – then:
  - Since $\sum_{i \in I} a_i \,.\, P_i + a^{\dagger} .\, Q \stackrel{\tau^*}{\Longrightarrow} \stackrel{\tau}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'$ comes from $Q \stackrel{\tau^*}{\Longrightarrow} \stackrel{\tau}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} Q'$ with $\mathit{to\_forward}(P') = \mathit{to\_forward}(Q')$ and Q initial and in F-nf, by the induction hypothesis $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash Q = Q + \tau \,.\, \mathit{to\_forward}(P')$.
  - $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + a \,.\, \mathit{to\_forward}(P')$ because:
    * $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + a \,.\, Q$ by $\mathcal{A}_9$ applied to $a \,.\, Q$ inside P and substitutivity.
    * $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + a \,.\, (Q + \tau \,.\, \mathit{to\_forward}(P'))$ by substitutivity and transitivity.
    * $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + a \,.\, (Q + \tau \,.\, \mathit{to\_forward}(P')) + a \,.\, \mathit{to\_forward}(P')$ by $\mathcal{A}^{\tau}_3$, substitutivity, and transitivity.
    * $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + a \,.\, Q + a \,.\, \mathit{to\_forward}(P')$ by substitutivity and transitivity.
    * $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + a \,.\, \mathit{to\_forward}(P')$ by $\mathcal{A}_9$ as P contains $a \,.\, Q$ as summand, substitutivity, and transitivity.

- If P is $\sum_{i \in I} a_i \,.\, P_i + \tau \,.\, Q$ and $\sum_{i \in I} a_i \,.\, P_i + \tau^{\dagger} .\, Q \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'$ – i.e., at least one τ-transition precedes the a-transition in $P \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'$ – then:
  - Since $\sum_{i \in I} a_i \,.\, P_i + \tau^{\dagger} .\, Q \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'$ comes from $Q \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} Q'$ with $\mathit{to\_forward}(P') = \mathit{to\_forward}(Q')$ and Q initial and in F-nf, by the induction hypothesis $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash Q = Q + a \,.\, \mathit{to\_forward}(P')$.
  - $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + a \,.\, \mathit{to\_forward}(P')$ because:
    * $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + \tau \,.\, Q$ by $\mathcal{A}_9$ applied to $\tau \,.\, Q$ inside P and substitutivity.
    * $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + \tau \,.\, Q + Q$ by $\mathcal{A}^{\tau}_2$, substitutivity, and transitivity.
    * $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + \tau \,.\, Q + Q + a \,.\, \mathit{to\_forward}(P')$ by substitutivity and transitivity.
    * $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + \tau \,.\, Q + a \,.\, \mathit{to\_forward}(P')$ by $\mathcal{A}^{\tau}_2$, substitutivity, and transitivity.
    * $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = P + a \,.\, \mathit{to\_forward}(P')$ by $\mathcal{A}_9$ as P contains $\tau \,.\, Q$ as summand, substitutivity, and transitivity. ■

Proof of Lemma 5.7.
We proceed by induction on the syntactical structure of P in F-nf:

- If P is $\underline{0}$, then it is sufficient to take Q equal to $\underline{0}$.

- If P is $\sum_{i \in I} a_i \,.\, P_i$, then by the induction hypothesis for all $i \in I$ there is Q_i in F-snf such that $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P_i = Q_i$, hence $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = \sum_{i \in I} a_i \,.\, Q_i$ by substitutivity with respect to action prefix and alternative composition.

  Suppose that $\sum_{i \in I} a_i \,.\, Q_i \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} Q'$ but there is no Q'' such that $\sum_{i \in I} a_i \,.\, Q_i \stackrel{a}{\longrightarrow} Q''$ with $Q' \approx_{\mathrm{FB:ps}} Q''$. Since $\sum_{i \in I} a_i \,.\, Q_i$ is initial, from Lemma 5.6 we get $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash \sum_{i \in I} a_i \,.\, Q_i = \sum_{i \in I} a_i \,.\, Q_i + a \,.\, \mathit{to\_forward}(Q')$, hence $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = \sum_{i \in I} a_i \,.\, Q_i + a \,.\, \mathit{to\_forward}(Q')$ by transitivity, where $\mathit{to\_forward}(Q')$ is initial and in F-snf.

  Therefore $\sum_{i \in I} a_i \,.\, Q_i + a \,.\, \mathit{to\_forward}(Q') \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} Q'$ and moreover $\sum_{i \in I} a_i \,.\, Q_i + a \,.\, \mathit{to\_forward}(Q') \stackrel{a}{\longrightarrow} \sum_{i \in I} a_i \,.\, Q_i + a^{\dagger} .\, \mathit{to\_forward}(Q')$, where $Q' \approx_{\mathrm{FB:ps}} \sum_{i \in I} a_i \,.\, Q_i + a^{\dagger} .\, \mathit{to\_forward}(Q')$ as from Q' and $a^{\dagger} .\, \mathit{to\_forward}(Q')$ being both non-initial it follows that $Q' \approx_{\mathrm{FB:ps}} a^{\dagger} .\, \mathit{to\_forward}(Q')$, at which point we exploit the soundness of $\mathcal{A}_6$ (Theorem 5.2) on the righthand side and the fact that $\approx_{\mathrm{FB:ps}}$ is transitive.

- If P is $b^{\dagger} .\, \hat{P}$, then by the induction hypothesis there is $\hat{Q}$ in F-snf such that $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash \hat{P} = \hat{Q}$, hence $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = b^{\dagger} .\, \hat{Q}$ by substitutivity with respect to action prefix.

  Suppose that $b^{\dagger} .\, \hat{Q} \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} Q'$ but there is no Q'' such that $b^{\dagger} .\, \hat{Q} \stackrel{a}{\longrightarrow} Q''$ with $Q' \approx_{\mathrm{FB:ps}} Q''$. Since $\hat{Q}$ is initial, from Lemma 5.6 and substitutivity we get $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash b^{\dagger} .\, \hat{Q} = b^{\dagger} .\, (\hat{Q} + a \,.\, \mathit{to\_forward}(Q'))$, hence $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P = b^{\dagger} .\, (\hat{Q} + a \,.\, \mathit{to\_forward}(Q'))$ by transitivity, where $\mathit{to\_forward}(Q')$ is initial and in F-snf.

  Therefore $b^{\dagger} .\, (\hat{Q} + a \,.\, \mathit{to\_forward}(Q')) \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} Q'$ and moreover $b^{\dagger} .\, (\hat{Q} + a \,.\, \mathit{to\_forward}(Q')) \stackrel{a}{\longrightarrow} b^{\dagger} .\, (\hat{Q} + a^{\dagger} .\, \mathit{to\_forward}(Q'))$, where $Q' \approx_{\mathrm{FB:ps}} b^{\dagger} .\, (\hat{Q} + a^{\dagger} .\, \mathit{to\_forward}(Q'))$ as from Q' and $a^{\dagger} .\, \mathit{to\_forward}(Q')$ being both non-initial it follows that $Q' \approx_{\mathrm{FB:ps}} a^{\dagger} .\, \mathit{to\_forward}(Q')$, at which point we exploit the soundness of $\mathcal{A}_6$ and $\mathcal{A}_4$ (Theorem 5.2) on the righthand side and the fact that $\approx_{\mathrm{FB:ps}}$ is transitive. ■

Proof of Theorem 5.8.
Suppose that P_1 and P_2 are both in F-snf. Should this not be the case, thanks to Lemmas 5.4 and 5.7 we could find Q_1 and Q_2 in F-snf such that $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P_1 = Q_1$ and $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P_2 = Q_2$, hence $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash Q_2 = P_2$ by symmetry. Due to soundness (Theorem 5.2), we would get $P_1 \approx_{\mathrm{FB:ps}} Q_1$, hence $Q_1 \approx_{\mathrm{FB:ps}} P_1$ as $\approx_{\mathrm{FB:ps}}$ is symmetric, and $P_2 \approx_{\mathrm{FB:ps}} Q_2$. Since $P_1 \approx_{\mathrm{FB:ps}} P_2$, we would also get $Q_1 \approx_{\mathrm{FB:ps}} Q_2$ as $\approx_{\mathrm{FB:ps}}$ is transitive. Proving $Q_1 \approx_{\mathrm{FB:ps}} Q_2 \implies \mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash Q_1 = Q_2$ would finally entail $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P_1 = P_2$ by transitivity.
We proceed by induction on $k = \mathit{size}(P_1) + \mathit{size}(P_2) \in \mathbb{N}_{\geq 2}$:

- If $k = 2$, then from $P_1 \approx_{\mathrm{FB:ps}} P_2$ and P_1 and P_2 in F-snf we derive that both P_1 and P_2 are equal to $\underline{0}$, from which the result follows by reflexivity.

- Let $k > 2$ with P_1 being $\sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}$ and P_2 being $\sum_{i \in I_2} a_{2,i} \,.\, P_{2,i}$, where every $P_{1,i}$ and every $P_{2,i}$ is initial and in F-snf. Since $P_1 \approx_{\mathrm{FB:ps}} P_2$, whenever for some $a_{1,i_1} = a$ we have $P_1 \stackrel{a}{\longrightarrow} a^{\dagger} .\, P_{1,i_1} + \sum_{i \in I_1 \setminus \{i_1\}} a_{1,i} \,.\, P_{1,i}$, then for some $a_{2,i_2} = a$ we have $P_2 \stackrel{a}{\longrightarrow} a^{\dagger} .\, P_{2,i_2} + \sum_{i \in I_2 \setminus \{i_2\}} a_{2,i} \,.\, P_{2,i}$ as P_2 is in F-snf where $a^{\dagger} .\, P_{1,i_1} + \sum_{i \in I_1 \setminus \{i_1\}} a_{1,i} \,.\, P_{1,i} \approx_{\mathrm{FB:ps}} a^{\dagger} .\, P_{2,i_2} + \sum_{i \in I_2 \setminus \{i_2\}} a_{2,i} \,.\, P_{2,i}$, and vice versa. Since $P_{1,i_1} =$

$\mathit{to\_forward}(a^{\dagger} . P_{1,i_1} + \sum_{i \in I_1 \setminus \{i_1\}} a_{1,i} . P_{1,i})$ and $P_{2,i_2} = \mathit{to\_forward}(a^{\dagger} . P_{2,i_2} + \sum_{i \in I_2 \setminus \{i_2\}} a_{2,i} . P_{2,i})$, from the third property in Proposition 5.1 two cases arise:

– If $P_{1,i_1} \approx_{\mathrm{FB:ps}} P_{2,i_2}$, then from the induction hypothesis we obtain $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P_{1,i_1} = P_{2,i_2}$, hence $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash a_{1,i_1} . P_{1,i_1} = a_{2,i_2} . P_{2,i_2}$ by substitutivity with respect to action prefix.

– If $P_{1,i_1} \approx_{\mathrm{FB}} P_{2,i_2}$ but $P_{1,i_1} \not\approx_{\mathrm{FB:ps}} P_{2,i_2}$ – as is the case, e.g., when $a_{1,i_1} . P_{1,i_1}$ is $a . \tau . \underline{0}$ and $a_{2,i_2} . P_{2,i_2}$ is $a . \underline{0}$ – then P_{1,i_1} can execute τ-actions (thus reaching non-initial processes) to which P_{2,i_2} can respond only by idling (thus remaining in an initial process), or vice versa. If the considered summand of P_1 is $a_{1,i_1} . \tau . P'_{1,i_1}$, we exploit the soundness of $\mathcal{A}^{\tau}_1$ (Theorem 5.2) to obtain $a_{1,i_1} . \tau . P'_{1,i_1} \approx_{\mathrm{FB:ps}} a_{1,i_1} . P''_{1,i_1}$ where P''_{1,i_1} is a subprocess of P'_{1,i_1} that is initial, in F-snf, and not executing τ-actions, so that $P''_{1,i_1} \approx_{\mathrm{FB:ps}} P_{2,i_2}$ and we can then proceed like in the previous case where $\mathcal{A}^{\tau}_1$ is additionally applied.
  More generally, the considered summand of P_1 may be of the form $a_{1,i_1} . (\tau . P'_{1,i_1} + \dots)$, but then P'_{1,i_1}, after executing possible τ-actions, must offer all the alternative visible actions enabled by P_{2,i_2} and only those actions, otherwise $P_{1,i_1} \approx_{\mathrm{FB}} P_{2,i_2}$ cannot hold given that P_{2,i_2} can only idle whenever P_{1,i_1} executes a τ-action. As a consequence, for every subprocess alternative to $\tau . P'_{1,i_1}$:

  * If it starts with a τ-action, then for the same reason it must offer all the alternative visible actions enabled by P_{2,i_2} and only those actions, hence it must be $\approx_{\mathrm{FB:ps}}$-equivalent to $\tau . P'_{1,i_1}$ and can be absorbed by $\tau . P'_{1,i_1}$ by exploiting the soundness of $\mathcal{A}_9$ (Theorem 5.2).

  * If it starts with a visible action, then that action must be enabled by P_{2,i_2} in order for $P_{1,i_1} \approx_{\mathrm{FB}} P_{2,i_2}$ to hold and the considered subprocess can be absorbed within $\tau . P'_{1,i_1}$ as follows by exploiting the soundness of $\mathcal{A}_9$ and $\mathcal{A}^{\tau}_2$ (Theorem 5.2).

    · $\tau . P'_{1,i_1}$ is expanded to $P'_{1,i_1} + \tau . P'_{1,i_1}$ via $\mathcal{A}^{\tau}_2$, with its application being repeated in the case that P'_{1,i_1} starts with a τ-action and so on, until the considered subprocess appears in the expansion.

    · The original occurrence of the considered subprocess and the new one inside the expansion are merged into a single one via $\mathcal{A}_9$.

    · The resulting process is contracted back to $\tau . P'_{1,i_1}$ via as many applications of $\mathcal{A}^{\tau}_2$.

The result finally follows by substitutivity with respect to alternative composition and, in the presence of identical summands on the same side, axiom $\mathcal{A}_9$ possibly preceded by applications of axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ to move identical summands next to each other.

• Let $k > 2$ with P_1 being $a_1^{\dagger} . P'_1$ and P_2 being $a_2^{\dagger} . P'_2$, where P'_1 and P'_2 are both initial and in F-snf. Since $P'_1 = \mathit{to\_forward}(P_1)$ and $P'_2 = \mathit{to\_forward}(P_2)$, from $P_1 \approx_{\mathrm{FB:ps}} P_2$ and the third property in Proposition 5.1 two cases arise:

– If $P'_1 \approx_{\mathrm{FB:ps}} P'_2$, then from the induction hypothesis we obtain $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash P'_1 = P'_2$, hence $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash a^{\dagger} . P'_1 = a^{\dagger} . P'_2$ by substitutivity with respect to action prefix. Thanks to $\mathcal{A}_5$ we derive $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash a^{\dagger}_1 . P'_1 = a^{\dagger} . P'_1$ and $\mathcal{A}^{\tau}_{\mathrm{FB:ps}} \vdash a^{\dagger} . P'_2 = a^{\dagger}_2 . P'_2$, from which the result follows by transitivity.

– If $P'_1 \approx_{\mathrm{FB}} P'_2$ but $P'_1 \not\approx_{\mathrm{FB:ps}} P'_2$ – as is the case, e.g., when $a^{\dagger}_1 . P'_1$ is $a^{\dagger}_1 . \tau . \underline{0}$ and $a^{\dagger}_2 . P'_2$ is $a^{\dagger}_2 . \underline{0}$ – then P'_1 can execute τ-actions (thus reaching non-initial processes) to which P'_2 can respond only by idling (thus remaining in an initial process), or vice versa. If P_1 is $a^{\dagger}_1 . \tau . P''_1$, we exploit the soundness of $\mathcal{A}^{\tau}_4$ (Theorem 5.2) to obtain $P_1 \approx_{\mathrm{FB:ps}} a^{\dagger}_1 . P'''_1$ where P'''_1 is a subprocess of P''_1 that is initial, in F-snf, and not executing τ-actions, so that $P'''_1 \approx_{\mathrm{FB:ps}} P'_2$ and we can then proceed like in the previous case where $\mathcal{A}^{\tau}_4$ is additionally applied.
More generally, the considered summand of P_1 may be of the form $a^{\dagger}_{1,i_1} . (\tau . P'_{1,i_1} + \dots)$, but then every subprocess alternative to $\tau . P'_{1,i_1}$ can be suitably absorbed as shown before.

Note that the case $k > 2$ with P_1 being $\sum_{i \in I_1} a_{1,i} . P_{1,i}$ or $\underline{0}$ and P_2 being $a^{\dagger}_2 . P'_2$, or vice versa, cannot occur because the former is initial while the latter is not. Likewise, the case $k > 2$ with P_1 being $\sum_{i \in I_1} a_{1,i} . P_{1,i}$ and P_2 being $\underline{0}$, or vice versa, would contradict $P_1 \approx_{\mathrm{FB:ps}} P_2$. ■

Proof of Theorem 5.9.
A straightforward consequence of the axioms and inference rules behind $\vdash$ together with the fact that $\approx_{\mathrm{RB}}$ is an equivalence relation and a congruence (Theorem 4.3) and the fact that the lefthand side process of each additional axiom in $\mathcal{A}^{\tau}_{\mathrm{RB}}$ is $\approx_{\mathrm{RB}}$-equivalent to the righthand side process of the same axiom. ■

Proof of Lemma 5.11.
Similar to the proof of [9, Lemma 2] (which uses axioms $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_7$, $\mathcal{A}_8$) because, in the considered normal form, τ-actions do not play a role different from the one of visible actions; in particular, executed τ-actions are not abstracted away. ■

Proof of Theorem 5.12.
Suppose that P_1 and P_2 are both in R-nf. Should this not be the case, thanks to Lemma 5.11 we could find Q_1 and Q_2 in R-nf such that $\mathcal{A}^{\tau}_{\mathrm{RB}} \vdash P_1 = Q_1$ and $\mathcal{A}^{\tau}_{\mathrm{RB}} \vdash P_2 = Q_2$, hence $\mathcal{A}^{\tau}_{\mathrm{RB}} \vdash Q_2 = P_2$ by symmetry. Due to soundness (Theorem 5.9), we would get $P_1 \approx_{\mathrm{RB}} Q_1$, hence $Q_1 \approx_{\mathrm{RB}} P_1$ as $\approx_{\mathrm{RB}}$ is symmetric, and $P_2 \approx_{\mathrm{RB}} Q_2$. Since $P_1 \approx_{\mathrm{RB}} P_2$, we would also derive $Q_1 \approx_{\mathrm{RB}} Q_2$ as $\approx_{\mathrm{RB}}$ is transitive. Proving $Q_1 \approx_{\mathrm{RB}} Q_2 \implies \mathcal{A}^{\tau}_{\mathrm{RB}} \vdash Q_1 = Q_2$ would finally entail $\mathcal{A}^{\tau}_{\mathrm{RB}} \vdash P_1 = P_2$ by transitivity.
We proceed by induction on $k = \mathit{size}(P_1) + \mathit{size}(P_2) \in \mathbb{N}_{\geq 2}$:

- If $k = 2$, then from $P_1 \approx_{\mathrm{RB}} P_2$ and P_1 and P_2 in R-nf we derive that both P_1 and P_2 are equal to $\underline{0}$, from which the result follows by reflexivity.

- If $k > 2$, then from $P_1 \approx_{\mathrm{RB}} P_2$ and P_1 and P_2 in R-nf we derive that P_1 is $a^{\dagger}_1 . P'_1$ and P_2 is $a^{\dagger}_2 . P'_2$. There are three cases:

- If $a_1 \neq \tau \neq a_2$, then $a_1 = a_2$ and $P'_1 \approx_{\mathrm{RB}} P'_2$ otherwise $P_1 \approx_{\mathrm{RB}} P_2$ could not hold. From the induction hypothesis we obtain $\mathcal{A}^\tau_{\mathrm{RB}} \vdash P'_1 = P'_2$, hence $\mathcal{A}^\tau_{\mathrm{RB}} \vdash a_1^\dagger . P'_1 = a_2^\dagger . P'_2$ by substitutivity with respect to action prefix.
- If $a_1 = \tau$, then $\mathcal{A}^\tau_{\mathrm{RB}} \vdash a_1^\dagger . P'_1 = P'_1$ by axiom $\mathcal{A}^\tau_5$, hence $a_1^\dagger . P'_1 \approx_{\mathrm{RB}} P'_1$ due to soundness (Theorem 5.9) and $P'_1 \approx_{\mathrm{RB}} a_1^\dagger . P'_1$ as $\approx_{\mathrm{RB}}$ is symmetric. From $P_1 \approx_{\mathrm{RB}} P_2$ it then follows that $P'_1 \approx_{\mathrm{RB}} a_2^\dagger . P'_2$ as $\approx_{\mathrm{RB}}$ is transitive. From the induction hypothesis we obtain $\mathcal{A}^\tau_{\mathrm{RB}} \vdash P'_1 = a_2^\dagger . P'_2$, hence $\mathcal{A}^\tau_{\mathrm{RB}} \vdash a_1^\dagger . P'_1 = a_1^\dagger . a_2^\dagger . P'_2$ by substitutivity with respect to action prefix and $\mathcal{A}^\tau_{\mathrm{RB}} \vdash a_1^\dagger . P'_1 = a_2^\dagger . P'_2$ by axiom $\mathcal{A}^\tau_5$ applied to $a_1^\dagger . a_2^\dagger . P'_2$ and transitivity.
- The case $a_2 = \tau$ is similar to the previous one. ■

Proof of Theorem 5.13.
A straightforward consequence of the axioms and inference rules behind $\vdash$ together with the fact that $\approx_{\mathrm{FRB:ps}}$ is an equivalence relation and a congruence (Theorem 4.3) and the fact that the lefthand side process of each additional axiom in $\mathcal{A}^\tau_{\mathrm{FRB:ps}}$ is $\approx_{\mathrm{FRB:ps}}$-equivalent to the righthand side process of the same axiom. ■

Proof of Lemma 5.15.
Similar to the proof of [9, Lemma 3] (which uses axioms $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$) because, in the considered normal form, τ-actions do not play a role different from the one of visible actions; in particular, neither unexecuted τ-actions nor executed τ-actions are abstracted away. ■

Proof of Theorem 5.17.
The proof is divided into two parts:

- Suppose that $P_1 \approx_{\mathrm{FRB:c}} P_2$. There are two cases:
  - If P_1 and P_2 are initial, it holds that, for all $a \in A$, whenever $P_1 \xrightarrow{a} P'_1$, then $P_2 \xrightarrow{a} P'_2$ and $P'_1 \approx_{\mathrm{FRB}} P'_2$, and vice versa. Since every pair P'_1 and P'_2 is composed of two $\approx_{\mathrm{FRB}}$-equivalent non-initial processes whose only incoming transitions are identically labeled and respectively depart from the two initial processes P_1 and P_2, it follows that $P_1 \approx_{\mathrm{FRB:ps}} P_2$ (and $P'_1 \approx_{\mathrm{FRB:ps}} P'_2$ for all those pairs).
  - If P_1 and P_2 are not initial, then $P_1 \approx_{\mathrm{FRB}} P_2$ and *to_initial*$(P_1) \approx_{\mathrm{FRB:c}}$ *to_initial*(P_2). While stepwise mimicking each other behavior in the forward direction, P_1 and P_2 can only encounter pairs of non-initial processes related by $\approx_{\mathrm{FRB}}$. By virtue of *to_initial*$(P_1) \approx_{\mathrm{FRB:c}}$ *to_initial*(P_2), while stepwise mimicking each other behavior in the backward direction, there is a way for P_1 and P_2 not to respectively end up in an initial process and a non-initial process. In conclusion, $P_1 \approx_{\mathrm{FRB:ps}} P_2$.
- Suppose that $P_1 \approx_{\mathrm{FRB:ps}} P_2$. There are two cases:
  - If P_1 and P_2 are initial, whenever P_1 has a τ-transition to a non-initial process that is $\approx_{\mathrm{FRB}}$-equivalent to P_2, then P_2 must have a τ-transition to a non-initial process that is $\approx_{\mathrm{FRB}}$-equivalent to P_1, and vice versa, otherwise $P_1 \approx_{\mathrm{FRB:ps}} P_2$ would be contradicted. Therefore, for all $a \in A$, whenever $P_1 \xrightarrow{a} P'_1$, then $P_2 \xrightarrow{a} P'_2$ and $P'_1 \approx_{\mathrm{FRB}} P'_2$, and vice versa, i.e., $P_1 \approx_{\mathrm{FRB:c}} P_2$.

- Let P_1 and P_2 be not initial. On the one hand, we have that $P_1 \approx_{\rm FRB:ps} P_2$ implies $P_1 \approx_{\rm FRB} P_2$. On the other hand, from $P_1 \approx_{\rm FRB:ps} P_2$ it follows that, while stepwise mimicking each other behavior in the backward direction, there is a way for P_1 and P_2 not to respectively end up in an initial process and a non-initial process. Therefore $\mathit{to\_initial}(P_1) \approx_{\rm FRB:ps} \mathit{to\_initial}(P_2)$ and hence $\mathit{to\_initial}(P_1) \approx_{\rm FRB:c} \mathit{to\_initial}(P_2)$ due to what we have proved in the first case of the first part of the proof. In conclusion, $P_1 \approx_{\rm FRB:c} P_2$. ■

Proof of Lemma 5.18.
Suppose that P_1 and P_2 are both in FR-nf. Should this not be the case, thanks to Lemma 5.15 we could find Q_1 and Q_2 in FR-nf such that $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash P_1 = Q_1$ and $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash P_2 = Q_2$, hence $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash Q_2 = P_2$ by symmetry. Due to soundness (Theorem 5.13), we would get $P_1 \approx_{\rm FRB:ps} Q_1$, hence $Q_1 \approx_{\rm FRB:ps} P_1$ as $\approx_{\rm FRB:ps}$ is symmetric, and $P_2 \approx_{\rm FRB:ps} Q_2$. Therefore $Q_1 \approx_{\rm FRB} P_1$ and $P_2 \approx_{\rm FRB} Q_2$ because $\approx_{\rm FRB:ps}$ is contained in $\approx_{\rm FRB}$. From $P_1 \approx_{\rm FRB} P_2$ we would then get $Q_1 \approx_{\rm FRB} Q_2$ as $\approx_{\rm FRB}$ is transitive. Since P_1, P_2, Q_1, Q_2 are initial and $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash P_1 = Q_1 \implies a \,.\, P_1 = a \,.\, Q_1$ and $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash Q_2 = P_2 \implies a \,.\, Q_2 = a \,.\, P_2$ by substitutivity with respect to action prefix, proving $Q_1 \approx_{\rm FRB} Q_2 \implies \mathcal{A}^{\tau}_{\rm FRB:ps} \vdash a \,.\, Q_1 = a \,.\, Q_2$ would finally entail $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash a \,.\, P_1 = a \,.\, P_2$ by transitivity.
We proceed by induction on $k = \mathit{size}(P_1) + \mathit{size}(P_2) \in \mathbb{N}_{\geq 2}$:

- If $k = 2$, then from $P_1 \approx_{\rm FRB} P_2$ and P_1 and P_2 in FR-nf we derive that both P_1 and P_2 are equal to $\underline{0}$, from which the result follows by reflexivity and substitutivity with respect to action prefix.

- Let $k > 2$, so that P_1 is $\sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}$ or $\underline{0}$ and P_2 is $\sum_{i \in I_2} a_{2,i} \,.\, P_{2,i}$ or $\underline{0}$, where every $P_{1,i}$ and every $P_{2,i}$ is initial and in FR-nf (when either process is $\underline{0}$, all the actions of the other process must be τ). For the sake of uniformity, also $\underline{0}$ will be denoted as a summation, in which the index set is empty. Consider the following two conditions:

  1. There exists $i \in I_1$ such that $a_{1,i} = \tau$ and $P_{1,i} \approx_{\rm FRB} P_2$.
  2. There exists $i \in I_2$ such that $a_{2,i} = \tau$ and $P_{2,i} \approx_{\rm FRB} P_1$.

  We distinguish three cases:

  - Suppose that neither condition 1 nor condition 2 holds. Since $P_1 \approx_{\rm FRB} P_2$, whenever for some $a_{1,i_1} = b$ we have $P_1 \xrightarrow{b} b^{\dagger} \,.\, P_{1,i_1} + \sum_{i \in I_1 \setminus \{i_1\}} a_{1,i} \,.\, P_{1,i}$, then for some $a_{2,i_2} = b$ it must be $P_2 \xrightarrow{b} b^{\dagger} \,.\, P_{2,i_2} + \sum_{i \in I_2 \setminus \{i_2\}} a_{2,i} \,.\, P_{2,i}$ where $b^{\dagger} \,.\, P_{1,i_1} + \sum_{i \in I_1 \setminus \{i_1\}} a_{1,i} \,.\, P_{1,i} \approx_{\rm FRB} b^{\dagger} \,.\, P_{2,i_2} + \sum_{i \in I_2 \setminus \{i_2\}} a_{2,i} \,.\, P_{2,i}$, and vice versa (note that P_2 – resp. P_1 – cannot idle when $b = \tau$).
    Since every pair of $\approx_{\rm FRB}$-equivalent reached processes is composed of two non-initial processes whose only incoming transitions are identically labeled and respectively depart from the two $\approx_{\rm FRB}$-equivalent initial processes P_1 and P_2, we have that $P_{1,i_1} = \mathit{to\_forward}(b^{\dagger} \,.\, P_{1,i_1} + \sum_{i \in I_1 \setminus \{i_1\}} a_{1,i} \,.\, P_{1,i}) \approx_{\rm FRB} \mathit{to\_forward}(b^{\dagger} \,.\, P_{2,i_2} + \sum_{i \in I_2 \setminus \{i_2\}} a_{2,i} \,.\, P_{2,i}) = P_{2,i_2}$. From the induction hypothesis it follows that $\mathcal{A}^{\tau}_{\rm FRB:ps}$

$\vdash a_{1,i_1} \,.\, P_{1,i_1} = a_{2,i_2} \,.\, P_{2,i_2}$, hence $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_1 = P_2$ by substitutivity with respect to alternative composition and, in the presence of identical summands on the same side, axiom $\mathcal{A}_{10}$ possibly preceded by applications of axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ to move identical summands next to each other. Finally $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash a \,.\, P_1 = a \,.\, P_2$ by substitutivity with respect to action prefix.

– Suppose that both condition 1 and condition 2 hold. Then there exist $i_1 \in I_1$ and $i_2 \in I_2$ such that $a_{1,i_1} = \tau = a_{2,i_2}$ and $P_{1,i_1} \approx_{\mathrm{FRB}} P_2 \approx_{\mathrm{FRB}} P_1 \approx_{\mathrm{FRB}} P_{2,i_2}$, hence $P_{1,i_1} \approx_{\mathrm{FRB}} P_{2,i_2}$, where we have exploited the fact that $\approx_{\mathrm{FRB}}$ is symmetric and transitive. Since the considered chain of equalities can be rewritten as $P_1 \approx_{\mathrm{FRB}} P_{2,i_2} \approx_{\mathrm{FRB}} P_{1,i_1} \approx_{\mathrm{FRB}} P_2$ by virtue of the same two properties, from the induction hypothesis and transitivity it follows that $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash a \,.\, P_1 = a \,.\, P_{2,i_2} = a \,.\, P_{1,i_1} = a \,.\, P_2$.

– Suppose that only one of the two conditions holds, say condition 1. For every summand $\tau \,.\, P_{1,i}$ of P_1 such that $P_{1,i} \approx_{\mathrm{FRB}} P_2$ it holds that $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash \tau \,.\, P_{1,i} = \tau \,.\, P_2$ by the induction hypothesis. Indicating with P'_1 the summation of all the other summands of P_1 – for each of which $a_{1,i} \neq \tau$ or $P_{1,i} \not\approx_{\mathrm{FRB}} P_2$ – we obtain $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_1 = \tau \,.\, P_2 + P'_1$ by substitutivity with respect to alternative composition and, in the presence of identical summands on the righthand side, axiom $\mathcal{A}_{10}$ possibly preceded by applications of axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ to move identical summands next to each other, hence $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash a \,.\, P_1 = a \,.\, (\tau \,.\, P_2 + P'_1)$ by substitutivity with respect to action prefix.
Since $P_1 \approx_{\mathrm{FRB}} P_2$, condition 1 does not hold over P'_1, and condition 2 does not hold (over P_2), similar to the first case for each summand $a_{1,i_1} \,.\, P_{1,i_1}$ of P'_1 there must be a summand $a_{2,i_2} \,.\, P_{2,i_2}$ of P_2 such that $a_{1,i_1} = a_{2,i_2}$ and $P_{1,i_1} \approx_{\mathrm{FRB}} P_{2,i_2}$, and vice versa, hence $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash a_{1,i_1} \,.\, P_{1,i_1} = a_{2,i_2} \,.\, P_{2,i_2}$ by the induction hypothesis. Indicating with P'_2 the summation of all the other summands of P_2 – none of which matches a summand of P'_1 – we obtain $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_2 = P'_2 + P'_1$ by substitutivity with respect to alternative composition.
Therefore $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash a \,.\, P_1 = a \,.\, (\tau \,.\, P_2 + P'_1) = a \,.\, (\tau \,.\, (P'_2 + P'_1) + P'_1) = a \,.\, (P'_2 + P'_1) = a \,.\, P_2$ by substitutivity, $\mathcal{A}^{\tau}_6$, and transitivity.
[Example: $P_1 \triangleq \tau \,.\, (b \,.\, \underline{0} + c \,.\, \underline{0} + d \,.\, \underline{0}) + d \,.\, \underline{0}$, $P_2 \triangleq b \,.\, \underline{0} + c \,.\, \underline{0} + d \,.\, \underline{0}$.] ■

Proof of Theorem 5.19.
Suppose that P_1 and P_2 are both in FR-nf. Should this not be the case, thanks to Lemma 5.15 we could find Q_1 and Q_2 in FR-nf such that $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_1 = Q_1$ and $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_2 = Q_2$, hence $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash Q_2 = P_2$ by symmetry. Due to soundness (Theorem 5.13), we would get $P_1 \approx_{\mathrm{FRB:ps}} Q_1$, hence $Q_1 \approx_{\mathrm{FRB:ps}} P_1$ as $\approx_{\mathrm{FRB:ps}}$ is symmetric, and $P_2 \approx_{\mathrm{FRB:ps}} Q_2$. Since $P_1 \approx_{\mathrm{FRB:ps}} P_2$, we would also get $Q_1 \approx_{\mathrm{FRB:ps}} Q_2$ as $\approx_{\mathrm{FRB:ps}}$ is transitive. Proving $Q_1 \approx_{\mathrm{FRB:ps}} Q_2 \implies \mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash Q_1 = Q_2$ would finally entail $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_1 = P_2$ by transitivity.
There are two cases based on $k = \mathit{size}(P_1) + \mathit{size}(P_2) \in \mathbb{N}_{\geq 2}$:

- If $k = 2$, then from $P_1 \approx_{\mathrm{FRB:ps}} P_2$ and P_1 and P_2 in FR-nf we derive that both P_1 and P_2 are equal to $\underline{0}$, from which the result follows by reflexivity.

- If $k > 2$, then from $P_1 \approx_{\mathrm{FRB:ps}} P_2$ and P_1 and P_2 in FR-nf we derive that P_1 is $\sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}$ and P_2 is $\sum_{i \in I_2} a_{2,i} \,.\, P_{2,i}$, where every $P_{1,i}$ and every $P_{2,i}$ is initial and in FR-nf. Since $P_1 \approx_{\mathrm{FRB:ps}} P_2$ is the same as $P_1 \approx_{\mathrm{FRB:c}} P_2$ due to Theorem 5.17, whenever for some $a_{1,i_1} = a$ we have $P_1 \xrightarrow{a} a^{\dagger} . P_{1,i_1} + \sum_{i \in I_1 \setminus \{i_1\}} a_{1,i} \,.\, P_{1,i}$, then for some $a_{2,i_2} = a$ we have $P_2 \xrightarrow{a} a^{\dagger} . P_{2,i_2} + \sum_{i \in I_2 \setminus \{i_2\}} a_{2,i} \,.\, P_{2,i}$ where $a^{\dagger} . P_{1,i_1} + \sum_{i \in I_1 \setminus \{i_1\}} a_{1,i} \,.\, P_{1,i} \approx_{\mathrm{FRB}} a^{\dagger} . P_{2,i_2} + \sum_{i \in I_2 \setminus \{i_2\}} a_{2,i} \,.\, P_{2,i}$, and vice versa.
  Since every pair of $\approx_{\mathrm{FRB}}$-equivalent reached processes is composed of two non-initial processes whose only incoming transitions are identically labeled and respectively depart from the two equivalent initial processes P_1 and P_2, we have that $P_{1,i_1} = \mathit{to\_forward}(a^{\dagger} . P_{1,i_1} + \sum_{i \in I_1 \setminus \{i_1\}} a_{1,i} \,.\, P_{1,i}) \approx_{\mathrm{FRB}} \mathit{to\_forward}(a^{\dagger} . P_{2,i_2} + \sum_{i \in I_2 \setminus \{i_2\}} a_{2,i} \,.\, P_{2,i}) = P_{2,i_2}$. Since P_{1,i_1} and P_{2,i_2} are initial, $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash a_{1,i_1} \,.\, P_{1,i_1} = a_{2,i_2} \,.\, P_{2,i_2}$ by Lemma 5.18 and hence $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_1 = P_2$ by substitutivity with respect to alternative composition and, in the presence of identical summands on the same side, axiom $\mathcal{A}_{10}$ possibly preceded by applications of axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ to move identical summands next to each other. ■

Proof of Lemma 5.20.
Suppose that P_1 and P_2 are both in FR-nf. Should this not be the case, thanks to Lemma 5.15 we could find Q_1 and Q_2 in FR-nf such that $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_1 = Q_1$ and $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_2 = Q_2$, hence $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash Q_2 = P_2$ by symmetry. Due to soundness (Theorem 5.13), we would get $P_1 \approx_{\mathrm{FRB:ps}} Q_1$, hence $Q_1 \approx_{\mathrm{FRB:ps}} P_1$ as $\approx_{\mathrm{FRB:ps}}$ is symmetric, and $P_2 \approx_{\mathrm{FRB:ps}} Q_2$. Therefore $Q_1 \approx_{\mathrm{FRB}} P_1$ and $P_2 \approx_{\mathrm{FRB}} Q_2$ because $\approx_{\mathrm{FRB:ps}}$ is contained in $\approx_{\mathrm{FRB}}$. From $P_1 \approx_{\mathrm{FRB}} P_2$ we would then get $Q_1 \approx_{\mathrm{FRB}} Q_2$ as $\approx_{\mathrm{FRB}}$ is transitive. Since $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash P_1 = Q_1 \implies a^{\dagger} . P_1 = a^{\dagger} . Q_1$ and $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash Q_2 = P_2 \implies a^{\dagger} . Q_2 = a^{\dagger} . P_2$ by substitutivity with respect to action prefix, proving $Q_1 \approx_{\mathrm{FRB}} Q_2 \implies \mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash a^{\dagger} . Q_1 = a^{\dagger} . Q_2$ would finally entail $\mathcal{A}^{\tau}_{\mathrm{FRB:ps}} \vdash a^{\dagger} . P_1 = a^{\dagger} . P_2$ by transitivity.
We proceed by induction on $k = \mathit{size}(P_1) + \mathit{size}(P_2) \in \mathbb{N}_{\geq 2}$:

- If $k = 2$, then from $P_1 \approx_{\mathrm{FRB}} P_2$ and P_1 and P_2 in FR-nf we derive that both P_1 and P_2 are equal to $\underline{0}$, from which the result follows by reflexivity and substitutivity with respect to action prefix.

- Let $k > 2$ with P_1 being $\sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}$ or $\underline{0}$ and P_2 being $\sum_{i \in I_2} a_{2,i} \,.\, P_{2,i}$ or $\underline{0}$, where every $P_{1,i}$ and every $P_{2,i}$ is initial and in FR-nf (when either process is $\underline{0}$, all the actions of the other process must be τ). The proof is similar to the one of the corresponding case in the proof of Lemma 5.18, with the use of $a^{\dagger}$ in place of a and the final application of $\mathcal{A}^{\tau}_{7}$ in lieu of $\mathcal{A}^{\tau}_{6}$.

- Let $k > 2$ with P_1 being $a_1^{\dagger} . P'_1$ or $a_1^{\dagger} . P'_1 + \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}$ and P_2 being $a_2^{\dagger} . P'_2$ or $a_2^{\dagger} . P'_2 + \sum_{i \in I_2} a_{2,i} \,.\, P_{2,i}$, where P'_1 and P'_2 are in FR-nf, every $P_{1,i}$ and every $P_{2,i}$ is initial and in FR-nf, $\mathit{to\_initial}(a_1^{\dagger} . P'_1) \approx_{\mathrm{FRB}} \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}$ so that $a_1^{\dagger} . P'_1 + \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i} \approx_{\mathrm{FRB}} a_1^{\dagger} . P'_1$ by the soundness of axiom $\mathcal{A}_{10}$ (Theorem 5.13) as $\approx_{\mathrm{FRB:ps}}$ is contained in $\approx_{\mathrm{FRB}}$, and $\mathit{to\_initial}(a_2^{\dagger} . P'_2) \approx_{\mathrm{FRB}} \sum_{i \in I_2} a_{2,i} \,.\, P_{2,i}$ so that $a_2^{\dagger} . P'_2 + \sum_{i \in I_2} a_{2,i} \,.\, P_{2,i} \approx_{\mathrm{FRB}} a_2^{\dagger} . P'_2$ for the same reason. There are two cases:

– If $a_1 = a_2$, then $P'_1 \approx_{\rm FRB} P'_2$ otherwise $P_1 \approx_{\rm FRB} P_2$ could not hold. Therefore $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash a^{\dagger}_1 . P'_1 = a^{\dagger}_2 . P'_2$ by the induction hypothesis.

– If $a_1 \neq a_2$, from $P_1 \approx_{\rm FRB} P_2$ it follows that either action is τ, say a_1, while the other action is observable. Then $P'_1 \approx_{\rm FRB} P_2$ otherwise $P_1 \approx_{\rm FRB} P_2$ could not hold. Therefore $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash \tau^{\dagger} . P'_1 = \tau^{\dagger} . P_2$ by the induction hypothesis, hence $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash a^{\dagger} . \tau^{\dagger} . P'_1 = a^{\dagger} . \tau^{\dagger} . P_2$ by substitutivity with respect to action prefix and then $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash a^{\dagger} . P_1 = a^{\dagger} . P_2$ by axiom $\mathcal{A}^{\tau}_8$ applied to the righthand side and transitivity.

- Let $k > 2$ with P_1 being $a^{\dagger}_1 . P'_1 + \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}$ and P_2 being $a^{\dagger}_2 . P'_2 + \sum_{i \in I_2} a_{2,i} \,.\, P_{2,i}$, where P'_1 and P'_2 are in FR-nf, every $P_{1,i}$ and every $P_{2,i}$ is initial and in FR-nf, *to_initial*$(a^{\dagger}_1 . P'_1) \not\approx_{\rm FRB} \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}$, and *to_initial*$(a^{\dagger}_2 . P'_2) \not\approx_{\rm FRB} \sum_{i \in I_2} a_{2,i} \,.\, P_{2,i}$ (note that if it were $\not\approx_{\rm FRB}$ inside either process, then $P_1 \approx_{\rm FRB} P_2$ could not hold). Observing that only $a^{\dagger}_1 . P'_1$ and $a^{\dagger}_2 . P'_2$ can move but, after going back to P_1 and P_2, also $\sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}$ and $\sum_{i \in I_2} a_{2,i} \,.\, P_{2,i}$ can move, there are two cases:

  – If every τ-summand of *to_initial*(P_1) has a $\approx_{\rm FRB}$-matching τ-summand of *to_initial*(P_2) and vice versa, then $a^{\dagger}_1 . P'_1 \approx_{\rm FRB:ps} a^{\dagger}_2 . P'_2$, hence $a_1 = a_2$ and $P'_1 \approx_{\rm FRB:ps} P'_2$, as well as $\sum_{i \in I_1} a_{1,i} \,.\, P_{1,i} \approx_{\rm FRB:ps} \sum_{i \in I_2} a_{2,i} \,.\, P_{2,i}$, hence $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i} = \sum_{i \in I_2} a_{2,i} \,.\, P_{2,i}$ by completeness (Theorem 5.19). Therefore $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash a^{\dagger}_1 . P'_1 = a^{\dagger}_2 . P'_2$ by the induction hypothesis, hence $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash a^{\dagger}_1 . P'_1 + \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i} = a^{\dagger}_2 . P'_2 + \sum_{i \in I_2} a_{2,i} \,.\, P_{2,i}$ by substitutivity with respect to alternative composition and then $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash a^{\dagger} . (a^{\dagger}_1 . P'_1 + \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}) = a^{\dagger} . (a^{\dagger}_2 . P'_2 + \sum_{i \in I_2} a_{2,i} \,.\, P_{2,i})$ by substitutivity with respect to action prefix.

  – Otherwise any other τ-summand of *to_initial*(P_1) must be such that its continuation is $\approx_{\rm FRB}$-equivalent to *to_initial*(P_2) or vice versa, where we can exploit the soundness of axiom $\mathcal{A}_{10}$ (Theorem 5.13) as $\approx_{\rm FRB:ps}$ is contained in $\approx_{\rm FRB}$ to reduce the summation of all such τ-summands to a single one. Such a single τ-summand can occur in either process and each of the other summands in that process must be $\approx_{\rm FRB:ps}$-equivalent to one of the summands of the other process. There are two subcases:

    * If $a_1 = \tau$ and $a_2 \neq \tau$, so that $P'_1 \approx_{\rm FRB} P_2$, or vice versa, we have that $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash \tau^{\dagger} . P'_1 = \tau^{\dagger} . P_2$ by the induction hypothesis, hence $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash \tau^{\dagger} . P'_1 + \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i} = \tau^{\dagger} . P_2 + \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}$ by substitutivity with respect to alternative composition and then $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash a^{\dagger} . (\tau^{\dagger} . P'_1 + \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}) = a^{\dagger} . (\tau^{\dagger} . P_2 + \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i})$ by substitutivity with respect to action prefix. Due to completeness (Theorem 5.19) and substitutivity with respect to alternative composition, $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash P_2 = a^{\dagger}_2 . P'_2 + P''_2 + \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}$ where P''_2 is the summation of the initial summands of P_2 not $\approx_{\rm FRB:ps}$-equivalent to any of the initial summands of P_1. Therefore $\mathcal{A}^{\tau}_{\rm FRB:ps} \vdash a^{\dagger} . (\tau^{\dagger} . P'_1 + \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}) = a^{\dagger} . (\tau^{\dagger} . (a^{\dagger}_2 . P'_2 + P''_2 + \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}) + \sum_{i \in I_1} a_{1,i} \,.\, P_{1,i}) =$

$a^\dagger . (a_2^\dagger . P_2' + P_2'' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i})$ by substitutivity, axiom $\mathcal{A}_8^\tau$, and transitivity.
[Example: $P_1 \triangleq \tau^\dagger . (b^\dagger . \underline{0} + c \, . \, \underline{0} + d \, . \, \underline{0}) + d \, . \, \underline{0}$, $P_2 \triangleq b^\dagger . \underline{0} + c \, . \, \underline{0} + d \, . \, \underline{0}$.]

* If $a_1 = a_2$, so that $P_1' \approx_{\rm FRB} P_2'$, and the aforementioned single τ-summand occurs in *to_initial*(P_1), or viceversa, we have that $\mathcal{A}_{\rm FRB:ps}^\tau \vdash a_1^\dagger . P_1' = a_2^\dagger . P_2'$ by the induction hypothesis. Since the occurrence of that τ-summand in *to_initial*(P_1), specifically in $\sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}$, implies $\sum_{i \in I_1} a_{1,i} \, . \, P_{1,i} \approx_{\rm FRB:ps} \tau \, . \, (\textit{to\_initial}(a_2^\dagger . P_2') + \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i})$, we have that $\mathcal{A}_{\rm FRB:ps}^\tau \vdash \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i} = \tau \, . \, (\textit{to\_initial}(a_2^\dagger . P_2') + \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i})$ by completeness (Theorem 5.19). Therefore $\mathcal{A}_{\rm FRB:ps}^\tau \vdash a_1^\dagger . P_1' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i} = a_2^\dagger . P_2' + \tau \, . \, (\textit{to\_initial}(a_2^\dagger . P_2') + \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i})$ by substitutivity with respect to alternative composition, hence $\mathcal{A}_{\rm FRB:ps}^\tau \vdash a^\dagger . (a_1^\dagger . P_1' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}) = a^\dagger . (a_2^\dagger . P_2' + \tau \, . \, (\textit{to\_initial}(a_2^\dagger . P_2') + \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i}))$ by substitutivity with respect to action prefix and then $\mathcal{A}_{\rm FRB:ps}^\tau \vdash a^\dagger . (a_1^\dagger . P_1' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}) = a^\dagger . (a_2^\dagger . P_2' + \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i})$ by axiom $\mathcal{A}_7^\tau$ applied to the righthand side and transitivity.
[Example: $P_1 \triangleq d^\dagger . \underline{0} + \tau \, . \, (d \, . \, \underline{0} + b \, . \, \underline{0} + c \, . \, \underline{0})$, $P_2 \triangleq d^\dagger . \underline{0} + b \, . \, \underline{0} + c \, . \, \underline{0}$.]

- Let $k > 2$ with P_1 being $a_1^\dagger . P_1' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}$ and P_2 being $\sum_{i \in I_2} a_{2,i} \, . \, P_{2,i}$, or vice versa, where P_1' is in FR-nf, every $P_{1,i}$ and every $P_{2,i}$ is initial and in FR-nf, and with abuse of notation I_1 and I_2 can be empty in which case the I_1-related summation disappears while the I_2-related summation is $\underline{0}$.
It must be $a_1 = \tau$ otherwise $P_1 \approx_{\rm FRB} P_2$ could not hold, so that $P_1' \approx_{\rm FRB} P_2$ and each of the initial summands of P_1 must be $\approx_{\rm FRB:ps}$-equivalent to one of the initial summands of P_2. Therefore $\mathcal{A}_{\rm FRB:ps}^\tau \vdash \tau^\dagger . P_1' = \tau^\dagger . P_2$ by the induction hypothesis, hence $\mathcal{A}_{\rm FRB:ps}^\tau \vdash \tau^\dagger . P_1' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i} = \tau^\dagger . P_2 + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}$ by substitutivity with respect to alternative composition and then $\mathcal{A}_{\rm FRB:ps}^\tau \vdash a^\dagger . (\tau^\dagger . P_1' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}) = a^\dagger . (\tau^\dagger . P_2 + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i})$ by substitutivity with respect to action prefix.
Due to completeness (Theorem 5.19) and substitutivity with respect to alternative composition, $\mathcal{A}_{\rm FRB:ps}^\tau \vdash P_2 = P_2'' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}$ where P_2'' is the summation of the initial summands of P_2 not $\approx_{\rm FRB:ps}$-equivalent to any of the initial summands of P_1. Therefore $\mathcal{A}_{\rm FRB:ps}^\tau \vdash a^\dagger . (\tau^\dagger . P_1' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}) = a^\dagger . (\tau^\dagger . (P_2'' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}) + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}) = a^\dagger . (P_2'' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i})$ by substitutivity, axiom $\mathcal{A}_8^\tau$, and transitivity.
[Example: $P_1 \triangleq \tau^\dagger . (b \, . \, \underline{0} + c \, . \, \underline{0} + d \, . \, \underline{0}) + d \, . \, \underline{0}$, $P_2 \triangleq b \, . \, \underline{0} + c \, . \, \underline{0} + d \, . \, \underline{0}$.] ■

Proof of Theorem 5.21.
Suppose that P_1 and P_2 are both in FR-nf as done in the proof of Theorem 5.19. There are two cases:

- Let P_1 be $a_1^\dagger . P_1'$ or $a_1^\dagger . P_1' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}$ and P_2 be $a_2^\dagger . P_2'$ or $a_2^\dagger . P_2' + \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i}$, where P_1' and P_2' are in FR-nf, every $P_{1,i}$ and every $P_{2,i}$ is initial and in FR-nf, *to_initial*$(a_1^\dagger . P_1') \approx_{\rm FRB:ps} \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}$ so that $a_1^\dagger . P_1' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i} \approx_{\rm FRB:ps} a_1^\dagger . P_1'$ by the soundness of axiom $\mathcal{A}_{10}$ (Theorem 5.13), and *to_initial*$(a_2^\dagger . P_2') \approx_{\rm FRB:ps} \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i}$ so that

$a_2^\dagger . P_2' + \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i} \approx_{\mathrm{FRB:ps}} a_2^\dagger . P_2'$ for the same reason.
Since $P_1 \approx_{\mathrm{FRB:ps}} P_2$ is the same as $P_1 \approx_{\mathrm{FRB:c}} P_2$ due to Theorem 5.17, from the fact that P_1 and P_2 are not initial it follows that *to_initial*$(P_1) \approx_{\mathrm{FRB:c}}$ *to_initial*(P_2) and hence $a_1 = a_2$ with *to_initial*$(P_1') \approx_{\mathrm{FRB}}$ *to_initial*(P_2'), so that $P_1' \approx_{\mathrm{FRB}} P_2'$ otherwise $P_1 \approx_{\mathrm{FRB:ps}} P_2$ could not hold. As a consequence $\mathcal{A}^\tau_{\mathrm{FRB:ps}} \vdash a_1^\dagger . P_1' = a_2^\dagger . P_2'$ by Lemma 5.20.

- Let P_1 be $a_1^\dagger . P_1' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}$ and P_2 be $a_2^\dagger . P_2' + \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i}$, where P_1' and P_2' are in FR-nf, every $P_{1,i}$ and every $P_{2,i}$ is initial and in FR-nf, *to_initial*$(a_1^\dagger . P_1') \not\approx_{\mathrm{FRB:ps}} \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}$, and *to_initial*$(a_2^\dagger . P_2') \not\approx_{\mathrm{FRB:ps}} \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i}$ (note that if it were $\not\approx_{\mathrm{FRB:ps}}$ inside either process, then $P_1 \approx_{\mathrm{FRB:ps}} P_2$ could not hold).
  Observing that only $a_1^\dagger . P_1'$ and $a_2^\dagger . P_2'$ can move and, after going back to P_1 and P_2, also $\sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}$ and $\sum_{i \in I_2} a_{2,i} \, . \, P_{2,i}$ can move but it holds that *to_initial*$(a_1^\dagger . P_1') \not\approx_{\mathrm{FRB:ps}} \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i}$ and *to_initial*$(a_2^\dagger . P_2') \not\approx_{\mathrm{FRB:ps}} \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i}$, from $P_1 \approx_{\mathrm{FRB:ps}} P_2$ it follows that $a_1 = a_2$ with $P_1' \approx_{\mathrm{FRB}} P_2'$ and $\sum_{i \in I_1} a_{1,i} \, . \, P_{1,i} \approx_{\mathrm{FRB:ps}} \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i}$. Therefore $\mathcal{A}^\tau_{\mathrm{FRB:ps}} \vdash a_1^\dagger . P_1' = a_2^\dagger . P_2'$ by Lemma 5.20 and $\mathcal{A}^\tau_{\mathrm{FRB:ps}} \vdash \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i} = \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i}$ by Theorem 5.19, hence $\mathcal{A}^\tau_{\mathrm{FRB:ps}} \vdash a_1^\dagger . P_1' + \sum_{i \in I_1} a_{1,i} \, . \, P_{1,i} = a_2^\dagger . P_2' + \sum_{i \in I_2} a_{2,i} \, . \, P_{2,i}$ by substitutivity with respect to alternative composition. ■