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A Survey of Modal Logics Characterizing Behavioral Equivalences for Nondeterministic and Stochastic Systems

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Behavioral equivalences are a means to establish whether computing systems possess the same properties. The specific set of properties that are preserved by a specific behavioral equivalence clearly depends on how the system behavior is observed and can usually be characterized by means of a modal logic. In this paper we consider three different approaches to the definition of behavioral equivalences – bisimulation, testing, and trace – applied to three different classes of systems – nondeterministic, probabilistic, and Markovian – and we survey the nine resulting modal logic characterizations, each of which stems from the Hennessy-Milner logic. We then compare the nine characterizations with respect to the logical operators, in order to emphasize differences across the three approaches to the definition of behavioral equivalences and regularities within each of the three approaches. In the probabilistic and Markovian cases we also address the issue of whether the probabilistic and temporal aspects should be treated in a local or global way and consequently whether the modal logic interpretation should be qualitative or quantitative.

Contents

1. Introduction

Behavioral equivalences are a means to establish whether computing systems possess the same properties. Among the numerous approaches to the definition of such behavioral equivalences (van Glabbeek 2001), three of them have received a particular attention: bisimulation (Milner 1989), testing (De Nicola and Hennessy 1983), and trace (Hoare 1985). The basic idea behind bisimulation equivalence is to capture whether two systems are able to mimic each other's behavior stepwise. In the case of testing equivalence, two systems are considered to be equivalent if an external observer cannot distinguish between them, with the only way that the observer has to infer information about the behavior of a system being to interact with it by means of tests. Trace equivalence directly considers the computations of the systems taken in isolation, thus abstracting from the branching points of their behavior.

In the literature we find several variants of these three approaches originally defined for nondeterministic systems, which take into account additional information about the probabilistic and temporal aspects of the system behavior. Among them we mention the probabilistic variants of bisimulation, testing and trace equivalences – see e.g. (Larsen and Skou 1991; van Glabbeek et al. 1995; Christoff 1990; Cleaveland et al. 1999; Jou and Smolka 1990) – as well as their Markovian variants – see e.g. (Hillston 1996; Hermanns 2002; Bernardo and Bravetti 2003; Bernardo and Cleaveland 2000; Wolf et al. 2005; Bernardo 2007).

The specific set of properties that are preserved by a specific behavioral equivalence clearly depends on how the system behavior is observed and can usually be characterized by means of a modal logic. In this paper we consider the nine behavioral equivalences resulting from the combination of the bisimulation, testing and trace approaches with the nondeterministic, probabilistic and Markovian classes of systems and we present the nine corresponding modal languages.

The purpose of this paper is to survey in a uniform framework the modal logic characterizations that are known in the literature, and to contribute with some new results for probabilistic and Markovian testing/trace equivalences. All the nine modal logic characterizations, which stem from the Hennessy-Milner logic (Hennessy and Milner 1985), are then compared with respect to the logical operators they rely on, in order to emphasize differences across the three approaches to the definition of behavioral equivalences and regularities within each of the three approaches. We shall see that the number of logical operators that are needed tends to diminish as the distinguishing power decreases. Furthermore we shall observe that, unlike bisimulation and trace equivalences, the nondeterministic, probabilistic and Markovian versions of testing equivalence are not characterized with the same logical operators.

In the probabilistic and Markovian cases we also address the issue of whether the probabilistic and temporal aspects should be treated in a local or global way. According to the local view, the probabilistic and temporal aspects should decorate the modal operators and the resulting modal logic should be given the usual qualitative interpretation. By contrast, in the global view the probabilistic and temporal aspects should be associated with the overall system computations, so the modal logic syntax should be unchanged but its interpretation should become quantitative. We shall see that the local view is appropriate for bisimulation equivalence, while the global view is appropriate for testing and trace equivalences.

This paper, which is an extended and revised version of (Bernardo and Botta 2006), is organized as follows. In Sect. 2 we introduce three process calculi, which generate all the nondeterministic, probabilistic and Markovian finite-state processes with as few operators as possible. In Sect. 3 we recall the bisimulation, testing, and trace approaches to the definition of behavioral equivalences, by showing for each of them the nondeterministic version, the probabilistic version, and the Markovian version. In Sect. 4, 5, and 6 we present the modal logic characterizations of the nine resulting behavioral equivalences. Finally, in Sect. 7 we compare the nine modal logic characterizations with respect to the logical operators they rely on, then we conclude by mentioning some future work.

2. Process Calculi

In this section we introduce a nondeterministic process calculus, a probabilistic process calculus, and a Markovian process calculus, which generate all the nondeterministic, probabilistic and Markovian finite-state processes with as few operators as possible: the null term, the action prefix operator, the alternative composition operator, and recursion. For each calculus we shall present the syntax and the structural operational semantics. As far as actions are concerned, in the following we denote by *Name* the set of the action names and we assume that all the actions are observable.

2.1. *Nondeterministic Process Calculus*

In the nondeterministic process calculus (NPC for short) the choice among all the actions that are simultaneously enabled at the same state is nondeterministic. We denote by $Act_{\rm N} = Name$ the set of the actions of NPC.

Definition 2.1. The set of the process terms of NPC is generated by the following syntax:

$$P ::= \underline{0} \mid a.P \mid P + P \mid A$$

where $\underline{0}$ is the null term, $a \in Act_{\rm N}$, and A is a process constant defined through the (possibly recursive) equation $A \triangleq P$. We denote by $\mathcal{P}_{\rm N}$ the set of the closed and guarded process terms of NPC. ■

The semantics for NPC can be defined in the usual operational style. As a consequence, the behavior of each process term is given by a transition system, whose states correspond to process terms and whose transitions are labeled with actions. Observed that the null term $\underline{0}$ cannot execute any action – hence the corresponding labeled transition system is just a state with no transitions – we now provide the semantic rules for the various operators of NPC:

— Action prefix: $a.P$ can execute action a and then behaves as term P:

$$a.P \xrightarrow{a}_{\rm N} P$$

— Alternative composition: $P_1 + P_2$ behaves as either P_1 or P_2 depending on whether P_1 or P_2 executes an action first:

$$\frac{P_1 \xrightarrow{a}_{\rm N} P'}{P_1 + P_2 \xrightarrow{a}_{\rm N} P'} \qquad \frac{P_2 \xrightarrow{a}_{\rm N} P'}{P_1 + P_2 \xrightarrow{a}_{\rm N} P'}$$

— Process constant: A behaves as the right-hand side process term in its defining equation:

$$\frac{P \xrightarrow{a}_{\rm N} P' \quad A \triangleq P}{A \xrightarrow{a}_{\rm N} P'}$$

2.2. *Probabilistic Process Calculus*

In the probabilistic process calculus (PPC for short) every action has a probability associated with it, hence it is represented as a pair composed of the name of the action and the probability of the action. As a consequence, the choice among all the actions that are simultaneously enabled at the same state is probabilistic. We denote by $Act_{\rm P} = Name \times \mathbb{R}_{]0,1]}$ the set of the actions of PPC. In order to ensure that the probabilities of the actions enabled at any non-deadlocked state sum up to 1, we replace the action prefix operator and the binary alternative composition operator with a set of n-ary guarded alternative composition operators, with n ranging over the whole $\mathbb{N}_{>0}$.

Definition 2.2. The set of the process terms of PPC is generated by the following syntax:

$$P ::= \underline{0} \mid \sum_{i \in I} <a_i, p_i>.P_i \mid A$$

where I is a non-empty finite index set, $<a_i, p_i> \in Act_{\rm P}$ for all $i \in I$, and $\sum_{i \in I} p_i = 1$. We denote by $\mathcal{P}_{\rm P}$ the set of the closed and guarded process terms of PPC. ■

The semantics for PPC can be defined in the usual operational style, provided that the multiplicity of the transitions is taken into account. The reason is that idempotency (i.e. $P + P = P$) no longer holds when moving from nondeterministic processes to probabilistic ones. As an example, a term like $<a, 0.5>.P + <a, 0.5>.P$ cannot be equated to $<a, 0.5>.P$, as we have to take into account the fact that there are two 0.5-probability transitions both labeled with a and reaching P, not only one. Here are the semantic rules for generating labeled multitransition systems for PPC:

— Probabilistic guarded alternative composition:

$$\sum_{i \in I} <a_i, p_i>.P_i \xrightarrow{a_i, p_i}_{\rm P} P_i \qquad i \in I$$

— Process constant:

$$\frac{P \xrightarrow{a, p}_{\rm P} P' \qquad A \triangleq P}{A \xrightarrow{a, p}_{\rm P} P'}$$

2.3. *Markovian Process Calculus*

In the Markovian process calculus (MPC for short) every action is durational, hence it is represented as a pair composed of the name of the action and the rate of the exponential distribution quantifying the duration of the action. The choice among all the actions that are simultaneously enabled at the same state is governed by the race policy. As a consequence, the execution probability of each action is proportional to its rate and the average sojourn time in the state is quantified by an exponentially distributed random variable whose rate is the sum of the rates of the actions. We denote by $Act_{\mathrm{M}} = Name \times \mathbb{R}_{>0}$ the set of the actions of MPC.

Definition 2.3. The set of the process terms of MPC is generated by the following syntax:

$$P ::= \underline{0} \mid {<}a, \lambda{>}.P \mid P + P \mid A$$

where ${<}a, \lambda{>} \in Act_{\mathrm{M}}$. We denote by $\mathcal{P}_{\mathrm{M}}$ the set of the closed and guarded process terms of MPC. ■

Similarly to PPC, idempotency no longer holds and the multiplicity of the transitions has to be taken into account. As an example, a term like ${<}a, 4.6{>}.P + {<}a, 4.6{>}.P$ is not equivalent to ${<}a, 4.6{>}.P$ but to ${<}a, 9.2{>}.P$, because rates sum up due to the race policy. Here are the semantic rules for MPC:

— Exponentially timed action prefix:

$${<}a, \lambda{>}.P \xrightarrow{a,\lambda}_{\mathrm{M}} P$$

— Alternative composition:

$$\frac{P_1 \xrightarrow{a,\lambda}_{\mathrm{M}} P'}{P_1 + P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'} \qquad \frac{P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'}{P_1 + P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'}$$

— Process constant:

$$\frac{P \xrightarrow{a,\lambda}_{\mathrm{M}} P' \qquad A \triangleq P}{A \xrightarrow{a,\lambda}_{\mathrm{M}} P'}$$

3. Behavioral Equivalences

In this section we recall the three major approaches to the definition of behavioral equivalences: bisimulation, testing, and trace. For each of the three approaches, we shall present the nondeterministic version, the probabilistic version, and the Markovian version. The definition of the nine behavioral equivalences is preceded by the introduction of some notation that will be used several times in the rest of the paper.

3.1. *Exit Probabilities, Exit Rates, and Computations*

The exit probability of a probabilistic process term is the probability with which the process term can execute actions of a given name that lead to a given set of terms.

Definition 3.1. Let $P \in \mathcal{P}_{\mathrm{P}}$, $a \in \mathit{Name}$, and $C \subseteq \mathcal{P}_{\mathrm{P}}$. The exit probability of P when executing actions of name a that lead to C is defined through the following $\mathbb{R}_{[0,1]}$-valued function:

$$\mathit{prob}(P, a, C) = \sum\{\!| \, p \in \mathbb{R}_{]0,1]} \mid \exists P' \in C.\, P \xrightarrow{a,p}_{\mathrm{P}} P' \, |\!\}$$

where the summation is taken to be zero whenever its multiset is empty. ■

Definition 3.2. Let $P \in \mathcal{P}_{\mathrm{P}}$ and $\mathcal{E} \subseteq \mathit{Name}$. The conditional exit probability of P with respect to $\mathcal{E}$ is defined through the following $\mathbb{R}_{[0,1]}$-valued function:

$$\mathit{prob}_{\mathrm{c}}(P|\mathcal{E}) = \sum_{a \in \mathcal{E}} \mathit{prob}(P, a, \mathcal{P}_{\mathrm{P}})$$

where $\mathit{prob}(P, a, \mathcal{P}_{\mathrm{P}})$ is called the total exit probability of P with respect to a. ■

The exit rate of a Markovian process term is the rate at which it is possible to leave the term. We distinguish among the rate at which the process term can execute actions of a given name that lead to a given set of terms, the total rate at which the process term can execute actions of a given name, and the total exit rate of the process term. The latter coincides with the reciprocal of the average sojourn time in the state corresponding to the process term.

Definition 3.3. Let $P \in \mathcal{P}_{\mathrm{M}}$, $a \in \mathit{Name}$, and $C \subseteq \mathcal{P}_{\mathrm{M}}$. The exit rate of P when executing actions of name a that lead to C is defined through the following non-negative real function:

$$\mathit{rate}(P, a, C) = \sum\{\!| \, \lambda \in \mathbb{R}_{>0} \mid \exists P' \in C.\, P \xrightarrow{a,\lambda}_{\mathrm{M}} P' \, |\!\}$$

where the summation is taken to be zero whenever its multiset is empty. ■

Definition 3.4. Let $P \in \mathcal{P}_{\mathrm{M}}$ and $\mathcal{E} \subseteq \mathit{Name}$. The total exit rate of P is defined through the following non-negative real function:

$$\mathit{rate}_{\mathrm{t}}(P) = \sum_{a \in \mathit{Name}} \mathit{rate}(P, a, \mathcal{P}_{\mathrm{M}})$$

while the conditional exit rate of P with respect to $\mathcal{E}$ is defined through the following non-negative real function:

$$\boxed{rate_{\rm c}(P|\mathcal{E}) = \sum_{a \in \mathcal{E}} rate(P, a, \mathcal{P}_{\rm M})}$$

where $rate(P, a, \mathcal{P}_{\rm M})$ is called the total exit rate of P with respect to a. ■

A computation of a process term is a sequence of transitions that can be executed starting from the term. The length of a computation is given by the number of transitions occurring in it. We say that two distinct computations are independent of each other if it is not the case that one of them is a prefix of the other one. In the following, we denote by $\mathcal{C}_{\rm f}(P)$ and $\mathcal{I}_{\rm f}(P)$ the multisets of the finite-length computations and independent computations of process term P. Below we inductively define the trace, the execution probability, and the average duration of an element of $\mathcal{C}_{\rm f}(P)$.

Definition 3.5. Let $P \in \mathcal{P}_{\rm N} \cup \mathcal{P}_{\rm P} \cup \mathcal{P}_{\rm M}$ and $c \in \mathcal{C}_{\rm f}(P)$. The trace associated with the execution of c is the sequence of the action names labeling the transitions of c, which is defined by induction on the length of c through the following $Name^*$-valued function:

$$\boxed{trace(c) = \begin{cases} \varepsilon & \text{if } length(c) = 0 \\ a \circ trace(c') & \text{if } c \equiv P \xrightarrow{a}_{\rm N} c' \text{ with } P \in \mathcal{P}_{\rm N} \text{ or} \\ & \quad c \equiv P \xrightarrow{a,p}_{\rm P} c' \text{ with } P \in \mathcal{P}_{\rm P} \text{ or} \\ & \quad c \equiv P \xrightarrow{a,\lambda}_{\rm M} c' \text{ with } P \in \mathcal{P}_{\rm M} \end{cases}}$$

where ε is the empty trace. ■

Definition 3.6. Let $P \in \mathcal{P}_{\rm P} \cup \mathcal{P}_{\rm M}$ and $c \in \mathcal{C}_{\rm f}(P)$. The probability of executing c is the product of the execution probabilities of the transitions of c, which is defined by induction on the length of c through the following $\mathbf{R}_{]0,1]}$-valued function:

$$\boxed{prob(c) = \begin{cases} 1 & \text{if } length(c) = 0 \\ p \cdot prob(c') & \text{if } c \equiv P \xrightarrow{a,p}_{\rm P} c' \text{ with } P \in \mathcal{P}_{\rm P} \\ \frac{\lambda}{rate_{\rm t}(P)} \cdot prob(c') & \text{if } c \equiv P \xrightarrow{a,\lambda}_{\rm M} c' \text{ with } P \in \mathcal{P}_{\rm M} \end{cases}}$$

We also define the probability of executing a computation of C as:

$$\boxed{prob(C) = \sum_{c \in C} prob(c)}$$

for all $C \subseteq \mathcal{I}_{\rm f}(P)$. ■

Definition 3.7. Let $P \in \mathcal{P}_{\mathrm{M}}$ and $c \in \mathcal{C}_{\mathrm{f}}(P)$. The average duration of c is the sequence of the average sojourn times in the states traversed by c, which is defined by induction on the length of c through the following $(\mathbb{R}_{>0})^*$-valued function:

$$time(c) = \begin{cases} \varepsilon & \text{if } length(c) = 0 \\ \frac{1}{rate_{\mathrm{t}}(P)} \circ time(c') & \text{if } c \equiv P \xrightarrow{a,\lambda}_{\mathrm{M}} c' \end{cases}$$

where ε is the empty average duration. We also define the set of the computations of C whose average duration is not greater than θ as:

$$C_{\leq\theta} = \{c \in C \mid length(c) \leq length(\theta) \wedge \forall i = 1, \ldots, length(c).\, time(c)[i] \leq \theta[i]\}$$

for all $C \subseteq \mathcal{C}_{\mathrm{f}}(P)$ and $\theta \in (\mathbb{R}_{>0})^*$. ■

3.2. *Bisimulation Equivalence*

The basic idea behind bisimulation equivalence is to capture whether two process terms are able to mimic each other's behavior stepwise. In this section we recall the nondeterministic, probabilistic and Markovian versions of bisimulation equivalence, which have been originally defined in (Milner 1989; Larsen and Skou 1991; Hillston 1996).

3.2.1. *Nondeterministic Bisimulation Equivalence* Two nondeterministic process terms are bisimulation equivalent if they can play a game such that, whenever one of them executes an action, then the other one can respond by executing the same action so that the game can go on at the two derivative terms.

Definition 3.8. A relation $\mathcal{B} \subseteq \mathcal{P}_{\mathrm{N}} \times \mathcal{P}_{\mathrm{N}}$ is a nondeterministic bisimulation iff, whenever $(P_1, P_2) \in \mathcal{B}$, then for all actions $a \in Act_{\mathrm{N}}$:

— Whenever $P_1 \xrightarrow{a}_{\mathrm{N}} P'_1$ for some $P'_1 \in \mathcal{P}_{\mathrm{N}}$, then $P_2 \xrightarrow{a}_{\mathrm{N}} P'_2$ for some $P'_2 \in \mathcal{P}_{\mathrm{N}}$ such that $(P'_1, P'_2) \in \mathcal{B}$.

— Whenever $P_2 \xrightarrow{a}_{\mathrm{N}} P'_2$ for some $P'_2 \in \mathcal{P}_{\mathrm{N}}$, then $P_1 \xrightarrow{a}_{\mathrm{N}} P'_1$ for some $P'_1 \in \mathcal{P}_{\mathrm{N}}$ such that $(P'_1, P'_2) \in \mathcal{B}$. ■

Definition 3.9. Nondeterministic bisimulation equivalence, denoted by $\sim_{\mathrm{NB}}$, is the union of all the nondeterministic bisimulations. ■

3.2.2. *Probabilistic Bisimulation Equivalence* In the probabilistic case we can no longer reason in terms of single transitions and derivatives. Instead, we have to consider the cumulative probability with which a class of equivalent terms is reached when executing actions with the same name.

Definition 3.10. An equivalence relation $\mathcal{B} \subseteq \mathcal{P}_{\mathrm{P}} \times \mathcal{P}_{\mathrm{P}}$ is a probabilistic bisimulation iff, whenever $(P_1, P_2) \in \mathcal{B}$, then for all action names $a \in Name$ and equivalence classes $C \in \mathcal{P}_{\mathrm{P}}/\mathcal{B}$:

$$prob(P_1, a, C) = prob(P_2, a, C)$$ ■

Definition 3.11. Probabilistic bisimulation equivalence, denoted by $\sim_{\mathrm{PB}}$, is the union of all the probabilistic bisimulations. ■

3.2.3. *Markovian Bisimulation Equivalence* Similarly to the probabilistic case, in the Markovian case we have to consider the cumulative rate with which a class of equivalent terms is reached when executing actions with the same name.

Definition 3.12. An equivalence relation $\mathcal{B} \subseteq \mathcal{P}_{\mathrm{M}} \times \mathcal{P}_{\mathrm{M}}$ is a Markovian bisimulation iff, whenever $(P_1, P_2) \in \mathcal{B}$, then for all action names $a \in Name$ and equivalence classes $C \in \mathcal{P}_{\mathrm{M}}/\mathcal{B}$:

$$rate(P_1, a, C) = rate(P_2, a, C)$$ ■

Definition 3.13. Markovian bisimulation equivalence, denoted by $\sim_{\mathrm{MB}}$, is the union of all the Markovian bisimulations. ■

3.3. *Testing Equivalence*

In the case of testing equivalence two process terms are considered to be equivalent if an external observer cannot distinguish between them, with the only way that the observer has to infer information about the behavior of a process term being to interact with it by means of tests. In this section we recall the nondeterministic, probabilistic and Markovian versions of testing equivalence, which have been originally defined in (De Nicola and Hennessy 1983; Christoff 1990; Bernardo and Cleaveland 2000).

3.3.1. *Formalization of Test Interaction* The most convenient way to represent a test is through another process term, which interacts with the term to be tested by means of a parallel composition operator that enforces synchronization on all action names. Since a test should be conducted in a finite amount of time, for the test formalization we restrict ourselves to process terms that are finite state and acyclic, hence no recursion is admitted within the tests. In other words, the state-transition semantic models underlying the tests must have a finite dag-like structure.

In order to represent the fact that a test is passed or not, each of the terminal nodes of the dag-like semantic model underlying a test must be suitably labeled so as to establish whether it is a success or failure state. At the process calculus level, this amounts to replace $\underline{0}$ with the two zeroary operators "s" (for success) and "f" (for failure). Ambiguous terms like $\mathrm{s} + \mathrm{f}$ will be avoided in the test syntax by replacing the action prefix operator and the binary alternative composition operator with a set of n-ary guarded alternative composition operators, with n ranging over the whole $\mathbb{N}_{>0}$.

In the nondeterministic case, the tests are made out of the same kind of actions that can occur in the nondeterministic process terms.

Definition 3.14. The set $\mathcal{T}_{\mathrm{N}}$ of the nondeterministic tests is generated by the following syntax:

$$T ::= \mathrm{f} \mid \mathrm{s} \mid \sum_{i \in I} a_i.T_i$$

where I is a non-empty finite index set and $a_i \in \mathit{Name}$ for all $i \in I$. ■

In order to deal with nondeterministic tests, we extend $\longrightarrow_{\mathrm{N}}$ as follows:

$$\sum_{i \in I} a_i.T_i \xrightarrow{a_i}_{\mathrm{N}} T_i \qquad i \in I$$

The following operational rule defines the interaction of $P \in \mathcal{P}_{\mathrm{N}}$ and $T \in \mathcal{T}_{\mathrm{N}}$:

$$\frac{P \xrightarrow{a}_{\mathrm{N}} P' \qquad T \xrightarrow{a}_{\mathrm{N}} T'}{P \parallel T \xrightarrow{a}_{\mathrm{N}} P' \parallel T'}$$

In the probabilistic and Markovian cases, instead, the tests are made out of passive actions, each equipped with a weight $w \in \mathbf{R}_{>0}$. The idea is that, in any of its states, a process term to be tested probabilistically generates the proposal of an action to be executed among those enabled in that state, then the test reacts by probabilistically selecting a passive action (if any) with the same name as the proposed one.

Definition 3.15. The set $\mathcal{T}_{\mathrm{R}}$ of the reactive tests is generated by the following syntax:

$$T ::= \mathrm{f} \mid \mathrm{s} \mid \sum_{i \in I} <a_i, *_{w_i}>.T_i$$

where I is a non-empty finite index set and $a_i \in \mathit{Name}, w_i \in \mathbf{R}_{>0}$ for all $i \in I$. ■

The only semantic rule for reactive tests is the following:

$$\sum_{i \in I} <a_i, *_{w_i}>.T_i \xrightarrow{a_i, *_{w_i}}_{\mathrm{R}} T_i \qquad i \in I$$

Definition 3.16. Let $T \in \mathcal{T}_{\mathrm{R}}$ and $a \in \mathit{Name}$. The set of the names of the actions initially enabled by T is defined as follows:

$$\mathit{init}(T) = \{a \in \mathit{Name} \mid \exists w \in \mathbf{R}_{>0}. \exists T' \in \mathcal{T}_{\mathrm{R}}. T \xrightarrow{a, *_w}_{\mathrm{R}} T'\}$$

while the total weight of T with respect to a is defined as follows:

$$weight(T, a) = \sum\{\!|\, w \in \mathbb{R}_{>0} \mid \exists T' \in \mathcal{T}_{\mathrm{R}}.\, T \xrightarrow{a,*_w}_{\mathrm{R}} T' \,|\!\}$$

■

The following operational rule defines the probabilistic generative-reactive interaction (Bravetti and Aldini 2003) of $P \in \mathcal{P}_{\mathrm{P}}$ and $T \in \mathcal{T}_{\mathrm{R}}$:

$$\frac{P \xrightarrow{a,p}_{\mathrm{P}} P' \qquad T \xrightarrow{a,*_w}_{\mathrm{R}} T'}{P \,\|\, T \xrightarrow{a,\, \frac{p}{prob_{\mathrm{C}}(P|init(T))} \cdot \frac{w}{weight(T,a)}}_{\mathrm{P}} P' \,\|\, T'}$$

while the following operational rule defines the Markovian generative-reactive interaction (Bernardo and Bravetti 2003) of $P \in \mathcal{P}_{\mathrm{M}}$ and $T \in \mathcal{T}_{\mathrm{R}}$:

$$\frac{P \xrightarrow{a,\lambda}_{\mathrm{M}} P' \qquad T \xrightarrow{a,*_w}_{\mathrm{R}} T'}{P \,\|\, T \xrightarrow{a,\lambda \cdot \frac{w}{weight(T,a)}}_{\mathrm{M}} P' \,\|\, T'}$$

Definition 3.17. Let $P \in \mathcal{P}_{\mathrm{N}}$ and $T \in \mathcal{T}_{\mathrm{N}}$, or $P \in \mathcal{P}_{\mathrm{P}} \cup \mathcal{P}_{\mathrm{M}}$ and $T \in \mathcal{T}_{\mathrm{R}}$. The interaction system of P and T is process term $P \,\|\, T$, where we say that:

— A configuration is a state of the semantic model underlying $P \,\|\, T$.
— A configuration is successful (resp. failed) iff its test component is "s" (resp. "f").
— A computation is successful (resp. failed) iff so is its last configuration. A computation that is neither successful nor failed is said to be interrupted.

We denote by $\mathcal{SC}(P, T)$ the multiset of the successful computations of $\mathcal{C}_{\mathrm{f}}(P \,\|\, T)$. ■

Note that $\mathcal{SC}(P, T) \subseteq \mathcal{I}_{\mathrm{f}}(P \,\|\, T)$, because of the maximality of the successful test-driven computations, and that $\mathcal{SC}(P, T)$ is finite, because of the finitely-branching structure of the considered terms.

3.3.2. *Nondeterministic Testing Equivalence* Two nondeterministic process terms are testing equivalent if they have the same capability with respect to the possibility and the necessity of passing an arbitrary test.

Definition 3.18. Let $P \in \mathcal{P}_{\mathrm{N}}$ and $T \in \mathcal{T}_{\mathrm{N}}$. We say that:

— P may pass T, written P *may* T, iff at least one test-driven computation is successful:

$$\mathcal{SC}(P, T) \neq \emptyset$$

— P must pass T, written $P\ \mathit{must}\ T$, iff all maximal test-driven computations are successful:

$$\mathcal{SC}(P, T) = \mathcal{I}_{\mathrm{f}}(P \parallel T) \qquad \blacksquare$$

Definition 3.19. Let $P_1, P_2 \in \mathcal{P}_{\mathrm{N}}$. We say that:

— P_1 is may-testing equivalent to P_2, written $P_1 \sim_{\mathrm{T,may}} P_2$, iff for all nondeterministic tests $T \in \mathcal{T}_{\mathrm{N}}$:

$$P_1\ \mathit{may}\ T \iff P_2\ \mathit{may}\ T$$

— P_1 is must-testing equivalent to P_2, written $P_1 \sim_{\mathrm{T,must}} P_2$, iff for all nondeterministic tests $T \in \mathcal{T}_{\mathrm{N}}$:

$$P_1\ \mathit{must}\ T \iff P_2\ \mathit{must}\ T$$

— P_1 is nondeterministic testing equivalent to P_2, written $P_1 \sim_{\mathrm{NT}} P_2$, iff:

$$P_1 \sim_{\mathrm{T,may}} P_2 \ \wedge\ P_1 \sim_{\mathrm{T,must}} P_2 \qquad \blacksquare$$

3.3.3. *Probabilistic Testing Equivalence* In the probabilistic case, the possibility and the necessity of passing tests are subsumed by the probability of passing tests.

Definition 3.20. Let $P_1, P_2 \in \mathcal{P}_{\mathrm{P}}$. We say that P_1 is probabilistic testing equivalent to P_2, written $P_1 \sim_{\mathrm{PT}} P_2$, iff for all reactive tests $T \in \mathcal{T}_{\mathrm{R}}$:

$$\mathit{prob}(\mathcal{SC}(P_1, T)) = \mathit{prob}(\mathcal{SC}(P_2, T)) \qquad \blacksquare$$

3.3.4. *Markovian Testing Equivalence* In the Markovian case, we have to consider the probability of passing the same tests within a certain average amount of time.

Definition 3.21. Let $P_1, P_2 \in \mathcal{P}_{\mathrm{M}}$. We say that P_1 is Markovian testing equivalent to P_2, written $P_1 \sim_{\mathrm{MT}} P_2$, iff for all reactive tests $T \in \mathcal{T}_{\mathrm{R}}$ and sequences $\theta \in (\mathbf{R}_{>0})^*$ of average amounts of time:

$$\mathit{prob}(\mathcal{SC}_{\leq \theta}(P_1, T)) = \mathit{prob}(\mathcal{SC}_{\leq \theta}(P_2, T)) \qquad \blacksquare$$

3.4. *Trace Equivalence*

Unlike testing equivalence, in the case of trace equivalence we no longer consider tests that interact with the process terms. Instead, we directly consider the finite-length computations of the process terms taken in isolation, thus abstracting from the branching points of their behavior. In this section we recall the nondeterministic, probabilistic and Markovian versions of trace equivalence, which have been originally defined in (Hoare 1985; Jou and Smolka 1990; Wolf et al. 2005).

3.4.1. *Formalization of Trace Compatibility* A trace is an element of Name^*, i.e. a finite-length sequence of action names. The compatibility of a computation with a trace depends on whether the computation exhibits the trace or not.

Definition 3.22. Let $P \in \mathcal{P}_{\mathrm{N}} \cup \mathcal{P}_{\mathrm{P}} \cup \mathcal{P}_{\mathrm{M}}$, $c \in \mathcal{C}_{\mathrm{f}}(P)$, and $\alpha \in \mathit{Name}^*$. We say that c is compatible with α iff:

$$trace(c) = \alpha$$

We denote by $\mathcal{CC}(P, \alpha)$ the multiset of the finite-length computations of P that are compatible with α. ■

Note that $\mathcal{CC}(P, \alpha) \subseteq \mathcal{I}_{\rm f}(P)$, because of the compatibility of the computations with the same trace α, and that $\mathcal{CC}(P, \alpha)$ is finite, because of the finitely-branching structure of the considered terms.

3.4.2. *Nondeterministic Trace Equivalence* Two nondeterministic process terms are trace equivalent if they can execute the same traces.

Definition 3.23. Let $P \in \mathcal{P}_{\rm N}$ and $\alpha \in \mathit{Name}^*$. We say that P executes α, written $P \; \mathit{execute} \, \alpha$, iff at least one computation is compatible with α:

$$\mathcal{CC}(P, \alpha) \neq \emptyset$$ ■

Definition 3.24. Let $P_1, P_2 \in \mathcal{P}_{\rm N}$. We say that P_1 is nondeterministic trace equivalent to P_2, written $P_1 \sim_{\rm NTr} P_2$, iff for all traces $\alpha \in \mathit{Name}^*$:

$$P_1 \; \mathit{execute} \, \alpha \iff P_2 \; \mathit{execute} \, \alpha$$ ■

3.4.3. *Probabilistic Trace Equivalence* In the probabilistic case, we have to consider the probability of executing the same traces.

Definition 3.25. Let $P_1, P_2 \in \mathcal{P}_{\rm P}$. We say that P_1 is probabilistic trace equivalent to P_2, written $P_1 \sim_{\rm PTr} P_2$, iff for all traces $\alpha \in \mathit{Name}^*$:

$$\mathit{prob}(\mathcal{CC}(P_1, \alpha)) = \mathit{prob}(\mathcal{CC}(P_2, \alpha))$$ ■

3.4.4. *Markovian Trace Equivalence* In the Markovian case, we have to consider the probability of executing the same traces within a certain average amount of time.

Definition 3.26. Let $P_1, P_2 \in \mathcal{P}_{\rm M}$. We say that P_1 is Markovian trace equivalent to P_2, written $P_1 \sim_{\rm MTr} P_2$, iff for all traces $\alpha \in \mathit{Name}^*$ and sequences $\theta \in (\mathbf{R}_{>0})^*$ of average amounts of time:

$$\mathit{prob}(\mathcal{CC}_{\leq \theta}(P_1, \alpha)) = \mathit{prob}(\mathcal{CC}_{\leq \theta}(P_2, \alpha))$$ ■

4. Modal Characterizations of Bisimulation Equivalence

In this section we recall from (Hennessy and Milner 1985; Larsen and Skou 1991; Clark et al. 1999) the modal logic characterizations of the nondeterministic, probabilistic and Markovian versions of bisimulation equivalence.

4.1. *Characterization of Nondeterministic Bisimulation Equivalence*

Nondeterministic bisimulation equivalence is precisely characterized by the Hennessy-Milner logic (Hennessy and Milner 1985). This is propositional logic extended with the so-called diamond operator, which is a modal operator expressing the possibility of performing an action with a given name and reaching a state that satisfies a certain formula.

Definition 4.1. The set of the formulas of the Hennessy-Milner logic (HML) is generated by the following syntax:

$$\phi ::= \mathit{true} \mid \neg\phi \mid \phi \wedge \phi \mid \langle a \rangle \phi$$

where $a \in \mathit{Name}$. ■

Definition 4.2. The satisfaction relation $\models$ of HML over $\mathcal{P}_{\mathrm{N}}$ is defined by structural induction as follows:

$$\begin{array}{lll} P & \models & \mathit{true} \\ P & \models & \neg\phi & \text{if } P \not\models \phi \\ P & \models & \phi_1 \wedge \phi_2 & \text{if } P \models \phi_1 \text{ and } P \models \phi_2 \\ P & \models & \langle a \rangle \phi & \text{if } P \xrightarrow{a}_{\mathrm{N}} P' \text{ with } P' \models \phi \text{ for some } P' \end{array}$$

■

Theorem 4.3. Let $P_1, P_2 \in \mathcal{P}_{\mathrm{N}}$. Then:

$$P_1 \sim_{\mathrm{NB}} P_2 \iff (\forall \phi \in \mathrm{HML}.\, P_1 \models \phi \iff P_2 \models \phi)$$ ■

4.2. *Local Characterization of Probabilistic Bisimulation Equivalence*

Probabilistic bisimulation equivalence is characterized by a probabilistic extension of HML in which the diamond operator is decorated with a positive real number. This number expresses a lower bound to the probability of performing an action with a given name and reaching a state that satisfies a certain formula. Since this number is strictly positive, an additional modal operator is needed to express the fact that an action having a given name cannot be executed at all.

Definition 4.4. The set of the formulas of $\mathrm{HML}_{\mathrm{PB}}$ is generated by the following syntax:

$$\phi ::= \mathit{true} \mid \neg\phi \mid \phi \wedge \phi \mid \nabla_a \mid \langle a \rangle_p \phi$$

where $a \in \mathit{Name}$ and $p \in \mathbb{R}_{]0,1]}$. ■

Definition 4.5. The satisfaction relation $\models_{\text{PB}}$ of HML_{PB} over $\mathcal{P}_{\text{P}}$ is defined by structural induction as follows:

$$
\begin{array}{llll}
P & \models_{\text{PB}} & \mathit{true} & \\
P & \models_{\text{PB}} & \neg\phi & \text{if } P \not\models_{\text{PB}} \phi \\
P & \models_{\text{PB}} & \phi_1 \wedge \phi_2 & \text{if } P \models_{\text{PB}} \phi_1 \text{ and } P \models_{\text{PB}} \phi_2 \\
P & \models_{\text{PB}} & \nabla_a & \text{if } \mathit{prob}(P, a, \mathcal{P}_{\text{P}}) = 0 \\
P & \models_{\text{PB}} & \langle a \rangle_p \phi & \text{if } \mathit{prob}(P, a, \{P' \in \mathcal{P}_{\text{P}} \mid P' \models_{\text{PB}} \phi\}) \geq p
\end{array}
$$

■

Theorem 4.6. Let $P_1, P_2 \in \mathcal{P}_{\text{P}}$. Then:

$$P_1 \sim_{\text{PB}} P_2 \iff (\forall \phi \in \text{HML}_{\text{PB}}.\, P_1 \models_{\text{PB}} \phi \Longleftrightarrow P_2 \models_{\text{PB}} \phi)$$ ■

4.3. *Local Characterization of Markovian Bisimulation Equivalence*

The modal characterization of Markovian bisimulation equivalence is similar to the previous one, with the difference that the positive real number decorating the diamond operator is now interpreted as a rate lower bound rather than a probability lower bound.

Definition 4.7. The set of the formulas of HML_{MB} is generated by the following syntax:

$$\phi ::= \mathit{true} \mid \neg\phi \mid \phi \wedge \phi \mid \nabla_a \mid \langle a \rangle_\lambda \phi$$

where $a \in \mathit{Name}$ and $\lambda \in \mathbb{R}_{>0}$. ■

Definition 4.8. The satisfaction relation $\models_{\text{MB}}$ of HML_{MB} over $\mathcal{P}_{\text{M}}$ is defined by structural induction as follows:

$$
\begin{array}{llll}
P & \models_{\text{MB}} & \mathit{true} & \\
P & \models_{\text{MB}} & \neg\phi & \text{if } P \not\models_{\text{MB}} \phi \\
P & \models_{\text{MB}} & \phi_1 \wedge \phi_2 & \text{if } P \models_{\text{MB}} \phi_1 \text{ and } P \models_{\text{MB}} \phi_2 \\
P & \models_{\text{MB}} & \nabla_a & \text{if } \mathit{rate}(P, a, \mathcal{P}_{\text{M}}) = 0 \\
P & \models_{\text{MB}} & \langle a \rangle_\lambda \phi & \text{if } \mathit{rate}(P, a, \{P' \in \mathcal{P}_{\text{M}} \mid P' \models_{\text{MB}} \phi\}) \geq \lambda
\end{array}
$$

■

Theorem 4.9. Let $P_1, P_2 \in \mathcal{P}_{\text{M}}$. Then:

$$P_1 \sim_{\text{MB}} P_2 \iff (\forall \phi \in \text{HML}_{\text{MB}}.\, P_1 \models_{\text{MB}} \phi \Longleftrightarrow P_2 \models_{\text{MB}} \phi)$$ ■

5. Modal Characterizations of Testing Equivalence

In this section we recall from (Hennessy 1985) the modal logic characterization of nondeterministic testing equivalence, then we provide a modal logic characterization for the probabilistic and Markovian versions of testing equivalence.

5.1. *Characterization of Nondeterministic Testing Equivalence*

Nondeterministic testing equivalence is characterized by a restriction of HML in which negation does not occur. To be more precise, the modal language permits to ask simple questions after a trace has been executed. Therefore, the syntax of the modal language has a two-level definition. At the top level we have a modal operator on traces. At the bottom level we have constant true, logical disjunction, and a restriction of the diamond operator. Then two satisfaction relations are defined, which express the fact that a process term may or must satisfy a bottom-level formula after executing a trace.

Definition 5.1. The set of the formulas of $\mathrm{HML_{NT}}$ is generated by the following syntax:

$$\begin{array}{lcl} \phi & ::= & (\!|\alpha|\!)\varphi \\ \varphi & ::= & \mathit{true} \mid \varphi \vee \varphi \mid \langle a \rangle \end{array}$$

where $\alpha \in \mathit{Name}^*$ and $a \in \mathit{Name}$. ■

Definition 5.2. The may satisfaction relation $\models_{\mathrm{NT,may}}$ and the must satisfaction relation $\models_{\mathrm{NT,must}}$ of $\mathrm{HML_{NT}}$ over $\mathcal{P}_{\mathrm{N}}$ are defined by structural induction as follows:

$$\begin{array}{llll} P & \models_{\mathrm{NT,may}} & (\!|\alpha|\!)\varphi & \text{if } P \xrightarrow{\alpha}_{\mathrm{N}} P' \text{ with } P' \models_{\mathrm{NT}} \varphi \text{ for some } P' \\ P & \models_{\mathrm{NT,must}} & (\!|\alpha|\!)\varphi & \text{if } P' \models_{\mathrm{NT}} \varphi \text{ for all } P' \text{ such that } P \xrightarrow{\alpha}_{\mathrm{N}} P' \end{array}$$

where:

$$\begin{array}{llll} P & \models_{\mathrm{NT}} & \mathit{true} & \\ P & \models_{\mathrm{NT}} & \varphi_1 \vee \varphi_2 & \text{if } P \models_{\mathrm{NT}} \varphi_1 \text{ or } P \models_{\mathrm{NT}} \varphi_2 \\ P & \models_{\mathrm{NT}} & \langle a \rangle & \text{if } P \xrightarrow{a}_{\mathrm{N}} P' \text{ for some } P' \end{array}$$

■

Theorem 5.3. Let $P_1, P_2 \in \mathcal{P}_{\mathrm{N}}$. Then:

$$P_1 \sim_{\mathrm{NT}} P_2 \iff (\forall \phi \in \mathrm{HML_{NT}}.\, P_1 \models_{\mathrm{NT,may}} \phi \iff P_2 \models_{\mathrm{NT,may}} \phi \wedge P_1 \models_{\mathrm{NT,must}} \phi \iff P_2 \models_{\mathrm{NT,must}} \phi)$$ ■

5.2. *Global Characterization of Probabilistic Testing Equivalence*

In order to obtain a modal logic characterization of $\sim_{\mathrm{PT}}$, it is worth recalling from (Cleaveland et al. 1999) a fully abstract characterization of probabilistic testing equivalence, which is consistent with the definition of the equivalence itself provided in (Christoff 1990). This fully abstract characterization will allow us to concentrate on a set of canonical reactive tests having a more regular structure than the one of Def. 3.15.

Definition 5.4. The set $\mathcal{T}_{\mathrm{R,c}}$ of the canonical reactive tests is generated by the following syntax:

$$T ::= \mathrm{s} \mid <a, *_1>.T + \sum_{b \in \mathcal{E} - \{a\}} <b, *_1>.\mathrm{f}$$

where $a \in \mathit{Name}$, $\mathcal{E} \subseteq \mathit{Name}$ such that $a \in \mathcal{E}$, and the summation is absent whenever $\mathcal{E} - \{a\} = \emptyset$. ■

Theorem 5.5. Let $P_1, P_2 \in \mathcal{P}_{\mathrm{p}}$. Then $P_1 \sim_{\mathrm{PT}} P_2$ iff for all $T \in \mathcal{T}_{\mathrm{R,c}}$:

$$\mathit{prob}(\mathcal{SC}(P_1, T)) = \mathit{prob}(\mathcal{SC}(P_2, T))$$ ■

In each of its states, a non-trivial canonical reactive test enables a set $\mathcal{E}$ of passive actions – representing the environment from the point of view of a process term to be tested – such that only one of them leads to success, while all the others lead to failure in one step. Based on this structure, we now define a restriction of HML in which both negation and logical conjunction are ruled out, while the diamond operator is made dependent from the environment.

Definition 5.6. The set of the formulas of $\mathrm{HML}_{\mathrm{PMT}}$ is generated by the following syntax:

$$\phi ::= \mathit{true} \mid \langle a | \mathcal{E} \rangle \phi$$

where $a \in \mathit{Name}$ and $\mathcal{E} \subseteq \mathit{Name}$ such that $a \in \mathcal{E}$. ■

This modal language, which does not exhibit any probabilistic aspect, is equipped with a quantitative interpretation function inspired by (Kwiatkowska and Norman 1998), which establishes the probability with which a process term satisfies a formula.

Definition 5.7. The interpretation function $[\![.]\!]_{\mathrm{PT}}$ of $\mathrm{HML}_{\mathrm{PMT}}$ over $\mathcal{P}_{\mathrm{P}}$ is defined by structural induction as follows:

$$\begin{array}{rcl} [\![\mathit{true}]\!]_{\mathrm{PT}}(P) & = & 1 \\ [\![\langle a | \mathcal{E} \rangle \phi]\!]_{\mathrm{PT}}(P) & = & \sum\limits_{P \xrightarrow{a,p}_{\mathrm{P}} P'} \frac{p}{\mathit{prob}_{\mathrm{c}}(P|\mathcal{E})} \cdot [\![\phi]\!]_{\mathrm{PT}}(P') \end{array}$$

where the summation is taken to be zero whenever there are no a-transitions departing from P. ■

We now see that the formulas of $\mathrm{HML}_{\mathrm{PMT}}$ have a one-to-one correspondence with the canonical reactive tests, from which the modal logic characterization result for $\sim_{\mathrm{PT}}$ will immediately follow.

Lemma 5.8. For each $T \in \mathcal{T}_{\mathrm{R,c}}$ there exists $\phi_T \in \mathrm{HML}_{\mathrm{PMT}}$ such that for all $P \in \mathcal{P}_{\mathrm{P}}$:

$$[\![\phi_T]\!]_{\mathrm{PT}}(P) = \mathit{prob}(\mathcal{SC}(P, T))$$

Proof. We proceed by induction on the syntactical structure of T:

— Let $T \equiv \mathrm{s}$ and take $\phi_T \equiv \mathit{true}$. Then for all $P \in \mathcal{P}_{\mathrm{P}}$ we immediately derive:

$$[\![\phi_T]\!]_{\mathrm{PT}}(P) = 1 = \mathit{prob}(\mathcal{SC}(P, T))$$

— Let $T \equiv \,<a, *_1>.T' + \sum_{b \in \mathcal{E} - \{a\}} <b, *_1>.\mathrm{f}$ and take $\phi_T \equiv \langle a|\mathcal{E}\rangle \phi_{T'}$ such that $\phi_{T'}$ satisfies the induction hypothesis with respect to T'. In order to avoid trivial cases, consider $P \in \mathcal{P}_{\mathrm{P}}$ that can perform a-actions, otherwise:

$$[\![\phi_T]\!]_{\mathrm{PT}}(P) = 0 = \mathit{prob}(\mathcal{SC}(P, T))$$

Then we have:

$$[\![\phi_T]\!]_{\mathrm{PT}}(P) = \sum_{P \xrightarrow{a,p}_{\mathrm{P}} P'} \frac{p}{\mathit{prob}_{\mathrm{c}}(P|\mathcal{E})} \cdot [\![\phi_{T'}]\!]_{\mathrm{PT}}(P')$$

and:

$$\mathit{prob}(\mathcal{SC}(P, T)) = \sum_{P \xrightarrow{a,p}_{\mathrm{P}} P'} \frac{p}{\mathit{prob}_{\mathrm{c}}(P|\mathit{init}(T))} \cdot \mathit{prob}(\mathcal{SC}(P', T'))$$

where $\mathit{prob}_{\mathrm{c}}(P|\mathcal{E}) = \mathit{prob}_{\mathrm{c}}(P|\mathit{init}(T))$. By the induction hypothesis, for all P' reachable from P via an a-transition we have:

$$[\![\phi_{T'}]\!]_{\mathrm{PT}}(P') = \mathit{prob}(\mathcal{SC}(P', T'))$$

from which the result follows. ■

Lemma 5.9. For each $\phi \in \mathrm{HML}_{\mathrm{PMT}}$ there exists $T_\phi \in \mathcal{T}_{\mathrm{R,c}}$ such that for all $P \in \mathcal{P}_{\mathrm{P}}$:

$$\mathit{prob}(\mathcal{SC}(P, T_\phi)) = [\![\phi]\!]_{\mathrm{PT}}(P)$$

Proof. We proceed by induction on the syntactical structure of ϕ:

— Let $\phi \equiv \mathit{true}$ and take $T_\phi \equiv \mathrm{s}$. Then for all $P \in \mathcal{P}_{\mathrm{P}}$ we immediately derive:

$$\mathit{prob}(\mathcal{SC}(P, T_\phi)) = 1 = [\![\phi]\!]_{\mathrm{PT}}(P)$$

— Let $\phi \equiv \langle a|\mathcal{E}\rangle \phi'$ and take $T_\phi \equiv \,<a, *_1>.T_{\phi'} + \sum_{b \in \mathcal{E} - \{a\}} <b, *_1>.\mathrm{f}$ such that $T_{\phi'}$ satisfies the induction hypothesis with respect to ϕ'. In order to avoid trivial cases, consider $P \in \mathcal{P}_{\mathrm{P}}$ that can perform a-actions, otherwise:

$$\mathit{prob}(\mathcal{SC}(P, T_\phi)) = 0 = [\![\phi]\!]_{\mathrm{PT}}(P)$$

Then we have:

$$\mathit{prob}(\mathcal{SC}(P, T_\phi)) = \sum_{P \xrightarrow{a,p}_{\mathrm{P}} P'} \frac{p}{\mathit{prob}_{\mathrm{c}}(P|\mathit{init}(T_\phi))} \cdot \mathit{prob}(\mathcal{SC}(P', T_{\phi'}))$$

and:

$$[\![\phi]\!]_{\mathrm{PT}}(P) = \sum_{P \xrightarrow{a,p}_{\mathrm{P}} P'} \frac{p}{\mathit{prob}_{\mathrm{c}}(P|\mathcal{E})} \cdot [\![\phi']\!]_{\mathrm{PT}}(P')$$

where $\mathit{prob}_{\mathrm{c}}(P|\mathit{init}(T_\phi)) = \mathit{prob}_{\mathrm{c}}(P|\mathcal{E})$. By the induction hypothesis, for all P' reachable from P via an a-transition we have:

$$\mathit{prob}(\mathcal{SC}(P', T_{\phi'})) = [\![\phi']\!]_{\mathrm{PT}}(P')$$

from which the result follows. ■

Theorem 5.10. Let $P_1, P_2 \in \mathcal{P}_{\rm P}$. Then:

$$P_1 \sim_{\rm PT} P_2 \iff \forall \phi \in {\rm HML}_{\rm PMT}. [\![\phi]\!]_{\rm PT}(P_1) = [\![\phi]\!]_{\rm PT}(P_2)$$

Proof. The result is a straightforward consequence of Thm. 5.5 and of the bijective correspondence between canonical reactive tests and formulas of $\rm HML_{PMT}$ established by Lemmas 5.8 and 5.9. ■

5.3. *Global Characterization of Markovian Testing Equivalence*

Similarly to the probabilistic case, in order to derive a modal logic characterization of $\sim_{\rm MT}$, we recall from (Bernardo and Cleaveland 2000) a fully abstract characterization of Markovian testing equivalence, which again is based on the canonical reactive tests.

Theorem 5.11. Let $P_1, P_2 \in \mathcal{P}_{\rm M}$. Then $P_1 \sim_{\rm MT} P_2$ iff for all $T \in \mathcal{T}_{\rm R,c}$ and $\theta \in (\mathbb{R}_{>0})^*$:

$$prob(\mathcal{SC}_{\leq \theta}(P_1, T)) = prob(\mathcal{SC}_{\leq \theta}(P_2, T))$$ ■

The modal language for $\sim_{\rm MT}$ is the same as for $\sim_{\rm PT}$, i.e. $\rm HML_{PMT}$, but its quantitative interpretation is different as it has to take into account the temporal aspects as well. What has now to be computed is in fact the probability with which a process term satisfies a formula within a given average amount of time.

Definition 5.12. The interpretation function $[\![.]\!]_{\rm MT}$ of $\rm HML_{PMT}$ over $\mathcal{P}_{\rm M} \times (\mathbb{R}_{>0})^*$ is defined by structural induction as follows:

$$[\![true]\!]_{\rm MT}(P, \theta) = 1$$

$$[\![\langle a|\mathcal{E}\rangle\phi]\!]_{\rm MT}(P, \theta) = \begin{cases} 0 & \text{if } \theta = \varepsilon \vee \frac{1}{rate_{\rm c}(P|\mathcal{E})} > \theta[1] \\ \sum\limits_{P \xrightarrow{a,\lambda}_{\rm M} P'} \frac{\lambda}{rate_{\rm c}(P|\mathcal{E})} \cdot [\![\phi]\!]_{\rm MT}(P', \theta') & \text{if } \theta = t \circ \theta' \wedge \frac{1}{rate_{\rm c}(P|\mathcal{E})} \leq t \end{cases}$$

where the summation is taken to be zero whenever there are no a-transitions departing from P. ■

Like in the probabilistic case, we are able to establish a one-to-one correspondence between the formulas of $\rm HML_{PMT}$ and the canonical reactive tests, from which the modal logic characterization result for $\sim_{\rm MT}$ will immediately follow.

Lemma 5.13. For each $T \in \mathcal{T}_{\rm R,c}$ there exists $\phi_T \in \rm HML_{PMT}$ such that for all $P \in \mathcal{P}_{\rm M}$ and $\theta \in (\mathbb{R}_{>0})^*$:

$$[\![\phi_T]\!]_{\rm MT}(P, \theta) = prob(\mathcal{SC}_{\leq \theta}(P, T))$$

Proof. We proceed by induction on the syntactical structure of T:

— Let $T \equiv {\rm s}$ and take $\phi_T \equiv true$. Then for all $P \in \mathcal{P}_{\rm M}$ and $\theta \in (\mathbb{R}_{>0})^*$ we immediately derive:

$$[\![\phi_T]\!]_{\rm MT}(P, \theta) = 1 = prob(\mathcal{SC}_{\leq \theta}(P, T))$$

— Let $T \equiv \text{<}a, *_1\text{>}.T' + \sum_{b \in \mathcal{E}-\{a\}} \text{<}b, *_1\text{>}.\mathrm{f}$ and take $\phi_T \equiv \langle a|\mathcal{E}\rangle \phi_{T'}$ such that $\phi_{T'}$ satisfies the induction hypothesis with respect to T'. In order to avoid trivial cases, consider $P \in \mathcal{P}_\mathrm{M}$ that can perform a-actions and $\theta \in (\mathbb{R}_{>0})^*$ such that $\theta = t \circ \theta'$ and $\frac{1}{rate_\mathrm{c}(P|\mathcal{E})} \leq t$, otherwise:

$$[\![\phi_T]\!]_\mathrm{MT}(P, \theta) = 0 = prob(\mathcal{SC}_{\leq \theta}(P, T))$$

Then we have:

$$[\![\phi_T]\!]_\mathrm{MT}(P, \theta) = \sum_{P \xrightarrow{a,\lambda}_\mathrm{M} P'} \frac{\lambda}{rate_\mathrm{c}(P|\mathcal{E})} \cdot [\![\phi_{T'}]\!]_\mathrm{MT}(P', \theta')$$

and:

$$prob(\mathcal{SC}_{\leq \theta}(P, T)) = \sum_{P \xrightarrow{a,\lambda}_\mathrm{M} P'} \frac{\lambda}{rate_\mathrm{t}(P \parallel T)} \cdot prob(\mathcal{SC}_{\leq \theta'}(P', T'))$$

where $rate_\mathrm{c}(P|\mathcal{E}) = rate_\mathrm{t}(P \parallel T)$. By the induction hypothesis, for all P' reachable from P via an a-transition we have:

$$[\![\phi_{T'}]\!]_\mathrm{MT}(P', \theta') = prob(\mathcal{SC}_{\leq \theta'}(P', T'))$$

from which the result follows. ■

Lemma 5.14. For each $\phi \in \mathrm{HML}_\mathrm{PMT}$ there exists $T_\phi \in \mathcal{T}_\mathrm{R,c}$ such that for all $P \in \mathcal{P}_\mathrm{M}$ and $\theta \in (\mathbb{R}_{>0})^*$:

$$prob(\mathcal{SC}_{\leq \theta}(P, T_\phi)) = [\![\phi]\!]_\mathrm{MT}(P, \theta)$$

Proof. We proceed by induction on the syntactical structure of ϕ:

— Let $\phi \equiv \mathit{true}$ and take $T_\phi \equiv \mathrm{s}$. Then for all $P \in \mathcal{P}_\mathrm{M}$ and $\theta \in (\mathbb{R}_{>0})^*$ we immediately derive:

$$prob(\mathcal{SC}_{\leq \theta}(P, T_\phi)) = 1 = [\![\phi]\!]_\mathrm{MT}(P, \theta)$$

— Let $\phi \equiv \langle a|\mathcal{E}\rangle \phi'$ and take $T_\phi \equiv \text{<}a, *_1\text{>}.T_{\phi'} + \sum_{b \in \mathcal{E}-\{a\}} \text{<}b, *_1\text{>}.\mathrm{f}$ such that $T_{\phi'}$ satisfies the induction hypothesis with respect to ϕ'. In order to avoid trivial cases, consider $P \in \mathcal{P}_\mathrm{M}$ that can perform a-actions and $\theta \in (\mathbb{R}_{>0})^*$ such that $\theta = t \circ \theta'$ and $\frac{1}{rate_\mathrm{c}(P|\mathcal{E})} \leq t$, otherwise:

$$prob(\mathcal{SC}_{\leq \theta}(P, T_\phi)) = 0 = [\![\phi]\!]_\mathrm{MT}(P, \theta)$$

Then we have:

$$prob(\mathcal{SC}_{\leq \theta}(P, T_\phi)) = \sum_{P \xrightarrow{a,\lambda}_\mathrm{M} P'} \frac{\lambda}{rate_\mathrm{t}(P \parallel T_\phi)} \cdot prob(\mathcal{SC}_{\leq \theta'}(P', T_{\phi'}))$$

and:

$$[\![\phi]\!]_\mathrm{MT}(P, \theta) = \sum_{P \xrightarrow{a,\lambda}_\mathrm{M} P'} \frac{\lambda}{rate_\mathrm{c}(P|\mathcal{E})} \cdot [\![\phi']\!]_\mathrm{MT}(P', \theta')$$

where $rate_\mathrm{t}(P \parallel T_\phi) = rate_\mathrm{c}(P|\mathcal{E})$. By the induction hypothesis, for all P' reachable from P via an a-transition we have:

$$prob(\mathcal{SC}_{\leq \theta'}(P', T_{\phi'})) = [\![\phi']\!]_\mathrm{MT}(P', \theta')$$

from which the result follows. ■

Theorem 5.15. Let $P_1, P_2 \in \mathcal{P}_{\mathrm{M}}$. Then:

$$P_1 \sim_{\mathrm{MT}} P_2 \iff \forall \phi \in \mathrm{HML}_{\mathrm{PMT}}. \forall \theta \in (\mathbb{R}_{>0})^*. [\![\phi]\!]_{\mathrm{MT}}(P_1, \theta) = [\![\phi]\!]_{\mathrm{MT}}(P_2, \theta)$$

Proof. The result is a straightforward consequence of Thm. 5.11 and of the bijective correspondence between canonical reactive tests and formulas of $\mathrm{HML}_{\mathrm{PMT}}$ established by Lemmas 5.13 and 5.14. ■

6. Modal Characterizations of Trace Equivalence

In this section we recall from (van Glabbeek 2001) the modal logic characterization of nondeterministic trace equivalence, then we provide a modal logic characterization for the probabilistic and Markovian versions of trace equivalence.

6.1. *Characterization of Nondeterministic Trace Equivalence*

Nondeterministic trace equivalence is simply characterized by a restriction of HML in which neither negation nor logical conjunction occur.

Definition 6.1. The set of the formulas of $\mathrm{HML}_{\mathrm{NPMTr}}$ is generated by the following syntax:

$$\phi ::= \mathit{true} \mid \langle a \rangle \phi$$

where $a \in \mathit{Name}$. ■

Definition 6.2. The satisfaction relation $\models_{\mathrm{NTr}}$ of $\mathrm{HML}_{\mathrm{NPMTr}}$ over $\mathcal{P}_{\mathrm{N}}$ is defined by structural induction as follows:

$$\begin{array}{lll} P & \models_{\mathrm{NTr}} & \mathit{true} \\ P & \models_{\mathrm{NTr}} & \langle a \rangle \phi \qquad \text{if } P \xrightarrow{a}_{\mathrm{N}} P' \text{ with } P' \models_{\mathrm{NTr}} \phi \text{ for some } P' \end{array}$$

■

Theorem 6.3. Let $P_1, P_2 \in \mathcal{P}_{\mathrm{N}}$. Then:

$$P_1 \sim_{\mathrm{NTr}} P_2 \iff (\forall \phi \in \mathrm{HML}_{\mathrm{NPMTr}}. P_1 \models_{\mathrm{NTr}} \phi \iff P_2 \models_{\mathrm{NTr}} \phi)$$ ■

6.2. *Global Characterization of Probabilistic Trace Equivalence*

The logic characterization of probabilistic trace equivalence is based on the same modal language as before, i.e. $\mathrm{HML}_{\mathrm{NPMTr}}$, with the difference that a quantitative interpretation like the one of Sect. 5.2 is adopted in order to measure the probability with which a process term satisfies a formula.

Definition 6.4. The interpretation function $[\![.]\!]_{\mathrm{PTr}}$ of $\mathrm{HML}_{\mathrm{NPMTr}}$ over $\mathcal{P}_{\mathrm{P}}$ is defined by structural induction as follows:

$$\begin{array}{|lcl|} \hline [\![\mathit{true}]\!]_{\mathrm{PTr}}(P) & = & 1 \\ [\![\langle a\rangle\phi]\!]_{\mathrm{PTr}}(P) & = & \sum\limits_{P \xrightarrow{a,p}_{\mathrm{P}} P'} p \cdot [\![\phi]\!]_{\mathrm{PTr}}(P') \\ \hline \end{array}$$

where the summation is taken to be zero whenever there are no a-transitions departing from P. ■

We now see that the formulas of $\mathrm{HML}_{\mathrm{NPMTr}}$ have a one-to-one correspondence with the traces, from which the modal logic characterization result for $\sim_{\mathrm{PTr}}$ will immediately follow.

Lemma 6.5. For each $\alpha \in \mathit{Name}^*$ there exists $\phi_\alpha \in \mathrm{HML}_{\mathrm{NPMTr}}$ such that for all $P \in \mathcal{P}_{\mathrm{P}}$:

$$[\![\phi_\alpha]\!]_{\mathrm{PTr}}(P) = \mathit{prob}(\mathcal{CC}(P,\alpha))$$

Proof. We proceed by induction on the length of α:

— Let $\mathit{length}(\alpha) = 0$, i.e. $\alpha \equiv \varepsilon$, and take $\phi_\alpha \equiv \mathit{true}$. Then for all $P \in \mathcal{P}_{\mathrm{P}}$ we immediately derive:

$$[\![\phi_\alpha]\!]_{\mathrm{PTr}}(P) = 1 = \mathit{prob}(\mathcal{CC}(P,\alpha))$$

— Let $\mathit{length}(\alpha) > 0$, say $\alpha \equiv a \cdot \alpha'$, and take $\phi_\alpha \equiv \langle a\rangle\phi_{\alpha'}$ such that $\phi_{\alpha'}$ satisfies the induction hypothesis with respect to α'. In order to avoid trivial cases, consider $P \in \mathcal{P}_{\mathrm{P}}$ that can perform a-actions, otherwise:

$$[\![\phi_\alpha]\!]_{\mathrm{PTr}}(P) = 0 = \mathit{prob}(\mathcal{CC}(P,\alpha))$$

Then we have:

$$[\![\phi_\alpha]\!]_{\mathrm{PTr}}(P) = \sum_{P \xrightarrow{a,p}_{\mathrm{P}} P'} p \cdot [\![\phi_{\alpha'}]\!]_{\mathrm{PTr}}(P')$$

and:

$$\mathit{prob}(\mathcal{CC}(P,\alpha)) = \sum_{P \xrightarrow{a,p}_{\mathrm{P}} P'} p \cdot \mathit{prob}(\mathcal{CC}(P',\alpha'))$$

By the induction hypothesis, for all P' reachable from P via an a-transition we have:

$$[\![\phi_{\alpha'}]\!]_{\mathrm{PTr}}(P') = \mathit{prob}(\mathcal{CC}(P',\alpha'))$$

from which the result follows. ■

Lemma 6.6. For each $\phi \in \mathrm{HML}_{\mathrm{NPMTr}}$ there exists $\alpha_\phi \in \mathit{Name}^*$ such that for all $P \in \mathcal{P}_{\mathrm{P}}$:

$$\mathit{prob}(\mathcal{CC}(P,\alpha_\phi)) = [\![\phi]\!]_{\mathrm{PTr}}(P)$$

Proof. We proceed by induction on the syntactical structure of ϕ:

— Let $\phi \equiv \mathit{true}$ and take $\alpha_\phi \equiv \varepsilon$. Then for all $P \in \mathcal{P}_{\mathrm{P}}$ we immediately derive:

$$\mathit{prob}(\mathcal{CC}(P,\alpha_\phi)) = 1 = [\![\phi]\!]_{\mathrm{PTr}}(P)$$

— Let $\phi \equiv \langle a \rangle \phi'$ and take $\alpha_\phi \equiv a \cdot \alpha_{\phi'}$ such that $\alpha_{\phi'}$ satisfies the induction hypothesis with respect to ϕ'. In order to avoid trivial cases, consider $P \in \mathcal{P}_{\rm P}$ that can perform a-actions, otherwise:

$$prob(\mathcal{CC}(P, \alpha_\phi)) = 0 = [\![\phi]\!]_{\rm PTr}(P)$$

Then we have:

$$prob(\mathcal{CC}(P, \alpha_\phi)) = \sum_{P \xrightarrow{a,p}_{\rm P} P'} p \cdot prob(\mathcal{CC}(P', \alpha_{\phi'}))$$

and:

$$[\![\phi]\!]_{\rm PTr}(P) = \sum_{P \xrightarrow{a,p}_{\rm P} P'} p \cdot [\![\phi']\!]_{\rm PTr}(P')$$

By the induction hypothesis, for all P' reachable from P via an a-transition we have:

$$prob(\mathcal{CC}(P', \alpha_{\phi'})) = [\![\phi']\!]_{\rm PTr}(P')$$

from which the result follows. ■

Theorem 6.7. Let $P_1, P_2 \in \mathcal{P}_{\rm P}$. Then:

$$P_1 \sim_{\rm PTr} P_2 \iff \forall \phi \in {\rm HML}_{\rm NPMTr}.\, [\![\phi]\!]_{\rm PTr}(P_1) = [\![\phi]\!]_{\rm PTr}(P_2)$$

Proof. The result is a straightforward consequence of the bijective correspondence between traces and formulas of $\rm HML_{NPMTr}$ established by Lemmas 6.5 and 6.6. ■

6.3. *Global Characterization of Markovian Trace Equivalence*

The modal language for $\sim_{\rm MTr}$ is the same as for $\sim_{\rm PTr}$, i.e. $\rm HML_{NPMTr}$, with the difference that the quantitative interpretation has now to measure the probability with which a process term satisfies a formula within a given average amount of time.

Definition 6.8. The interpretation function $[\![.]\!]_{\rm MTr}$ of $\rm HML_{NPMTr}$ over $\mathcal{P}_{\rm M} \times (\mathbb{R}_{>0})^*$ is defined by structural induction as follows:

$$[\![true]\!]_{\rm MTr}(P, \theta) = 1$$

$$[\![\langle a \rangle \phi]\!]_{\rm MTr}(P, \theta) = \begin{cases} 0 & \text{if } \theta = \varepsilon \vee \frac{1}{rate_{\rm t}(P)} > \theta[1] \\ \sum_{P \xrightarrow{a,\lambda}_{\rm M} P'} \frac{\lambda}{rate_{\rm t}(P)} \cdot [\![\phi]\!]_{\rm MTr}(P', \theta') & \text{if } \theta = t \circ \theta' \wedge \frac{1}{rate_{\rm t}(P)} \leq t \end{cases}$$

where the summation is taken to be zero whenever there are no a-transitions departing from P. ■

Like in the probabilistic case, we are able to establish a one-to-one correspondence between the formulas of $\rm HML_{NPMTr}$ and the traces, from which the modal logic characterization result for $\sim_{\rm MTr}$ will immediately follow.

Lemma 6.9. For each $\alpha \in \mathit{Name}^*$ there exists $\phi_\alpha \in \mathrm{HML}_{\mathrm{NPMTr}}$ such that for all $P \in \mathcal{P}_{\mathrm{M}}$ and $\theta \in (\mathbb{R}_{>0})^*$:

$$[\![\phi_\alpha]\!]_{\mathrm{MTr}}(P, \theta) = \mathit{prob}(\mathcal{CC}_{\leq \theta}(P, \alpha))$$

Proof. We proceed by induction on the length of α:

— Let $\mathit{length}(\alpha) = 0$, i.e. $\alpha \equiv \varepsilon$, and take $\phi_\alpha \equiv \mathit{true}$. Then for all $P \in \mathcal{P}_{\mathrm{M}}$ and $\theta \in (\mathbb{R}_{>0})^*$ we immediately derive:

$$[\![\phi_\alpha]\!]_{\mathrm{MTr}}(P, \theta) = 1 = \mathit{prob}(\mathcal{CC}_{\leq \theta}(P, \alpha))$$

— Let $\mathit{length}(\alpha) > 0$, say $\alpha \equiv a \cdot \alpha'$, and take $\phi_\alpha \equiv \langle a \rangle \phi_{\alpha'}$ such that $\phi_{\alpha'}$ satisfies the induction hypothesis with respect to α'. In order to avoid trivial cases, consider $P \in \mathcal{P}_{\mathrm{M}}$ that can perform a-actions and $\theta \in (\mathbb{R}_{>0})^*$ such that $\theta = t \circ \theta'$ and $\frac{1}{\mathit{rate}_{\mathrm{t}}(P)} \leq t$, otherwise:

$$[\![\phi_\alpha]\!]_{\mathrm{MTr}}(P, \theta) = 0 = \mathit{prob}(\mathcal{CC}_{\leq \theta}(P, \alpha))$$

Then we have:

$$[\![\phi_\alpha]\!]_{\mathrm{MTr}}(P, \theta) = \sum_{P \xrightarrow{a,\lambda}_{\mathrm{M}} P'} \frac{\lambda}{\mathit{rate}_{\mathrm{t}}(P)} \cdot [\![\phi_{\alpha'}]\!]_{\mathrm{MTr}}(P', \theta')$$

and:

$$\mathit{prob}(\mathcal{CC}_{\leq \theta}(P, \alpha)) = \sum_{P \xrightarrow{a,\lambda}_{\mathrm{M}} P'} \frac{\lambda}{\mathit{rate}_{\mathrm{t}}(P)} \cdot \mathit{prob}(\mathcal{CC}_{\leq \theta'}(P', \alpha'))$$

By the induction hypothesis, for all P' reachable from P via an a-transition we have:

$$[\![\phi_{\alpha'}]\!]_{\mathrm{MTr}}(P', \theta') = \mathit{prob}(\mathcal{CC}_{\leq \theta'}(P', \alpha'))$$

from which the result follows. ■

Lemma 6.10. For each $\phi \in \mathrm{HML}_{\mathrm{NPMTr}}$ there exists $\alpha_\phi \in \mathit{Name}^*$ such that for all $P \in \mathcal{P}_{\mathrm{M}}$ and $\theta \in (\mathbb{R}_{>0})^*$:

$$\mathit{prob}(\mathcal{CC}_{\leq \theta}(P, \alpha_\phi)) = [\![\phi]\!]_{\mathrm{MTr}}(P, \theta)$$

Proof. We proceed by induction on the syntactical structure of ϕ:

— Let $\phi \equiv \mathit{true}$ and take $\alpha_\phi \equiv \varepsilon$. Then for all $P \in \mathcal{P}_{\mathrm{M}}$ and $\theta \in (\mathbb{R}_{>0})^*$ we immediately derive:

$$\mathit{prob}(\mathcal{CC}_{\leq \theta}(P, \alpha_\phi)) = 1 = [\![\phi]\!]_{\mathrm{MTr}}(P, \theta)$$

— Let $\phi \equiv \langle a \rangle \phi'$ and take $\alpha_\phi \equiv a \cdot \alpha_{\phi'}$ such that $\alpha_{\phi'}$ satisfies the induction hypothesis with respect to ϕ'. In order to avoid trivial cases, consider $P \in \mathcal{P}_{\mathrm{M}}$ that can perform a-actions and $\theta \in (\mathbb{R}_{>0})^*$ such that $\theta = t \circ \theta'$ and $\frac{1}{\mathit{rate}_{\mathrm{t}}(P)} \leq t$, otherwise:

$$\mathit{prob}(\mathcal{CC}_{\leq \theta}(P, \alpha_\phi)) = 0 = [\![\phi]\!]_{\mathrm{MTr}}(P, \theta)$$

Then we have:

$$\mathit{prob}(\mathcal{CC}_{\leq \theta}(P, \alpha_\phi)) = \sum_{P \xrightarrow{a,\lambda}_{\mathrm{M}} P'} \frac{\lambda}{\mathit{rate}_{\mathrm{t}}(P)} \cdot \mathit{prob}(\mathcal{CC}_{\leq \theta'}(P', \alpha_{\phi'}))$$

and:

$$[\![\phi]\!]_{\rm MTr}(P, \theta) = \sum_{P \xrightarrow{a,\lambda}_{\rm M} P'} \frac{\lambda}{rate_{\rm t}(P)} \cdot [\![\phi']\!]_{\rm MTr}(P', \theta')$$

By the induction hypothesis, for all P' reachable from P via an a-transition we have:

$$prob(\mathcal{CC}_{\leq \theta'}(P', \alpha_{\phi'})) = [\![\phi']\!]_{\rm MTr}(P', \theta')$$

from which the result follows. ■

Theorem 6.11. Let $P_1, P_2 \in \mathcal{P}_{\rm M}$. Then:

$$P_1 \sim_{\rm MTr} P_2 \iff \forall \phi \in {\rm HML}_{\rm NPMTr}. \forall \theta \in (\mathbb{R}_{>0})^*. [\![\phi]\!]_{\rm MTr}(P_1, \theta) = [\![\phi]\!]_{\rm MTr}(P_2, \theta)$$

Proof. The result is a straightforward consequence of the bijective correspondence between traces and formulas of $\rm HML_{NPMTr}$ established by Lemmas 6.9 and 6.10. ■

7. Comparing the Modal Logic Characterizations

The modal logic characterizations for the nine behavioral equivalences considered in this paper are summarized in Fig. 1. On the horizontal axis we have the three approaches to the definition of behavioral equivalences, while on the vertical axis we have the three classes of systems.

7.1. *Differences across the Approaches*

When moving from bisimulation equivalence to trace equivalence, we see from Fig. 1 that the number of logical operators that are needed tends to diminish, in accordance with the decreasing distinguishing power of the three approaches. More precisely, in the case of bisimulation equivalence we have all the logical operators of HML, then negation is dropped in the case of testing equivalence, and finally logical conjunction is left out as well in the case of trace equivalence.

7.2. *Regularities within the Approaches*

The three logical characterizations for nondeterministic, probabilistic and Markovian bisimulation equivalence basically rely on the same modal language, which is HML. The only differences are that, in the probabilistic (resp. Markovian) case, the diamond operator is decorated with a probability (resp. rate) lower bound and, since this lower bound cannot be zero, an additional modal operator is present to express the fact that an action cannot be executed at all.

The situation is more varied for testing equivalence. In the nondeterministic case, two modal operators are needed – one for traces and one for individual actions – together with logical disjunction. When moving to the probabilistic or Markovian case, all the previous operators are replaced by a conditional diamond operator.

Finally, the three logical characterizations for nondeterministic, probabilistic and Markovian trace equivalence rely exactly on the same modal language, which comprises only the constant *true* and the diamond operator.

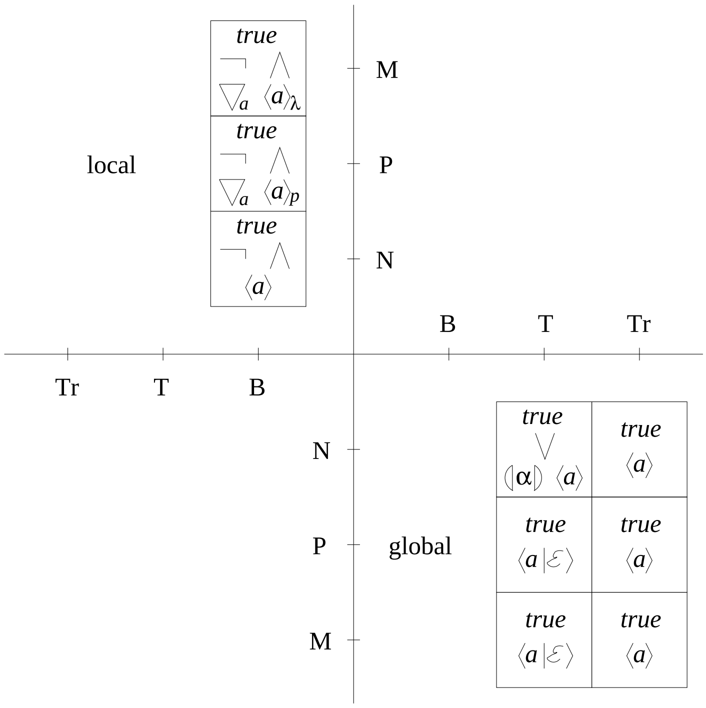

Fig. 1. Summary of results

7.3. *Local vs. Global Characterizations*

For probabilistic and Markovian systems the question arises about how to deal with the probabilistic and temporal aspects of the computations. There are in principle two opposite answers. The first one – which we call local – is that such aspects should be considered at the level of the individual actions occurring in the computations, while the second one – which we call global – is that they should be considered at the level of the overall computations.

The answer that is chosen has a deep impact on the syntax and the interpretation of the modal language. In the local case probabilistic and temporal parameters should be present in the syntax of the modal operators and the interpretation of the resulting formulas should be qualitative as usual, in the sense that it should return a truth value. By contrast, in the global case no probabilistic and temporal parameters should be present in the syntax but the interpretation of the usual formulas should be quantitative, in the sense that it should return a number that measures how much a formula is satisfied.

The local view is the one adopted in (Larsen and Skou 1991; Clark et al. 1999) for the logical characterizations of probabilistic and Markovian bisimulation equivalences, consistently with the fact that bisimilarity relates systems that behave the same step by step. The global view, originally proposed in (Kwiatkowska and Norman 1998), has instead been adopted for the logical characterizations of probabilistic and Markovian testing/trace equivalences, consistently with the fact that such equivalences take into account the probabilistic and temporal aspects of the overall, possibly test-driven computations.

We explicitly observe that a local characterization is not possible for probabilistic and Markovian testing/trace equivalences. As an example, in the probabilistic case one may think of using modal operators like $\langle a|\mathcal{E}\rangle_p\phi$ and $\langle a\rangle_p\phi$, with $p \in \mathbf{R}_{]0,1]}$ being a probability lower bound. Then e.g. the two probabilistic trace equivalent process terms (with $p_1 + p_2 = 1$):

$$\begin{array}{rcl} P & \equiv & <a, p_1>.<b, 1>.\underline{0} + <a, p_2>.<c, 1>.\underline{0} \\ Q & \equiv & <a, 1>.(<b, p_1>.\underline{0} + <c, p_2>.\underline{0}) \end{array}$$

would be distinguished by the following formula:

$$\phi \quad \equiv \quad \langle a\rangle_{p_1}\langle b\rangle_1 \, \mathit{true}$$

which is satisfied by P but not by Q. As another example, in the Markovian case one may think of using modal operators like $\langle a|\mathcal{E}\rangle_{p,t}\phi$ and $\langle a\rangle_{p,t}\phi$, with $t \in \mathbf{R}_{>0}$ being an average time upper bound. Then e.g. the two Markovian trace equivalent process terms:

$$\begin{array}{rcl} P' & \equiv & <a, \lambda_1>.<b, \mu>.\underline{0} + <a, \lambda_2>.<c, \mu>.\underline{0} \\ Q' & \equiv & <a, \lambda_1 + \lambda_2>.(<b, \frac{\lambda_1}{\lambda_1+\lambda_2} \cdot \mu>.\underline{0} + <c, \frac{\lambda_2}{\lambda_1+\lambda_2} \cdot \mu>.\underline{0}) \end{array}$$

would be distinguished by the following formula:

$$\phi' \quad \equiv \quad \langle a\rangle_{\frac{\lambda_1}{\lambda_1+\lambda_2}, \frac{1}{\lambda_1+\lambda_2}}\langle b\rangle_{1, \frac{1}{\mu}} \, \mathit{true}$$

which is satisfied by P' but not by Q'.

7.4. *Future Work*

As already observed, unlike bisimulation and trace equivalences, in the case of testing equivalence the logical characterizations for the nondeterministic, probabilistic and Markovian versions comprise different logical operators. It would thus be interesting to understand whether a more regular pattern can be found for the logical operators characterizing the various versions of testing equivalence.

In (Kwiatkowska and Norman 1998) a modal logic characterization is provided for a testing equivalence for reactive probabilistic processes. The resulting modal language is essentially HML without negation, which is then interpreted quantitatively. However, instead of classical tests à la (De Nicola and Hennessy 1983), the authors consider tests with copying, in the sense that multiple copies of a process can be taken at any stage of a test in order to experiment on one copy at a time (Abramsky 1987). Tests with the copying facility induce equivalences that are at most as coarse as $\frac{2}{3}$-bisimilarity (Larsen and Skou 1991), thus increasing the distinguishing power with respect to classical testing equivalence.

We conjecture that a suitable restriction of tests with copying may result exactly in classical testing equivalence à la (De Nicola and Hennessy 1983). We believe that a good candidate is the restriction considered in (Kwiatkowska and Norman 1998) to ensure probabilistic soundness, which amounts to impose that the tests that are applied to the various copies are independent of each other. At the modal language level, this means that, whenever a conjunction of diamond operators is encountered, the names of the actions occurring in them must be all different. Should our conjecture be verified, the nondeterministic, probabilistic and Markovian versions of testing equivalence would be uniformly characterized by a modal logic given by HML without negation.

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