The Spectrum of Strong Behavioral Equivalences for Nondeterministic and Probabilistic Processes

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We present a spectrum of trace-based, testing, and bisimulation equivalences for nondeterministic and probabilistic processes whose activities are all observable. For every equivalence under study, we examine the discriminating power of three variants stemming from three approaches that differ for the way probabilities of events are compared when nondeterministic choices are resolved via deterministic schedulers. We show that the first approach – which compares two resolutions relatively to the probability distributions of all considered events – results in a fragment of the spectrum compatible with the spectrum of behavioral equivalences for fully probabilistic processes. In contrast, the second approach – which compares the probabilities of the events of a resolution with the probabilities of the same events in possibly different resolutions – gives rise to another fragment composed of coarser equivalences that exhibits several analogies with the spectrum of behavioral equivalences for fully nondeterministic processes. Finally, the third approach – which only compares the extremal probabilities of each event stemming from the different resolutions – yields even coarser equivalences that, however, give rise to a hierarchy similar to that stemming from the second approach.

1 Introduction

Process algebras are mathematically rigorous languages that have been widely used to model and analyze the behavior of interacting systems. Their structural operational semantics associates with each process term a labeled transition system (LTS), whose states are the terms themselves and whose labels are the actions that each term can perform. In order to abstract from unwanted details, the operational semantics is often coupled with observational mechanisms that permit equating those systems that cannot be distinguished by external entities. The resulting behavioral equivalences heavily depend on how the specified systems are expected to be used. Indeed, there is still disagreement on which are the "reasonable" observations and how their outcomes can be used to distinguish or identify systems. Thus, many equivalences have been proposed and much work has been done to assess their discriminating power and mutual relationships.

The first study in this direction was done by [9]. There, most of the then known equivalences over LTS models were "ordered" and it was shown that *trace equivalences* (equating systems performing the same sequences of actions) are strictly coarser than *decorated-trace equivalences* (equating systems performing the same sequences of actions and refusing/accepting the same sets of actions after them), which in turn are strictly coarser than *bisimulation equivalences* (equating systems performing the same sequences of actions and recursively exhibiting the same behavior). It was also shown that the equivalence obtained by *testing* processes with external observers was coincident with *failure equivalence*

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obtained via traces decorated with refusal sets. Afterwards, [14] built the first spectrum that relates twelve different equivalences and set up a general testing scenario that could be used to generate many more equivalences.

When process algebras have been enriched with additional dimensions to deal with probabilistic, stochastic, and timed systems, new behavioral equivalences have been defined and possible classifications have been proposed. Here, we would like to concentrate on equivalences for probabilistic systems. For this class of systems, comparative results have been obtained only for so-called fully probabilistic systems [20, 17, 2] or only for bisimulation and testing relations [2, 22, 29, 34].

In this paper, we aim at a systematic account of the known probabilistic equivalences for nondeterministic *and* probabilistic systems and introduce, motivate, and relate some new ones. We shall consider an extension of the LTS model combining nondeterminism and probability that we call NPLTS, in which every action-labeled transition goes from a source state to a probability distribution over target states rather than to a single target state [21, 25]. Actions will be assumed to be visible (i.e., we shall not admit τ-actions) and, for the considered strong equivalences, resolutions of nondeterminism will be derived by applying memoryless deterministic (as opposed to randomized) schedulers.

When defining behavioral relations over NPLTS models, the idea is to compare resolutions on the basis of the probabilities of *equivalence-specific events*, like (i) performing certain sequences of actions, (ii) exhibiting certain decorated traces, or (iii) reaching certain sets of equivalent states via given actions.

The typical approach followed in the literature (see, e.g., [28, 26, 27]) consists of comparing the *probability distributions of all equivalence-specific events* of two resolutions. Two processes are considered as equivalent if, for each resolution of any of the two processes, there exists a resolution of the other process such that the probability of *each* equivalence-specific event is the same in the two resolutions (*fully matching resolutions*). For the known relations based on this approach, we have that the probabilistic bisimilarity in [28] implies the probabilistic failure equivalence in [27] that in turn implies the probabilistic trace equivalence in [26]. All these relations are conservative extensions of the corresponding relations defined over fully nondeterministic models [16, 6] and fully probabilistic models [13, 20, 17], but in many situations they turn out to have a high discriminating power.

A different approach has been followed in the literature for defining testing equivalences (see, e.g., [35, 19, 27, 11]). Instead of comparing individual resolutions of the parallel composition of processes and tests, the comparison is performed between the *extremal probabilities* of reaching success *over all resolutions* generated by the experiments on processes under test (*max-min-matching resolution sets*). In this case, it holds that the resulting probabilistic testing equivalence is implied by the probabilistic bisimilarity in [28], but it is related neither to the probabilistic failure equivalence in [27] nor to the probabilistic trace equivalence in [26] when restricting attention to deterministic schedulers. Moreover, the resulting probabilistic testing equivalence subsumes testing equivalence for fully probabilistic processes [7], but it is not a conservative extension of testing equivalence for fully nondeterministic processes [10].

Recently, in [8, 32, 31, 3, 5] a further approach has appeared that compares resolutions on the basis of the *probabilities of individual equivalence-specific events*. Thus, a resolution of any of the two processes can be matched, with respect to *different equivalence-specific events*, by *different resolutions* of the other process (*partially matching resolutions*). For the behavioral relations resulting from this approach, which weakens the impact of schedulers, we have that probabilistic bisimilarity implies probabilistic failure equivalence, which in turn implies probabilistic testing equivalence, which finally implies probabilistic trace equivalence. This approach has contributed to the development of new probabilistic bisimilarities in [8], [32, 5], and [31] that, unlike the one in [28], are characterized by standard probabilistic logics such as quantitative μ-calculus, PML, and PCTL/PCTL*, respectively. Moreover, in the case of testing equivalence this approach has the advantage of being conservative also for fully nondeterministic models [3], while in the case of trace equivalence it surprisingly results in a congruence with respect to parallel composition (full version of [3]).

In our view, the motivations behind the three approaches outlined above are all very reasonable. Indeed, when applied to fully nondeterministic processes or fully probabilistic processes, they give rise to well-studied relations that for the fully nondeterministic setting fit into the spectra in [9, 14] and for the fully probabilistic setting fit into the spectra in [20, 17]. The situation is significantly different when the three approaches are instantiated for nondeterministic *and* probabilistic processes, as in that case they give rise to a much wider variety of relations.

In this paper, we study the relationships between the equivalences for NPLTS models that stem from the three approaches. For each approach, we consider the three main families of equivalences, namely trace-based, testing, and bisimulation equivalences. To the best of our knowledge, this is the first comparative study of different kinds of behavioral equivalences over models featuring both nondeterministic and probabilistic aspects. Such a study is even more on demand after the recent introduction of new equivalences, like the ones in [8, 32, 31, 3, 5], that have interesting properties.

To have a full picture of the spectrum, the reader is referred to Fig. 4 in the concluding section. There, the equivalences stemming from the same approach are contained in boxes with the same shape (hexagonal, rounded, or rectangular) and the equivalences specifically introduced for the purposes of this paper are in dashed boxes. We would like to stress that the original contribution of the paper is not given by the equivalences that we introduce to fill in gaps, *but is the spectrum itself*.

We shall see that the family of equivalences that assign a central role to schedulers by requiring fully matching resolutions, yields a hierarchy that is in accordance with the one for fully probabilistic processes in [20, 17]. Conversely, the family of equivalences that assign a weaker role to schedulers by requiring partially matching resolutions, gives rise to relations that are coarser than the former and yields a hierarchy that is in accordance with the one for fully nondeterministic processes in [9, 14]. Finally, the family of equivalences that only consider extremal probabilities, has again several analogies with the fully nondeterministic spectrum and yields even coarser equivalences. There are however some noticeable anomalies in the last two families, given by a few equivalences suffering from isolation.

The rest of the paper is organized as follows. In Sect. 2, we introduce the NPLTS model. In Sects. 3 to 5, we define and compare, respectively, the trace-based, testing, and bisimulation equivalences that arise from the three approaches outlined above. Finally, in Sect. 6 we draw some conclusions and graphically summarize the results by depicting the spectrum of all the considered equivalences.

2 Nondeterministic and Probabilistic Processes

Processes combining nondeterminism and probability are typically described by means of extensions of the LTS model, in which every action-labeled transition goes from a source state to a *probability distribution over target states* rather than to a single target state. They are essentially Markov decision processes and are representative of a number of slightly different probabilistic computational models including internal nondeterminism such as, e.g., concurrent Markov chains [33], alternating probabilistic models [15, 35, 24], probabilistic automata in the sense of [25], and the denotational probabilistic models in [18] (see [30] for an overview). We formalize them as a variant of simple probabilistic automata [25].

Definition 2.1 A nondeterministic and probabilistic labeled transition system, NPLTS for short, is a triple $(S, A, \longrightarrow)$ where S is an at most countable set of states, A is a countable set of transition-labeling actions, and $\longrightarrow \subseteq S \times A \times Distr(S)$ is a transition relation with $Distr(S)$ being the set of discrete probability distributions over S. ■

A transition $(s, a, \mathcal{D})$ is written $s \xrightarrow{a} \mathcal{D}$. We say that $s' \in S$ is not reachable from s via that a-transition if $\mathcal{D}(s') = 0$, otherwise we say that it is reachable with probability $p = \mathcal{D}(s')$. The reachable states form the support of $\mathcal{D}$: $supp(\mathcal{D}) = \{s' \in S \mid \mathcal{D}(s') > 0\}$. We write $s \xrightarrow{a}$ to indicate that s has an a-transition. The choice among all the transitions departing from s is nondeterministic, while the choice of the target state for a specific transition is probabilistic. An NPLTS represents (i) a *fully nondeterministic process* when every transition leads to a distribution that concentrates all the probability mass into a single state or (ii) a *fully probabilistic process* when every state has at most one outgoing transition.

An NPLTS can be depicted as a directed graph-like structure in which vertices represent states and action-labeled edges represent action-labeled transitions. Given a transition $s \xrightarrow{a} \mathcal{D}$, the corresponding a-labeled edge goes from the vertex representing state s to a set of vertices linked by a dashed line, each of which represents a state $s' \in supp(\mathcal{D})$ and is labeled with $\mathcal{D}(s')$ – label omitted if $\mathcal{D}(s') = 1$. Figure 1 shows eighteen NPLTS models, nine of which are fully nondeterministic.

In this setting, a computation is a sequence of state-to-state steps, each denoted by $s \overset{a}{\longmapsto} s'$ and derived from a state-to-distribution transition. Formally, given an NPLTS $\mathcal{L} = (S, A, \longrightarrow)$ and $s, s' \in S$, we say that $c \equiv s_0 \overset{a_1}{\longmapsto} s_1 \overset{a_2}{\longmapsto} s_2 \dots s_{n-1} \overset{a_n}{\longmapsto} s_n$ is a computation of $\mathcal{L}$ of length n from $s = s_0$ to $s' = s_n$ iff for all $i = 1, \dots, n$ there exists a transition $s_{i-1} \xrightarrow{a_i} \mathcal{D}_i$ such that $s_i \in supp(\mathcal{D}_i)$, with $\mathcal{D}_i(s_i)$ being the execution probability of step $s_{i-1} \overset{a_i}{\longmapsto} s_i$ conditioned on the selection of transition $s_{i-1} \xrightarrow{a_i} \mathcal{D}_i$ of $\mathcal{L}$ at state s_{i-1}. We denote by $\mathit{first}(c)$ and $\mathit{last}(c)$ the initial state and the final state of c, respectively, and by $\mathcal{C}_{\text{fin}}(s)$ the set of finite-length computations from s.

We call resolution of s any possible way of resolving nondeterminism starting from s. Each resolution is a tree-like structure whose branching points represent probabilistic choices. This is obtained by unfolding from s the graph structure underlying $\mathcal{L}$ and by selecting at each state a single transition of $\mathcal{L}$ (*deterministic scheduler*) or a convex combination of equally labeled transitions of $\mathcal{L}$ (*randomized scheduler*) among all the transitions possible from that state. Below, we introduce the notion of resolution arising from a deterministic scheduler as a fully probabilistic NPLTS. Notice that, when $\mathcal{L}$ is fully nondeterministic, resolutions boil down to computations.

Definition 2.2 Let $\mathcal{L} = (S, A, \longrightarrow)$ be an NPLTS and $s \in S$. We say that an NPLTS $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}})$ is a resolution of s obtained via a deterministic scheduler iff there exists a state correspondence function $corr_{\mathcal{Z}} : Z \to S$ such that $s = corr_{\mathcal{Z}}(z_s)$, for some $z_s \in Z$, and for all $z \in Z$ it holds that:

- If $z \xrightarrow{a}_{\mathcal{Z}} \mathcal{D}$, then $corr_{\mathcal{Z}}(z) \xrightarrow{a} \mathcal{D}'$ with $\mathcal{D}(z') = \mathcal{D}'(corr_{\mathcal{Z}}(z'))$ for all $z' \in Z$.
- If $z \xrightarrow{a_1}_{\mathcal{Z}} \mathcal{D}_1$ and $z \xrightarrow{a_2}_{\mathcal{Z}} \mathcal{D}_2$, then $a_1 = a_2$ and $\mathcal{D}_1 = \mathcal{D}_2$. ■

We denote by $Res(s)$ the set of resolutions of s and by $Res_{\max}(s)$ the set of maximal resolutions of s, i.e., the resolutions of s that cannot be further extended in accordance with the graph structure of $\mathcal{L}$ and the constraints above. Since $\mathcal{Z} \in Res(s)$ is fully probabilistic, the probability $prob(c)$ of executing $c \in \mathcal{C}_{\text{fin}}(z_s)$ can be defined as the product of the (no longer conditional) execution probabilities of the individual steps of c, with $prob(c)$ being always equal to 1 if $\mathcal{L}$ is fully nondeterministic. This notion is lifted to $C \subseteq \mathcal{C}_{\text{fin}}(z_s)$ by letting $prob(C) = \sum_{c \in C} prob(c)$ whenever none of the computations in C is a proper prefix of one of the others.

We finally introduce a notion of fully synchronous parallel composition for NPLTS models that is instrumental to the definition of testing equivalences.

Definition 2.3 Let $\mathcal{L}_i = (S_i, A, \longrightarrow_i)$ be an NPLTS for $i = 1, 2$. The parallel composition of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the NPLTS $\mathcal{L}_1 \parallel \mathcal{L}_2 = (S_1 \times S_2, A, \longrightarrow)$ where $\longrightarrow \subseteq (S_1 \times S_2) \times A \times Distr(S_1 \times S_2)$ is such that $(s_1, s_2) \xrightarrow{a} \mathcal{D}$ iff $s_1 \xrightarrow{a}_1 \mathcal{D}_1$ and $s_2 \xrightarrow{a}_2 \mathcal{D}_2$ with $\mathcal{D}(s'_1, s'_2) = \mathcal{D}_1(s'_1) \cdot \mathcal{D}_2(s'_2)$ for each $(s'_1, s'_2) \in S_1 \times S_2$. ■

3 Trace-Based Equivalences for NPLTS Models

Trace-based equivalences examine the probability with which two states perform computations labeled with the same (decorated) traces for each possible way of resolving nondeterminism. As outlined in Sect. 1, there are three different approaches to defining them. The first approach is to match resolutions according to *trace-based distributions*, which means that for each resolution of one of the two states there must exist a resolution of the other state such that, *for every (decorated) trace*, the two resolutions have the same probability of performing a computation labeled with that (decorated) trace. In other words, matching resolutions of the two states are related by the fully probabilistic version of the trace-based equivalence (fully matching resolutions). The second approach is to consider *a single (decorated) trace at a time*, i.e., to anticipate the quantification over (decorated) traces with respect to the quantification over resolutions. In this way, differently labeled computations of a resolution of one of the two states are allowed to be matched by computations of several different resolutions of the other state (partially matching resolutions). The third approach is to compare only the *extremal probabilities* of performing each (decorated) trace over the various resolutions (max-min-matching resolution sets).

We say that a computation is compatible with a trace $\alpha \in A^*$ iff the sequence of actions labeling its steps is equal to α. Given an NPLTS $\mathscr{L} = (S, A, \longrightarrow)$, $s \in S$, and $\mathscr{Z} \in Res(s)$, we denote by $\mathscr{CC}(z_s, \alpha)$ the set of α-compatible computations in $\mathscr{C}_{\text{fin}}(z_s)$ and by $Res_\alpha(s)$ the set of resolutions in $Res(s)$ having no computations corresponding to proper prefixes of α-compatible computations of $\mathscr{L}$. In each of the following definitions, we assume $s_1, s_2 \in S$ and we explicitly add a reference whenever the defined equivalence has already appeared in the literature. In some definitions, we indicate with $\sqcup$/$\sqcap$ the supremum/infimum of a set of numbers in $\mathbb{R}_{[0,1]}$ and we assume it to be 0 when the set is empty.

Definition 3.1 (*Probabilistic trace-distribution equivalence* – $\sim_{\text{PTr,dis}}$ – [26])
$s_1 \sim_{\text{PTr,dis}} s_2$ iff for each $\mathscr{Z}_1 \in Res(s_1)$ there exists $\mathscr{Z}_2 \in Res(s_2)$ such that <u>for all $\alpha \in A^*$</u>:

$$prob(\mathscr{CC}(z_{s_1}, \alpha)) = prob(\mathscr{CC}(z_{s_2}, \alpha))$$

and symmetrically for each $\mathscr{Z}_2 \in Res(s_2)$. ■

Definition 3.2 (*Probabilistic trace equivalence* – $\sim_{\text{PTr}}$ – [3])
$s_1 \sim_{\text{PTr}} s_2$ iff <u>for all $\alpha \in A^*$</u> it holds that for each $\mathscr{Z}_1 \in Res(s_1)$ there exists $\mathscr{Z}_2 \in Res(s_2)$ such that:

$$prob(\mathscr{CC}(z_{s_1}, \alpha)) = prob(\mathscr{CC}(z_{s_2}, \alpha))$$

and symmetrically for each $\mathscr{Z}_2 \in Res(s_2)$. ■

Definition 3.3 (*Probabilistic $\sqcup\sqcap$-trace equivalence* – $\sim_{\text{PTr},\sqcup\sqcap}$)
$s_1 \sim_{\text{PTr},\sqcup\sqcap} s_2$ iff for all $\alpha \in A^*$:

$$\bigsqcup_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{CC}(z_{s_1}, \alpha)) = \bigsqcup_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{CC}(z_{s_2}, \alpha))$$

$$\bigsqcap_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{CC}(z_{s_1}, \alpha)) = \bigsqcap_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{CC}(z_{s_2}, \alpha))$$

■

A variant that additionally considers completed computations was introduced in the literature of fully nondeterministic models in order to equip trace equivalence with deadlock sensitivity. We denote by $\mathscr{CCC}(z_s, \alpha)$ the set of completed α-compatible computations from z_s. Each of these computations c belongs to $\mathscr{CC}(z_s, \alpha)$ and is such that $corr_{\mathscr{Z}}(last(c))$ has no outgoing transitions in $\mathscr{L}$.

Definition 3.4 (*Probabilistic completed-trace-distribution equivalence* – $\sim_{\text{PCTr,dis}}$)
$s_1 \sim_{\text{PCTr,dis}} s_2$ iff for each $\mathscr{Z}_1 \in Res(s_1)$ there exist $\mathscr{Z}_2, \mathscr{Z}'_2 \in Res(s_2)$ such that <u>for all $\alpha \in A^*$</u>:

$$prob(\mathscr{CC}(z_{s_1}, \alpha)) = prob(\mathscr{CC}(z_{s_2}, \alpha))$$

$$prob(\mathscr{CCC}(z_{s_1}, \alpha)) = prob(\mathscr{CCC}(z'_{s_2}, \alpha))$$

and symmetrically for each $\mathscr{Z}_2 \in Res(s_2)$. ■

Definition 3.5 (*Probabilistic completed-trace equivalence* – $\sim_{\mathrm{PCTr}}$)
$s_1 \sim_{\mathrm{PCTr}} s_2$ iff for all $\alpha \in A^*$ it holds that for each $\mathscr{Z}_1 \in Res(s_1)$ there exist $\mathscr{Z}_2, \mathscr{Z}_2' \in Res(s_2)$ such that:

$$\begin{aligned} prob(\mathscr{CC}(z_{s_1}, \alpha)) &= prob(\mathscr{CC}(z_{s_2}, \alpha)) \\ prob(\mathscr{CCC}(z_{s_1}, \alpha)) &= prob(\mathscr{CCC}(z'_{s_2}, \alpha)) \end{aligned}$$

and symmetrically for each $\mathscr{Z}_2 \in Res(s_2)$. ■

Definition 3.6 (*Probabilistic $\sqcup\sqcap$-completed-trace equivalence* – $\sim_{\mathrm{PCTr},\sqcup\sqcap}$)
$s_1 \sim_{\mathrm{PCTr},\sqcup\sqcap} s_2$ iff for all $\alpha \in A^*$:

$$\begin{aligned} \bigsqcup_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{CC}(z_{s_1}, \alpha)) &= \bigsqcup_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{CC}(z_{s_2}, \alpha)) \\ \bigsqcap_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{CC}(z_{s_1}, \alpha)) &= \bigsqcap_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{CC}(z_{s_2}, \alpha)) \end{aligned}$$

and:

$$\begin{aligned} \bigsqcup_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{CCC}(z_{s_1}, \alpha)) &= \bigsqcup_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{CCC}(z_{s_2}, \alpha)) \\ \bigsqcap_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{CCC}(z_{s_1}, \alpha)) &= \bigsqcap_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{CCC}(z_{s_2}, \alpha)) \end{aligned}$$

■

Failure semantics generalizes completed-trace equivalence towards arbitrary safety properties. A failure pair is an element $\varphi \in A^* \times 2^A$ formed by a trace α and a decoration F called failure set. We say that $c \in \mathscr{C}_{\mathrm{fin}}(z_s)$ is compatible with φ iff $c \in \mathscr{CC}(z_s, \alpha)$ and $corr_{\mathscr{Z}}(last(c))$ has no outgoing transitions in $\mathscr{L}$ labeled with an action in F. We denote by $\mathscr{FCC}(z_s, \varphi)$ the set of φ-compatible computations from z_s. Moreover, we call failure trace an element $\phi \in (A \times 2^A)^*$ given by a sequence of $n \in \mathbb{N}$ pairs of the form (a_i, F_i). We say that $c \in \mathscr{C}_{\mathrm{fin}}(z_s)$ is compatible with ϕ iff $c \in \mathscr{CC}(z_s, a_1 \dots a_n)$ and, denoting by z_i the state reached by c after the i-th step for all $i = 1, \dots, n$, $corr_{\mathscr{Z}}(z_i)$ has no outgoing transitions in $\mathscr{L}$ labeled with an action in F_i. We denote by $\mathscr{FTCC}(z_s, \phi)$ the set of ϕ-compatible computations from z_s.

Definition 3.7 (*Probabilistic failure-distribution equivalence* – $\sim_{\mathrm{PF,dis}}$ – [27])
$s_1 \sim_{\mathrm{PF,dis}} s_2$ iff for each $\mathscr{Z}_1 \in Res(s_1)$ there exists $\mathscr{Z}_2 \in Res(s_2)$ such that for all $\varphi \in A^* \times 2^A$:

$$prob(\mathscr{FCC}(z_{s_1}, \varphi)) = prob(\mathscr{FCC}(z_{s_2}, \varphi))$$

and symmetrically for each $\mathscr{Z}_2 \in Res(s_2)$. ■

Definition 3.8 (*Probabilistic failure equivalence* – $\sim_{\mathrm{PF}}$ – [3])
$s_1 \sim_{\mathrm{PF}} s_2$ iff for all $\varphi \in A^* \times 2^A$ it holds that for each $\mathscr{Z}_1 \in Res(s_1)$ there exists $\mathscr{Z}_2 \in Res(s_2)$ such that:

$$prob(\mathscr{FCC}(z_{s_1}, \varphi)) = prob(\mathscr{FCC}(z_{s_2}, \varphi))$$

and symmetrically for each $\mathscr{Z}_2 \in Res(s_2)$. ■

Definition 3.9 (*Probabilistic $\sqcup\sqcap$-failure equivalence* – $\sim_{\mathrm{PF},\sqcup\sqcap}$)
$s_1 \sim_{\mathrm{PF},\sqcup\sqcap} s_2$ iff for all $\varphi = (\alpha, F) \in A^* \times 2^A$:

$$\begin{aligned} \bigsqcup_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{FCC}(z_{s_1}, \varphi)) &= \bigsqcup_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{FCC}(z_{s_2}, \varphi)) \\ \bigsqcap_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{FCC}(z_{s_1}, \varphi)) &= \bigsqcap_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{FCC}(z_{s_2}, \varphi)) \end{aligned}$$

■

Definition 3.10 (*Probabilistic failure-trace-distribution equivalence* – $\sim_{\mathrm{PFTr,dis}}$)
Same as Def. 3.7 with $\phi \in (A \times 2^A)^*$ and $\mathscr{FTCC}$ in place of $\varphi \in A^* \times 2^A$ and $\mathscr{FCC}$, respectively. ■

Definition 3.11 (*Probabilistic failure-trace equivalence* – $\sim_{\mathrm{PFTr}}$)
Same as Def. 3.8 with $\phi \in (A \times 2^A)^*$ and $\mathscr{FTCC}$ in place of $\varphi \in A^* \times 2^A$ and $\mathscr{FCC}$, respectively. ■

Definition 3.12 (*Probabilistic $\sqcup\sqcap$-failure-trace equivalence* – $\sim_{\mathrm{PFTr},\sqcup\sqcap}$)
Same as Def. 3.9 with $\phi \in (A \times 2^A)^*$ and $\mathscr{FTCC}$ in place of $\varphi \in A^* \times 2^A$ and $\mathscr{FCC}$, respectively. ■

A different generalization towards liveness properties is readiness semantics. A ready pair is an element $\varrho \in A^* \times 2^A$ formed by a trace α and a decoration R called ready set. We say that c is compatible with ϱ iff $c \in \mathscr{CC}(z_s, \alpha)$ and the set of actions labeling the transitions in $\mathscr{L}$ departing from $corr_{\mathscr{Z}}(last(c))$ is precisely R. We denote by $\mathscr{RCC}(z_s, \varrho)$ the set of ϱ-compatible computations from z_s. Moreover, we call ready trace an element $\rho \in (A \times 2^A)^*$ given by a sequence of $n \in \mathbb{N}$ pairs of the form (a_i, R_i). We say that $c \in \mathscr{C}_{\text{fin}}(z_s)$ is compatible with ρ iff $c \in \mathscr{CC}(z_s, a_1 \dots a_n)$ and, denoting by z_i the state reached by c after the i-th step for all $i = 1, \dots, n$, the set of actions labeling the transitions in $\mathscr{L}$ departing from $corr_{\mathscr{Z}}(z_i)$ is precisely R_i. We denote by $\mathscr{RTCC}(z_s, \rho)$ the set of ρ-compatible computations from z_s.

Definition 3.13 (*Probabilistic readiness-distribution equivalence* – $\sim_{\text{PR,dis}}$)
$s_1 \sim_{\text{PR,dis}} s_2$ iff for each $\mathscr{Z}_1 \in Res(s_1)$ there exists $\mathscr{Z}_2 \in Res(s_2)$ such that $\underline{\text{for all } \varrho \in A^* \times 2^A}$:

$$prob(\mathscr{RCC}(z_{s_1}, \varrho)) = prob(\mathscr{RCC}(z_{s_2}, \varrho))$$

and symmetrically for each $\mathscr{Z}_2 \in Res(s_2)$. ■

Definition 3.14 (*Probabilistic readiness equivalence* – $\sim_{\text{PR}}$)
$s_1 \sim_{\text{PR}} s_2$ iff $\underline{\text{for all } \varrho \in A^* \times 2^A}$ it holds that for each $\mathscr{Z}_1 \in Res(s_1)$ there exists $\mathscr{Z}_2 \in Res(s_2)$ such that:

$$prob(\mathscr{RCC}(z_{s_1}, \varrho)) = prob(\mathscr{RCC}(z_{s_2}, \varrho))$$

and symmetrically for each $\mathscr{Z}_2 \in Res(s_2)$. ■

Definition 3.15 (*Probabilistic $\sqcup\sqcap$-readiness equivalence* – $\sim_{\text{PR},\sqcup\sqcap}$)
$s_1 \sim_{\text{PR},\sqcup\sqcap} s_2$ iff for all $\varrho = (\alpha, R) \in A^* \times 2^A$:

$$\bigsqcup_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{RCC}(z_{s_1}, \varrho)) = \bigsqcup_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{RCC}(z_{s_2}, \varrho))$$

$$\bigsqcap_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{RCC}(z_{s_1}, \varrho)) = \bigsqcap_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{RCC}(z_{s_2}, \varrho))$$

■

Definition 3.16 (*Probabilistic ready-trace-distribution equivalence* – $\sim_{\text{PRTr,dis}}$)
Same as Def. 3.13 with $\rho \in (A \times 2^A)^*$ and $\mathscr{RTCC}$ in place of $\varrho \in A^* \times 2^A$ and $\mathscr{RCC}$, respectively. ■

Definition 3.17 (*Probabilistic ready-trace equivalence* – $\sim_{\text{PRTr}}$)
Same as Def. 3.14 with $\rho \in (A \times 2^A)^*$ and $\mathscr{RTCC}$ in place of $\varrho \in A^* \times 2^A$ and $\mathscr{RCC}$, respectively. ■

Definition 3.18 (*Probabilistic $\sqcup\sqcap$-ready-trace equivalence* – $\sim_{\text{PRTr},\sqcup\sqcap}$)
Same as Def. 3.15 with $\rho \in (A \times 2^A)^*$ and $\mathscr{RTCC}$ in place of $\varrho \in A^* \times 2^A$ and $\mathscr{RCC}$, respectively. ■

The eighteen trace-based equivalences defined above are all backward compatible with the corresponding trace-based equivalences respectively defined in [6, 23] for fully nondeterministic processes and in [20, 17] for fully probabilistic processes.

Theorem 3.19 Let $\sigma \in \{\text{RTr}, \text{FTr}, \text{R}, \text{F}, \text{CTr}, \text{Tr}\}$ with $\sim_{\text{P}\sigma,\text{dis}}$, $\sim_{\text{P}\sigma}$, and $\sim_{\text{P}\sigma,\sqcup\sqcap}$ being the equivalences defined above, $\sim_{\sigma,\text{fnd}}$ being the corresponding equivalence defined for fully nondeterministic processes, and $\sim_{\sigma,\text{fpr}}$ being the corresponding equivalence defined for fully probabilistic processes. Then:

1. $\sim_{\text{P}\sigma,\text{dis}} = \sim_{\text{P}\sigma} = \sim_{\text{P}\sigma,\sqcup\sqcap} = \sim_{\sigma,\text{fnd}}$ over fully nondeterministic NPLTS models.
2. $\sim_{\text{P}\sigma,\text{dis}} = \sim_{\text{P}\sigma} = \sim_{\text{P}\sigma,\sqcup\sqcap} = \sim_{\sigma,\text{fpr}}$ over fully probabilistic NPLTS models. ■

We now investigate the relationships among the eighteen trace-based equivalences. As expected, each equivalence relying on trace-based distributions is finer than the corresponding equivalence considering a single (decorated) trace at a time, which in turn is finer than the corresponding equivalence based on extremal probabilities of (decorated) traces. For the equivalences of the first type, similar to the fully probabilistic spectrum in [20, 17] it turns out that the readiness semantics coincides with the failure semantics. In contrast, for the other two types of equivalences, unlike the fully nondeterministic spectrum in [14] no connection can be established between readiness semantics and failure semantics.

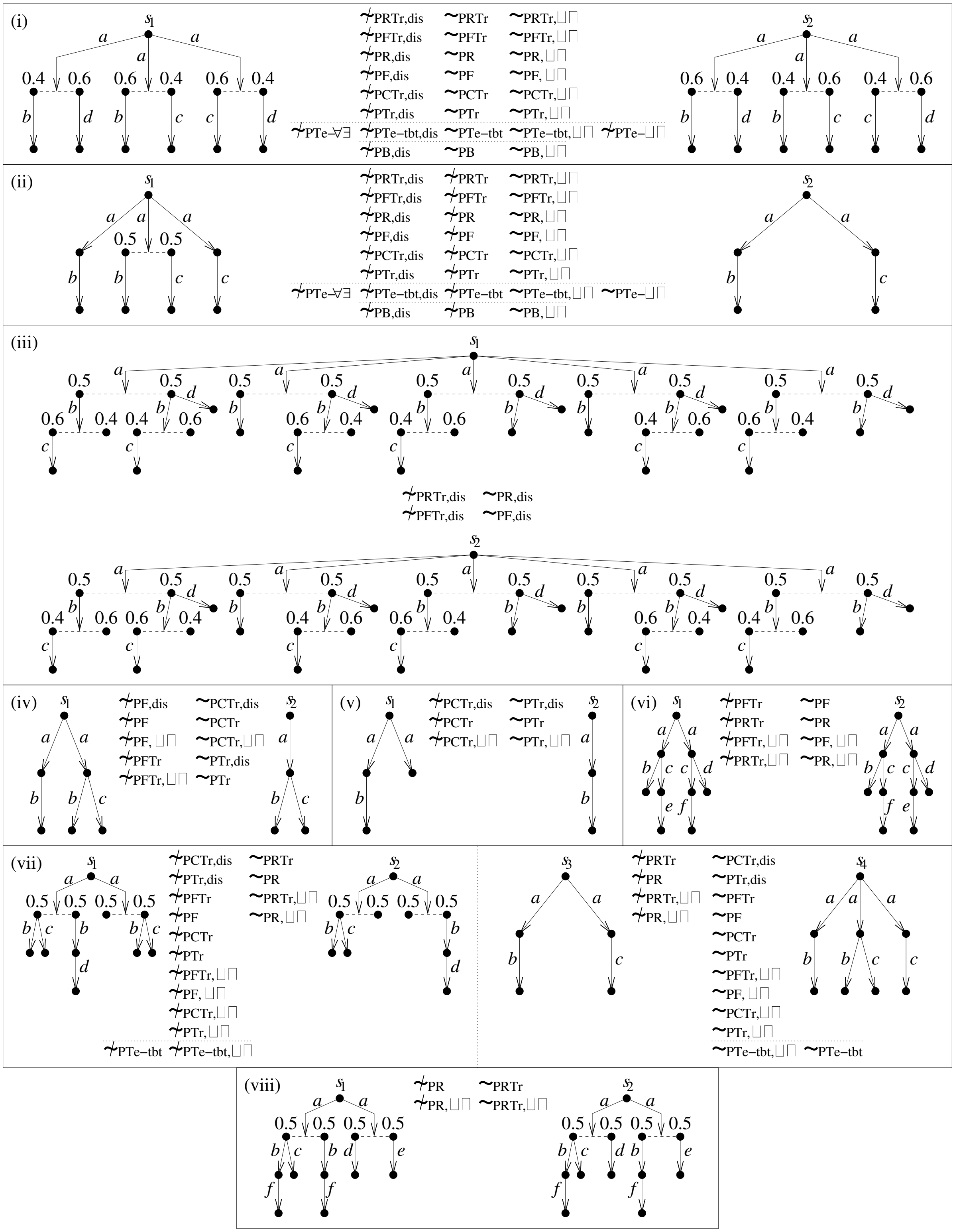

Figure 1: Counterexamples for strict inclusion and incomparability of the trace-based equivalences

Theorem 3.20 It holds that:

1. $\sim_{\pi,\mathrm{dis}} \subseteq \sim_{\pi} \subseteq \sim_{\pi,\sqcup\sqcap}$ for all $\pi \in \{\mathrm{PRTr}, \mathrm{PFTr}, \mathrm{PR}, \mathrm{PF}, \mathrm{PCTr}, \mathrm{PTr}\}$.
2. $\sim_{\mathrm{PRTr,dis}} = \sim_{\mathrm{PFTr,dis}}$ over finitely-branching NPLTS models.
3. $\sim_{\mathrm{PR,dis}} = \sim_{\mathrm{PF,dis}}$ over finitely-branching NPLTS models.
4. $\sim_{\mathrm{PFTr,dis}} \subseteq \sim_{\mathrm{PF,dis}} \subseteq \sim_{\mathrm{PCTr,dis}} \subseteq \sim_{\mathrm{PTr,dis}}$.
5. $\sim_{\mathrm{PFTr}} \subseteq \sim_{\mathrm{PF}} \subseteq \sim_{\mathrm{PCTr}} \subseteq \sim_{\mathrm{PTr}}$.
6. $\sim_{\mathrm{PFTr},\sqcup\sqcap} \subseteq \sim_{\mathrm{PF},\sqcup\sqcap} \subseteq \sim_{\mathrm{PCTr},\sqcup\sqcap} \subseteq \sim_{\mathrm{PTr},\sqcup\sqcap}$. ■

All the inclusions above are strict, as shown in Figs. 1(i) to (vi). It is worth noting the isolation of $\sim_{\mathrm{PRTr}}$, $\sim_{\mathrm{PR}}$, $\sim_{\mathrm{PRTr},\sqcup\sqcap}$, and $\sim_{\mathrm{PR},\sqcup\sqcap}$, each of which is incomparable with $\sim_{\mathrm{PCTr,dis}}$, $\sim_{\mathrm{PTr,dis}}$, $\sim_{\mathrm{PFTr}}$, $\sim_{\mathrm{PF}}$, $\sim_{\mathrm{PCTr}}$, $\sim_{\mathrm{PTr}}$, $\sim_{\mathrm{PFTr},\sqcup\sqcap}$, $\sim_{\mathrm{PF},\sqcup\sqcap}$, $\sim_{\mathrm{PCTr},\sqcup\sqcap}$, and $\sim_{\mathrm{PTr},\sqcup\sqcap}$, as shown in Fig. 1(vii). Moreover, Figs. 1(i) and (iv) show that $\sim_{\mathrm{PFTr}}$, $\sim_{\mathrm{PF}}$, $\sim_{\mathrm{PFTr},\sqcup\sqcap}$, and $\sim_{\mathrm{PF},\sqcup\sqcap}$ are incomparable with $\sim_{\mathrm{PCTr,dis}}$ and $\sim_{\mathrm{PTr,dis}}$, while Figs. 1(ii) and (iv) show that $\sim_{\mathrm{PFTr},\sqcup\sqcap}$ and $\sim_{\mathrm{PF},\sqcup\sqcap}$ are also incomparable with $\sim_{\mathrm{PCTr}}$ and $\sim_{\mathrm{PTr}}$. Finally, Figs. 1(vi) and (viii) show that $\sim_{\mathrm{PRTr}}$ and $\sim_{\mathrm{PRTr},\sqcup\sqcap}$ are incomparable with $\sim_{\mathrm{PR}}$ and $\sim_{\mathrm{PR},\sqcup\sqcap}$, Figs. 1(ii) and (vi) show that $\sim_{\mathrm{PFTr},\sqcup\sqcap}$ is incomparable with $\sim_{\mathrm{PF}}$, Figs. 1(i) and (v) show that $\sim_{\mathrm{PCTr}}$ and $\sim_{\mathrm{PCTr},\sqcup\sqcap}$ are incomparable with $\sim_{\mathrm{PTr,dis}}$, and Figs. 1(ii) and (v) show that $\sim_{\mathrm{PCTr},\sqcup\sqcap}$ is incomparable with $\sim_{\mathrm{PTr}}$.

4 Testing Equivalences for NPLTS Models

Testing equivalences consider the probability of two processes of performing computations along which the same tests are passed. Tests specify which actions of a process are permitted at each step and, in this setting, can be formalized as NPLTS models equipped with a success state. For the sake of simplicity, we restrict ourselves to finite tests, each of which has finitely many states, finitely many outgoing transitions from each state, an acyclic graph structure, and hence finitely many computations leading to success.

Definition 4.1 A nondeterministic and probabilistic test, NPT for short, is a finite NPLTS $\mathscr{T} = (O, A, \longrightarrow)$ where O contains a distinguished success state denoted by ω that has no outgoing transitions. We say that a computation of $\mathscr{T}$ is successful iff its last state is ω. ■

Definition 4.2 Let $\mathscr{L} = (S, A, \longrightarrow)$ be an NPLTS and $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ be an NPT. The interaction system of $\mathscr{L}$ and $\mathscr{T}$ is the NPLTS $\mathscr{I}(\mathscr{L}, \mathscr{T}) = \mathscr{L} \parallel \mathscr{T}$ where:

- Every element $(s, o) \in S \times O$ is called a configuration and is said to be successful iff $o = \omega$.
- A computation of $\mathscr{I}(\mathscr{L}, \mathscr{T})$ is said to be successful iff its last configuration is successful. Given $s \in S$, $o \in O$, and $\mathscr{Z} \in \mathit{Res}(s, o)$, we denote by $\mathscr{SC}(z_{s,o})$ the set of successful computations from the state $z_{s,o}$ of $\mathscr{Z}$ corresponding to the configuration (s, o) of $\mathscr{I}(\mathscr{L}, \mathscr{T})$. ■

Due to the possible presence of equally labeled transitions departing from the same state, there is not necessarily a single probability value with which an NPLTS passes a test. Thus, given two states s_1 and s_2 of the NPLTS under test and the initial state o of the test, we need to compute the probability of performing a successful computation from the two configurations (s_1, o) and (s_2, o) in every maximal resolution of the interaction system. One option is comparing, for the two configurations, *only the extremal values of these success probabilities* over all maximal resolutions of the interaction system. An alternative option is comparing *all the success probabilities* and requiring that for each maximal resolution of either configuration there is a matching maximal resolution of the other configuration.

Definition 4.3 (*Probabilistic $\sqcup\sqcap$-testing equivalence* – $\sim_{\text{PTe-}\sqcup\sqcap}$ – [35, 19, 27, 11])
$s_1 \sim_{\text{PTe-}\sqcup\sqcap} s_2$ iff for every NPT $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ with initial state $o \in O$:

$$\bigsqcup_{\mathscr{Z}_1 \in Res_{\max}(s_1,o)} prob(\mathscr{SC}(z_{s_1,o})) = \bigsqcup_{\mathscr{Z}_2 \in Res_{\max}(s_2,o)} prob(\mathscr{SC}(z_{s_2,o}))$$
$$\bigsqcap_{\mathscr{Z}_1 \in Res_{\max}(s_1,o)} prob(\mathscr{SC}(z_{s_1,o})) = \bigsqcap_{\mathscr{Z}_2 \in Res_{\max}(s_2,o)} prob(\mathscr{SC}(z_{s_2,o}))$$

■

Definition 4.4 (*Probabilistic $\forall\exists$-testing equivalence* – $\sim_{\text{PTe-}\forall\exists}$ – [3])
$s_1 \sim_{\text{PTe-}\forall\exists} s_2$ iff for every NPT $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ with initial state $o \in O$ it holds that for each $\mathscr{Z}_1 \in Res_{\max}(s_1, o)$ there exists $\mathscr{Z}_2 \in Res_{\max}(s_2, o)$ such that:

$$prob(\mathscr{SC}(z_{s_1,o})) = prob(\mathscr{SC}(z_{s_2,o}))$$

and symmetrically for each $\mathscr{Z}_2 \in Res_{\max}(s_2, o)$. ■

Neither $\sim_{\text{PTe-}\sqcup\sqcap}$ nor $\sim_{\text{PTe-}\forall\exists}$ is backward compatible with the testing equivalence defined in [10] for fully nondeterministic processes. For instance, Fig. 2(i) shows two such processes related by classical testing equivalence that are distinguished by $\sim_{\text{PTe-}\sqcup\sqcap}$ and $\sim_{\text{PTe-}\forall\exists}$. The reason of the higher discriminating power of the latter two equivalences arises from the presence of probabilistic choices within tests, which results in the capability of making copies of the process under test [1] and hence in an unrealistic estimation of success probabilities [12]. In order to counterbalance this strong discriminating power, as illustrated in [3] the idea is to consider *success probabilities in a trace-by-trace fashion* rather than on entire resolutions. Since traces come again into play, the idea can be implemented in three different ways by following the three approaches used in Sect. 3.

In the following, given a state s of an NPLTS, a state o of an NPT, and a trace $\alpha \in A^*$, we denote by $Res_{\max,\mathscr{C},\alpha}(s,o)$ the set of resolutions $\mathscr{Z} \in Res_{\max}(s,o)$ such that $\mathscr{CCC}(z_{s,o}, \alpha) \neq \emptyset$, i.e., the maximal resolutions of $z_{s,o}$ having at least one completed α-compatible computation. Moreover, for each such resolution $\mathscr{Z}$, we denote by $\mathscr{SCC}(z_{s,o}, \alpha)$ the set of successful α-compatible computations from $z_{s,o}$.

Definition 4.5 (*Probabilistic trace-by-trace-distribution testing equivalence* – $\sim_{\text{PTe-tbt,dis}}$)
$s_1 \sim_{\text{PTe-tbt,dis}} s_2$ iff for every NPT $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ with initial state $o \in O$ it holds that for each $\mathscr{Z}_1 \in Res_{\max}(s_1, o)$ there exists $\mathscr{Z}_2 \in Res_{\max}(s_2, o)$ such that <u>for all $\alpha \in A^*$</u> it holds that $\mathscr{CCC}(z_{s_1,o}, \alpha) \neq \emptyset$ implies $\mathscr{CCC}(z_{s_2,o}, \alpha) \neq \emptyset$ and:

$$prob(\mathscr{SCC}(z_{s_1,o}, \alpha)) = prob(\mathscr{SCC}(z_{s_2,o}, \alpha))$$

and symmetrically for each $\mathscr{Z}_2 \in Res_{\max}(s_2, o)$. ■

Definition 4.6 (*Probabilistic trace-by-trace testing equivalence* – $\sim_{\text{PTe-tbt}}$ – [3])
$s_1 \sim_{\text{PTe-tbt}} s_2$ iff for every NPT $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ with initial state $o \in O$ and <u>for all $\alpha \in A^*$</u> it holds that for each $\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1, o)$ there exists $\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2, o)$ such that:

$$prob(\mathscr{SCC}(z_{s_1,o}, \alpha)) = prob(\mathscr{SCC}(z_{s_2,o}, \alpha))$$

and symmetrically for each $\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2, o)$. ■

Definition 4.7 (*Probabilistic $\sqcup\sqcap$-trace-by-trace testing equivalence* – $\sim_{\text{PTe-tbt,}\sqcup\sqcap}$)
$s_1 \sim_{\text{PTe-tbt,}\sqcup\sqcap} s_2$ iff for every NPT $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ it holds that $Res_{\max,\mathscr{C},\alpha}(s_1, o) \neq \emptyset$ iff $Res_{\max,\mathscr{C},\alpha}(s_2, o) \neq \emptyset$ and:

$$\bigsqcup_{\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)} prob(\mathscr{SCC}(z_{s_1,o}, \alpha)) = \bigsqcup_{\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2,o)} prob(\mathscr{SCC}(z_{s_2,o}, \alpha))$$
$$\bigsqcap_{\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)} prob(\mathscr{SCC}(z_{s_1,o}, \alpha)) = \bigsqcap_{\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2,o)} prob(\mathscr{SCC}(z_{s_2,o}, \alpha))$$

■

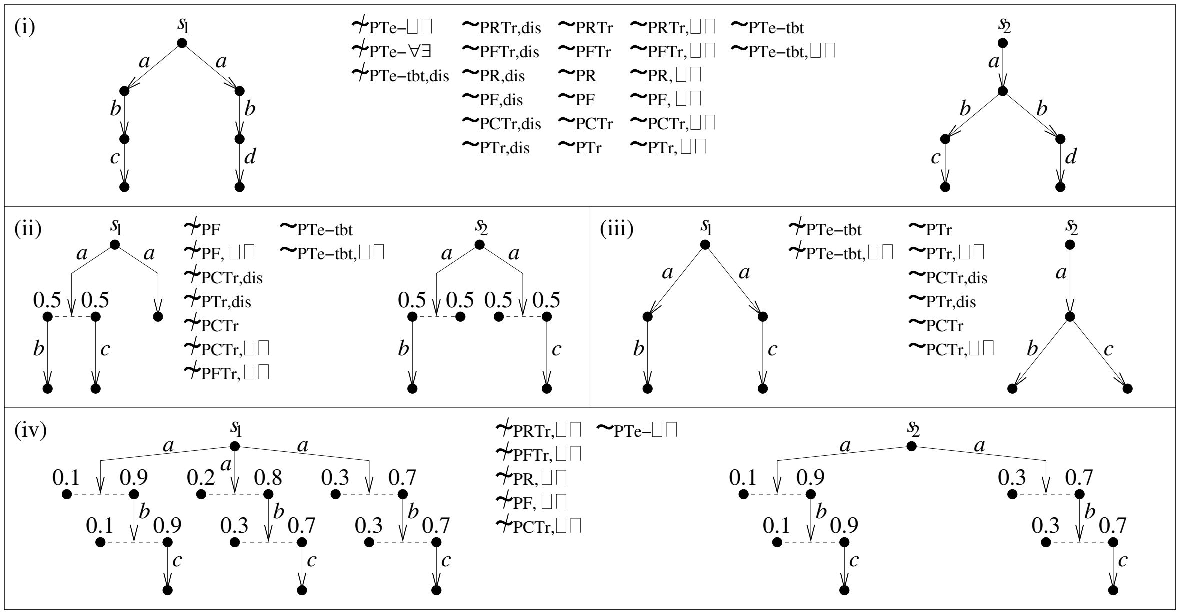

Figure 2: Counterexamples for strict inclusion and incomparability of the testing equivalences

While only $\sim_{\text{PTe-tbt}}$ and $\sim_{\text{PTe-tbt},\sqcup\sqcap}$ are backward compatible with the testing equivalence defined in [10] for fully nondeterministic processes – which we denote by $\sim_{\text{Te,fnd}}$ (see Fig. 2(i) for the counterexamples) – all the five testing equivalences defined above are backward compatible with the testing equivalence defined in [7] for fully probabilistic processes – which we denote by $\sim_{\text{Te,fpr}}$.

Theorem 4.8 It holds that:

1. $\sim_{\text{PTe-tbt}} = \sim_{\text{PTe-tbt},\sqcup\sqcap} = \sim_{\text{Te,fnd}}$ over fully nondeterministic NPLTS models.
2. $\sim_{\text{PTe-}\sqcup\sqcap} = \sim_{\text{PTe-}\forall\exists} = \sim_{\text{PTe-tbt,dis}} = \sim_{\text{PTe-tbt}} = \sim_{\text{PTe-tbt},\sqcup\sqcap} = \sim_{\text{Te,fpr}}$ over fully probabilistic NPLTS models. ■

We now investigate the relationships of the five testing equivalences among themselves (first two properties below) and with the eighteen trace-based equivalences (last three properties below). It turns out that $\sim_{\text{PTe-}\forall\exists}$ and $\sim_{\text{PTe-tbt,dis}}$ perform exactly the same identifications. Unlike the fully nondeterministic spectrum – where the testing semantics coincides with the failure semantics when all actions are observable [9] – here $\sim_{\text{PTe-tbt,dis}}$ is finer than $\sim_{\text{PFTr,dis}}$ while $\sim_{\text{PTe-tbt}}$ and $\sim_{\text{PTe-tbt},\sqcup\sqcap}$ are coarser than $\sim_{\text{PF}}$ and $\sim_{\text{PF},\sqcup\sqcap}$, respectively. In contrast, $\sim_{\text{PTe-}\sqcup\sqcap}$ has no inclusion relationship with the failure semantics.

Theorem 4.9 It holds that:

1. $\sim_{\text{PTe-}\forall\exists} \subseteq \sim_{\text{PTe-}\sqcup\sqcap} \subseteq \sim_{\text{PTe-tbt},\sqcup\sqcap}$.
2. $\sim_{\text{PTe-}\forall\exists} = \sim_{\text{PTe-tbt,dis}} \subseteq \sim_{\text{PTe-tbt}} \subseteq \sim_{\text{PTe-tbt},\sqcup\sqcap}$.
3. $\sim_{\text{PTe-tbt,dis}} \subseteq \sim_{\text{PRTr,dis}}$.
4. $\sim_{\text{PF}} \subseteq \sim_{\text{PTe-tbt}} \subseteq \sim_{\text{PTr}}$.
5. $\sim_{\text{PF},\sqcup\sqcap} \subseteq \sim_{\text{PTe-tbt},\sqcup\sqcap} \subseteq \sim_{\text{PTr},\sqcup\sqcap}$. ■

All the inclusions above are strict, as shown in Figs. 1(i) and (ii) and Figs. 2(i) to (iii). It is worth noting the isolation of $\sim_{\text{PTe-}\sqcup\sqcap}$, which is incomparable with $\sim_{\text{PRTr,dis}}$, $\sim_{\text{PFTr,dis}}$, $\sim_{\text{PR,dis}}$, $\sim_{\text{PF,dis}}$, $\sim_{\text{PCTr,dis}}$, $\sim_{\text{PTr,dis}}$, $\sim_{\text{PRTr}}$, $\sim_{\text{PFTr}}$, $\sim_{\text{PR}}$, $\sim_{\text{PF}}$, $\sim_{\text{PCTr}}$, $\sim_{\text{PTr}}$, and $\sim_{\text{PTe-tbt}}$, as shown in Fig. 1(ii) and Fig. 2(i), and with $\sim_{\text{PRTr},\sqcup\sqcap}$, $\sim_{\text{PFTr},\sqcup\sqcap}$, $\sim_{\text{PR},\sqcup\sqcap}$, $\sim_{\text{PF},\sqcup\sqcap}$, and $\sim_{\text{PCTr},\sqcup\sqcap}$, as shown in Fig. 1(i) and Fig. 2(iv). Furthermore, $\sim_{\text{PTe-tbt}}$ and $\sim_{\text{PTe-tbt},\sqcup\sqcap}$ are incomparable with $\sim_{\text{PRTr}}$, $\sim_{\text{PR}}$, $\sim_{\text{PRTr},\sqcup\sqcap}$, and $\sim_{\text{PR},\sqcup\sqcap}$, as shown in Fig. 1(vii), and with $\sim_{\text{PCTr,dis}}$, $\sim_{\text{PTr,dis}}$, $\sim_{\text{PCTr}}$, and $\sim_{\text{PCTr},\sqcup\sqcap}$, as shown in Figs. 2(ii) and (iii). Finally, Figs. 1(ii) and 2(ii) show that $\sim_{\text{PTe-tbt}}$ is also incomparable with $\sim_{\text{PFTr},\sqcup\sqcap}$ and $\sim_{\text{PF},\sqcup\sqcap}$, while Figs. 1(ii) and 2(iii) show that $\sim_{\text{PTe-tbt},\sqcup\sqcap}$ is also incomparable with $\sim_{\text{PTr}}$.

5 Bisimulation Equivalences for NPLTS Models

Bisimulation equivalences capture the ability of two processes of mimicking each other's behavior stepwise. Similar to the trace-based case, given two states there are three different approaches to the definition of these bisimilarities, each following the style of [21] based on equivalence relations. The first approach is to match transitions on the basis of *class distributions*, which means that for each transition of one of the two states there must exist an equally labeled transition of the other state such that, *for every equivalence class*, the two transitions have the same probability of reaching a state in that class. In other words, matching transitions of the two states are related by the fully probabilistic version of bisimilarity (fully matching transitions). The second approach is to consider *a single equivalence class at a time*, i.e., to anticipate the quantification over classes. In this way, a transition departing from one of the two states is allowed to be matched, with respect to the probabilities of reaching different classes, by several different transitions departing from the other state (partially matching transitions). The third approach is to compare only the *extremal probabilities* of reaching each class over all possible transitions labeled with a certain action (max-min-matching transition sets).

Unlike [21], we will consider *groups of equivalence classes* rather than individual equivalence classes. This does not change the discriminating power in the case of the first approach, while it increases the discriminating power thereby resulting in desirable logical characterizations in the case of the other two approaches [8, 32, 31, 5]. Given an NPLTS $(S, A, \longrightarrow)$ and a distribution $\mathcal{D} \in \mathit{Distr}(S)$, in the following we let $\mathcal{D}(S') = \sum_{s \in S'} \mathcal{D}(s)$ for $S' \subseteq S$. Moreover, given an equivalence relation $\mathcal{B}$ over S and a group of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$, we also let $\bigcup \mathcal{G} = \bigcup_{C \in \mathcal{G}} C$.

Definition 5.1 (*Probabilistic group-distribution bisimilarity* – $\sim_{\text{PB,dis}}$ – [28])
$s_1 \sim_{\text{PB,dis}} s_2$ iff (s_1, s_2) belongs to the largest probabilistic group-distribution bisimulation. An equivalence relation $\mathcal{B}$ over S is a *probabilistic group-distribution bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for each $s_1 \xrightarrow{a} \mathcal{D}_1$ there exists $s_2 \xrightarrow{a} \mathcal{D}_2$ such that $\underline{\text{for all } \mathcal{G} \in 2^{S/\mathcal{B}}}$ it holds that $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$. ■

Definition 5.2 (*Probabilistic bisimilarity* – $\sim_{\text{PB}}$ – [5])
$s_1 \sim_{\text{PB}} s_2$ iff (s_1, s_2) belongs to the largest probabilistic bisimulation. An equivalence relation $\mathcal{B}$ over S is a *probabilistic bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then $\underline{\text{for all } \mathcal{G} \in 2^{S/\mathcal{B}}}$ it holds that for each $s_1 \xrightarrow{a} \mathcal{D}_1$ there exists $s_2 \xrightarrow{a} \mathcal{D}_2$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) = \mathcal{D}_2(\bigcup \mathcal{G})$. ■

Definition 5.3 (*Probabilistic $\sqcup\sqcap$-bisimilarity* – $\sim_{\text{PB},\sqcup\sqcap}$ – [5])
$s_1 \sim_{\text{PB},\sqcup\sqcap} s_2$ iff (s_1, s_2) belongs to the largest probabilistic $\sqcup\sqcap$-bisimulation. An equivalence relation $\mathcal{B}$ over S is a *probabilistic $\sqcup\sqcap$-bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ and $a \in A$ it holds that $s_1 \xrightarrow{a}$ iff $s_2 \xrightarrow{a}$ and:

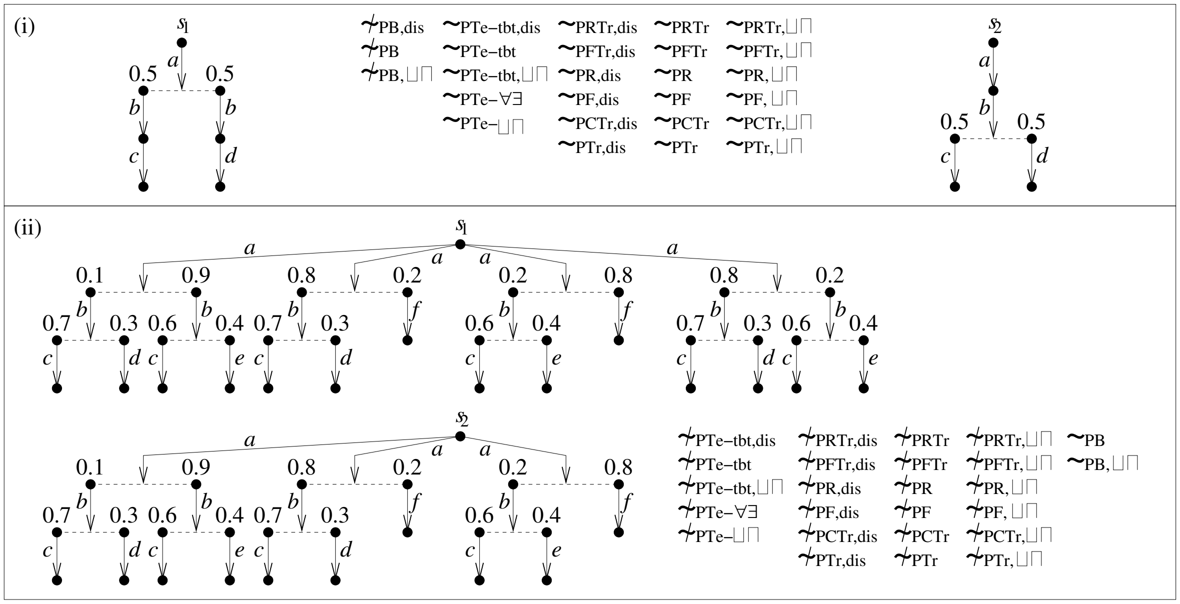

Figure 3: Counterexamples for strict inclusion and incomparability of the bisimulation equivalences

$$\bigsqcup_{s_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcup_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
$$\bigsqcap_{s_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) = \bigsqcap_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})$$
■

The three bisimulation equivalences defined above are all backward compatible with the bisimulation equivalences respectively defined in [16] for fully nondeterministic processes – which we denote by $\sim_{\rm B,fnd}$ – and in [13] for fully probabilistic processes – which we denote by $\sim_{\rm B,fpr}$.

Theorem 5.4 It holds that:

1. $\sim_{\rm PB,dis} = \sim_{\rm PB} = \sim_{\rm PB,\sqcup\sqcap} = \sim_{\rm B,fnd}$ over fully nondeterministic NPLTS models.
2. $\sim_{\rm PB,dis} = \sim_{\rm PB} = \sim_{\rm PB,\sqcup\sqcap} = \sim_{\rm B,fpr}$ over fully probabilistic NPLTS models. ■

We now investigate the relationships of the three bisimulation equivalences among themselves (first property below) and with the five testing equivalences and the eighteen trace-based equivalences (second property below).

Theorem 5.5 It holds that:

1. $\sim_{\rm PB,dis} \subsetneq \sim_{\rm PB} \subsetneq \sim_{\rm PB,\sqcup\sqcap}$.
2. $\sim_{\rm PB,dis} \subsetneq \sim_{\rm PTe\text{-}tbt,dis}$. ■

All the inclusions above are strict, as shown in Figs. 1(i) and (ii) and Fig. 3(i). It is worth noting the isolation of $\sim_{\rm PB}$ and $\sim_{\rm PB,\sqcup\sqcap}$, which are incomparable with all the five testing equivalences and all the eighteen trace-based equivalences, as shown in Figs. 3(i) and (ii).

6 Conclusion

We have studied the relationships among the equivalences that stem from three significantly different approaches to the definition of behavioral relations for NPLTS models. The specificity of the three approaches is determined by the way they deal with the probabilities associated with the resolutions of nondeterminism. For each approach, we have considered the families of strong trace-based, testing, and bisimulation equivalences under deterministic schedulers. The relationships among the equivalences for finitely-branching NPLTS models are summarized in Fig. 4. In the spectrum, the absence of (chains of) arrows represents incomparability, adjacency of boxes within the same fragment and double arrows connecting boxes of different fragments indicate coincidence, and single arrows stand for the strictly-more-discriminating-than relation.

Continuous hexagonal boxes contain equivalences studied in the last twenty years [28, 26, 27], which compare probability distributions of all equivalence-specific events. In contrast, continuous rounded boxes contain equivalences assigning a weaker role to schedulers that have been recently introduced in [8, 32, 31, 3, 5], which compare separately the probabilities of individual equivalence-specific events. Continuous rectangular boxes instead contain old equivalences [35, 19, 27, 11] and new equivalences [5] based on extremal probabilities. The only hybrid box is the one containing $\sim_{\text{PTe-}\forall\exists}$, as this equivalence does not follow any of the three definitional approaches. Finally, dashed boxes contain equivalences defined for the first time in this paper to better assess the different impact of the approaches themselves.

Figure 4 evidences that the top fragment of the spectrum collapses several equivalences, whilst the middle fragment and the bottom fragment do not. Indeed, like in the spectrum for fully probabilistic processes [20, 17], we have that the top variants of ready-trace and failure-trace equivalences and of readiness and failure equivalences respectively induce the same identifications. In contrast, the more liberal variants in the middle fragment and the bottom fragment, which guarantee a higher degree of flexibility in determining the matching resolutions and are in general coarser, do not flatten the specificity of the intuition behind the original definition of the behavioral equivalences for LTS models. Therefore, those two fragments preserve much of the original spectrum of [14] for fully nondeterministic processes. We finally stress again the isolation of $\sim_{\text{PB}}$, $\sim_{\text{PB},\sqcup\sqcap}$, $\sim_{\text{PTe-}\sqcup\sqcap}$, $\sim_{\text{PRTr}}$, $\sim_{\text{PR}}$, $\sim_{\text{PRTr},\sqcup\sqcap}$, and $\sim_{\text{PR},\sqcup\sqcap}$.

As future work, we intend first of all to enrich the spectrum with simulation equivalences. Secondly, we plan to address how the spectrum changes if randomized schedulers are used. Thirdly, we would like to investigate the spectrum of weak behavioral equivalences, for which the choice of randomized schedulers is more appropriate. Finally, it would be interesting to compare the discriminating power of the various equivalences after defining them more abstractly on a parametric model. A suitable framework might be that of ULTRAS [4], as it has been shown to encompass trace, testing, and bisimulation equivalences for models such as labeled transition systems, discrete-/continuous-time Markov chains, and discrete-/continuous-time Markov decision processes without/with internal nondeterminism.

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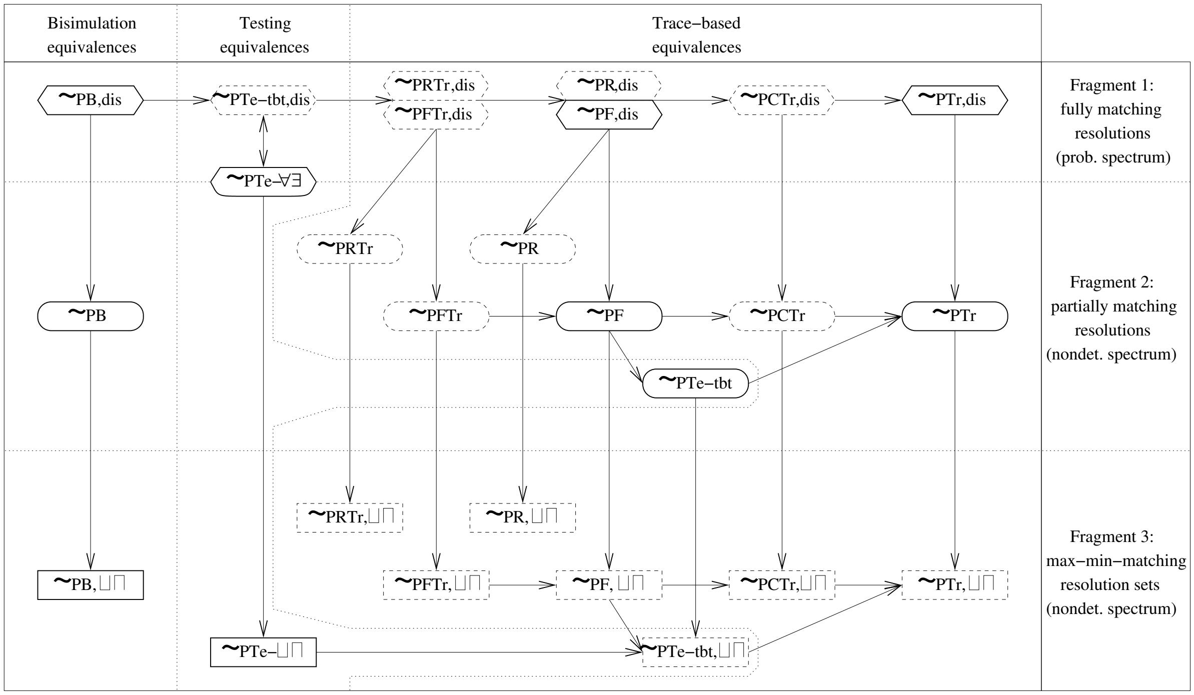

Figure 4: Spectrum of strong behavioral equivalences for NPLTS models (deterministic schedulers)

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Appendix: Proofs of Results

Proof of Thm. 3.19. Given $\sigma \in \{\mathrm{RTr}, \mathrm{FTr}, \mathrm{R}, \mathrm{F}, \mathrm{CTr}, \mathrm{Tr}\}$, the reader is referred to [6, 23] for the definition of $\sim_{\sigma,\mathrm{fnd}}$ and to [20, 17] for the definition of $\sim_{\sigma,\mathrm{fpr}}$ for the various trace-based equivalences:

1. The result over fully nondeterministic NPLTS models is a straightforward consequence of the fact that the resolutions of these models correspond to the computations of the models themselves, hence the probability of performing within a resolution of one of these models a computation compatible with a ready trace, a failure trace, a ready pair, a failure pair, a completed trace, or a trace can only be either 1 or 0.
2. The result over fully probabilistic NPLTS models is a straightforward consequence of the fact that each of these models has a single maximal resolution, which corresponds to the model itself. ■

Proof of Thm. 3.20. Let $(S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$:

1. A straightforward consequence of the definitions of the trace-based equivalences. In particular, if $s_1 \sim_{\pi,\mathrm{dis}} s_2$, then $s_1 \sim_\pi s_2$ by taking the same fully matching resolutions considered in $\sim_{\pi,\mathrm{dis}}$. Likewise, if $s_1 \sim_\pi s_2$, then the two sets of probability values over all resolutions considered in $\sim_{\pi,\sqcup\sqcap}$ coincide and hence have the same supremum and infimum, i.e., $s_1 \sim_{\pi,\sqcup\sqcap} s_2$.
2. We preliminarily observe that for all $s \in S$, $\mathscr{Z} \in \mathit{Res}(s)$, $n \in \mathbb{N}$, $\alpha = a_1 \dots a_n \in A^*$, and $F_1, \dots, F_n, R_1, \dots, R_n \in 2^A$ it holds that:
$$\mathit{prob}(\mathscr{FTCC}(z_s, (a_1, F_1) \dots (a_n, F_n))) = \\ = \textstyle\sum_{R'_1,\dots,R'_n \in 2^A \text{ s.t. } R'_i \cap F_i = \emptyset \text{ for all } i=1,\dots,n} \mathit{prob}(\mathscr{RTCC}(z_s, (a_1, R'_1) \dots (a_n, R'_n)))$$
$$\mathit{prob}(\mathscr{RTCC}(z_s, (a_1, R_1) \dots (a_n, R_n))) = \mathit{prob}(\mathscr{FTCC}(z_s, (a_1, \overline{R}_1) \dots (a_n, \overline{R}_n))) \\ - \textstyle\sum^{R'_j \subset R_j \text{ for some } j=1,\dots,n}_{R'_1,\dots,R'_n \in 2^A \text{ s.t. } R'_i \subseteq R_i \text{ for all } i=1,\dots,n} \mathit{prob}(\mathscr{RTCC}(z_s, (a_1, R'_1) \dots (a_n, R'_n)))$$
where $\overline{R}_i = A \setminus R_i$ for all $i = 1, \dots, n$.
Suppose that $s_1 \sim_{\mathrm{PRTr,dis}} s_2$. Then we immediately derive that:
   - For each $\mathscr{Z}_1 \in \mathit{Res}(s_1)$ there exists $\mathscr{Z}_2 \in \mathit{Res}(s_2)$ such that for all $(a_1, F_1) \dots (a_n, F_n) \in (A \times 2^A)^*$:
$$\mathit{prob}(\mathscr{FTCC}(z_{s_1}, (a_1, F_1) \dots (a_n, F_n))) = \\ = \textstyle\sum_{R'_1,\dots,R'_n \in 2^A \text{ s.t. } R'_i \cap F_i = \emptyset \text{ for all } i=1,\dots,n} \mathit{prob}(\mathscr{RTCC}(z_{s_1}, (a_1, R'_1) \dots (a_n, R'_n))) \\ = \textstyle\sum_{R'_1,\dots,R'_n \in 2^A \text{ s.t. } R'_i \cap F_i = \emptyset \text{ for all } i=1,\dots,n} \mathit{prob}(\mathscr{RTCC}(z_{s_2}, (a_1, R'_1) \dots (a_n, R'_n))) \\ = \mathit{prob}(\mathscr{FTCC}(z_{s_2}, (a_1, F_1) \dots (a_n, F_n)))$$
   - Symmetrically for each $\mathscr{Z}_2 \in \mathit{Res}(s_2)$.

   This means that $s_1 \sim_{\mathrm{PFTr,dis}} s_2$.
Suppose now that $s_1 \sim_{\mathrm{PFTr,dis}} s_2$. For each ready trace $\rho \in (A \times 2^A)^*$ including at least one infinite ready set, it trivially holds that for all $\mathscr{Z}_1 \in \mathit{Res}(s_1)$ and $\mathscr{Z}_2 \in \mathit{Res}(s_2)$:
$$\mathit{prob}(\mathscr{RTCC}(z_{s_1}, \rho)) = 0 = \mathit{prob}(\mathscr{RTCC}(z_{s_2}, \rho))$$
whenever the considered NPLTS is finitely branching. Thus, in order to prove that $s_1 \sim_{\mathrm{PRTr,dis}} s_2$, we can restrict ourselves to ready traces including only finite ready sets. Given an arbitrary $\mathscr{Z}_1 \in \mathit{Res}(s_1)$ that is matched by some $\mathscr{Z}_2 \in \mathit{Res}(s_2)$ according to $\sim_{\mathrm{PFTr,dis}}$, we show that the matching holds also under $\sim_{\mathrm{PRTr,dis}}$ by proceeding by induction on the sum $k \in \mathbb{N}$ of the cardinalities of the ready sets occurring in ready traces including only finite ready sets:

- Let $k = 0$, i.e., consider ready traces whose ready sets are all empty. Then for all $\alpha = a_1 \dots a_n \in A^*$:

$$\begin{aligned} prob(\mathcal{RTCC}(z_{s_1}, (a_1, \emptyset) \dots (a_n, \emptyset))) &= prob(\mathcal{FTCC}(z_{s_1}, (a_1, A) \dots (a_n, A))) \\ &= prob(\mathcal{FTCC}(z_{s_2}, (a_1, A) \dots (a_n, A))) \\ &= prob(\mathcal{RTCC}(z_{s_2}, (a_1, \emptyset) \dots (a_n, \emptyset))) \end{aligned}$$

- Let $k \in \mathbb{N}_{>0}$ and suppose that the result holds for all ready traces for which the sum of the cardinalities of the ready sets is less than k. Then for all $(a_1, R_1) \dots (a_n, R_n) \in (A \times 2^A)^*$ such that $\sum_{1 \le i \le n} |R_i| = k$:

$$\begin{aligned} prob(\mathcal{RTCC}(z_{s_1}, (a_1, R_1) \dots (a_n, R_n))) &= prob(\mathcal{FTCC}(z_{s_1}, (a_1, \overline{R}_1) \dots (a_n, \overline{R}_n))) \\ &\quad - \sum_{R'_1, \dots, R'_n \in 2^A \text{ s.t. } R'_i \subseteq R_i \text{ for all } i = 1, \dots, n}^{R'_j \subset R_J \text{ for some } j = 1, \dots, n} prob(\mathcal{RTCC}(z_{s_1}, (a_1, R'_1) \dots (a_n, R'_n))) \\ &= prob(\mathcal{FTCC}(z_{s_2}, (a_1, \overline{R}_1) \dots (a_n, \overline{R}_n))) \\ &\quad - \sum_{R'_1, \dots, R'_n \in 2^A \text{ s.t. } R'_i \subseteq R_i \text{ for all } i = 1, \dots, n}^{R'_j \subset R_J \text{ for some } j = 1, \dots, n} prob(\mathcal{RTCC}(z_{s_2}, (a_1, R'_1) \dots (a_n, R'_n))) \\ &= prob(\mathcal{RTCC}(z_{s_2}, (a_1, R_1) \dots (a_n, R_n))) \end{aligned}$$

A similar result holds also starting from an arbitrary $\mathcal{Z}_2 \in Res(s_2)$ that is matched by some $\mathcal{Z}_1 \in Res(s_1)$ according to $\sim_{\text{PFTr,dis}}$. Therefore, we can conclude that $s_1 \sim_{\text{PRTr,dis}} s_2$.

3. We preliminarily observe that for all $s \in S$, $\mathcal{Z} \in Res(s)$, $\alpha \in A^*$, and $F, R \in 2^A$ it holds that:

$$\begin{aligned} prob(\mathcal{FCC}(z_s, (\alpha, F))) &= \sum_{R' \in 2^A \text{ s.t. } R' \cap F = \emptyset} prob(\mathcal{RCC}(z_s, (\alpha, R'))) \\ prob(\mathcal{RCC}(z_s, (\alpha, R))) &= prob(\mathcal{FCC}(z_s, (\alpha, A \setminus R))) - \sum_{R' \subset R} prob(\mathcal{RCC}(z_s, (\alpha, R'))) \end{aligned}$$

Suppose that $s_1 \sim_{\text{PR,dis}} s_2$. Then we immediately derive that:

- For each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$ such that for all $(\alpha, F) \in A^* \times 2^A$:

$$\begin{aligned} prob(\mathcal{FCC}(z_{s_1}, (\alpha, F))) &= \sum_{R' \in 2^A \text{ s.t. } R' \cap F = \emptyset} prob(\mathcal{RCC}(z_{s_1}, (\alpha, R'))) \\ &= \sum_{R' \in 2^A \text{ s.t. } R' \cap F = \emptyset} prob(\mathcal{RCC}(z_{s_2}, (\alpha, R'))) \\ &= prob(\mathcal{FCC}(z_{s_2}, (\alpha, F))) \end{aligned}$$

- Symmetrically for each $\mathcal{Z}_2 \in Res(s_2)$.

This means that $s_1 \sim_{\text{PF,dis}} s_2$.

Suppose now that $s_1 \sim_{\text{PF,dis}} s_2$. For each ready pair $(\alpha, R) \in A^* \times 2^A$ such that R is infinite, it trivially holds that for all $\mathcal{Z}_1 \in Res(s_1)$ and $\mathcal{Z}_2 \in Res(s_2)$:

$$prob(\mathcal{RCC}(z_{s_1}, (\alpha, R))) = 0 = prob(\mathcal{RCC}(z_{s_2}, (\alpha, R)))$$

whenever the considered NPLTS is finitely branching. Thus, in order to prove that $s_1 \sim_{\text{PR,dis}} s_2$, we can restrict ourselves to ready pairs whose ready set is finite. Given an arbitrary $\mathcal{Z}_1 \in Res(s_1)$ that is matched by some $\mathcal{Z}_2 \in Res(s_2)$ according to $\sim_{\text{PF,dis}}$, we show that the matching holds also under $\sim_{\text{PR,dis}}$ by proceeding by induction on the cardinality $k \in \mathbb{N}$ of the ready set of ready pairs whose ready set is finite:

- Let $k = 0$, i.e., consider ready pairs whose ready set is empty. Then for all $\alpha \in A^*$:

$$\begin{aligned} prob(\mathcal{RCC}(z_{s_1}, (\alpha, \emptyset))) &= prob(\mathcal{FCC}(z_{s_1}, (\alpha, A))) = \\ &= prob(\mathcal{FCC}(z_{s_2}, (\alpha, A))) = prob(\mathcal{RCC}(z_{s_2}, (\alpha, \emptyset))) \end{aligned}$$

- Let $k \in \mathbb{N}_{>0}$ and suppose that the result holds for all ready pairs whose ready set has cardinality less than k. Then for all $(\alpha, R) \in A^* \times 2^A$ such that $|R| = k$:

$$\begin{aligned} prob(\mathcal{RCC}(z_{s_1}, (\alpha, R))) &= prob(\mathcal{FCC}(z_{s_1}, (\alpha, A \setminus R))) - \sum_{R' \subset R} prob(\mathcal{RCC}(z_{s_1}, (\alpha, R'))) \\ &= prob(\mathcal{FCC}(z_{s_2}, (\alpha, A \setminus R))) - \sum_{R' \subset R} prob(\mathcal{RCC}(z_{s_2}, (\alpha, R'))) \\ &= prob(\mathcal{RCC}(z_{s_2}, (\alpha, R))) \end{aligned}$$

A similar result holds also starting from an arbitrary $\mathscr{Z}_2 \in Res(s_2)$ that is matched by some $\mathscr{Z}_1 \in Res(s_1)$ according to $\sim_{\rm PF,dis}$. Therefore, we can conclude that $s_1 \sim_{\rm PR,dis} s_2$.

4. We preliminarily observe that for all $s \in S$, $\mathscr{Z} \in Res(s)$, $n \in \mathbb{N}$, $\alpha = a_1 \dots a_n \in A^*$, and $F \in 2^A$ it holds that:

$$\begin{aligned} prob(\mathscr{FCC}(z_s, (\alpha, F))) &= prob(\mathscr{FTCC}(z_s, (a_1, \emptyset) \dots (a_{n-1}, \emptyset)(a_n, F))) \\ prob(\mathscr{CC}(z_s, \alpha)) &= prob(\mathscr{FCC}(z_s, (\alpha, \emptyset))) \\ prob(\mathscr{CCC}(z_s, \alpha)) &= prob(\mathscr{FCC}(z_s, (\alpha, A))) \end{aligned}$$

Suppose that $s_1 \sim_{\rm PFTr,dis} s_2$. Then we immediately derive that:

- For each $\mathscr{Z}_1 \in Res(s_1)$ there exists $\mathscr{Z}_2 \in Res(s_2)$ such that for all $(a_1 \dots a_n, F) \in A^* \times 2^A$:
$$\begin{aligned} prob(\mathscr{FCC}(z_{s_1}, (a_1 \dots a_n, F))) &= prob(\mathscr{FTCC}(z_{s_1}, (a_1, \emptyset) \dots (a_{n-1}, \emptyset)(a_n, F))) \\ &= prob(\mathscr{FTCC}(z_{s_2}, (a_1, \emptyset) \dots (a_{n-1}, \emptyset)(a_n, F))) \\ &= prob(\mathscr{FCC}(z_{s_2}, (a_1 \dots a_n, F))) \end{aligned}$$
- Symmetrically for each $\mathscr{Z}_2 \in Res(s_2)$.

This means that $s_1 \sim_{\rm PF,dis} s_2$.
Suppose now that $s_1 \sim_{\rm PF,dis} s_2$. Then we immediately derive that:

- For each $\mathscr{Z}_1 \in Res(s_1)$ there exists $\mathscr{Z}_2 \in Res(s_2)$ such that for all $\alpha \in A^*$:
$$\begin{aligned} prob(\mathscr{CC}(z_{s_1}, \alpha)) &= prob(\mathscr{FCC}(z_{s_1}, (\alpha, \emptyset))) = \\ &= prob(\mathscr{FCC}(z_{s_2}, (\alpha, \emptyset))) = prob(\mathscr{CC}(z_{s_2}, \alpha)) \\ prob(\mathscr{CCC}(z_{s_1}, \alpha)) &= prob(\mathscr{FCC}(z_{s_1}, (\alpha, A))) = \\ &= prob(\mathscr{FCC}(z_{s_2}, (\alpha, A))) = prob(\mathscr{CCC}(z_{s_2}, \alpha)) \end{aligned}$$
- Symmetrically for each $\mathscr{Z}_2 \in Res(s_2)$.

This means that $s_1 \sim_{\rm PCTr,dis} s_2$.
The fact that $s_1 \sim_{\rm PCTr,dis} s_2$ implies $s_1 \sim_{\rm PTr,dis} s_2$ is a straightforward consequence of the definition of the two equivalences.

5. Suppose that $s_1 \sim_{\rm PFTr} s_2$. Then we immediately derive that for all $(a_1 \dots a_n, F) \in A^* \times 2^A$:

- For each $\mathscr{Z}_1 \in Res(s_1)$ there exists $\mathscr{Z}_2 \in Res(s_2)$ such that:
$$\begin{aligned} prob(\mathscr{FCC}(z_{s_1}, (a_1 \dots a_n, F))) &= prob(\mathscr{FTCC}(z_{s_1}, (a_1, \emptyset) \dots (a_{n-1}, \emptyset)(a_n, F))) \\ &= prob(\mathscr{FTCC}(z_{s_2}, (a_1, \emptyset) \dots (a_{n-1}, \emptyset)(a_n, F))) \\ &= prob(\mathscr{FCC}(z_{s_2}, (a_1 \dots a_n, F))) \end{aligned}$$
- Symmetrically for each $\mathscr{Z}_2 \in Res(s_2)$.

This means that $s_1 \sim_{\rm PF} s_2$.
Suppose now that $s_1 \sim_{\rm PF} s_2$. Then we immediately derive that for all $\alpha \in A^*$:

- For each $\mathscr{Z}_1 \in Res(s_1)$ there exist $\mathscr{Z}_2 \in Res(s_2)$ such that:
$$\begin{aligned} prob(\mathscr{CC}(z_{s_1}, \alpha)) &= prob(\mathscr{FCC}(z_{s_1}, (\alpha, \emptyset))) = \\ &= prob(\mathscr{FCC}(z_{s_2}, (\alpha, \emptyset))) = prob(\mathscr{CC}(z_{s_2}, \alpha)) \end{aligned}$$
and $\mathscr{Z}'_2 \in Res(s_2)$ such that:
$$\begin{aligned} prob(\mathscr{CCC}(z_{s_1}, \alpha)) &= prob(\mathscr{FCC}(z_{s_1}, (\alpha, A))) = \\ &= prob(\mathscr{FCC}(z'_{s_2}, (\alpha, A))) = prob(\mathscr{CCC}(z'_{s_2}, \alpha)) \end{aligned}$$
- Symmetrically for each $\mathscr{Z}_2 \in Res(s_2)$.

This means that $s_1 \sim_{\mathrm{PCTr}} s_2$.
The fact that $s_1 \sim_{\mathrm{PCTr}} s_2$ implies $s_1 \sim_{\mathrm{PTr}} s_2$ is a straightforward consequence of the definition of the two equivalences.

6. Suppose that $s_1 \sim_{\mathrm{PFTr},\sqcup\sqcap} s_2$. Then we immediately derive that for all $\varphi = (\alpha, F) \in A^* \times 2^A$:

$$\begin{aligned}
\bigsqcup_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{FCC}(z_{s_1}, \varphi)) &= \bigsqcup_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{FTCC}(z_{s_1}, (a_1, \emptyset) \ldots (a_{n-1}, \emptyset)(a_n, F))) \\
&= \bigsqcup_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{FTCC}(z_{s_2}, (a_1, \emptyset) \ldots (a_{n-1}, \emptyset)(a_n, F))) \\
&= \bigsqcup_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{FCC}(z_{s_2}, \varphi)) \\
\bigsqcap_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{FCC}(z_{s_1}, \varphi)) &= \bigsqcap_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{FTCC}(z_{s_1}, (a_1, \emptyset) \ldots (a_{n-1}, \emptyset)(a_n, F))) \\
&= \bigsqcap_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{FTCC}(z_{s_2}, (a_1, \emptyset) \ldots (a_{n-1}, \emptyset)(a_n, F))) \\
&= \bigsqcap_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{FCC}(z_{s_2}, \varphi))
\end{aligned}$$

where $a_1 \ldots a_n = \alpha$. This means that $s_1 \sim_{\mathrm{PF},\sqcup\sqcap} s_2$.
Suppose now that $s_1 \sim_{\mathrm{PF},\sqcup\sqcap} s_2$. Then we immediately derive that for all $\alpha \in A^*$:

$$\begin{aligned}
\bigsqcup_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{CC}(z_{s_1}, \alpha)) &= \bigsqcup_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{FCC}(z_{s_1}, (\alpha, \emptyset))) \\
&= \bigsqcup_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{FCC}(z_{s_2}, (\alpha, \emptyset))) \\
&= \bigsqcup_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{CC}(z_{s_2}, \alpha)) \\
\bigsqcap_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{CC}(z_{s_1}, \alpha)) &= \bigsqcap_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{FCC}(z_{s_1}, (\alpha, \emptyset))) \\
&= \bigsqcap_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{FCC}(z_{s_2}, (\alpha, \emptyset))) \\
&= \bigsqcap_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{CC}(z_{s_2}, \alpha))
\end{aligned}$$

and:

$$\begin{aligned}
\bigsqcup_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{CCC}(z_{s_1}, \alpha)) &= \bigsqcup_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{FCC}(z_{s_1}, (\alpha, A))) \\
&= \bigsqcup_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{FCC}(z_{s_2}, (\alpha, A))) \\
&= \bigsqcup_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{CCC}(z_{s_2}, \alpha)) \\
\bigsqcap_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{CCC}(z_{s_1}, \alpha)) &= \bigsqcap_{\mathscr{Z}_1 \in Res_\alpha(s_1)} prob(\mathscr{FCC}(z_{s_1}, (\alpha, A))) \\
&= \bigsqcap_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{FCC}(z_{s_2}, (\alpha, A))) \\
&= \bigsqcap_{\mathscr{Z}_2 \in Res_\alpha(s_2)} prob(\mathscr{CCC}(z_{s_2}, \alpha))
\end{aligned}$$

This means that $s_1 \sim_{\mathrm{PCTr},\sqcup\sqcap} s_2$.
The fact that $s_1 \sim_{\mathrm{PCTr},\sqcup\sqcap} s_2$ implies $s_1 \sim_{\mathrm{PTr},\sqcup\sqcap} s_2$ is a straightforward consequence of the definition of the two equivalences. ■

Proof of Thm. 4.8. Let $(S,A,\longrightarrow)$ be an NPLTS and $s_1,s_2 \in S$:

1. Suppose that the NPLTS is fully nondeterministic. We preliminarily recall from [10] that $s_1 \sim_{\text{Te,fnd}} s_2$ means that for every *fully nondeterministic* test $\mathscr{T} = (O,A,\longrightarrow_{\mathscr{T}})$ with initial state $o \in O$:

   - There exists a successful computation from (s_1,o) iff there exists a successful computation from (s_2,o).
   - All completed computations from (s_1,o) are successful iff all completed computations from (s_2,o) are successful.

   The proof is divided into two parts:

   (a) Suppose that $s_1 \sim_{\text{PTe-tbt}} s_2$. Then, in particular, for every fully nondeterministic test $\mathscr{T} = (O,A,\longrightarrow_{\mathscr{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$:

   - For each $\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)$ there exists $\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2,o)$ such that: $$prob(\mathscr{SCC}(z_{s_1,o},\alpha)) = prob(\mathscr{SCC}(z_{s_2,o},\alpha))$$
   - For each $\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2,o)$ there exists $\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)$ such that: $$prob(\mathscr{SCC}(z_{s_2,o},\alpha)) = prob(\mathscr{SCC}(z_{s_1,o},\alpha))$$

   Since the NPLTS under test and the considered tests are all fully nondeterministic, the resulting interaction systems are fully nondeterministic too, and hence their resolutions correspond to their computations and each of the probability values above is either 1 or 0. As a consequence, the previous relationships among maximal resolutions can be rephrased as follows:

   - For each completed α-compatible computation from (s_1,o) there exists a completed α-compatible computation from (s_2,o) such that the two computations are both successful or both unsuccessful.
   - For each completed α-compatible computation from (s_2,o) there exists a completed α-compatible computation from (s_1,o) such that the two computations are both successful or both unsuccessful.

   From this, we immediately derive that:

   - There exists a successful computation from (s_1,o) iff there exists a successful computation from (s_2,o).
   - All completed computations from (s_1,o) are successful iff all completed computations from (s_2,o) are successful. In fact, assume that all completed computations from, e.g., (s_1,o) are successful. Then at least one completed computation from (s_2,o) is successful. Assume that (s_2,o) has at least two completed computations and that one of them is not successful. Then at least one completed computation from (s_1,o) would not be successful, thus contradicting the assumption that all completed computations from (s_1,o) are successful. Therefore, whenever all completed computations from (s_1,o) are successful, then all completed computations from (s_2,o) are successful. Likewise, whenever all completed computations from (s_2,o) are successful, then all completed computations from (s_1,o) are successful.

   This means that $s_1 \sim_{\text{Te,fnd}} s_2$.
   Suppose now that $s_1 \sim_{\text{Te,fnd}} s_2$ and consider an arbitrary NPT $\mathscr{T} = (O,A,\longrightarrow_{\mathscr{T}})$ with initial state $o \in O$, an arbitrary trace $\alpha \in A^*$ such that $Res_{\max,\mathscr{C},\alpha}(s_1,o) \neq \emptyset$, and an arbitrary resolution $\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)$.

Assume that $Res_{\max,\mathscr{C},\alpha}(s_2,o)=\emptyset$, i.e., assume that for all $\mathscr{Z}_2\in Res_{\max}(s_2,o)$ it holds that $\mathscr{CCC}(z_{s_2,o},\alpha)=\emptyset$. Let $\mathscr{T}_\alpha=(O,A,\longrightarrow_{\mathscr{T}_\alpha})$ be a fully nondeterministic test obtained from $\mathscr{T}$ in which (i) only the completed α-compatible computations reach ω and (ii) each transition $o'\xrightarrow{a}_{\mathscr{T}}\mathscr{D}$ such that the set $O'=\{o''\in O\mid \mathscr{D}(o'')>0\}$ has cardinality greater than 1 is transformed into $|O'|$ transitions $o'\xrightarrow{a}_{\mathscr{T}_\alpha}\mathscr{D}_{o''}$, $o''\in O'$, where $\mathscr{D}_{o''}(o'')=1$ and $\mathscr{D}_{o''}(o''')=0$ for all $o'''\in O\setminus\{o''\}$. Observing that $\mathscr{T}_\alpha$ yields the same α-compatible computations as $\mathscr{T}$ in the interaction systems, the test $\mathscr{T}_\alpha$ would violate $s_1\sim_{\mathrm{Te,fnd}}s_2$ because at least one completed computation from (s_1,o) is successful whilst there are no completed computations from (s_2,o) that are successful. We have thus deduced that, whenever $s_1\sim_{\mathrm{Te,fnd}}s_2$, then the existence of $\mathscr{Z}_1\in Res_{\max,\mathscr{C},\alpha}(s_1,o)$ implies the existence of $\mathscr{Z}_2\in Res_{\max,\mathscr{C},\alpha}(s_2,o)$.

Assume now that for all $\mathscr{Z}_2\in Res_{\max,\mathscr{C},\alpha}(s_2,o)$ it holds that:

$$prob(\mathscr{SCC}(z_{s_1,o},\alpha))\neq prob(\mathscr{SCC}(z_{s_2,o},\alpha))$$

Observing that $\mathscr{T}$ must have a successful α-compatible computation – otherwise it would hold that $prob(\mathscr{SCC}(z_{s_1,o},\alpha))=0=prob(\mathscr{SCC}(z_{s_2,o},\alpha))$ for all $\mathscr{Z}_2\in Res_{\max,\mathscr{C},\alpha}(s_2,o)$ – from $\mathscr{CCC}(z_{s_1,o},\alpha)\neq\emptyset$ and $\mathscr{CCC}(z_{s_2,o},\alpha)\neq\emptyset$ we derive that $prob(\mathscr{SCC}(z_{s_1,o},\alpha))>0$ and $prob(\mathscr{SCC}(z_{s_2,o},\alpha))>0$. Denoting by $\mathscr{Z}_1'$ the element of $Res_{\max}(s_1)$ that originates $\mathscr{Z}_1$, we would then have that for each $\mathscr{Z}_2'\in Res_{\max}(s_2)$ originating $\mathscr{Z}_2$:

$$\begin{aligned} prob(\mathscr{CC}(z'_{s_1},\alpha)) \;&=\; prob(\mathscr{SCC}(z_{s_1,o},\alpha))/p \;\neq\\ &\neq\; prob(\mathscr{SCC}(z_{s_2,o},\alpha))/p \;=\; prob(\mathscr{CC}(z'_{s_2},\alpha))\end{aligned}$$

where p is the probability of performing a successful α-compatible computation in the element $\mathscr{Z}$ of $Res_{\max}(o)$ that originates $\mathscr{Z}_1$. However, since the NPLTS under test is fully nondeterministic, $\mathscr{Z}_1'$ and $\mathscr{Z}_2'$ boil down to two α-compatible computations and it holds that:

$$prob(\mathscr{CC}(z'_{s_1},\alpha))=1=prob(\mathscr{CC}(z'_{s_2},\alpha))$$

which contradicts what established before.

In conclusion, whenever $s_1\sim_{\mathrm{Te,fnd}}s_2$, then for each $\mathscr{Z}_1\in Res_{\max,\mathscr{C},\alpha}(s_1,o)$ there exists $\mathscr{Z}_2\in Res_{\max,\mathscr{C},\alpha}(s_2,o)$ such that:

$$prob(\mathscr{SCC}(z_{s_1,o},\alpha))=prob(\mathscr{SCC}(z_{s_2,o},\alpha))$$

With a similar argument, we can prove that, whenever $s_1\sim_{\mathrm{Te,fnd}}s_2$, then for each $\mathscr{Z}_2\in Res_{\max,\mathscr{C},\alpha}(s_2,o)$ there exists $\mathscr{Z}_1\in Res_{\max,\mathscr{C},\alpha}(s_1,o)$ such that:

$$prob(\mathscr{SCC}(z_{s_2,o},\alpha))=prob(\mathscr{SCC}(z_{s_1,o},\alpha))$$

This means that $s_1\sim_{\text{PTe-tbt}}s_2$.

(b) Suppose that $s_1\sim_{\text{PTe-tbt},\sqcup\sqcap}s_2$. Then, in particular, for every fully nondeterministic test $\mathscr{T}=(O,A,\longrightarrow_{\mathscr{T}})$ with initial state $o\in O$ and for all $\alpha\in A^*$ it holds that $Res_{\max,\mathscr{C},\alpha}(s_1,o)\neq\emptyset$ iff $Res_{\max,\mathscr{C},\alpha}(s_2,o)\neq\emptyset$ and:

$$\begin{aligned}\bigsqcup_{\mathscr{Z}_1\in Res_{\max,\mathscr{C},\alpha}(s_1,o)} prob(\mathscr{SCC}(z_{s_1,o},\alpha)) \;&=\; \bigsqcup_{\mathscr{Z}_2\in Res_{\max,\mathscr{C},\alpha}(s_2,o)} prob(\mathscr{SCC}(z_{s_2,o},\alpha))\\ \bigsqcap_{\mathscr{Z}_1\in Res_{\max,\mathscr{C},\alpha}(s_1,o)} prob(\mathscr{SCC}(z_{s_1,o},\alpha)) \;&=\; \bigsqcap_{\mathscr{Z}_2\in Res_{\max,\mathscr{C},\alpha}(s_2,o)} prob(\mathscr{SCC}(z_{s_2,o},\alpha))\end{aligned}$$

Since the NPLTS under test and the considered tests are all fully nondeterministic, the resulting interaction systems are fully nondeterministic too, and hence their resolutions correspond to their computations and each of the extremal probability values above is either 1 or 0. As a consequence, the previous relationships among extremal probability values over maximal resolutions can be rephrased as follows:

($\bigsqcup$) There exists a successful α-compatible computation from (s_1,o) iff there exists a successful α-compatible computation from (s_2,o).

($\bigsqcap$) All completed α-compatible computations from (s_1,o) are successful iff all completed

α-compatible computations from (s_2, o) are successful.

From this, we immediately derive that:

($\bigsqcup$) There exists a successful computation from (s_1, o) iff there exists a successful computation from (s_2, o).

($\bigsqcap$) All completed computations from (s_1, o) are successful iff all completed computations from (s_2, o) are successful.

This means that $s_1 \sim_{\text{Te,fnd}} s_2$.
Suppose now that $s_1 \sim_{\text{Te,fnd}} s_2$. Then $s_1 \sim_{\text{PTe-tbt}} s_2$ – as we have proved in the first part – and hence $s_1 \sim_{\text{PTe-tbt},\sqcup\sqcap} s_2$ – as a consequence of Thm. 4.9.

2. Suppose that the NPLTS is fully probabilistic. We preliminarily recall from [7] that $s_1 \sim_{\text{Te,fpr}} s_2$ means that for every *fully probabilistic* test $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ with initial state $o \in O$:
$$prob(\mathscr{SC}(s_1, o)) = prob(\mathscr{SC}(s_2, o))$$
Since the NPLTS under test is fully probabilistic and hence each of its states has at most one outgoing transition, when checking for $\sim_{\text{PTe-}\sqcup\sqcap}$, $\sim_{\text{PTe-}\forall\exists}$, $\sim_{\text{PTe-tbt,dis}}$, $\sim_{\text{PTe-tbt}}$, or $\sim_{\text{PTe-tbt},\sqcup\sqcap}$ possible nondeterministic choices within tests disappear in the resulting interaction systems, which are thus fully probabilistic too. Since each such system has a single maximal resolution that corresponds to the system itself, the fact that $\sim_{\text{PTe-}\sqcup\sqcap} = \sim_{\text{PTe-}\forall\exists} = \sim_{\text{PTe-tbt,dis}} = \sim_{\text{PTe-tbt}} = \sim_{\text{PTe-tbt},\sqcup\sqcap} = \sim_{\text{Te,fpr}}$ over fully probabilistic NPLTS models is straightforward.
In particular, for $\sim_{\text{PTe-tbt,dis}}$, $\sim_{\text{PTe-tbt}}$, and $\sim_{\text{PTe-tbt},\sqcup\sqcap}$ we observe that, given $s \in S$ and the initial state o of a test, it holds that:
$$prob(\mathscr{SC}(s, o)) = \sum_{\alpha \in A^* \text{ s.t. } \mathscr{CCC}((s,o),\alpha) \neq \emptyset} prob(\mathscr{SCC}((s, o), \alpha))$$
Therefore, the fact that all of $s_1 \sim_{\text{PTe-tbt,dis}} s_2$, $s_1 \sim_{\text{PTe-tbt}} s_2$, and $s_1 \sim_{\text{PTe-tbt},\sqcup\sqcap} s_2$ imply $s_1 \sim_{\text{Te,fpr}} s_2$ is trivial. Likewise, in order to prove that $s_1 \sim_{\text{Te,fpr}} s_2$ implies all of $s_1 \sim_{\text{PTe-tbt,dis}} s_2$, $s_1 \sim_{\text{PTe-tbt}} s_2$, and $s_1 \sim_{\text{PTe-tbt},\sqcup\sqcap} s_2$, given an arbitrary NPT $\mathscr{T}$ it is sufficient to consider fully probabilistic tests $\mathscr{T}_\alpha$, $\alpha \in A^*$, obtained from $\mathscr{T}$ by making unsuccessful all the successful computations of $\mathscr{T}$ not compatible with α, so that $prob(\mathscr{SC}(s, o))$ reduces to $prob(\mathscr{SCC}((s, o), \alpha))$. ■

Proof of Thm. 4.9. Let $(S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$:

1. Suppose that $s_1 \sim_{\text{PTe-}\forall\exists} s_2$. Then we immediately derive that for every NPT $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ with initial state $o \in O$:
$$\{prob(\mathscr{SC}(z_{s_1,o})) \mid \mathscr{Z}_1 \in Res_{\max}(s_1, o)\} \subseteq \{prob(\mathscr{SC}(z_{s_2,o})) \mid \mathscr{Z}_2 \in Res_{\max}(s_2, o)\}$$
and:
$$\{prob(\mathscr{SC}(z_{s_2,o})) \mid \mathscr{Z}_2 \in Res_{\max}(s_2, o)\} \subseteq \{prob(\mathscr{SC}(z_{s_1,o})) \mid \mathscr{Z}_1 \in Res_{\max}(s_1, o)\}$$
As a consequence, for every NPT $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ with initial state $o \in O$:
$$\{prob(\mathscr{SC}(z_{s_1,o})) \mid \mathscr{Z}_1 \in Res_{\max}(s_1, o)\} = \{prob(\mathscr{SC}(z_{s_2,o})) \mid \mathscr{Z}_2 \in Res_{\max}(s_2, o)\}$$
and hence:
$$\bigsqcup_{\mathscr{Z}_1 \in Res_{\max}(s_1,o)} prob(\mathscr{SC}(z_{s_1,o})) = \bigsqcup_{\mathscr{Z}_2 \in Res_{\max}(s_2,o)} prob(\mathscr{SC}(z_{s_2,o}))$$
$$\bigsqcap_{\mathscr{Z}_1 \in Res_{\max}(s_1,o)} prob(\mathscr{SC}(z_{s_1,o})) = \bigsqcap_{\mathscr{Z}_2 \in Res_{\max}(s_2,o)} prob(\mathscr{SC}(z_{s_2,o}))$$
This means that $s_1 \sim_{\text{PTe-}\sqcup\sqcap} s_2$.
Suppose now that $s_1 \sim_{\text{PTe-}\sqcup\sqcap} s_2$ and consider an arbitrary NPT $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ with initial state $o \in O$. Given $s \in S$ and $\mathscr{Z} \in Res_{\max}(s, o)$, it holds that:
$$prob(\mathscr{SC}(z_{s,o})) = \sum_{\alpha \in A^* \text{ s.t. } \mathscr{CCC}(z_{s,o},\alpha) \neq \emptyset} prob(\mathscr{SCC}(z_{s,o}, \alpha))$$

Therefore, to derive that $s_1 \sim_{\text{PTe-tbt},\sqcup\sqcap} s_2$, it is sufficient to consider tests $\mathscr{T}_\alpha$, $\alpha \in A^*$, obtained from $\mathscr{T}$ by making unsuccessful all the successful computations of $\mathscr{T}$ not compatible with α, so that $prob(\mathscr{SC}(z_{s,o}))$ reduces to $prob(\mathscr{SCC}(z_{s,o},\alpha))$.

2. Let us prove the contrapositive of $s_1 \sim_{\text{PTe-}\forall\exists} s_2 \Longrightarrow s_1 \sim_{\text{PTe-tbt,dis}} s_2$. Suppose that $s_1 \not\sim_{\text{PTe-tbt,dis}} s_2$. Then there exist an NPT $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ with initial state $o \in O$ and, say, a resolution $\mathscr{Z}_1 \in Res_{\max}(s_1, o)$ such that for each $\mathscr{Z}_2 \in Res_{\max}(s_2, o)$ there is $\alpha_2 \in A^*$ such that $\mathscr{CCC}(z_{s_1,o},\alpha_2) \neq \emptyset$ and (i) $\mathscr{CCC}(z_{s_2,o},\alpha_2) = \emptyset$ or (ii) $prob(\mathscr{SCC}(z_{s_1,o},\alpha_2)) \neq prob(\mathscr{SCC}(z_{s_2,o},\alpha_2))$. We show that from this fact it follows that $s_1 \not\sim_{\text{PTe-}\forall\exists} s_2$ by proceeding by induction on the number n of traces labeling the successful computations from o (note that n is finite – because $\mathscr{T}$ is finite state, finitely branching, and acyclic – and greater than 0 – otherwise $\mathscr{T}$ cannot distinguish s_1 from s_2 with respect to $\sim_{\text{PTe-tbt,dis}}$):

   - Let $n = 1$ and denote by α the only trace labeling the successful computations from o. Then $\mathscr{CCC}(z_{s_1,o},\alpha) \neq \emptyset$ and (i) $\mathscr{CCC}(z_{s_2,o},\alpha) = \emptyset$ in which case:
   $$prob(\mathscr{SC}(z_{s_1,o})) > 0 = prob(\mathscr{SC}(z_{s_2,o}))$$
   or (ii) it holds that:
   $$\begin{aligned} prob(\mathscr{SC}(z_{s_1,o})) &= prob(\mathscr{SCC}(z_{s_1,o},\alpha)) \neq \\ &\neq prob(\mathscr{SCC}(z_{s_2,o},\alpha)) = prob(\mathscr{SC}(z_{s_2,o})) \end{aligned}$$
   As a consequence, in both cases $s_1 \not\sim_{\text{PTe-}\forall\exists} s_2$.
   - Let $n \in \mathbb{N}_{>1}$ and suppose that the result holds for all $m = 1, \dots, n-1$. Given a trace α labeling some of the successful computations from o, we denote by $\mathscr{T}_{\downarrow\alpha}$ the NPT obtained from $\mathscr{T}$ by transforming into a normal terminal state every success state reached by a completed α-compatible computation and by $\mathscr{T}_{\uparrow\alpha}$ the NPT obtained from $\mathscr{T}$ by transforming into a normal terminal state every success state reached by a completed computation not compatible with α. Since $\mathscr{T}$ distinguishes s_1 from s_2 with respect to $\sim_{\text{PTe-tbt,dis}}$, $\mathscr{T}_{\downarrow\alpha}$ and $\mathscr{T}_{\uparrow\alpha}$ have the same structure as $\mathscr{T}$, and α labels some of the successful computations of $\mathscr{T}$, either $\mathscr{T}_{\downarrow\alpha}$ or $\mathscr{T}_{\uparrow\alpha}$ still distinguishes s_1 from s_2 with respect to $\sim_{\text{PTe-tbt,dis}}$. Since $\mathscr{T}_{\downarrow\alpha}$ has $n-1$ traces labeling its successful computations and $\mathscr{T}_{\uparrow\alpha}$ has a single trace labeling its successful computations, by the induction hypothesis it follows that $s_1 \not\sim_{\text{PTe-}\forall\exists} s_2$.

Suppose now that $s_1 \sim_{\text{PTe-tbt,dis}} s_2$ and consider an arbitrary NPT $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ with initial state $o \in O$. Since for all $s \in S$ and $\mathscr{Z} \in Res_{\max}(s, o)$ it holds that:
$$prob(\mathscr{SC}(z_{s,o})) = \sum_{\alpha \in A^* \text{ s.t. } \mathscr{CCC}(z_{s,o},\alpha) \neq \emptyset} prob(\mathscr{SCC}(z_{s,o},\alpha))$$
from $s_1 \sim_{\text{PTe-tbt,dis}} s_2$ it follows that:

- For each $\mathscr{Z}_1 \in Res_{\max}(s_1, o)$ there exists $\mathscr{Z}_2 \in Res_{\max}(s_2, o)$ such that:
$$\begin{aligned} prob(\mathscr{SC}(z_{s_1,o})) &= \sum_{\alpha \in A^* \text{ s.t. } \mathscr{CCC}(z_{s_1,o},\alpha) \neq \emptyset} prob(\mathscr{SCC}(z_{s_1,o},\alpha)) = \\ &= \sum_{\alpha \in A^* \text{ s.t. } \mathscr{CCC}(z_{s_2,o},\alpha) \neq \emptyset} prob(\mathscr{SCC}(z_{s_2,o},\alpha)) = prob(\mathscr{SC}(z_{s_2,o})) \end{aligned}$$
- Symmetrically for each $\mathscr{Z}_2 \in Res_{\max}(s_2, o)$.

This means that $s_1 \sim_{\text{PTe-}\forall\exists} s_2$. In conclusion, $\sim_{\text{PTe-}\forall\exists} = \sim_{\text{PTe-tbt,dis}}$.
The fact that $s_1 \sim_{\text{PTe-tbt,dis}} s_2$ implies $s_1 \sim_{\text{PTe-tbt}} s_2$ is easily seen by taking the same fully matching resolutions considered in $\sim_{\text{PTe-tbt,dis}}$.
Suppose now that $s_1 \sim_{\text{PTe-tbt}} s_2$. This means that for every NPT $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ it holds that:

- For each $\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)$ there exists $\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2,o)$ such that:
$$prob(\mathscr{SCC}(z_{s_1,o},\alpha)) = prob(\mathscr{SCC}(z_{s_2,o},\alpha))$$
- For each $\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2,o)$ there exists $\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)$ such that:
$$prob(\mathscr{SCC}(z_{s_2,o},\alpha)) = prob(\mathscr{SCC}(z_{s_1,o},\alpha))$$

This is to say that:

- $Res_{\max,\mathscr{C},\alpha}(s_1,o) \neq \emptyset$ implies $Res_{\max,\mathscr{C},\alpha}(s_2,o) \neq \emptyset$ and:
$$\bigcup_{\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)} \{prob(\mathscr{SCC}(z_{s_1,o},\alpha))\} \subseteq \bigcup_{\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2,o)} \{prob(\mathscr{SCC}(z_{s_2,o},\alpha))\}$$
- $Res_{\max,\mathscr{C},\alpha}(s_2,o) \neq \emptyset$ implies $Res_{\max,\mathscr{C},\alpha}(s_1,o) \neq \emptyset$ and:
$$\bigcup_{\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2,o)} \{prob(\mathscr{SCC}(z_{s_2,o},\alpha))\} \subseteq \bigcup_{\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)} \{prob(\mathscr{SCC}(z_{s_1,o},\alpha))\}$$

Equivalently, $Res_{\max,\mathscr{C},\alpha}(s_1,o) \neq \emptyset$ iff $Res_{\max,\mathscr{C},\alpha}(s_2,o) \neq \emptyset$ and:
$$\bigcup_{\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)} \{prob(\mathscr{SCC}(z_{s_1,o},\alpha))\} = \bigcup_{\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2,o)} \{prob(\mathscr{SCC}(z_{s_2,o},\alpha))\}$$
which implies:
$$\bigsqcup_{\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)} prob(\mathscr{SCC}(z_{s_1,o},\alpha)) = \bigsqcup_{\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2,o)} prob(\mathscr{SCC}(z_{s_2,o},\alpha))$$
$$\bigsqcap_{\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)} prob(\mathscr{SCC}(z_{s_1,o},\alpha)) = \bigsqcap_{\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2,o)} prob(\mathscr{SCC}(z_{s_2,o},\alpha))$$
This means that $s_1 \sim_{\text{PTe-tbt},\sqcup\sqcap} s_2$.

3. We show that $s_1 \sim_{\text{PTe-tbt,dis}} s_2$ implies $s_1 \sim_{\text{PRTr,dis}} s_2$ by building a test that permits to reason about all ready traces at once for each resolution of s_1 and s_2.
We start by deriving a new NPLTS $(S_\text{r}, A_\text{r}, \longrightarrow_\text{r})$ that is isomorphic to the given one up to transition labels and terminal states. A transition $s \xrightarrow{a} \mathscr{D}$ becomes $s_\text{r} \xrightarrow{a \triangleleft R}_\text{r} \mathscr{D}_\text{r}$ where $R \subseteq A$ is the set of actions labeling the outgoing transitions of s and $\mathscr{D}_\text{r}(s_\text{r}) = \mathscr{D}(s)$ for all $s \in S$. If s is a terminal state, i.e., it has no outgoing transitions, then we add a transition $s_\text{r} \xrightarrow{\circ \triangleleft \emptyset}_\text{r} \delta_{s_\text{r}}$ where $\delta_{s_\text{r}}(s_\text{r}) = 1$ and $\delta_{s_\text{r}}(s'_\text{r}) = 0$ for all $s' \in S \setminus \{s\}$. Transition relabeling preserves $\sim_{\text{PTe-tbt,dis}}$, i.e., $s_1 \sim_{\text{PTe-tbt,dis}} s_2$ implies $s_{1,\text{r}} \sim_{\text{PTe-tbt,dis}} s_{2,\text{r}}$, because $\sim_{\text{PTe-tbt,dis}}$ is able to distinguish a state that has a single α-compatible computation reaching a state with a nondeterministic branching formed by a b-transition and a c-transition from a state that has two α-compatible computations such that one of them reaches a state with only one outgoing transition labeled with b and the other one reaches a state with only one outgoing transition labeled with c (e.g., use a test that has a single α-compatible computation whose last step leads to a distribution whose support contains only a state with only one outgoing transition labeled with b that reaches success and a state with only one outgoing transition labeled with c that reaches success).
For each $\alpha_\text{r} \in (A_\text{r})^*$ and $R \subseteq A$, we build an NPT $\mathscr{T}_{\alpha_\text{r},R} = (O_{\alpha_\text{r},R}, A_\text{r}, \longrightarrow_{\alpha_\text{r},R})$ having a single α_r-compatible computation that goes from the initial state $o_{\alpha_\text{r},R}$ to a state having a single transition to ω labeled with (i) $\circ \triangleleft \emptyset$ if $R = \emptyset$ or (ii) $\_ \triangleleft R$ if $R \neq \emptyset$. Since we compare individual states (like s_1 and s_2) rather than state distributions, the distinguishing power of $\sim_{\text{PTe-tbt,dis}}$ does not change if we additionally consider tests starting with a single τ-transition that can initially evolve autonomously in any interaction system. We thus build a further NPT $\mathscr{T} = (O, A_\text{r}, \longrightarrow_{\mathscr{T}})$ that has an initial τ-transition and then behaves as one of the tests $\mathscr{T}_{\alpha_\text{r},R}$, i.e., its initial τ-transition goes from the initial state o to a state distribution whose support is the set $\{o_{\alpha_\text{r},R} \mid \alpha_\text{r} \in (A_\text{r})^* \land R \subseteq A\}$, with the probability $p_{\alpha_\text{r},R}$ associated with $o_{\alpha_\text{r},R}$ being taken from the distribution whose values are of

the form $1/2^i$, $i \in \mathbb{N}_{>0}$. Note that $\mathscr{T}$ is not finite state, but this affects only the initial step, whose only purpose is to internally select a specific ready trace.
After this step, $\mathscr{T}$ interacts with the process under test. Let $\rho \in (A \times 2^A)^*$ be a ready trace of the form $(a_1, R_1) \dots (a_n, R_n)$, where $n \in \mathbb{N}$. Given $s \in S$, consider the trace $\alpha_{\rho,\mathrm{r}} \in (A_{\mathrm{r}})^*$ of length $n+1$ in which the first element is $a_1 \triangleleft R$, with $R \subseteq A$ being the set of actions labeling the outgoing transitions of s, the subsequent elements are of the form $a_i \triangleleft R_{i-1}$ for $i = 2, \dots, n$, and the last element is (i) $\circ \triangleleft \emptyset$ if $R_n = \emptyset$ or (ii) $\_ \triangleleft R_n$ if $R_n \neq \emptyset$. Then for all $\mathscr{Z} \in Res(s)$ it holds that:

$$prob(\mathscr{RTCC}(z_s, \rho)) = 0$$

if there is no $a_1 \dots a_n$-compatible computation from z_s, otherwise:

$$prob(\mathscr{RTCC}(z_s, \rho)) = prob(\mathscr{SCC}(z_{s_{\mathrm{r}},o}, \alpha_{\rho,\mathrm{r}})) / p_{\alpha'_{\rho,\mathrm{r}}, R_n}$$

where $\alpha'_{\rho,\mathrm{r}}$ is $\alpha_{\rho,\mathrm{r}}$ without its last element.
Suppose that $s_1 \sim_{\text{PTe-tbt,dis}} s_2$, which implies that s_1 and s_2 have the same set R of actions labeling their outgoing transitions and $s_{1,\mathrm{r}} \sim_{\text{PTe-tbt,dis}} s_{2,\mathrm{r}}$. Then:

- For each $\mathscr{Z}_1 \in Res(s_1)$ there exists $\mathscr{Z}_2 \in Res(s_2)$ such that for all ready traces $\rho = (a_1, R_1) \dots (a_n, R_n) \in (A \times 2^A)^*$ either:
$$prob(\mathscr{RTCC}(z_{s_1}, \rho)) = 0 = prob(\mathscr{RTCC}(z_{s_2}, \rho))$$
or:
$$\begin{aligned} prob(\mathscr{RTCC}(z_{s_1}, \rho)) &= prob(\mathscr{SCC}(z_{s_{1,\mathrm{r}},o}, \alpha_{\rho,\mathrm{r}})) / p_{\alpha'_{\rho,\mathrm{r}}, R_n} = \\ &= prob(\mathscr{SCC}(z_{s_{2,\mathrm{r}},o}, \alpha_{\rho,\mathrm{r}})) / p_{\alpha'_{\rho,\mathrm{r}}, R_n} = prob(\mathscr{RTCC}(z_{s_2}, \rho)) \end{aligned}$$
- Symmetrically for each $\mathscr{Z}_2 \in Res(s_2)$.

This means that $s_1 \sim_{\text{PRTr,dis}} s_2$.

4. Let us prove the contrapositive of $s_1 \sim_{\text{PF}} s_2 \Longrightarrow s_1 \sim_{\text{PTe-tbt}} s_2$. Suppose that $s_1 \not\sim_{\text{PTe-tbt}} s_2$. This means that there exist an NPT $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$ with initial state $o \in O$, a trace $\alpha \in A^*$, and, say, a resolution $\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1, o)$ such that $Res_{\max,\mathscr{C},\alpha}(s_2, o) = \emptyset$ or for all $\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2, o)$ it holds that:
$$prob(\mathscr{SCC}(z_{s_1,o}, \alpha)) \neq prob(\mathscr{SCC}(z_{s_2,o}, \alpha))$$
Observing that $Res_{\max,\mathscr{C},\alpha}(s_1, o) \neq \emptyset$, in the case that $Res_{\max,\mathscr{C},\alpha}(s_2, o) = \emptyset$ either s_2 cannot perform α at all – let $\varphi = (\alpha, \emptyset)$ – or, after performing α, the states reached by s_2 can always synchronize with the states reached by o on a set F of actions whereas the states reached by s_1 cannot – let $\varphi = (\alpha, F)$. The failure pair φ shows that $s_1 \not\sim_{\text{PF}} s_2$ in this case because, denoting by $\mathscr{Z}'_1$ the element of $Res(s_1)$ that originates $\mathscr{Z}_1$, we have that for all $\mathscr{Z}'_2 \in Res(s_2)$:
$$prob(\mathscr{FCC}(z'_{s_1}, \varphi)) > 0 = prob(\mathscr{FCC}(z'_{s_2}, \varphi))$$
In the case that $Res_{\max,\mathscr{C},\alpha}(s_2, o) \neq \emptyset$, the failure pair $\varphi = (\alpha, \emptyset)$ shows that $s_1 \not\sim_{\text{PF}} s_2$. In fact, without loss of generality we can assume that the only α-compatible computations in $\mathscr{T}$ are the ones exercised by $\mathscr{Z}_1$ – note that they must belong to the same element $\mathscr{Z}$ of $Res(o)$ – as the only effect of this assumption is that of possibly reducing the number of resolutions in $Res_{\max,\mathscr{C},\alpha}(s_2, o)$. At least one of these computations must be successful – and hence maximal – in $\mathscr{T}$ because otherwise the success probabilities of the considered resolutions would all be equal to 0. Denoting by $\mathscr{Z}'_1$ the element of $Res(s_1)$ that originates $\mathscr{Z}_1$, we then have that for all $\mathscr{Z}'_2 \in Res(s_2)$ originating some $\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2, o)$:
$$\begin{aligned} prob(\mathscr{FCC}(z'_{s_1}, \varphi)) &= prob(\mathscr{SCC}(z_{s_1,o}, \alpha)) / p \neq \\ &\neq prob(\mathscr{SCC}(z_{s_2,o}, \alpha)) / p = prob(\mathscr{FCC}(z'_{s_2}, \varphi)) \end{aligned}$$
where p is the probability of performing the α-compatible computations in the only element $\mathscr{Z}$ of $Res(o)$ that originates $\mathscr{Z}_1$ and all the resolutions $\mathscr{Z}_2$.

Suppose now that $s_1 \sim_{\text{PTe-tbt}} s_2$. Then, in particular, for every $\alpha \in A^*$ and NPT $\mathcal{T}_\alpha = (O, A, \longrightarrow_{\mathcal{T}_\alpha})$ with initial state $o \in O$ having a single maximal α-compatible computation that reaches success, it holds that:

- For each $\mathcal{Z}_1 \in Res_{\max,\mathcal{C},\alpha}(s_1, o)$ there exists $\mathcal{Z}_2 \in Res_{\max,\mathcal{C},\alpha}(s_2, o)$ such that: $$prob(\mathcal{SCC}(z_{s_1,o}, \alpha)) = prob(\mathcal{SCC}(z_{s_2,o}, \alpha))$$
- For each $\mathcal{Z}_2 \in Res_{\max,\mathcal{C},\alpha}(s_2, o)$ there exists $\mathcal{Z}_1 \in Res_{\max,\mathcal{C},\alpha}(s_1, o)$ such that: $$prob(\mathcal{SCC}(z_{s_2,o}, \alpha)) = prob(\mathcal{SCC}(z_{s_1,o}, \alpha))$$

Since for all $s \in S$, $\mathcal{Z} \in Res_{\max,\mathcal{C},\alpha}(s, o)$, and $\mathcal{Z}' \in Res(s)$ originating $\mathcal{Z}$ in the interaction with $\mathcal{T}_\alpha$ it holds that:
$$prob(\mathcal{SCC}(z_{s,o}, \alpha)) = prob(\mathcal{CC}(z'_s, \alpha))$$
due to the structure of $\mathcal{T}_\alpha$, we immediately derive that for all $\alpha \in A^*$:

- For each $\mathcal{Z}'_1 \in Res(s_1)$ there exists $\mathcal{Z}'_2 \in Res(s_2)$ such that: $$prob(\mathcal{CC}(z'_{s_1}, \alpha)) = prob(\mathcal{CC}(z'_{s_2}, \alpha))$$
- For each $\mathcal{Z}'_2 \in Res(s_2)$ there exists $\mathcal{Z}'_1 \in Res(s_1)$ such that: $$prob(\mathcal{CC}(z'_{s_2}, \alpha)) = prob(\mathcal{CC}(z'_{s_1}, \alpha))$$

This means that $s_1 \sim_{\text{PTr}} s_2$.

5. Let us prove the contrapositive of $s_1 \sim_{\text{PF},\sqcup\sqcap} s_2 \Longrightarrow s_1 \sim_{\text{PTe-tbt},\sqcup\sqcap} s_2$. Suppose that $s_1 \not\sim_{\text{PTe-tbt},\sqcup\sqcap} s_2$. This means that there exist an NPT $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and a trace $\alpha \in A^*$ such that, for instance, $Res_{\max,\mathcal{C},\alpha}(s_1, o) \neq \emptyset$ and:
$$\bigsqcup_{\mathcal{Z}_1 \in Res_{\max,\mathcal{C},\alpha}(s_1,o)} prob(\mathcal{SCC}(z_{s_1,o}, \alpha)) > \bigsqcup_{\mathcal{Z}_2 \in Res_{\max,\mathcal{C},\alpha}(s_2,o)} prob(\mathcal{SCC}(z_{s_2,o}, \alpha))$$
Since $Res_{\max,\mathcal{C},\alpha}(s_1, o) \neq \emptyset$, in the case that $Res_{\max,\mathcal{C},\alpha}(s_2, o) = \emptyset$ either s_2 cannot perform α at all – let $\varphi = (\alpha, \emptyset)$ – or, after performing α, the states reached by s_2 can always synchronize with the states reached by o on a set F of actions whereas the states reached by s_1 cannot – let $\varphi = (\alpha, F)$. The failure pair φ shows that $s_1 \not\sim_{\text{PF},\sqcup\sqcap} s_2$ in this case because, denoting by $\mathcal{Z}'_1$ the element of $Res_\alpha(s_1)$ that originates some $\mathcal{Z}_1 \in Res_{\max,\mathcal{C},\alpha}(s_1, o)$, we have that:
$$\bigsqcup_{\mathcal{Z}'_1 \in Res_\alpha(s_1)} prob(\mathcal{FCC}(z'_{s_1}, \varphi)) > 0 = \bigsqcup_{\mathcal{Z}'_2 \in Res_\alpha(s_2)} prob(\mathcal{FCC}(z'_{s_2}, \varphi))$$
In the case that $Res_{\max,\mathcal{C},\alpha}(s_2, o) \neq \emptyset$, the failure pair $\varphi = (\alpha, \emptyset)$ shows that $s_1 \not\sim_{\text{PF},\sqcup\sqcap} s_2$. In fact, without loss of generality we can assume that the only α-compatible computations in $\mathcal{T}$ are the ones resulting in the supremum of the success probabilities of the α-compatible computations from (s_1, o) – note that they must belong to the same element $\mathcal{Z}$ of $Res(o)$ – as this assumption has no effect on the relationship between the two suprema. At least one of these computations must be successful – and hence maximal – in $\mathcal{T}$ because otherwise the success probabilities of the considered resolutions would all be equal to 0. Denoting by $\mathcal{Z}'_1$ the element of $Res_\alpha(s_1)$ that originates some $\mathcal{Z}_1 \in Res_{\max,\mathcal{C},\alpha}(s_1, o)$ and by $\mathcal{Z}'_2$ the element of $Res_\alpha(s_2)$ that originates some $\mathcal{Z}_2 \in Res_{\max,\mathcal{C},\alpha}(s_2, o)$, we then have that:
$$\begin{aligned}
\bigsqcup_{\mathcal{Z}'_1 \in Res_\alpha(s_1)} prob(\mathcal{FCC}(z'_{s_1}, \varphi)) &= \bigsqcup_{\mathcal{Z}_1 \in Res_{\max,\mathcal{C},\alpha}(s_1,o)} prob(\mathcal{SCC}(z_{s_1,o}, \alpha))/p \\
&> \bigsqcup_{\mathcal{Z}_2 \in Res_{\max,\mathcal{C},\alpha}(s_2,o)} prob(\mathcal{SCC}(z_{s_2,o}, \alpha))/p \\
&= \bigsqcup_{\mathcal{Z}'_2 \in Res_\alpha(s_2)} prob(\mathcal{FCC}(z'_{s_2}, \varphi))
\end{aligned}$$
where p is the probability of performing the α-compatible computations in the only element $\mathcal{Z}$

of $Res(o)$ that originates all the resolutions $\mathscr{Z}_1$ and $\mathscr{Z}_2$.
Suppose now that $s_1 \sim_{\text{PTe-tbt},\sqcup\sqcap} s_2$. Then, in particular, for every $\alpha \in A^*$ and NPT $\mathscr{T}_\alpha = (O, A, \longrightarrow_{\mathscr{T}_\alpha})$ with initial state $o \in O$ having a single maximal α-compatible computation that reaches success, it holds that $Res_{\max,\mathscr{C},\alpha}(s_1, o) \neq \emptyset$ iff $Res_{\max,\mathscr{C},\alpha}(s_2, o) \neq \emptyset$ and:

$$\bigsqcup_{\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)} prob(\mathscr{SCC}(z_{s_1,o}, \alpha)) = \bigsqcup_{\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2,o)} prob(\mathscr{SCC}(z_{s_2,o}, \alpha))$$
$$\bigsqcap_{\mathscr{Z}_1 \in Res_{\max,\mathscr{C},\alpha}(s_1,o)} prob(\mathscr{SCC}(z_{s_1,o}, \alpha)) = \bigsqcap_{\mathscr{Z}_2 \in Res_{\max,\mathscr{C},\alpha}(s_2,o)} prob(\mathscr{SCC}(z_{s_2,o}, \alpha))$$

Since for all $s \in S$, $\mathscr{Z} \in Res_{\max,\mathscr{C},\alpha}(s, o)$, and $\mathscr{Z}' \in Res_\alpha(s)$ originating $\mathscr{Z}$ in the interaction with $\mathscr{T}_\alpha$ it holds that:

$$prob(\mathscr{SCC}(z_{s,o}, \alpha)) = prob(\mathscr{CC}(z'_s, \alpha))$$

due to the structure of $\mathscr{T}_\alpha$, we immediately derive that for all $\alpha \in A^*$:

$$\bigsqcup_{\mathscr{Z}'_1 \in Res_\alpha(s_1)} prob(\mathscr{CC}(z'_{s_1}, \alpha)) = \bigsqcup_{\mathscr{Z}'_2 \in Res_\alpha(s_2)} prob(\mathscr{CC}(z'_{s_2}, \alpha))$$
$$\bigsqcap_{\mathscr{Z}'_1 \in Res_\alpha(s_1)} prob(\mathscr{CC}(z'_{s_1}, \alpha)) = \bigsqcap_{\mathscr{Z}'_2 \in Res_\alpha(s_2)} prob(\mathscr{CC}(z'_{s_2}, \alpha))$$

This means that $s_1 \sim_{\text{PTr},\sqcup\sqcap} s_2$. ■

Proof of Thm. 5.4. Let $(S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$:

1. Suppose that the NPLTS is fully nondeterministic. We preliminarily recall from [16] (and adapt to the NPLTS setting) that $s_1 \sim_{\text{B,fnd}} s_2$ means that there exists an fnd-bisimulation $\mathscr{B}$ over S such that $(s_1, s_2) \in \mathscr{B}$. Let us denote by δ_s the Dirac distribution for $s \in S$, i.e., let $\delta_s(s) = 1$ and $\delta_s(s') = 0$ for all $s' \in S \setminus \{s\}$. A relation $\mathscr{B}$ over S is an fnd-bisimulation iff, whenever $(s'_1, s'_2) \in \mathscr{B}$, then:

   - For each $s'_1 \xrightarrow{a} \delta_{s''_1}$ there exists $s'_2 \xrightarrow{a} \delta_{s''_2}$ such that $(s''_1, s''_2) \in \mathscr{B}$.
   - For each $s'_2 \xrightarrow{a} \delta_{s''_2}$ there exists $s'_1 \xrightarrow{a} \delta_{s''_1}$ such that $(s''_1, s''_2) \in \mathscr{B}$.

   The property $\sim_{\text{PB,dis}} = \sim_{\text{PB}} = \sim_{\text{PB},\sqcup\sqcap} = \sim_{\text{B,fnd}}$ over the considered fully nondeterministic NPLTS is a straightforward consequence of the fact that, since in this model every transition can reach with probability greater than 0 only one state and hence only one class of any equivalence relation – which are thus reached with probability 1 – the reflexive, symmetric, and transitive closure of an fnd-bisimulation is trivially a probabilistic group-distribution bisimulation, a probabilistic bisimulation, and a probabilistic $\sqcup\sqcap$-bisimulation.

2. Suppose that the NPLTS is fully probabilistic. We preliminarily recall from [13] (and adapt to the NPLTS setting) that $s_1 \sim_{\text{B,fpr}} s_2$ means that there exists an fpr-bisimulation $\mathscr{B}$ over S such that $(s_1, s_2) \in \mathscr{B}$. An equivalence relation $\mathscr{B}$ over S is an fpr-bisimulation iff, whenever $(s'_1, s'_2) \in \mathscr{B}$, then for all equivalence classes $C \in S/\mathscr{B}$ it holds that for each $s'_1 \xrightarrow{a} \mathscr{D}_1$ there exists $s'_2 \xrightarrow{a} \mathscr{D}_2$ such that $\mathscr{D}_1(C) = \mathscr{D}_2(C)$.
   The property $\sim_{\text{PB,dis}} = \sim_{\text{PB}} = \sim_{\text{PB},\sqcup\sqcap} = \sim_{\text{B,fpr}}$ over the considered fully probabilistic NPLTS is a straightforward consequence of the fact that, since in this model every state has at most one outgoing transition, an fpr-bisimulation is trivially a probabilistic group-distribution bisimulation, a probabilistic bisimulation, and a probabilistic $\sqcup\sqcap$-bisimulation. ■

Proof of Thm. 5.5. Let $(S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$:

1. The fact that $s_1 \sim_{\mathrm{PB,dis}} s_2$ implies $s_1 \sim_{\mathrm{PB}} s_2$ is a straightforward consequence of the fact that a probabilistic group-distribution bisimulation is a probabilistic bisimulation, as can be easily seen by taking the same fully matching transitions considered in the group-distribution bisimulation.
   Suppose now that $s_1 \sim_{\mathrm{PB}} s_2$. This means that there exists a probabilistic bisimulation $\mathscr{B}$ over S such that $(s_1, s_2) \in \mathscr{B}$. In other words, whenever $(s'_1, s'_2) \in \mathscr{B}$, then for all $\mathscr{G} \in 2^{S/\mathscr{B}}$ it holds that:

   - For each $s'_1 \stackrel{a}{\longrightarrow} \mathscr{D}_1$ there exists $s'_2 \stackrel{a}{\longrightarrow} \mathscr{D}_2$ such that $\mathscr{D}_1(\bigcup \mathscr{G}) = \mathscr{D}_2(\bigcup \mathscr{G})$.
   - For each $s'_2 \stackrel{a}{\longrightarrow} \mathscr{D}_2$ there exists $s'_1 \stackrel{a}{\longrightarrow} \mathscr{D}_1$ such that $\mathscr{D}_2(\bigcup \mathscr{G}) = \mathscr{D}_1(\bigcup \mathscr{G})$.

   This is to say that, whenever $(s'_1, s'_2) \in \mathscr{B}$, then for all $\mathscr{G} \in 2^{S/\mathscr{B}}$ and $a \in A$ it holds that:

   - If $s'_1 \stackrel{a}{\longrightarrow}$, then $s'_2 \stackrel{a}{\longrightarrow}$ and $\bigcup_{s'_1 \stackrel{a}{\longrightarrow} \mathscr{D}_1} \{\mathscr{D}_1(\bigcup \mathscr{G})\} \subseteq \bigcup_{s'_2 \stackrel{a}{\longrightarrow} \mathscr{D}_2} \{\mathscr{D}_2(\bigcup \mathscr{G})\}$.
   - If $s'_2 \stackrel{a}{\longrightarrow}$, then $s'_1 \stackrel{a}{\longrightarrow}$ and $\bigcup_{s'_2 \stackrel{a}{\longrightarrow} \mathscr{D}_2} \{\mathscr{D}_2(\bigcup \mathscr{G})\} \subseteq \bigcup_{s'_1 \stackrel{a}{\longrightarrow} \mathscr{D}_1} \{\mathscr{D}_1(\bigcup \mathscr{G})\}$.

   Equivalently, $s'_1 \stackrel{a}{\longrightarrow}$ iff $s'_2 \stackrel{a}{\longrightarrow}$ and:
   $$\bigcup_{s'_1 \stackrel{a}{\longrightarrow} \mathscr{D}_1} \{\mathscr{D}_1(\bigcup \mathscr{G})\} = \bigcup_{s'_2 \stackrel{a}{\longrightarrow} \mathscr{D}_2} \{\mathscr{D}_2(\bigcup \mathscr{G})\}$$
   which implies:
   $$\begin{aligned} \bigsqcup_{s'_1 \stackrel{a}{\longrightarrow} \mathscr{D}_1} \mathscr{D}_1(\bigcup \mathscr{G}) &= \bigsqcup_{s'_2 \stackrel{a}{\longrightarrow} \mathscr{D}_2} \mathscr{D}_2(\bigcup \mathscr{G}) \\ \bigsqcap_{s'_1 \stackrel{a}{\longrightarrow} \mathscr{D}_1} \mathscr{D}_1(\bigcup \mathscr{G}) &= \bigsqcap_{s'_2 \stackrel{a}{\longrightarrow} \mathscr{D}_2} \mathscr{D}_2(\bigcup \mathscr{G}) \end{aligned}$$
   Therefore, $\mathscr{B}$ is also a probabilistic $\sqcup\sqcap$-bisimulation, i.e., $s_1 \sim_{\mathrm{PB},\sqcup\sqcap} s_2$.

2. Suppose that $s_1 \sim_{\mathrm{PB,dis}} s_2$. Notice that states related by $\sim_{\mathrm{PB,dis}}$ have the same set of actions labeling their outgoing transitions, and that states not enjoying this property are trivially distinguished by $\sim_{\mathrm{PTe\text{-}tbt,dis}}$. Consider an arbitrary NPT $\mathscr{T} = (O, A, \longrightarrow_{\mathscr{T}})$. Since $\sim_{\mathrm{PB,dis}}$ is a congruence with respect to parallel composition [28], for all $s'_1, s'_2 \in S$ such that $s'_1 \sim_{\mathrm{PB,dis}} s'_2$ and for all $o \in O$ it holds that $(s'_1, o) \sim_{\mathrm{PB,dis}} (s'_2, o)$ due to some probabilistic group-distribution bisimulation $\mathscr{B}$ over $S \times O$. Since configurations related by $\sim_{\mathrm{PB,dis}}$ have the same set of actions labeling their outgoing transitions, this induces projections of $\mathscr{B}$ that are fpr-bisimulations [13] over pairs of matching resolutions of the interaction system that are both maximal. As a consequence, whenever $((s'_1, o), (s'_2, o)) \in \mathscr{B}$, then:

   - For each $\mathscr{Z}_1 \in Res_{\max}(s'_1, o)$ there exists $\mathscr{Z}_2 \in Res_{\max}(s'_2, o)$ such that the equivalence relation $\mathscr{B}_{1,2}$ over $Z = Z_1 \cup Z_2$ corresponding to $\mathscr{B}$ projected onto $Z \times Z$ is an fpr-bisimulation, i.e., whenever $(z_{s''_1, o'}, z_{s''_2, o'}) \in \mathscr{B}_{1,2}$, then for each $z_{s''_1, o'} \stackrel{a}{\longrightarrow} \mathscr{D}_1$ there exists $z_{s''_2, o'} \stackrel{a}{\longrightarrow} \mathscr{D}_2$ such that for all equivalence classes $C \in Z/\mathscr{B}_{1,2}$ it holds that $\mathscr{D}_1(C) = \mathscr{D}_2(C)$.
   - Symmetrically for each $\mathscr{Z}_2 \in Res_{\max}(s'_2, o)$.

   In particular, it holds that:

   - For each $\mathscr{Z}_1 \in Res_{\max}(s'_1, o)$ there exists $\mathscr{Z}_2 \in Res_{\max}(s'_2, o)$ such that for all $\alpha \in A^*$ it holds that $\mathscr{CCC}(z_{s'_1, o}, \alpha) \neq \emptyset$ implies $\mathscr{CCC}(z_{s'_2, o}, \alpha) \neq \emptyset$.
   - Symmetrically for each $\mathscr{Z}_2 \in Res_{\max}(s'_2, o)$.

Given $s'_1,\ s'_2 \in S$ and $o \in O$ such that $((s'_1, o), (s'_2, o)) \in \mathscr{B}$, and given $\mathscr{Z}_1 \in Res_{\max}(s'_1, o)$ and $\mathscr{Z}_2 \in Res_{\max}(s'_2, o)$ such that $z_{s'_1,o}$ and $z_{s'_2,o}$ are related by one of the projections of $\mathscr{B}$, we prove that for all $\alpha \in A^*$ such that $\mathscr{CCC}(z_{s'_1,o}, \alpha) \neq \emptyset \neq \mathscr{CCC}(z_{s'_2,o}, \alpha)$ it holds that:

$$prob(\mathscr{SCC}(z_{s'_1,o}, \alpha)) = prob(\mathscr{SCC}(z_{s'_2,o}, \alpha))$$

by proceeding by induction on the length n of α:

- If $n = 0$, i.e., $\alpha = \varepsilon$, then:

$$prob(\mathscr{SCC}(z_{s'_1,o}, \alpha)) = prob(\mathscr{SCC}(z_{s'_2,o}, \alpha)) = \begin{cases} 1 & \text{if } o = \omega \\ 0 & \text{if } o \neq \omega \end{cases}$$

- Let $n \in \mathbb{N}_{>0}$ and suppose that the result holds for all traces of length $m = 0, \dots, n-1$ that label completed computations starting from pairs of states of Z related by one of the projections of $\mathscr{B}$. Assume that $\alpha = a\,\alpha'$. Given $s \in S$ and $\mathscr{Z} \in Res_{\max}(s, o)$ such that $\mathscr{CCC}(z_{s,o}, \alpha) \neq \emptyset$, it holds that, whenever $z_{s,o} \xrightarrow{a} \mathscr{D}$, then:

$$\begin{aligned} prob(\mathscr{SCC}(z_{s,o}, \alpha)) &= \sum_{z_{s',o'} \in Z} \mathscr{D}(z_{s',o'}) \cdot prob(\mathscr{SCC}(z_{s',o'}, \alpha')) \\ &= \sum_{[z_{s',o'}] \in Z/\mathscr{B}'} \mathscr{D}([z_{s',o'}]) \cdot prob(\mathscr{SCC}(z_{s',o'}, \alpha')) \end{aligned}$$

  where $\mathscr{B}'$ is a projection of $\mathscr{B}$ and the factorization of $prob(\mathscr{SCC}(z_{s',o'}, \alpha'))$ with respect to the specific representative $z_{s',o'}$ of the equivalence class $[z_{s',o'}]$ stems from the application of the induction hypothesis on α' to all states of that equivalence class. Since $z_{s'_1,o}$ and $z_{s'_2,o}$ are related by a projection $\mathscr{B}_{1,2}$ of $\mathscr{B}$, it follows that, whenever $z_{s'_1,o} \xrightarrow{a} \mathscr{D}_1$, then $z_{s'_2,o} \xrightarrow{a} \mathscr{D}_2$ and:

$$\begin{aligned} prob(\mathscr{SCC}(z_{s'_1,o}, \alpha)) &= \sum_{[z_{s',o'}] \in Z/\mathscr{B}_{1,2}} \mathscr{D}_1([z_{s',o'}]) \cdot prob(\mathscr{SCC}(z_{s',o'}, \alpha')) \\ &= \sum_{[z_{s',o'}] \in Z/\mathscr{B}_{1,2}} \mathscr{D}_2([z_{s',o'}]) \cdot prob(\mathscr{SCC}(z_{s',o'}, \alpha')) \\ &= prob(\mathscr{SCC}(z_{s'_2,o}, \alpha)) \end{aligned}$$

Therefore $s_1 \sim_{\text{PTe-tbt,dis}} s_2$. ■