Noninterference Analysis of Deterministically Timed Reversible Systems

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Abstract. Information flow theory aims at guaranteeing the absence of covert channels among different security levels. As for the verification of noninterference via equivalence checking, in nondeterministic and probabilistic settings weak bisimilarity is adequate only for forward-computing systems, while branching bisimilarity has turned out to be appropriate for reversible systems too. In this paper we investigate noninterference for deterministically timed systems based on the model of Moller and Tofts. After recasting a selection of noninterference properties via timed variants of weak and branching bisimilarities, we analyze their preservation and compositionality aspects, establish their taxonomy, and compare it with the nondeterministic taxonomy for (ir)reversible systems. We illustrate the adequacy of our proposal on real-time database transactions.

1 Introduction

The notion of noninterference was introduced in [34] to reason about the way in which illegitimate information flows can occur in multi-level security systems due to covert channels from high-level agents to low-level ones. Since the first definition, conceived for deterministic systems, a lot of work has been done to extend the approach to a variety of more expressive domains, such as nondeterministic systems, systems in which quantitative aspects like time and probability play a central role, and reversible systems; see, e.g., [26,3,44,35,65,58,9,6,4,37,25,23] and the references therein. Likewise, to verify information-flow security properties based on noninterference, several different approaches have been proposed ranging from the application of type theory [70] and abstract interpretation [30] to control flow and equivalence or model checking [27,45,5].

Noninterference guarantees that low-level agents cannot infer from their observations what high-level ones are doing. Regardless of its specific definition, noninterference is closely tied to the notion of behavioral equivalence [32] because, given a multi-level security system, the idea is to compare the system behavior with high-level actions being prevented and the system behavior with the same actions being hidden. A natural framework in which to study system behavior is given by process algebra [46]. In this setting, weak bisimilarity has been employed in [26] to reason formally about covert channels and illegitimate information flows as well as to study a classification of noninterference properties for nondeterministic forward-computing systems.

Noninterference analysis has been recently extended to reversible systems – which feature forward and backward computations – both in the nondeterministic setting [25] and in the probabilistic one [23]. Reversibility has started to gain attention in computing since it has been shown that it may achieve lower levels of energy consumption [40,10]. Its applications range from biochemical reaction modeling [54,55] and parallel discrete-event simulation [52,60] to robotics [43], wireless communications [61], fault-tolerant systems [19,66,41,64], program debugging [29,42], and distributed algorithms [68,13].

As shown in [25,23], noninterference properties based on weak bisimilarity are not adequate in a reversible context because they fail to detect information flows emerging when backward computations are triggered. A more appropriate semantics turns out to be branching bisimilarity [33] because it coincides with weak back-and-forth bisimilarity [21]. The latter behavioral equivalence requires systems to be able to mimic each other's behavior stepwise not only when performing actions in the standard forward direction, but also when undoing those actions in the backward direction. Formally, weak back-and-forth bisimilarity is defined on computation paths instead of states thus preserving not only causality but also history, as backward moves are constrained to take place along the same path followed in the forward direction even in the presence of concurrency.

In this paper we extend the approach of [25,23] to a deterministically timed setting, in which delays are fixed (as opposed to being subject to stochastic fluctuations), so as to address noninterference properties in a framework featuring nondeterminism, time, and reversibility. To accomplish this we move to a model combining nondeterminism and time inspired by [47,48], in which transitions are divided into action transitions, each labeled with an action, and timed transitions, each labeled with a positive natural number that expresses a delay. The reason for choosing – in the vast realm of timed process calculi [57,47,69,8,15,2,36,50,56,17,49,63] – this model in which time passing is orthogonal to action execution instead of a model in which action execution and time passing are integrated (see [11] for encodings between integrated-time and orthogonal-time calculi) is that the former naturally supports the definition of behavioral equivalences abstracting from unobservable actions [48] – which are necessary for noninterference analysis – whereas this is not the case in the latter.

Following [47] we build a process calculus featuring action prefix separated from delay prefix. As for behavioral equivalences, we adopt the weak timed bisimilarity of [48] and introduce a novel timed branching bisimilarity. By using these two equivalences we recast the noninterference properties of [26,28] for irreversible systems and the noninterference properties of [25] for reversible systems, respectively, to study their preservation and compositionality aspects as well as to provide a taxonomy similar to those in [26,25,23]. Reversibility comes into play by extending one of the results of [21] to our orthogonal-time model; we show that a timed variant of weak back-and-forth bisimilarity coincides with our timed branching bisimilarity.

This paper is organized as follows. In Section 2 we recall the orthogonal-time model of [47] along with various definitions of strong and weak bisimilarities for it and a process calculus interpreted on it. In Section 3 we recast in our timed framework a selection of noninterference properties taken from [26,28,25]. In Section 4 we study their preservation and compositionality characteristics as well as their taxonomy, which in Section 5 we relate to the nondeterministic taxonomy of [25]. In Section 6 we establish a connection with reversibility by introducing a weak timed back-and-forth bisimilarity and proving that it coincides with timed branching bisimilarity. In Section 7 we present a real-time database management system example to show the adequacy of our approach when dealing with information flows in reversible systems featuring nondeterminism and time. Finally, in Section 8 we provide some concluding remarks.

2 Background Definitions and Results

In this section we recall the timed model of [47] (Section 2.1) along with weak timed bisimilarity [48] and define timed branching bisimilarity (Section 2.2). Then we introduce a timed process language inspired by [47] through which we will express bisimulation-based information-flow security properties accounting for nondeterminism and time (Section 2.3).

2.1 Timed Labeled Transition Systems

To represent the behavior of a process featuring nondeterminism and time, we use a timed labeled transition system. This is a variant of a labeled transition system [39] whose transitions are labeled with actions or positive natural numbers expressing delays [47]. We assume that the action set $\mathcal{A}_\tau$ contains a set $\mathcal{A}$ of observable actions and a single action $\tau \notin \mathcal{A}$ representing unobservable actions.

Definition 1. *A* timed labeled transition system (TLTS) *is a triple* $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ *where* $\mathcal{S} \neq \emptyset$ *is an at most countable set of states,* $\mathcal{A}_\tau = \mathcal{A} \cup \{\tau\}$ *is a countable set of actions, and* $\longrightarrow \, = \, \longrightarrow_{\rm a} \cup \longrightarrow_{\rm t}$ *is the transition relation, with* $\longrightarrow_{\rm a} \subseteq \mathcal{S} \times \mathcal{A}_\tau \times \mathcal{S}$ *being the action transition relation whilst* $\longrightarrow_{\rm t} \subseteq \mathcal{S} \times \mathbb{N}_{>0} \times \mathcal{S}$ *being the timed transition relation.* ■

An action transition (s, a, s') is written $s \xrightarrow{a}_{\rm a} s'$ while a timed transition (s, t, s') is written $s \xrightarrow{t}_{\rm t} s'$, where s is the source state and s' is the target state. We say that s' is reachable from s, written $s' \in \mathit{reach}(s)$, iff $s' = s$ or there exists a sequence of finitely many transitions such that the target state of each of them coincides with the source state of the subsequent one, with the source of the first transition being s and the target of the last one being s'.

Following [47] we assume that timed transitions are subject to *time determinism*, i.e., every state has at most one outgoing timed transition, and *time additivity*, i.e., a timed transition can be split into a sequence of timed transitions whose overall duration is equal to the duration of the original transition, as well as a sequence of timed transitions can be merged into a single timed transition whose duration is equal to the sum of the durations of the original transitions.

As for the interplay between action transitions and timed ones, we assume *eagerness*, i.e., actions must be performed as soon as they become enabled without any delay, thereby implying that their execution is urgent. Moreover τ-transitions, which cannot be disabled by the environment where the system executes, take precedence over timed ones; this property is called *maximal progress*.

2.2 Bisimulation Equivalences

Bisimilarity [51,46] identifies processes mimicking each other's behavior stepwise, i.e., having the same branching structure. In our setting this extends to timed behavior [47]. Due to maximal progress, timed transitions are compared only in states s with no outgoing τ-transitions, which is denoted by $s \xrightarrow{\tau}\!\!\!\!\!\!/\ _{\mathrm{a}}$.

Definition 2. *Let $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ be a TLTS. We say that $s_1, s_2 \in \mathcal{S}$ are* strongly timed bisimilar*, written $s_1 \sim_{\mathrm{t}} s_2$, iff $(s_1, s_2) \in \mathcal{B}$ for some strong timed bisimulation $\mathcal{B}$. A symmetric relation $\mathcal{B}$ over $\mathcal{S}$ is a* strong timed bisimulation *iff, whenever $(s_1, s_2) \in \mathcal{B}$, then:*

- *For each $s_1 \xrightarrow{a}_{\mathrm{a}} s_1'$ there exists $s_2 \xrightarrow{a}_{\mathrm{a}} s_2'$ such that $(s_1', s_2') \in \mathcal{B}$.*
- *If $s_1 \xrightarrow{\tau}\!\!\!\!\!\!/\ _{\mathrm{a}}$, for each $s_1 \xrightarrow{t}_{\mathrm{t}} s_1'$ there exists $s_2 \xrightarrow{t}_{\mathrm{t}} s_2'$ such that $(s_1', s_2') \in \mathcal{B}$.* ■

Weak bisimilarity [46] is additionally capable of abstracting from unobservable actions. Let $\Longrightarrow_{\mathrm{a}}$ be the reflexive and transitive closure of $\xrightarrow{\tau}_{\mathrm{a}}$. Moreover let $\xRightarrow{\hat{a}}_{\mathrm{a}}$ stand for $\Longrightarrow_{\mathrm{a}}$ if $a = \tau$ or $\Longrightarrow_{\mathrm{a}} \xrightarrow{a}_{\mathrm{a}} \Longrightarrow_{\mathrm{a}}$ if $a \neq \tau$, while $\xRightarrow{t}_{\mathrm{t}}$ stands for $\Longrightarrow_{\mathrm{a}} \xrightarrow{t_1}_{\mathrm{t}} \Longrightarrow_{\mathrm{a}} \dots \Longrightarrow_{\mathrm{a}} \xrightarrow{t_n}_{\mathrm{t}} \Longrightarrow_{\mathrm{a}}$ where $\sum_{1 \leq i \leq n} t_i = t$ and every t_i-transition departs from a state with no outgoing τ-transitions. The weak timed bisimilarity below is taken from [48].

Definition 3. *Let $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ be a TLTS. We say that $s_1, s_2 \in \mathcal{S}$ are* weakly timed bisimilar*, written $s_1 \approx_{\mathrm{tw}} s_2$, iff $(s_1, s_2) \in \mathcal{B}$ for some weak timed bisimulation $\mathcal{B}$. A symmetric relation $\mathcal{B}$ over $\mathcal{S}$ is a* weak timed bisimulation *iff, whenever $(s_1, s_2) \in \mathcal{B}$, then:*

- *For each $s_1 \xrightarrow{a}_{\mathrm{a}} s_1'$ there exists $s_2 \xRightarrow{\hat{a}}_{\mathrm{a}} s_2'$ such that $(s_1', s_2') \in \mathcal{B}$.*
- *If $s_1 \xrightarrow{\tau}\!\!\!\!\!\!/\ _{\mathrm{a}}$ then there exists $s_2 \Longrightarrow_{\mathrm{a}} \bar{s}_2$ such that $\bar{s}_2 \xrightarrow{\tau}\!\!\!\!\!\!/\ _{\mathrm{a}}$, $(s_1, \bar{s}_2) \in \mathcal{B}$, and for each $s_1 \xrightarrow{t}_{\mathrm{t}} s_1'$ there exists $\bar{s}_2 \xRightarrow{t}_{\mathrm{t}} s_2'$ such that $(s_1', s_2') \in \mathcal{B}$.* ■

Branching bisimilarity [33] is finer than weak bisimilarity as it preserves the branching structure of processes even when abstracting from τ-actions – see condition $(s_1, \bar{s}_2) \in \mathcal{B}$ in the action transitions matching of the definition below. We adapt it to the timed setting as follows.

Definition 4. *Let $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ be a TLTS. We say that $s_1, s_2 \in \mathcal{S}$ are* timed branching bisimilar*, written $s_1 \approx_{\mathrm{tb}} s_2$, iff $(s_1, s_2) \in \mathcal{B}$ for some timed branching bisimulation $\mathcal{B}$. A symmetric relation $\mathcal{B}$ over $\mathcal{S}$ is a* timed branching bisimulation *iff, whenever $(s_1, s_2) \in \mathcal{B}$, then:*

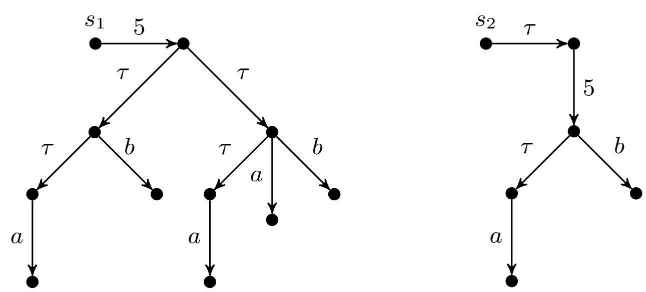

Fig. 1. States s_1 and s_2 are related by $\approx_{\rm tw}$ but distinguished by $\approx_{\rm tb}$

– *For each* $s_1 \xrightarrow{a}_{\rm a} s'_1$*:*
  - *either* $a = \tau$ *and* $(s'_1, s_2) \in \mathcal{B}$*;*
  - *or there exists* $s_2 \Longrightarrow_{\rm a} \bar{s}_2 \xrightarrow{a}_{\rm a} s'_2$ *such that* $(s_1, \bar{s}_2) \in \mathcal{B}$ *and* $(s'_1, s'_2) \in \mathcal{B}$*.*

– *If* $s_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ *then there exists* $s_2 \Longrightarrow_{\rm a} \bar{s}_2$ *such that* $\bar{s}_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, $(s_1, \bar{s}_2) \in \mathcal{B}$*, and for each* $s_1 \xrightarrow{t}_{\rm t} s'_1$ *there exists* $\bar{s}_2 \stackrel{t}{\Longrightarrow}_{\rm t} s'_2$ *such that* $(s'_1, s'_2) \in \mathcal{B}$*.* ■

It is worth noting that the clause for timed transitions in the two definitions above implies that the states along $\bar{s}_2 \stackrel{t}{\Longrightarrow}_{\rm t} s'_2$ that are connected by τ-transitions, for which time is not progressing, belong to the same equivalence class. This feature, which is a piecewise variant of the *stuttering property* for τ-computations of [33], is established by the following proposition.

Proposition 1. *Let* $s_1, s_2 \in \mathcal{S}$ *and* $\approx \in \{\approx_{\rm tw}, \approx_{\rm tb}\}$*. Suppose that* $s_1 \approx s_2$, $s_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, $s_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, $s_1 \xrightarrow{t}_{\rm t} s'_1$, $s_2 \xrightarrow{t_1}_{\rm t} s_{2,1} \Longrightarrow_{\rm a} \dots \Longrightarrow_{\rm a} s'_{2,n-1} \xrightarrow{t_n}_{\rm t} s_{2,n} \Longrightarrow_{\rm a} s'_2$, $\sum_{1 \leq i \leq n} t_i = t$*, and* $s'_1 \approx s'_2$*. Then* $s_{2,i} \approx s'_{2,i}$ *for all* $s_{2,i} \Longrightarrow_{\rm a} s'_{2,i}$*.* ■

It may be argued that the weak bisimilarity of Definition 3 is already very close to branching bisimilarity, because maximal progress forces a check on the branching structure of the considered processes. We show that our novel Definition 4, which sticks to the original one of [33], is more discriminating. Consider Figure 1, where every TLTS is depicted as a directed graph in which vertices represent states and action- or delay-labeled edges represent transitions. The initial states s_1 and s_2 of the two TLTSs are weakly timed bisimilar but not timed branching bisimilar. On the one hand, each of the two states reachable from s_1 after 5 time units and a τ-transition and the state reachable from s_2 after a τ-transition and 5 time units are all weakly timed bisimilar. On the other hand, the two states reachable from s_1 are not timed branching bisimilar, because if the one on the right performs a then the one on the left cannot respond by performing τ followed by a because the state reached after τ no longer enables b. Thus, with respect to timed branching bisimilarity, s_1 reaches two inequivalent states, while s_2 reaches only one of them.

Prefix	$a\,.\,P \xrightarrow{a}_{\mathrm{a}} P$	
Choice	$\dfrac{P_1 \xrightarrow{a}_{\mathrm{a}} P'_1}{P_1 + P_2 \xrightarrow{a}_{\mathrm{a}} P'_1}$	$\dfrac{P_2 \xrightarrow{a}_{\mathrm{a}} P'_2}{P_1 + P_2 \xrightarrow{a}_{\mathrm{a}} P'_2}$
Parallel	$\dfrac{P_1 \xrightarrow{a}_{\mathrm{a}} P'_1 \quad a \notin L}{P_1 \parallel_L P_2 \xrightarrow{a}_{\mathrm{a}} P'_1 \parallel_L P_2}$	$\dfrac{P_2 \xrightarrow{a}_{\mathrm{a}} P'_2 \quad a \notin L}{P_1 \parallel_L P_2 \xrightarrow{a}_{\mathrm{a}} P_1 \parallel_L P'_2}$
Synch	$\dfrac{P_1 \xrightarrow{a}_{\mathrm{a}} P'_1 \quad P_2 \xrightarrow{a}_{\mathrm{a}} P'_2 \quad a \in L}{P_1 \parallel_L P_2 \xrightarrow{a}_{\mathrm{a}} P'_1 \parallel_L P'_2}$	
Restriction	$\dfrac{P \xrightarrow{a}_{\mathrm{a}} P' \quad a \notin L}{P \setminus L \xrightarrow{a}_{\mathrm{a}} P' \setminus L}$	
Hiding	$\dfrac{P \xrightarrow{a}_{\mathrm{a}} P' \quad a \in L}{P\,/\,L \xrightarrow{\tau}_{\mathrm{a}} P'\,/\,L}$	$\dfrac{P \xrightarrow{a}_{\mathrm{a}} P' \quad a \notin L}{P\,/\,L \xrightarrow{a}_{\mathrm{a}} P'\,/\,L}$

Table 1. Operational semantic rules for action transitions

2.3 A Timed Process Calculus with High and Low Actions

We now introduce a timed process calculus to formalize the security properties of interest. To address two security levels, we partition the set $\mathcal{A}$ of observable actions into $\mathcal{A}_\mathcal{H} \cup \mathcal{A}_\mathcal{L}$, with $\mathcal{A}_\mathcal{H} \cap \mathcal{A}_\mathcal{L} = \emptyset$, where $\mathcal{A}_\mathcal{H}$ is the set of high-level actions, ranged over by h, and $\mathcal{A}_\mathcal{L}$ is the set of low-level actions, ranged over by l. Note that $\tau \notin \mathcal{A}_\mathcal{H} \cup \mathcal{A}_\mathcal{L}$.

The set $\mathbb{P}$ of process terms is obtained by considering typical operators from CCS [46] and CSP [16] together with delay prefix from [47]. In addition to prefix, choice, and parallel composition – taken from CSP so as not to turn synchronizations among high-level actions into τ as would happen with the CCS parallel composition – we include restriction and hiding as they are necessary to formalize noninterference properties. The syntax for $\mathbb{P}$ is:

$$P \;::=\; \underline{0} \mid a\,.\,P \mid (t)\,.\,P \mid P + P \mid P \parallel_L P \mid P \setminus L \mid P\,/\,L$$

where:

- $\underline{0}$ is the terminated process.
- $a\,.\,\_$, for $a \in \mathcal{A}_\tau$, is the action prefix operator describing a process that can initially perform action a.
- $(t)\,.\,\_$, for $t \in \mathbb{N}_{>0}$, is the delay prefix operator describing a process that can initially let t time units pass.
- $\_ + \_$ is the alternative composition operator expressing a choice between two processes, which is nondeterministic in case of actions, governed by time determinism in case of delays [47], or subject to maximal progress otherwise.
- $\_ \parallel_L \_$, for $L \subseteq \mathcal{A}$, is the parallel composition operator allowing two processes to proceed independently on any action not in L and forcing them to synchronize on every action in L and on delays due to time determinism [47].
- $\_ \setminus L$, for $L \subseteq \mathcal{A}$, is the restriction operator, which prevents the execution of all actions belonging to L.

TimedPrefix	$(t)\,.\,P \xrightarrow{t}_{\mathrm{t}} P$
TimedSplit	$\dfrac{t = t_1 + t_2 \quad t_1, t_2 \in \mathbb{N}_{>0}}{(t)\,.\,P \xrightarrow{t_1}_{\mathrm{t}} (t_2)\,.\,P}$
TimedMerge	$\dfrac{P \xrightarrow{t_2}_{\mathrm{t}} P' \quad t = t_1 + t_2}{(t_1)\,.\,P \xrightarrow{t}_{\mathrm{t}} P'}$
TimedChoice	$\dfrac{P_1 \xrightarrow{t}_{\mathrm{t}} P'_1 \quad P_2 \xrightarrow{t}_{\mathrm{t}} P'_2}{P_1 + P_2 \xrightarrow{t}_{\mathrm{t}} P'_1 + P'_2}$
TimedSynch	$\dfrac{P_1 \xrightarrow{t}_{\mathrm{t}} P'_1 \quad P_2 \xrightarrow{t}_{\mathrm{t}} P'_2}{P_1 \,\|_L\, P_2 \xrightarrow{t}_{\mathrm{t}} P'_1 \,\|_L\, P'_2}$
TimedRestriction	$\dfrac{P \xrightarrow{t}_{\mathrm{t}} P'}{P \setminus L \xrightarrow{t}_{\mathrm{t}} P' \setminus L}$
TimedHiding	$\dfrac{P \xrightarrow{t}_{\mathrm{t}} P'}{P \,/\, L \xrightarrow{t}_{\mathrm{t}} P' \,/\, L}$

Table 2. Operational semantic rules for timed transitions

- $\_ \,/\, L$, for $L \subseteq \mathcal{A}$, is the hiding operator, which turns all the executed actions belonging to L into the unobservable action τ.

The operational semantic rules for the process language are shown in Tables 1 and 2 for action and timed transitions respectively. Together they produce the TLTS $(\mathbb{P}, \mathcal{A}_\tau, \longrightarrow)$ where $\longrightarrow \; = \; \longrightarrow_{\mathrm{a}} \cup \longrightarrow_{\mathrm{t}}$, with $\longrightarrow_{\mathrm{a}} \subseteq \mathbb{P} \times \mathcal{A}_\tau \times \mathbb{P}$ and $\longrightarrow_{\mathrm{t}} \subseteq \mathbb{P} \times \mathbb{N}_{>0} \times \mathbb{P}$, to which the bisimulation equivalences defined in Section 2.2 are applicable. Following [47], rules *TimedSplit* and *TimedMerge* implement time additivity, while rules *TimedChoice* and *TimedSynch* implement time determinism, according to which time does not solve choices and does not decide which subprocess advances in a parallel composition.

3 Timed Information-Flow Security Properties

The intuition behind noninterference in a two-level security system is that, if a group of agents at the high level performs some actions, the effect of those actions should not be seen by any agent at the low level. To formalize this, the restriction and hiding operators play a central role.

In this section we recast the noninteference properties defined in [26,28,25] – *Nondeterministic Non-Interference* (NNI) and *Non-Deducibility on Composition* (NDC) – by taking as behavioral equivalence the weak or branching bisimilarity of Section 2.2. In the acronyms of the following variants of NNI and NDC properties, B stands for bisimulation-based, S stands for strong, and P stands for persistent.

Definition 5. *Let* $P \in \mathbb{P}$ *and* $\approx \in \{\approx_{\rm tw}, \approx_{\rm tb}\}$*:*

- $P \in \mathrm{BSNNI}_{\approx} \iff P \setminus \mathcal{A_H} \approx P \,/\, \mathcal{A_H}$.
- $P \in \mathrm{BNDC}_{\approx} \iff$ *for all* $Q \in \mathbb{P}$ *such that all of its action prefixes belongs to* $\mathcal{A_H}$ *whilst its timed prefixes match the ones in* P *and for all* $L \subseteq \mathcal{A_H}$*,* $P \setminus \mathcal{A_H} \approx ((P \,\|_L\, Q) \,/\, L) \setminus \mathcal{A_H}$.
- $P \in \mathrm{SBSNNI}_{\approx} \iff$ *for all* $P' \in \mathit{reach}(P)$*,* $P' \in \mathrm{BSNNI}_{\approx}$.
- $P \in \mathrm{P\_BNDC}_{\approx} \iff$ *for all* $P' \in \mathit{reach}(P)$*,* $P' \in \mathrm{BNDC}_{\approx}$.
- $P \in \mathrm{SBNDC}_{\approx} \iff$ *for all* $P', P'' \in \mathit{reach}(P)$ *such that* $P' \xrightarrow{h}_{\rm a} P''$*,* $P' \setminus \mathcal{A_H} \approx P'' \setminus \mathcal{A_H}$. ■

Bisimulation-based Strong Nondeterministic Non-Interference (BSNNI) has been one of the first and most intuitive proposals. Basically, it is satisfied by any process P that behaves the same when its high-level actions are prevented (as modeled by $P \setminus \mathcal{A_H}$) or when they are considered as hidden, unobservable actions (as modeled by $P \,/\, \mathcal{A_H}$). The equivalence between these two low-level views of P states that a low-level agent cannot deduce the high-level behavior of the system. For instance, in our timed setting, a low-level agent that observes the execution of l in $(t) \,.\, l \,.\, \underline{0} + h \,.\, h \,.\, (t) \,.\, l \,.\, \underline{0}$. cannot infer anything about the execution of h. Indeed, a low-level user always observes the execution of l after a delay of t units of time. Formally, $P \setminus \{h\} \approx P \,/\, \{h\}$ because $(t) \,.\, l \,.\, \underline{0} \approx (t) \,.\, l \,.\, \underline{0} + \tau \,.\, \tau \,.\, (t) \,.\, l \,.\, \underline{0}$.

$\mathrm{BSNNI}_{\approx}$ is not powerful enough to capture covert channels that derive from the behavior of a high-level agent interacting with the system. For instance, $(t) \,.\, l \,.\, \underline{0} + h_1 \,.\, h_2 \,.\, (t) \,.\, l \,.\, \underline{0}$ is $\mathrm{BSNNI}_{\approx}$ for the same reason discussed above. However, a high-level agent could decide to enable only h_1, thus yielding the low-level view of the system $(t) \,.\, l \,.\, \underline{0} + \tau \,.\, \underline{0}$, which is clearly distinguishable from $(t) \,.\, l \,.\, \underline{0}$ as the former is forced, due to maximal progress, to perform τ and reach a terminal state, while the latter can let t units of time pass and then perform l. To avoid such a limitation, the most obvious solution consists of checking explicitly the interaction on any action set $L \subseteq \mathcal{A_H}$ between the system and every possible high-level agent Q. The resulting property is the *Bisimulation-based Non-Deducibility on Composition* (BNDC), which features a universal quantification over Q containing only high-level actions.

Note that in our timed setting the high-level agent Q must allow the same amount of time as P to pass, otherwise the property BNDC would never be satisfied. To see why, consider the trivially safe process $(1) \,.\, l \,.\, \underline{0}$ and the high-level agent $h \,.\, \underline{0}$. The processes $((1) \,.\, l \,.\, \underline{0}) \setminus \mathcal{A_H}$ and $(((1) \,.\, l \,.\, \underline{0} \,\|_L\, h \,.\, \underline{0}) \,/\, L) \setminus \mathcal{A_H}$ are not equivalent, regardless of the specific $L \subseteq \mathcal{A_H}$ chosen, because the former can let time pass, while the latter cannot, as it is blocked by the process $h \,.\, \underline{0}$.

To overcome the verification problems related to the quantification over Q, several properties have been proposed that are stronger than BNDC. They all express some persistency conditions, stating that the security checks have to be extended to all the processes reachable from a secure one. Three of the most representative ones among such properties are the variant of BSNNI that requires every reachable process to satisfy BSNNI itself, called *Strong* BSNNI (SBSNNI), the variant of BNDC that requires every reachable process to satisfy BNDC itself, called *Persistent* BNDC (P_BNDC), and *Strong* BNDC (SBNDC), which requires the low-level view of every reachable process to be the same before and after the execution of any high-level action, meaning that the execution of high-level actions must be completely transparent to low-level agents. In the nondeterministic and probabilistic settings, P_BNDC and SBSNNI have been proven to coincide in the case of both weak bisimilarity and branching bisimilarity [28,25,23].

4 Characteristics of Timed Security Properties

In this section we investigate preservation and compositionality characteristics of the noninterference properties introduced in the previous section (Section 4.1) as well as the inclusion relationships between the ones based on $\approx_{\rm tw}$ and the ones based on $\approx_{\rm tb}$ (Section 4.2).

4.1 Preservation and Compositionality

All the timed noninterference properties of Definition 5 turn out to be preserved by the bisimilarity employed in their definition. This means that if a process P_1 is secure under any of such properties, then every other equivalent process P_2 is secure too according to the same property. This is very useful for automated property verification, as it allows us to work with the process with the smallest state space among the equivalent ones.

These results immediately follow from the next lemma, which states that $\approx_{\rm tw}$ and $\approx_{\rm tb}$ are congruences with respect to action prefix, delay prefix, parallel composition, restriction, and hiding. Some of these results were already proven in [48] for weak timed bisimilarity. Here we extend those results to the operators of our calculus as well as to timed branching bisimilarity.

Lemma 1. *Let $P_1, P_2 \in \mathbb{P}$ and $\approx \in \{\approx_{\rm tw}, \approx_{\rm tb}\}$. If $P_1 \approx P_2$ then:*

1. $a \,.\, P_1 \approx a \,.\, P_2$ *for all* $a \in \mathcal{A}_\tau$.
2. $(t) \,.\, P_1 \approx (t) \,.\, P_2$ *for all* $t \in \mathbb{N}_{>0}$.
3. $P_1 \,\|_L\, P \approx P_2 \,\|_L\, P$ *and* $P \,\|_L\, P_1 \approx P \,\|_L\, P_2$ *for all* $L \subseteq \mathcal{A}$ *and* $P \in \mathbb{P}$.
4. $P_1 \setminus L \approx P_2 \setminus L$ *for all* $L \subseteq \mathcal{A}$.
5. $P_1 \,/\, L \approx P_2 \,/\, L$ *for all* $L \subseteq \mathcal{A}$. ■

Theorem 1. *Let $P_1, P_2 \in \mathbb{P}$, $\approx \in \{\approx_{\rm tw}, \approx_{\rm tb}\}$, and $\mathcal{P} \in \{\mathrm{BSNNI}_\approx, \mathrm{BNDC}_\approx, \mathrm{SBSNNI}_\approx, \mathrm{P\_BNDC}_\approx, \mathrm{SBNDC}_\approx\}$. If $P_1 \approx P_2$ then $P_1 \in \mathcal{P} \iff P_2 \in \mathcal{P}$.* ■

As far as modular verification is concerned, like in the nondeterministic and probabilistic settings [26,25,23] only the local properties $\mathrm{SBSNNI}_\approx$, $\mathrm{P\_BNDC}_\approx$, and $\mathrm{SBNDC}_\approx$ are compositional, i.e., are preserved by some operators of the calculus in certain circumstances. Moreover, similar to [25,23] compositionality with respect to parallel composition is limited, for $\mathrm{SBSNNI}_{\approx_{\rm tb}}$ and $\mathrm{P\_BNDC}_{\approx_{\rm tb}}$, to the case in which synchronizations can take place only among low-level actions, i.e., $L \subseteq \mathcal{A}_{\mathcal{L}}$. A limitation to low-level actions applies to action prefix and hiding as well, whilst this is not the case for restriction. Another analogy with the nondeterministic and probabilistic settings [26,25,23] is that none of the considered noninterference properties is compositional with respect to alternative composition. As an example, let us examine processes $P_1 = l\,.\,\underline{0}$ and $P_2 = h\,.\,\underline{0}$. Both processes are $\mathrm{BSNNI}_{\approx}$, as $(l\,.\,\underline{0}) \setminus \{h\} \approx (l\,.\,\underline{0}) \,/\, \{h\}$ and $(h\,.\,\underline{0}) \setminus \{h\} \approx (h\,.\,\underline{0}) \,/\, \{h\}$, but $P_1 + P_2 \notin \mathrm{BSNNI}_{\approx}$, because $(l\,.\,\underline{0} + h\,.\,\underline{0}) \setminus \{h\} \approx l\,.\,\underline{0} \not\approx l\,.\,\underline{0} + \tau\,.\,\underline{0} \approx (l\,.\,\underline{0} + h\,.\,\underline{0}) \,/\, \{h\}$. It is easy to check that $P_1 + P_2 \notin \mathcal{P}$ also for $\mathcal{P} \in \{\mathrm{BNDC}_{\approx}, \mathrm{SBSNNI}_{\approx}, \mathrm{SBNDC}_{\approx}\}$.

Theorem 2. *Let* $P, P_1, P_2 \in \mathbb{P}$, $\approx \in \{\approx_{\mathrm{tw}}, \approx_{\mathrm{tb}}\}$, $\mathcal{P} \in \{\mathrm{SBSNNI}_{\approx}, \mathrm{P\_BNDC}_{\approx}, \mathrm{SBNDC}_{\approx}\}$. *Then:*

1. $P \in \mathcal{P} \Longrightarrow a\,.\,P \in \mathcal{P}$ *for all* $a \in \mathcal{A}_{\mathcal{L}} \cup \{\tau\}$.
2. $P \in \mathcal{P} \Longrightarrow (t)\,.\,P \in \mathcal{P}$ *for all* $t \in \mathbb{N}_{>0}$.
3. $P_1, P_2 \in \mathcal{P} \Longrightarrow P_1 \|_L P_2 \in \mathcal{P}$ *for all* $L \subseteq \mathcal{A}_{\mathcal{L}}$ *if* $\mathcal{P} \in \{\mathrm{SBSNNI}_{\approx_{\mathrm{tb}}}, \mathrm{P\_BNDC}_{\approx_{\mathrm{tb}}}\}$ *or for all* $L \subseteq \mathcal{A}$ *if* $\mathcal{P} \in \{\mathrm{SBSNNI}_{\approx_{\mathrm{tw}}}, \mathrm{P\_BNDC}_{\approx_{\mathrm{tw}}}, \mathrm{SBNDC}_{\approx_{\mathrm{tw}}}, \mathrm{SBNDC}_{\approx_{\mathrm{tb}}}\}$.
4. $P \in \mathcal{P} \Longrightarrow P \setminus L \in \mathcal{P}$ *for all* $L \subseteq \mathcal{A}$.
5. $P \in \mathcal{P} \Longrightarrow P \,/\, L \in \mathcal{P}$ *for all* $L \subseteq \mathcal{A}_{\mathcal{L}}$. ■

As for the limitation to $L \subseteq \mathcal{A}_{\mathcal{L}}$ for parallel composition under $\mathrm{SBSNNI}_{\approx_{\mathrm{tb}}}$, for example both $P_1 = h\,.\,\underline{0} + l_1\,.\,\underline{0} + \tau\,.\,\underline{0}$ and $P_2 = h\,.\,\underline{0} + l_2\,.\,\underline{0} + \tau\,.\,\underline{0}$ are $\mathrm{SBSNNI}_{\approx_{\mathrm{tb}}}$, but $P_1 \|_{\{h\}} P_2$ is not because the transition $(P_1 \|_{\{h\}} P_2) \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} (\underline{0} \|_{\{h\}} \underline{0}) \,/\, \mathcal{A}_{\mathcal{H}}$ arising from the synchronization between the two h-actions cannot be matched by $(P_1 \|_{\{h\}} P_2) \setminus \mathcal{A}_{\mathcal{H}}$ in the timed branching bisimulation game. Indeed, the only two possibilities are $(P_1 \|_{\{h\}} P_2) \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow_{\mathrm{a}} (P_1 \|_{\{h\}} P_2) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} (\underline{0} \|_{\{h\}} P_2) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} (\underline{0} \|_{\{h\}} \underline{0}) \setminus \mathcal{A}_{\mathcal{H}}$ and $(P_1 \|_{\{h\}} P_2) \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow_{\mathrm{a}} (P_1 \|_{\{h\}} P_2) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} (P_1 \|_{\{h\}} \underline{0}) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} (\underline{0} \|_{\{h\}} \underline{0}) \setminus \mathcal{A}_{\mathcal{H}}$. However, neither $(\underline{0} \|_{\{h\}} P_2) \setminus \mathcal{A}_{\mathcal{H}}$ nor $(P_1 \|_{\{h\}} \underline{0}) \setminus \mathcal{A}_{\mathcal{H}}$ is timed branching bisimilar to $(P_1 \|_{\{h\}} P_2) \setminus \mathcal{A}_{\mathcal{H}}$ when $l_1 \neq l_2$. Note that $(P_1 \|_{\{h\}} P_2) \,/\, \mathcal{A}_{\mathcal{H}} \approx (P_1 \|_{\{h\}} P_2) \setminus \mathcal{A}_{\mathcal{H}}$ because $(P_1 \|_{\{h\}} P_2) \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} (\underline{0} \|_{\{h\}} \underline{0}) \,/\, \mathcal{A}_{\mathcal{H}}$ is matched by $(P_1 \|_{\{h\}} P_2) \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow_{\mathrm{a}} (\underline{0} \|_{\{h\}} \underline{0}) \setminus \mathcal{A}_{\mathcal{H}}$. Similar to [25,23], it is not only a matter of the higher discriminating power of $\approx_{\mathrm{tb}}$ with respect to $\approx_{\mathrm{tw}}$. If we used the CCS parallel composition operator [46], which turns into τ the synchronization of two actions thus combining communication with hiding, then the parallel composition of P_1 and P_2 with restriction on $\mathcal{A}_{\mathcal{H}}$ would be able to respond with a single τ-transition reaching the parallel composition of $\underline{0}$ and $\underline{0}$ with restriction on $\mathcal{A}_{\mathcal{H}}$.

4.2 Taxonomy of Security Properties

First of all, similar to the nondeterministic and probabilistic settings [26,25,23] the properties in Definition 5 turn out to be increasingly finer. This result holds for both those based on $\approx_{\mathrm{tw}}$ and those based on $\approx_{\mathrm{tb}}$.

Theorem 3. *Let* $\approx \in \{\approx_{\mathrm{tw}}, \approx_{\mathrm{tb}}\}$. *Then:*

$$\mathrm{SBNDC}_{\approx} \subsetneq \mathrm{SBSNNI}_{\approx} = \mathrm{P\_BNDC}_{\approx} \subsetneq \mathrm{BNDC}_{\approx} \subsetneq \mathrm{BSNNI}_{\approx}$$ ■

All the inclusions are strict as shown by the following counterexamples:

- The process $\tau \,.\, l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} + h \,.\, l \,.\, \underline{0}$ is $\text{SBSNNI}_{\approx}$ (resp. $\text{P\_BDNC}_{\approx}$) because $(\tau \,.\, l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} + h \,.\, l \,.\, \underline{0}) \setminus \{h\} \approx (\tau \,.\, l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} + h \,.\, l \,.\, \underline{0}) / \{h\}$ and action h is enabled only by the initial process so every derivative is $\text{BSNNI}_{\approx}$ (resp. $\text{BNDC}_{\approx}$). It is not $\text{SBNDC}_{\approx}$ because the low-level view of the process reached after action h, i.e., $(l \,.\, \underline{0}) \setminus \{h\}$, is not $\approx$-equivalent to $(\tau \,.\, l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} + h \,.\, l \,.\, \underline{0}) \setminus \{h\}$.
- The process $l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} + l \,.\, h \,.\, l \,.\, \underline{0}$ is $\text{BNDC}_{\approx}$ because, whether there are synchronizations with high-level actions or not, the overall process can always perform either an l-action or a sequence of two l-actions. It is not $\text{SBSNNI}_{\approx}$ (resp. $\text{P\_BNDC}_{\approx}$) because the reachable process $h \,.\, l \,.\, \underline{0}$ is not $\text{BSNNI}_{\approx}$ (resp. $\text{BNDC}_{\approx}$).
- The process $l \,.\, \underline{0} + h \,.\, h \,.\, l \,.\, \underline{0}$ is $\text{BSNNI}_{\approx}$ as $(l \,.\, \underline{0} + h \,.\, h \,.\, l \,.\, \underline{0}) \setminus \{h\} \approx (l \,.\, \underline{0} + h \,.\, h \,.\, l \,.\, \underline{0}) / \{h\}$. It is not $\text{BNDC}_{\approx}$ as $(((l \,.\, \underline{0} + h \,.\, h \,.\, l \,.\, \underline{0}) \parallel_{\{h\}} (h \,.\, \underline{0})) / \{h\}) \setminus \{h\} \not\approx (l \,.\, \underline{0} + h \,.\, h \,.\, l \,.\, \underline{0}) \setminus \{h\}$ in that the former behaves as $l \,.\, \underline{0} + \tau \,.\, \underline{0}$ while the latter behaves as $l \,.\, \underline{0}$.

Secondly, we observe that all the $\approx_{\text{tb}}$-based noninterference properties imply the corresponding $\approx_{\text{tw}}$-based ones, due to the fact that $\approx_{\text{tb}}$ is finer than $\approx_{\text{tw}}$.

Theorem 4. *The following inclusions hold:*

1. $\text{BSNNI}_{\approx_{\text{tb}}} \subsetneq \text{BSNNI}_{\approx_{\text{tw}}}$.
2. $\text{BNDC}_{\approx_{\text{tb}}} \subsetneq \text{BNDC}_{\approx_{\text{tw}}}$.
3. $\text{SBSNNI}_{\approx_{\text{tb}}} \subsetneq \text{SBSNNI}_{\approx_{\text{tw}}}$.
4. $\text{P\_BNDC}_{\approx_{\text{tb}}} \subsetneq \text{P\_BNDC}_{\approx_{\text{tw}}}$.
5. $\text{SBNDC}_{\approx_{\text{tb}}} \subsetneq \text{SBNDC}_{\approx_{\text{tw}}}$. ■

All the inclusions above are strict by virtue of the following result; for an example of P_1 and P_2 below, see Figure 1.

Theorem 5. *Let $P_1, P_2 \in \mathbb{P}$ be such that $P_1 \approx_{\text{tw}} P_2$ but $P_1 \not\approx_{\text{tb}} P_2$. If no high-level actions occur in P_1 and P_2, then $Q \in \{P_1 + h \,.\, P_2, P_2 + h \,.\, P_1\}$ is such that:*

1. $Q \in \text{BSNNI}_{\approx_{\text{tw}}}$ *but* $Q \notin \text{BSNNI}_{\approx_{\text{tb}}}$.
2. $Q \in \text{BNDC}_{\approx_{\text{tw}}}$ *but* $Q \notin \text{BNDC}_{\approx_{\text{tb}}}$.
3. $Q \in \text{SBSNNI}_{\approx_{\text{tw}}}$ *but* $Q \notin \text{SBSNNI}_{\approx_{\text{tb}}}$.
4. $Q \in \text{P\_BNDC}_{\approx_{\text{tw}}}$ *but* $Q \notin \text{P\_BNDC}_{\approx_{\text{tb}}}$.
5. $Q \in \text{SBNDC}_{\approx_{\text{tw}}}$ *but* $Q \notin \text{SBNDC}_{\approx_{\text{tb}}}$. ■

The diagram in Figure 2, which follows the same pattern as the nondeterministic and probabilistic settings [25,23], summarizes the relationships among the various noninterference properties based on the results in Theorems 3 and 4. In the diagram, $\mathcal{P} \to \mathcal{Q}$ means that $\mathcal{P}$ is strictly included in $\mathcal{Q}$, while missing arrows express incomparability and are justified by the following counterexamples:

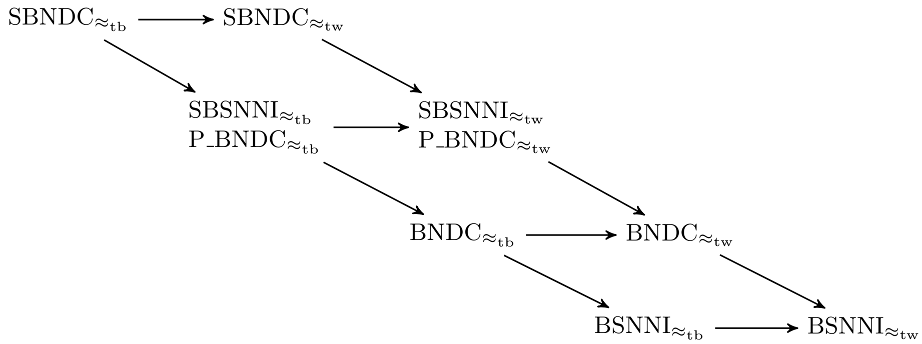

Fig. 2. Taxonomy of security properties based on timed bisimilarities

- $\mathrm{SBNDC}_{\approx_{\mathrm{tw}}}$ vs. $\mathrm{SBSNNI}_{\approx_{\mathrm{tb}}}$. The process $\tau \,.\, l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} + h \,.\, l \,.\, \underline{0}$ is $\mathrm{BSNNI}_{\approx_{\mathrm{tb}}}$ as $\tau \,.\, l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} \approx_{\mathrm{tb}} \tau \,.\, l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} + \tau \,.\, l \,.\, \underline{0}$. It is also $\mathrm{SBSNNI}_{\approx_{\mathrm{tb}}}$ because every reachable process does not enable any more high-level actions. However, it is not $\mathrm{SBNDC}_{\approx_{\mathrm{tw}}}$, because after executing the high-level action h it can perform a single action l, while the original process with the restriction on high-level actions can go along a path where it performs two l-actions. On the other hand, the process Q mentioned in Theorem 5 is $\mathrm{SBNDC}_{\approx_{\mathrm{tw}}}$ but neither $\mathrm{BSNNI}_{\approx_{\mathrm{tb}}}$ nor $\mathrm{SBSNNI}_{\approx_{\mathrm{tb}}}$.
- $\mathrm{SBSNNI}_{\approx_{\mathrm{tw}}}$ vs. $\mathrm{BNDC}_{\approx_{\mathrm{tb}}}$. The process $l \,.\, h \,.\, l \,.\, \underline{0} + l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0}$ is $\mathrm{BSNNI}_{\approx_{\mathrm{tb}}}$ as $l \,.\, \underline{0} + l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0} \approx_{\mathrm{tb}} l \,.\, \tau \,.\, l \,.\, \underline{0} + l \,.\, \underline{0} + l \,.\, l \,.\, \underline{0}$. The same process is $\mathrm{BNDC}_{\approx_{\mathrm{tb}}}$ too as it includes only one high-level action, hence the only possible high-level strategy coincides with the check conducted by $\mathrm{BSNNI}_{\approx_{\mathrm{tb}}}$. However, the process is not $\mathrm{SBSNNI}_{\approx_{\mathrm{tw}}}$ because of the reachable process $h \,.\, l \,.\, \underline{0}$, which is not $\mathrm{BSNNI}_{\approx_{\mathrm{tw}}}$. On the other hand, the process Q mentioned in Theorem 5 is $\mathrm{SBSNNI}_{\approx_{\mathrm{tw}}}$ but not $\mathrm{BSNNI}_{\approx_{\mathrm{tb}}}$ and, therefore, cannot be $\mathrm{BNDC}_{\approx_{\mathrm{tb}}}$.
- $\mathrm{BNDC}_{\approx_{\mathrm{tw}}}$ vs. $\mathrm{BSNNI}_{\approx_{\mathrm{tb}}}$. The process $(t) \,.\, l \,.\, \underline{0} + h_1 \,.\, h_2 \,.\, (t) \,.\, l \,.\, \underline{0}$ is $\mathrm{BSNNI}_{\approx_{\mathrm{tb}}}$ as discussed in Section 3, but it is not $\mathrm{BNDC}_{\approx_{\mathrm{tw}}}$. In contrast, the process Q mentioned in Theorem 5 is both $\mathrm{BSNNI}_{\approx_{\mathrm{tw}}}$ and $\mathrm{BNDC}_{\approx_{\mathrm{tw}}}$, but not $\mathrm{BSNNI}_{\approx_{\mathrm{mb}}}$.

5 Relating Nondeterministic and Timed Taxonomies

Let us compare our timed taxonomy with the nondeterministic one of [25]. In the following, we assume that $\approx_{\mathrm{w}}$ denotes the weak nondeterministic bisimilarity of [46] and $\approx_{\mathrm{b}}$ denotes the nondeterministic branching bisimilarity of [33]. These can also be obtained from the corresponding definitions in Section 2.2 by ignoring the clause about timed transitions. Since we are abstracting from delays, given a process $P \in \mathbb{P}$ we can obtain its nondeterministic variant, denoted by $nd(P)$, by replacing every occurrence of $(t) \,.\, P'$ with $\tau \,.\, P'$. However, to respect maximal progress, first we have to eliminate every subprocess starting with a delay prefix that is alternative to a subprocess starting with a τ-prefix. To accomplish this transformation syntactically, we focus on the set $\mathbb{P}_{\rm seq}$ of sequential processes, i.e., without parallel composition; this is not too restrictive because, in the absence of recursion, parallel composition can be eliminated by repeatedly applying a timed variant of the expansion law [47].

The next proposition states that if two sequential processes are equivalent according to any of the weak bisimilarities in Section 2.2, then their nondeterministic variants are equivalent according to the corresponding nondeterministic weak bisimilarity. The inverse does not hold; e.g., processes $P_1 = (1) \,.\, a \,.\, \underline{0}$ and $P_2 = (2) \,.\, a \,.\, \underline{0}$ are such that $P_1 \not\approx_{\rm tw} P_2$ and $P_1 \not\approx_{\rm tb} P_2$, but their nondeterministic counterparts coincide as both of them are equal to $\tau \,.\, a \,.\, \underline{0}$.

Proposition 2. *Let $P_1, P_2 \in \mathbb{P}_{\rm seq}$. Then:*

- $P_1 \approx_{\rm tw} P_2 \Longrightarrow nd(P_1) \approx_{\rm w} nd(P_2)$.
- $P_1 \approx_{\rm tb} P_2 \Longrightarrow nd(P_1) \approx_{\rm b} nd(P_2)$. ■

An immediate consequence is that if a sequential process is secure under any of the timed noninterference properties of Section 3, then its nondeterministic variant is secure under the corresponding nondeterministic property. The taxonomy of Figure 2 thus extends to the left the one in [25], as each of the properties of Section 3 is finer than its nondeterministic counterpart.

Corollary 1. *Let $\mathcal{P}_{\rm tm} \in \{{\rm BSNNI}_{\approx_{\rm tm}}, {\rm BNDC}_{\approx_{\rm tm}}, {\rm SBSNNI}_{\approx_{\rm tm}}, {\rm P\_BNDC}_{\approx_{\rm tm}}, {\rm SBNDC}_{\approx_{\rm tm}}\}$ and $\mathcal{P}_{\rm nd} \in \{{\rm BSNNI}_{\approx_{\rm nd}}, {\rm BNDC}_{\approx_{\rm nd}}, {\rm SBSNNI}_{\approx_{\rm nd}}, {\rm P\_BNDC}_{\approx_{\rm nd}}, {\rm SBNDC}_{\approx_{\rm nd}}\}$ for $\approx_{\rm tm} \in \{\approx_{\rm tw}, \approx_{\rm tb}\}$ and $\approx_{\rm nd} \in \{\approx_{\rm w}, \approx_{\rm b}\}$, where $\mathcal{P}_{\rm nd}$ is the nondeterministic variant of $\mathcal{P}_{\rm tm}$. Then $P \in \mathcal{P}_{\rm tm} \Longrightarrow nd(P) \in \mathcal{P}_{\rm nd}$ for all $P \in \mathbb{P}_{\rm seq}$.* ■

6 Reversibility via Timed Back-and-Forth Bisimilarity

In [21] it was shown that, for nondeterministic processes, weak back-and-forth bisimilarity coincides with branching bisimilarity. We now extend that result so that timed branching bisimilarity can be employed in the noninterference analysis of reversible processes featuring nondeterminism and time.

A TLTS $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ represents a reversible process if each of its transitions is seen as bidirectional. When going backward, it is of paramount importance to respect causality, i.e., the last performed transition must be the first one to be undone. Following [21] we set up an equivalence that enforces not only causality but also history preservation. This means that, when going backward, a process can only move along the path representing the history that brought the process to the current state even in the presence of concurrency. To accomplish this, the equivalence has to be defined over computations, not over states, and the notion of transition has to be revised so that it has source and target paths instead of states. We start by adapting the notation of the nondeterministic setting of [21] to our nondeterministic and timed setting. We use ℓ for a label in $\mathcal{A}_\tau \cup \mathbb{N}_{>0}$.

Definition 6. *A sequence* $\xi = (s_0, \ell_1, s_1)(s_1, \ell_2, s_2) \dots (s_{n-1}, \ell_n, s_n) \in\ \longrightarrow^*$ *is a* path *of length* n *from state* s_0*. We let* $\mathit{first}(\xi) = s_0$ *and* $\mathit{last}(\xi) = s_n$*; the empty path is indicated with* ε*. We denote by* $\mathit{path}(s)$ *the set of paths from* s*.* ■

Definition 7. *A pair* $\rho = (s, \xi)$ *is called a* run *from state* s *iff* $\xi \in \mathit{path}(s)$*, in which case we let* $\mathit{path}(\rho) = \xi$*,* $\mathit{first}(\rho) = \mathit{first}(\xi) = s$*,* $\mathit{last}(\rho) = \mathit{last}(\xi)$*, with* $\mathit{first}(\rho) = \mathit{last}(\rho) = s$ *when* $\xi = \varepsilon$*. We denote by* $\mathit{run}(s)$ *the set of runs from state* s*. Given* $\rho = (s, \xi) \in \mathit{run}(s)$ *and* $\rho' = (s', \xi') \in \mathit{run}(s')$*, their composition* $\rho\rho' = (s, \xi\xi') \in \mathit{run}(s)$ *is defined iff* $\mathit{last}(\rho) = \mathit{first}(\rho') = s'$*. We write* $\rho \xrightarrow{\ell} \rho'$ *iff there exists* $\rho'' = (\bar{s}, (\bar{s}, \ell, s'))$ *with* $\bar{s} = \mathit{last}(\rho)$ *such that* $\rho' = \rho\rho''$*; note that* $\mathit{first}(\rho) = \mathit{first}(\rho')$*.* ■

In the considered TLTS we work with the set $\mathcal{U}$ of runs in lieu of $\mathcal{S}$. Following [21], given a run ρ, we distinguish between *outgoing* and *incoming* transitions of ρ during the weak bisimulation game, both for action transitions and for timed ones, depending on whether we examine the forward or backward direction.

Definition 8. *Let* $(\mathcal{S}, \mathcal{A}_\tau, \longrightarrow)$ *be a TLTS. We say that* $s_1, s_2 \in \mathcal{S}$ *are* weakly timed back-and-forth bisimilar*, written* $s_1 \approx_{\mathrm{tbf}} s_2$*, iff* $((s_1, \varepsilon), (s_2, \varepsilon)) \in \mathcal{B}$ *for some weak timed back-and-forth bisimulation* $\mathcal{B}$*. A symmetric relation* $\mathcal{B}$ *over* $\mathcal{U}$ *is a* weak timed back-and-forth bisimulation *iff, whenever* $(\rho_1, \rho_2) \in \mathcal{B}$*, then:*

- *For each* $\rho_1 \xrightarrow{a}_{\mathrm{a}} \rho'_1$ *there exists* $\rho_2 \stackrel{\hat{a}}{\Longrightarrow}_{\mathrm{a}} \rho'_2$ *such that* $(\rho'_1, \rho'_2) \in \mathcal{B}$*.*
- *For each* $\rho'_1 \xrightarrow{a}_{\mathrm{a}} \rho_1$ *there exists* $\rho'_2 \stackrel{\hat{a}}{\Longrightarrow}_{\mathrm{a}} \rho_2$ *such that* $(\rho'_1, \rho'_2) \in \mathcal{B}$*.*
- *For each* $\rho_1 \Longrightarrow_{\mathrm{a}} \rho'_1$ *with* $\rho'_1 \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$ *there exists* $\rho_2 \Longrightarrow_{\mathrm{a}} \rho'_2$ *with* $\rho'_2 \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$ *such that* $(\rho'_1, \rho'_2) \in \mathcal{B}$ *and for each* $\rho'_1 \xrightarrow{t}_{\mathrm{t}} \rho''_1$ *there exists* $\rho'_2 \stackrel{t}{\Longrightarrow}_{\mathrm{t}} \rho''_2$ *such that* $(\rho''_1, \rho''_2) \in \mathcal{B}$*.*
- *For each* $\rho'_1 \xrightarrow{t}_{\mathrm{t}} \rho_1$ *with* $\rho'_1 \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$ *there exists* $\rho'_2 \stackrel{t}{\Longrightarrow}_{\mathrm{t}} \rho_2$ *with* $\rho'_2 \stackrel{\tau}{\not\longrightarrow}_{\mathrm{a}}$ *such that* $(\rho'_1, \rho'_2) \in \mathcal{B}$*.* ■

We show that weak timed back-and-forth bisimilarity over runs coincides with $\approx_{\mathrm{tb}}$, the forward-only timed branching bisimilarity over states. We proceed by adopting the proof strategy followed in [21] to show that their weak back-and-forth bisimilarity over runs coincides with the forward-only branching bisimilarity over states of [33]. Therefore we start by proving that $\approx_{\mathrm{tbf}}$ satisfies the *cross property*. This means that, whenever two runs of two $\approx_{\mathrm{tbf}}$-equivalent states can perform a sequence of finitely many τ-transitions, such that each of the two target runs is $\approx_{\mathrm{tbf}}$-equivalent to the source run of the other sequence, then the two target runs are $\approx_{\mathrm{tbf}}$-equivalent to each other as well.

Lemma 2. *Let* $s_1, s_2 \in \mathcal{S}$ *with* $s_1 \approx_{\mathrm{tbf}} s_2$*. For all* $\rho'_1, \rho''_1 \in \mathit{run}(s_1)$ *such that* $\rho'_1 \Longrightarrow_{\mathrm{a}} \rho''_1$ *and for all* $\rho'_2, \rho''_2 \in \mathit{run}(s_2)$ *such that* $\rho'_2 \Longrightarrow_{\mathrm{a}} \rho''_2$*, if* $\rho'_1 \approx_{\mathrm{tbf}} \rho''_2$ *and* $\rho''_1 \approx_{\mathrm{tbf}} \rho'_2$ *then* $\rho''_1 \approx_{\mathrm{tbf}} \rho''_2$*.* ■

Theorem 6. *Let* $s_1, s_2 \in \mathcal{S}$*. Then* $s_1 \approx_{\mathrm{tbf}} s_2 \iff s_1 \approx_{\mathrm{tb}} s_2$*.* ■

In conclusion, the properties $\text{BSNNI}_{\approx_{\text{tb}}}$, $\text{BNDC}_{\approx_{\text{tb}}}$, $\text{SBSNNI}_{\approx_{\text{tb}}}$, $\text{P\_BNDC}_{\approx_{\text{tb}}}$, and $\text{SBNDC}_{\approx_{\text{tb}}}$ do not change if $\approx_{\text{tb}}$ is replaced by $\approx_{\text{tbf}}$. This allows us to study noninterference properties for reversible systems featuring nondeterminism and time by using $\approx_{\text{tb}}$ in a process calculus like the one of Section 2.3, without having to resort to external memories [18], communication keys [53], or executed action decorations [14,12] like in reversible process calculi.

7 Use Case: Real-Time Database Transactions

Integrating security in real-time systems is a critical issue in several application domains, ranging from database management systems (DBMS) [1] to cyber-physical [7] and embedded [67] systems. In particular, the processing of concurrent transactions in real-time, multi-level secure database systems has to respect noninterference security properties about *values* (i.e., data read by low-level users cannot be affected by actions performed by high-level transactions), *delays* (i.e., the delay experienced by low-level transactions cannot depend on the execution of high-level transactions), and *recovery* (the abort of low-level transactions, as well as the actions taken to recover, cannot be influenced by the presence of high-level transactions) [38,62]. The satisfaction of these conditions is even more complicated in systems where transactions with real-time requirements are assigned priorities, as they are served according to their priorities rather than on a first-come-first-served basis [1].

Let us first explain through some examples inspired by [62] the subtleties of potential covert channels in such a complex scenario. To this aim, consider a sequence of three transactions, each with its own security level, priority, arrival time for scheduling, and execution time. Depending on these parameters, we will show that covert channels may or may not arise. In the following, we assume that the first transaction, arriving at time 1, is the high-level transaction HT_1. Then, we have the two low-level transactions LT_2 and LT_3, arriving at time 8 and 11 respectively, such that the priority of LT_3 is higher than the priority of LT_2. Moreover, HT_1 requests read access to variable x, while the two low-level transactions request write access to that variable. The three transactions follow a classical two-phase locking (2PL) mechanism based on the acquisition and release of a read/write lock before and after the requested operation. By abstracting from the lock operations and denoting by hr_1, lw_2, and lw_3 the three access operations, we obtain the following process:

$$DBMS \,=\, (1)\,.\,(\tau\,.\,(7)\,.\,lw_2\,.\,(3)\,.\,lw_3\,.\,\underline{0} \,+\, hr_1\,.\,(7)\,.\,lw_2\,.\,(3)\,.\,lw_3\,.\,\underline{0})$$

which represents the case in which HT_1, if scheduled, terminates before the arrival of LT_2, which in turn terminates before the arrival of LT_3. This means that every low-level transaction does not experience any delay due to the other transactions. Both *value* and *delay* security hold and, in particular, it can be easily verified that $DBMS \setminus \{hr_1\}$ and $DBMS \,/\, \{hr_1\}$ enable weakly/branching bisimilar behaviors in this nondeterministic and timed setting.

Now, consider the following variant:

$$DBMS' \,=\, (1)\,.\,(\tau\,.\,(7)\,.\,lw_2\,.\,(3)\,.\,lw_3\,.\,\underline{0} \,+\, hr_1\,.\,(9)\,.\,lw_2\,.\,(1)\,.\,lw_3\,.\,\underline{0})$$

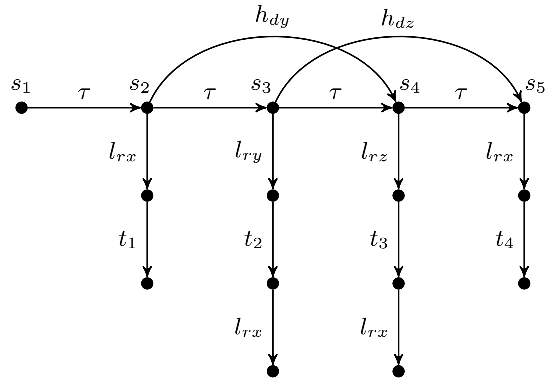

Fig. 3. Interleaving of two concurrent transactions (r for read, d for delete)

expressing that if HT_1 is scheduled then LT_2 is delayed by 2 time units with respect to its arrival time (e.g., because the duration of HT_1 exceeds 7 time units, thus blocking the lock acquisition for LT_2), while LT_3 does not experience any delay. *Delay* security does not hold anymore and, in particular, this can be verified via $DBMS' \setminus \{hr_1\}$ and $DBMS' \,/\, \{hr_1\}$ not being timed weakly/branching bisimilar. Note that, in contrast, their nondeterministic versions are still nondeterministic weakly/branching bisimilar as in the previous case.

Finally, consider the following further variant, where the execution of HT_1 requires 11 time units:

$$DBMS'' \;=\; (1) \,.\, (\tau \,.\, (7) \,.\, lw_2 \,.\, (3) \,.\, lw_3 \,.\, \underline{0} \;+\; hr_1 \,.\, (11) \,.\, lw_3 \,.\, (1) \,.\, lw_2 \,.\, \underline{0})$$

Note that, because of the latency due to HT_1, LT_3 arrives when LT_2 is still waiting, thus preempting it because of the higher priority level. Hence, in this case, both *value* and *delay* security do not hold anymore, as confirmed by the fact that $DBMS'' \setminus \{hr_1\}$ and $DBMS'' \,/\, \{hr_1\}$ are not nondeterministic/timed weakly/branching bisimilar.

Several approaches have been proposed to make 2PL robust with respect to these kinds of interferences. Here we consider a solution in which conflicting lock operations are not delayed thanks to the use of *virtual locks*. From the user viewpoint, such operations are transparent and the variables are accessed immediately on demand (see, e.g., [62,20] for details). In this scenario, we describe an example that emphasizes how the branching semantics helps to capture violations of *recovery* security whenever the DBMS supports reversible transactions [22,25]. Consider a low-level transaction LT and a high-level transaction HT accessing three variables x, y, and z. Their interleaving is shown in Figure 3, where actions of the form l_{rv} denote a read access on variable v by LT, while actions of the form h_{dv} denote the deletion of variable v by HT. Since the virtual lock operations are transparent from the user viewpoint, they are not modeled. The execution starts in s_1 with LT and HT that are activated. Then, in s_2, LT has to choose between some internal activity and the reading of variable x, after which a delay of t_1 time units follows, while at the same time HT may access variable y to delete it. The interpretation of the subsequent branches is analogous. Note that the high-level transaction departing from s_2 skips s_3, because variable y is deleted and, consequently, LT could not access it.

It is worth noting that the system in Figure 3, call it $DBMS_{vl}$, satisfies the SBSNNI$_{\approx_{\rm tw}}$ property. In particular, the most interesting case is given by the transition $s_2 \xrightarrow{\tau}_{\rm a} s_4$ in $DBMS_{vl} \,/\, \mathcal{A}_{\mathcal{H}}$, which is simulated by the sequence $s_2 \xrightarrow{\tau}_{\rm a} s_3 \xrightarrow{\tau}_{\rm a} s_4$ in $DBMS_{vl} \setminus \mathcal{A}_{\mathcal{H}}$ (we can reason analogously for the transition $s_3 \xrightarrow{\tau}_{\rm a} s_5$). However, this does not hold when considering the $\approx_{\rm tb}$-based semantics, because the intermediate state s_3 is not timed branching bisimilar to the departing state s_2. From the back-and-forth perspective, consider executing the run $\tau \,.\, \tau \,.\, l_{rz}$ of $DBMS_{vl} \,/\, \mathcal{A}_{\mathcal{H}}$, which can be matched by the run $\tau \,.\, \tau \,.\, \tau \,.\, l_{rz}$ of $DBMS_{vl} \setminus \mathcal{A}_{\mathcal{H}}$. By undoing the actions of the former run (e.g., due to the recovery following an abort on the reading operation), it is not possible to go back to a state enabling action l_{ry}. Instead, this is possible by undoing the latter run. This is enough to distinguish the two versions of the system in the setting of reversible transactions. As a consequence, it turns out that the BSNNI$_{\approx_{\rm tb}}$ property is not satisfied, thus revealing that the $\approx_{\rm tb}$-based semantics is adequate to verify *recovery* security.

8 Conclusions

In this paper we have extended to a deterministically timed setting our previous compositionality, preservation, and classification results about a selection of noninterfence properties for irreversible or reversible systems developed in a nondeterministic setting [25] and in a probabilistic one [23] (stochastic time has been recently addressed in [24]). To represent the passing of time, we have assumed time determinism and time additivity. The two behavioral equivalences – designed to comply with the further assumption of maximal progress – for those noninterference properties are weak timed bisimilarity [48] and a newly defined timed branching bisimilarity. Since we have shown that timed branching bisimilarity coincides with a timed variant of the weak back-and-forth bisimilarity of [21], noninterference properties based on this equivalence can be applied to reversible timed systems, thus extending the results in [25,23] for nondeterministic and probabilistic systems.

As for future work, we would like to include recursion in the considered process language, thus allowing one to model systems that may not terminate. This requires identifying adequate timed variants of the up-to technique for weak [59] and branching [31] bisimilarities, to be used in the proof of some results where we can now proceed by induction on the depth of the tree-like TLTS underlying the considered process term. Another direction that we want to pursue is addressing dense time [69].

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A Proofs of Results

Proof of Proposition 1. We start by noting that by repeatedly applying *TimeSplit* to $s_1 \xrightarrow{t}_{\rm t} s'_1$ we can obtain $s_1 \xrightarrow{t_1}_{\rm t} s_{1,1} \xrightarrow{t_2}_{\rm t} \dots \xrightarrow{t_{n-1}}_{\rm t} s_{1,n} \xrightarrow{t_n}_{\rm t} s'_1$ with $\sum_{1 \le i \le n} t_i = t$. Then, to prove the result, we reason by induction on the number n of timed transitions:

- If $n = 1$ then $t_1 = t$ and hence $P_1 \xrightarrow{t_1}_{\rm t} s_{1,1} \xrightarrow{t_2}_{\rm t} \dots \xrightarrow{t_{n-1}}_{\rm t} s_{1,n} \xrightarrow{t_n}_{\rm t} s'_1$ and $s_2 \Longrightarrow_{\rm a} s_{2,1} \xrightarrow{t_1}_{\rm t} s_{2,2} \Longrightarrow_{\rm a} \dots \Longrightarrow_{\rm a} s_{2,n-1} \xrightarrow{t_n}_{\rm t} s_{2,n} \Longrightarrow_{\rm a} s'_2$ boil down to $s_1 \xrightarrow{t}_{\rm t} s'_1$ and $s_2 \xrightarrow{t}_{\rm t} \bar{s}_2 \Longrightarrow_{\rm a} s'_2$, notice that s_2 cannot perform any τ action. Then from $s_1 \approx s_2$ we obtain that $s_1 \approx \bar{s}_2$ and by transitivity $\bar{s}_2 \approx s'_1 \approx s'_2$.
- If $n > 0$ then by focusing on the start of each sequence of transitions, i.e., $s_1 \xrightarrow{t_1}_{\rm t} s_{1,1} \xrightarrow{t_2}_{\rm t} \dots$ and $s_2 \Longrightarrow_{\rm a} s_{2,1} \xrightarrow{t_1}_{\rm t} s_{2,2} \Longrightarrow_{\rm a} \dots$ we can notice that from $s_1 \approx s_2$ and the fact the s_1 only enables one timed transition, we have $s_1 \approx s_{2,1}$. Then since we assume time determinism, i.e., each state can have at most one outgoing timed transition, from $s_1 \xrightarrow{t_1}_{\rm t} s_{1,1}$ and $s_{2,1} \xrightarrow{t_1}_{\rm t} s_{2,2}$ we obtain $s_{1,1} \approx s_{2,2}$. The remainder of the transition sequences are shorter, and hence we can apply the induction hypothesis to obtain the desired result.

■

Proof of Lemma 1. The congruence of $\approx_{\rm tw}$ with respect to the operators of our calculus has already been proven in [48], with the only differences being in the parallel composition operator considered, a CCS-style operator instead of CSP, and in the hiding operator, which is not presented directly but it is implemented throught a relabeling operator.
We then prove the four $\approx_{\rm tb}$-based properties. Let $\mathcal{B}$ be a timed branching bisimulation witnessing $P_1 \approx_{\rm tb} P_2$:

1. The symmetric relation $\mathcal{B}' = \mathcal{B} \cup \{(a \,.\, Q_1, a \,.\, Q_2) \mid (Q_1, Q_2) \in \mathcal{B} \wedge Q_1, Q_2 \in \mathbb{P}\}$ is a timed branching bisimulation too. The result immediately follows from the fact that, given $(a \,.\, Q_1, a \,.\, Q_2) \in \mathcal{B}'$, $a \,.\, Q_1 \xrightarrow{a}_{\rm a} Q_1$ is matched by $a \,.\, Q_2 \Longrightarrow a \,.\, Q_2 \xrightarrow{a}_{\rm a} Q_2$ with $(a \,.\, Q_1, a \,.\, Q_2) \in \mathcal{B}'$ and $(Q_1, Q_2) \in \mathcal{B}'$.
2. The symmetric relation $\mathcal{B}' = \mathcal{B} \cup \{((t) \,.\, Q_1, (t) \,.\, Q_2) \mid (Q_1, Q_2) \in \mathcal{B} \wedge Q_1, Q_2 \in \mathbb{P}\}$ is a timed branching bisimulation too. The result immediately follows from the fact that, given $((t) \,.\, Q_1, (t) \,.\, Q_2) \in \mathcal{B}'$, $(t) \,.\, Q_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, $(t) \,.\, Q_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ and $(t) \,.\, Q_1 \xrightarrow{t}_{\rm t} Q_1$ is matched by $(t) \,.\, Q_2 \Longrightarrow (t) \,.\, Q_2 \xrightarrow{t}_{\rm t} Q_2 \Longrightarrow Q_2$ with $(Q_1, Q_2) \in \mathcal{B}'$.
3. The symmetric relation $\mathcal{B}' = \mathcal{B} \cup \{(Q_1 \,\|_L\, Q, Q_2 \,\|_L\, Q) \mid (Q_1, Q_2) \in \mathcal{B}\}$ and its variant $\mathcal{B}''$ in which Q occurs to the left of parallel composition in each pair added with respect to $\mathcal{B}$ are timed branching bisimulations too. Let us focus on $\mathcal{B}'$ and consider $(Q_1 \,\|_L\, Q, Q_2 \,\|_L\, Q) \in \mathcal{B}'$. There are three cases for action transitions based on the operational semantic rules in Table 1:
   - If $Q_1 \,\|_L\, Q \xrightarrow{a}_{\rm a} Q'_1 \,\|_L\, Q$ with $Q_1 \xrightarrow{a}_{\rm a} Q'_1$ and $a \notin L$, then either $a = \tau$ and $(Q'_1, Q_2) \in \mathcal{B}$, or $Q_2 \Longrightarrow \bar{Q}_2 \xrightarrow{a}_{\rm a} Q'_2$ with $(Q_1, \bar{Q}_2) \in \mathcal{B}$ and $(Q'_1, Q'_2) \in \mathcal{B}$. Thus in the former subcase $Q_2 \,\|_L\, Q$ is allowed to stay idle with

$(Q'_1 \parallel_L Q, Q_2 \parallel_L Q) \in \mathcal{B}'$, while in the latter subcase $Q_2 \parallel_L Q \Longrightarrow \bar{Q}_2 \parallel_L Q \xrightarrow{a}_{\rm a} Q'_2 \parallel_L Q$ with $(Q_1 \parallel_L Q, \bar{Q}_2 \parallel_L Q) \in \mathcal{B}'$ and $(Q'_1 \parallel_L Q, Q'_2 \parallel_L Q) \in \mathcal{B}'$.
- The case $Q_1 \parallel_L Q \xrightarrow{a}_{\rm a} Q_1 \parallel_L Q'$ with $Q \xrightarrow{a}_{\rm a} Q'$ and $a \notin L$ is trivial.
- If $Q_1 \parallel_L Q \xrightarrow{a}_{\rm a} Q'_1 \parallel_L Q'$ with $Q_1 \xrightarrow{a}_{\rm a} Q'_1$, $Q \xrightarrow{a}_{\rm a} Q'$, and $a \in L$, then $Q_2 \Longrightarrow \bar{Q}_2 \xrightarrow{a}_{\rm a} Q'_2$ with $(Q_1, \bar{Q}_2) \in \mathcal{B}$ and $(Q'_1, Q'_2) \in \mathcal{B}$. Thus $Q_2 \parallel_L Q \Longrightarrow \bar{Q}_2 \parallel_L Q \xrightarrow{a}_{\rm a} Q'_2 \parallel_L Q'$ with $(Q_1 \parallel_L Q, \bar{Q}_2 \parallel_L Q) \in \mathcal{B}'$ and $(Q'_1 \parallel_L Q', Q'_2 \parallel_L Q') \in \mathcal{B}'$.

As for delays, suppose $Q_1 \parallel_L Q \;\, \not\xrightarrow{\tau}_{\rm a}$ so that $Q_1 \;\, \not\xrightarrow{\tau}_{\rm a}$, $Q \;\, \not\xrightarrow{\tau}_{\rm a}$, and $Q_1 \parallel_L Q \xrightarrow{t}_{\rm t} Q'_1 \parallel_L Q'$ with $Q_1 \xrightarrow{t}_{\rm t} Q'_1$ and $Q \xrightarrow{t}_{\rm t} Q'$ then from $(Q_1, Q_2) \in \mathcal{B}$ we have that $Q_2 \Longrightarrow \bar{Q}_2$ such that $\bar{Q}_2 \;\, \not\xrightarrow{\tau}_{\rm a}$, $(Q_1, \bar{Q}_2) \in \mathcal{B}$, and $\bar{Q}_2 \stackrel{t}{\Longrightarrow} Q'_2$ with $(Q'_1, Q'_2) \in \mathcal{B}$. Thus $Q_2 \parallel_L Q \Longrightarrow \bar{Q}_2 \parallel_L Q \stackrel{t}{\Longrightarrow} Q'_2 \parallel_L Q'$ with $\bar{Q}_2 \parallel_L Q) \not\xrightarrow{\tau}_{\rm a}$, $(Q_1 \parallel_L Q, \bar{Q}_2 \parallel_L Q) \in \mathcal{B}'$, and $(Q'_1 \parallel_L Q', Q'_2 \parallel_L Q') \in \mathcal{B}'$.

4. The symmetric relation $\mathcal{B}' = \mathcal{B} \cup \{(Q_1 \setminus L, Q_2 \setminus L) \mid (Q_1, Q_2) \in \mathcal{B}\}$ is a timed branching bisimulation too. Given $(Q_1 \setminus L, Q_2 \setminus L) \in \mathcal{B}'$, there are two cases for action transitions based on the operational semantic rules in Table 1:
   - If $Q_1 \setminus L \xrightarrow{\tau}_{\rm a} Q'_1 \setminus L$ with $Q_1 \xrightarrow{\tau}_{\rm a} Q'_1$, then either $(Q'_1, Q_2) \in \mathcal{B}$, or $Q_2 \Longrightarrow \bar{Q}_2 \xrightarrow{\tau}_{\rm a} Q'_2$ with $(Q_1, \bar{Q}_2) \in \mathcal{B}$ and $(Q'_1, Q'_2) \in \mathcal{B}$. Since the restriction operator does not apply to τ in the former subcase $Q_2 \setminus L$ is allowed to stay idle with $(Q'_1 \setminus L, Q_2 \setminus L) \in \mathcal{B}'$, while in the latter subcase $Q_2 \setminus L \Longrightarrow \bar{Q}_2 \setminus L \xrightarrow{\tau}_{\rm a} Q'_2 \setminus L$ with $(Q_1 \setminus L, \bar{Q}_2 \setminus L) \in \mathcal{B}'$ and $(Q'_1 \setminus L, Q'_2 \setminus L) \in \mathcal{B}'$.
   - If $Q_1 \setminus L \xrightarrow{a}_{\rm a} Q'_1 \setminus L$ with $Q_1 \xrightarrow{a}_{\rm a} Q'_1$ and $a \notin L \cup \{\tau\}$, then $Q_2 \Longrightarrow \bar{Q}_2 \xrightarrow{a}_{\rm a} Q'_2$ with $(Q_1, \bar{Q}_2) \in \mathcal{B}$ and $(Q'_1, Q'_2) \in \mathcal{B}$. Since the restriction operator does not apply to τ and $a \notin L$, it follows that $Q_2 \setminus L \Longrightarrow \bar{Q}_2 \setminus L \xrightarrow{a}_{\rm a} Q'_2 \setminus L$ with $(Q_1 \setminus L, \bar{Q}_2 \setminus L) \in \mathcal{B}'$ and $(Q'_1 \setminus L, Q'_2 \setminus L) \in \mathcal{B}'$.

   As for delays, suppose $Q_1 \setminus L \not\xrightarrow{\tau}_{\rm a}$ so that $Q_1 \not\xrightarrow{\tau}_{\rm a}$ and $Q_1 \setminus L \xrightarrow{t}_{\rm t} Q'_1 \setminus L$ with $Q_1 \xrightarrow{t}_{\rm t} Q'_1$, then from $(Q_1, Q_2) \in \mathcal{B}$ it follows that $Q_2 \Longrightarrow \bar{Q}_2$ such that $\bar{Q}_2 \not\xrightarrow{\tau}_{\rm a}$, $(Q_1, \bar{Q}_2) \in \mathcal{B}$, and $\bar{Q}_2 \stackrel{t}{\Longrightarrow} Q'_2$ with $(Q'_1, Q'_2) \in \mathcal{B}$. Since the restriction operator do not apply to τ or timed transistions, it follows that $Q_2 \setminus L \Longrightarrow \bar{Q}_2 \setminus L \stackrel{t}{\Longrightarrow} Q'_2 \setminus \mathcal{A_H}$ with $\bar{Q}_2 \setminus \mathcal{A_H} \not\xrightarrow{\tau}_{\rm a}$, $(Q_1 \setminus \mathcal{A_H}, \bar{Q}_2 \setminus \mathcal{A_H}) \in \mathcal{B}$, and $(Q'_1 \setminus L, Q'_2 \setminus L) \in \mathcal{B}'$.
5. The symmetric relation $\mathcal{B}' = \mathcal{B} \cup \{(Q_1 \,/\, L, Q_2 \,/\, L) \mid (Q_1, Q_2) \in \mathcal{B}\}$ is a timed branching bisimulation too. Given $(Q_1 \,/\, L, Q_2 \,/\, L) \in \mathcal{B}'$, there are two cases for action transitions based on the operational semantic rules in Table 1:
   - If $Q_1 \,/\, L \xrightarrow{\tau}_{\rm a} Q'_1 \,/\, L$ with $Q_1 \xrightarrow{\tau}_{\rm a} Q'_1$, then either $(Q'_1, Q_2) \in \mathcal{B}$, or $Q_2 \Longrightarrow \bar{Q}_2 \xrightarrow{\tau}_{\rm a} Q'_2$ with $(Q_1, \bar{Q}_2) \in \mathcal{B}$ and $(Q'_1, Q'_2) \in \mathcal{B}$. Since the hiding operator does not apply to τ in the former subcase $Q_2 \,/\, L$ is allowed to stay idle with $(Q'_1 \,/\, L, Q_2 \,/\, L) \in \mathcal{B}'$, while in the latter subcase $Q_2 \,/\, L \Longrightarrow \bar{Q}_2 \,/\, L \xrightarrow{\tau}_{\rm a} Q'_2 \,/\, L$ with $(Q_1 \,/\, L, \bar{Q}_2 \,/\, L) \in \mathcal{B}'$ and $(Q'_1 \,/\, L, Q'_2 \,/\, L) \in \mathcal{B}'$.
   - If $Q_1 \,/\, L \xrightarrow{a}_{\rm a} Q'_1 \,/\, L$ with $Q_1 \xrightarrow{b}_{\rm a} Q'_1$ and $b \in L \wedge a = \tau$ or $b \notin L \cup \{\tau\} \wedge a = b$, then $Q_2 \Longrightarrow \bar{Q}_2 \xrightarrow{b}_{\rm a} Q'_2$ with $(Q_1, \bar{Q}_2) \in \mathcal{B}$ and $(Q'_1, Q'_2) \in \mathcal{B}$.

Since the hiding operator does not apply to τ it follows that $Q_2 \, / \, L \Longrightarrow \bar{Q}_2 \, / \, L \xrightarrow{a}_{\rm a} Q'_2 \, / \, L$ with $(Q_1 \, / \, L, \bar{Q}_2 \, / \, L) \in \mathcal{B}'$ and $(Q'_1 \, / \, L, Q'_2 \, / \, L) \in \mathcal{B}'$.

As for delays, suppose $Q_1 \, / \, L \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ so that $Q_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ and $Q_1 \, / \, L \xrightarrow{t}_{\rm t} Q'_1 \, / \, L$ with $Q_1 \xrightarrow{t}_{\rm t} Q'_1$, then from $(Q_1, Q_2) \in \mathcal{B}$ we have $Q_2 \Longrightarrow \bar{Q}_2$ such that $\bar{Q}_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, $(Q_1, \bar{Q}_2) \in \mathcal{B}$, and $\bar{Q}_2 \stackrel{t}{\Longrightarrow} Q'_2$ with $(Q'_1, Q'_2) \in \mathcal{B}$. Since the hiding operator does not apply to τ or timed transistions, it follows that $Q_2 \, / \, L \Longrightarrow \bar{Q}_2 \stackrel{t}{\Longrightarrow} Q'_2 \, / \, L$ with $\bar{Q}_2 \setminus \mathcal{A_H} \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, $(Q'_1 \, / \, L, \bar{Q}_2 \, / \, L) \in \mathcal{B}'$, and $(Q'_1 \, / \, L, Q'_2 \, / \, L) \in \mathcal{B}'$. ∎

Proof of Theorem 1. The results immediately follow from the fact that $\approx_{\rm tw}$ and $\approx_{\rm tb}$ are congruences with respect to the parallel, restriction and hiding operators (see the proof of the Lemma 1). ∎

Proof of Theorem 2. We divide the proof into two parts. In the first part we prove the theorem for the $\approx_{\rm tw}$-based properties, and in the second part we do the same for the $\approx_{\rm tb}$-based properties. We first prove the results for $\text{SBSNNI}_{\approx_{\rm tw}}$, and hence for $\text{P\_BNDC}_{\approx_{\rm tw}}$ too by virtue of the forthcoming Theorem 3:

1. Given an arbitrary $P \in \text{SBSNNI}_{\approx_{\rm tw}}$ and an arbitrary $a \in \mathcal{A_L} \cup \{\tau\}$, from $P \setminus \mathcal{A_H} \approx_{\rm tw} P \, / \, \mathcal{A_H}$ we derive that $a \, . \, (P \setminus \mathcal{A_H}) \approx_{\rm tw} a \, . \, (P \, / \, \mathcal{A_H})$ because $\approx_{\rm tw}$ is a congruence with respect to action prefix (see proof of Lemma 1), from which it follows that $(a \, . \, P) \setminus \mathcal{A_H} \approx_{\rm tw} (a \, . \, P) \, / \, \mathcal{A_H}$, i.e., $a \, . \, P \in \text{BSNNI}_{\approx_{\rm tw}}$, because $a \notin \mathcal{A_H}$. To conclude the proof, it suffices to observe that all the processes reachable from $a \, . \, P$ after performing a are processes reachable from P, which are known to be $\text{BSNNI}_{\approx_{\rm tw}}$.
2. Given an arbitrary $P \in \text{SBSNNI}_{\approx_{\rm tw}}$ and an arbitrary $t \in \mathbb{N}_{>0}$, from $P \setminus \mathcal{A_H} \approx_{\rm tw} P \, / \, \mathcal{A_H}$ we derive that $(t) \, . \, (P \setminus \mathcal{A_H}) \approx_{\rm tw} (t) \, . \, (P \, / \, \mathcal{A_H})$ because $\approx_{\rm tw}$ is a congruence with respect to timed prefix (see proof of Lemma 1), from which it follows that $(t) \, . \, P \setminus \mathcal{A_H} \approx_{\rm tw} (t) \, . \, P \, / \, \mathcal{A_H}$, i.e., $(t) \, . \, P \in \text{BSNNI}_{\approx_{\rm tw}}$, because restriction and hiding do not apply to timed transitions. To conclude the proof, it suffices to observe that all the processes reachable from $(t) \, . \, P$ after performing t are processes reachable from P, which are known to be $\text{BSNNI}_{\approx_{\rm tw}}$.
3. Given two arbitrary $P_1, P_2 \in \mathbb{P}$ such that $Q_1, Q_2 \in \mathit{reach}(P_1)$, $R_1, R_2 \in \mathit{reach}(P_2)$, and arbitrary $L \subseteq \mathcal{A_L}$ the result follows by proving that the symmetric relation $\mathcal{B} = \{((Q_1 \parallel_L Q_2) \setminus \mathcal{A_H}, (R_1 \parallel_L R_2) \, / \, \mathcal{A_H}) \mid Q_1 \parallel_L Q_2) \in \mathit{reach}(P_1 \parallel_L P_2) \wedge (R_1 \parallel_L R_2 \in \mathit{reach}(P_1 \parallel_L P_2) \wedge Q_1 \setminus \mathcal{A_H} \approx_{\rm tw} R_1 \, / \, \mathcal{A_H} \wedge Q_2 \setminus \mathcal{A_H} \approx_{\rm tw} R_2 \, / \, \mathcal{A_H}\}$ by taking Q_1 identical to R_1 and Q_2 identical to R_2. thirteen cases for action transitions based on the operational semantic rules in Table 1. In the first five cases, it is $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H}$ to move first:
   - If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}$ with $Q_1 \xrightarrow{l}_{\rm a} Q'_1$ and $l \notin L$, then $Q_1 \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} Q'_1 \setminus \mathcal{A_H}$ as $l \notin \mathcal{A_H}$. From $Q_1 \setminus \mathcal{A_H} \approx_{\rm tw} R_1 \, / \, \mathcal{A_H}$ it follows that there exists R'_1 such that $R_1 \, / \, \mathcal{A_H} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow R'_1 \, / \, \mathcal{A_H}$ with $Q'_1 \setminus \mathcal{A_H} \approx_{\rm tw} R'_1 \, / \, \mathcal{A_H}$. Since synchronization does not apply to τ and l, we have that $(R_1 \parallel_L R_2) \, / \, \mathcal{A_H} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow (R'_1 \parallel_L R_2) \, / \, \mathcal{A_H}$ with $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}, (R'_1 \parallel_L R_2) \, / \, \mathcal{A_H}) \in \mathcal{B}$.

- If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (Q_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_2 \xrightarrow{l}_{\rm a} Q'_2$ and $l \notin L$, then the proof is similar to the one of the previous case.
- If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_i \xrightarrow{l}_{\rm a} Q'_i$ for $i \in \{1, 2\}$ and $l \in L$, then $Q_i \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} Q'_i \setminus \mathcal{A}_\mathcal{H}$ as $l \notin \mathcal{A}_\mathcal{H}$. From $Q_i \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} R_i \,/\, \mathcal{A}_\mathcal{H}$ it follows that there exists R'_i such that $R_i \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow R'_i \,/\, \mathcal{A}_\mathcal{H}$ with $Q'_i \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} R'_i \,/\, \mathcal{A}_\mathcal{H}$. Since synchronization does not apply to τ, we have that $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow (R'_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}$ with $((Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}, (R'_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
- If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_1 \xrightarrow{\tau}_{\rm a} Q'_1$, then $Q_1 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} Q'_1 \setminus \mathcal{A}_\mathcal{H}$ as $\tau \notin \mathcal{A}_\mathcal{H}$. From $Q_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} R_1 \,/\, \mathcal{A}_\mathcal{H}$ it follows that there exists R'_1 such that $R_1 \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow R'_1 \,/\, \mathcal{A}_\mathcal{H}$ with $Q'_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} R'_1 \,/\, \mathcal{A}_\mathcal{H}$. Since synchronization does not apply to τ, we have that $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow (R'_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}$ with $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (R'_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
- If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (Q_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_2 \xrightarrow{\tau}_{\rm a} Q'_2$, then the proof is similar to the one of the previous case.

In the other eight cases, instead, it is $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}$ to move first:

- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}$ with $R_1 \xrightarrow{l}_{\rm a} R'_1$ and $l \notin L$, then $R_1 \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} R'_1 \,/\, \mathcal{A}_\mathcal{H}$ as $l \notin \mathcal{A}_\mathcal{H}$. From $R_1 \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} Q_1 \setminus \mathcal{A}_\mathcal{H}$ it follows that there exists Q'_1 such that $Q_1 \setminus \mathcal{A}_\mathcal{H} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow Q'_1 \setminus \mathcal{A}_\mathcal{H}$ with $R_1 \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} Q'_1 \setminus \mathcal{A}_\mathcal{H}$ and $R'_1 \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} Q'_1 \setminus \mathcal{A}_\mathcal{H}$. Since synchronization does not apply to τ and l, we have that $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}$ with $((R'_1 \parallel_L R_2)/\mathcal{A}_\mathcal{H}, (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (R_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}$ with $R_2 \xrightarrow{l}_{\rm a} R'_2$ and $l \notin L$, then the proof is similar to the one of the previous case.
- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (R'_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}$ with $R_i \xrightarrow{l}_{\rm a} R'_i$ for $i \in \{1, 2\}$ and $l \in L$, then $R_i \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} R'_i \,/\, \mathcal{A}_\mathcal{H}$ as $l \notin \mathcal{A}_\mathcal{H}$. From $R_i \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} Q_i \setminus \mathcal{A}_\mathcal{H}$ it follows that there exists Q'_i such that $Q_i \setminus \mathcal{A}_\mathcal{H} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow Q'_i \setminus \mathcal{A}_\mathcal{H}$ with $R'_i \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} Q'_i \setminus \mathcal{A}_\mathcal{H}$. Since synchronization does not apply to τ, we have that $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}$ with $((R'_1 \parallel_L R'_2)/\mathcal{A}_\mathcal{H}, (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}$ with $R_1 \xrightarrow{\tau}_{\rm a} R'_1$, then $R_1 \,/ \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} R'_1 \,/ \mathcal{A}_\mathcal{H}$ as $\tau \notin \mathcal{A}_\mathcal{H}$. From $R_1 \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} Q_1 \setminus \mathcal{A}_\mathcal{H}$ it follows that there exists Q'_1 such that $Q_1 \setminus \mathcal{A}_\mathcal{H} \Longrightarrow Q'_1 \setminus \mathcal{A}_\mathcal{H}$ with $R'_1 \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} Q'_1 \setminus \mathcal{A}_\mathcal{H}$. Since synchronization does not apply to τ, we have that $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \Longrightarrow (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}$ with $((R'_1 \parallel_L R_2)/\mathcal{A}_\mathcal{H}, (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (R_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}$ with $R_2 \xrightarrow{\tau}_{\rm a} R'_2$, then the proof is similar to the one of the previous case.
- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}$ with $R_1 \xrightarrow{h}_{\rm a} R'_1$ and $h \notin L$, then $R_1 \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} R'_1 \,/\, \mathcal{A}_\mathcal{H}$ as $h \in \mathcal{A}_\mathcal{H}$. The rest of the proof is like the one of the fourth case.
- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (R_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}$ with $R_2 \xrightarrow{h}_{\rm a} R'_2$ and $h \notin L$, then the proof is similar to the one of the previous case.

- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (R'_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}$ with $R_i \xrightarrow{h}_{\rm a} R'_i$ for $i \in \{1, 2\}$ and $h \in L$, then $R_i \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} R'_i \,/\, \mathcal{A}_\mathcal{H}$ as $h \in \mathcal{A}_\mathcal{H}$. From $R_i \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} Q_i \setminus \mathcal{A}_\mathcal{H}$ it follows that there exists Q'_i such that $Q_i \setminus \mathcal{A}_\mathcal{H} \Longrightarrow Q'_i \setminus \mathcal{A}_\mathcal{H}$ with $R'_i \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} Q'_i \setminus \mathcal{A}_\mathcal{H}$. Since synchronization does not apply to τ, we have that $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \Longrightarrow (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}$ with $((R'_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}, (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.

As for delays, suppose $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \;\not\xrightarrow{\tau}_{\rm a}$ so that $Q_i \setminus \mathcal{A}_\mathcal{H} \;\not\xrightarrow{\tau}_{\rm a}$, then from $Q_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} R_1 \,/\, \mathcal{A}_\mathcal{H}$ and $Q_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} R_2 \,/\, \mathcal{A}_\mathcal{H}$ it follows that there exists $R_i \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow \bar{R}_i \,/\, \mathcal{A}_\mathcal{H}$ with $i \in \{1, 2\}$ such that $\bar{R}_i \;\not\xrightarrow{\tau}_{\rm a}$, $Q_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} \bar{R}_1 \,/\, \mathcal{A}_\mathcal{H}$, and $Q_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} \bar{R}_2 \,/\, \mathcal{A}_\mathcal{H}$. Since the sychronization operator does not apply to τ, we have that $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow (\bar{R}_1 \parallel_L \bar{R}_2) \,/\, \mathcal{A}_\mathcal{H}$ with $(\bar{R}_1 \parallel_L \bar{R}_2) \,/\, \mathcal{A}_\mathcal{H} \;\not\xrightarrow{\tau}_{\rm a}$ and $((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (\bar{R}_1 \parallel_L \bar{R}_2) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$. If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{t}_{\rm t} (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}$ then we have $Q_i \setminus \mathcal{A}_\mathcal{H} \xrightarrow{t}_{\rm t} Q'_i \setminus \mathcal{A}_\mathcal{H}$ and hence $\bar{R}_i \,/\, \mathcal{A}_\mathcal{H} \stackrel{t}{\Longrightarrow} R'_i \,/\, \mathcal{A}_\mathcal{H}$ with $Q'_i \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} R'_i \,/\, \mathcal{A}_\mathcal{H}$. Since the hiding operator does not apply to timed transitions, and since they can be splitted and merged by *TimeSplit* and *TimeMerge* we have that $(\bar{R}_1 \parallel_L \bar{R}_2) \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow (R'_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}$ with $((Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}, (R'_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
If we suppose that $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \;\not\xrightarrow{\tau}_{\rm a}$, then the reasoning is similar to the previous case.

4. Given an arbitrary $P \in \text{SBSNNI}_{\approx_{\rm tw}}$ and an arbitrary $L \subseteq \mathcal{A}$, the result follows by proving that the symmetric relation $\mathcal{B} = \{((Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L, (R \setminus L) \,/\, \mathcal{A}_\mathcal{H}), ((R \setminus L) \,/\, \mathcal{A}_\mathcal{H}, (Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L) \mid Q, R \in \mathit{reach}(P) \wedge Q \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} R \setminus \mathcal{A}_\mathcal{H}\}$ is a weak timed bisimulation, as can be seen by taking Q identical to R – which will be denoted by P' – because:

   - $(P' \setminus L) \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} (P' \setminus \mathcal{A}_\mathcal{H}) \setminus L$ as the order in which restriction sets are considered is unimportant.
   - $(P' \setminus \mathcal{A}_\mathcal{H}) \setminus L \approx_{\rm tw} (P' \,/\, \mathcal{A}_\mathcal{H}) \setminus L$ because $P' \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P' \,/\, \mathcal{A}_\mathcal{H}$ – as $P \in \text{SBSNNI}_{\approx_{\rm t}}$ and $P' \in \mathit{reach}(P)$ – and $\approx_{\rm tw}$ is a congruence with respect to the restriction operator due to Lemma 1(3).
   - $(P' \,/\, \mathcal{A}_\mathcal{H}) \setminus L \approx_{\rm tw} (P' \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ as $((P' \,/\, \mathcal{A}_\mathcal{H}) \setminus L, (P' \setminus L) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
   - From the transitivity of $\approx_{\rm tw}$ we obtain that $(P' \setminus L) \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} (P' \setminus L) \,/\, \mathcal{A}_\mathcal{H}$.

   Starting from $(Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L$ and $(R \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ related by $\mathcal{B}$, so that $Q \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} R \setminus \mathcal{A}_\mathcal{H}$, there are six cases for action transitions based on the operational semantic rules in Table 1. In the first three cases, it is $(Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L$ to move first:

   - If $(Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L \xrightarrow{l}_{\rm a} (Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L$ with $Q \xrightarrow{l}_{\rm a} Q'$ and $l \notin L$, then $Q \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} Q' \,/\, \mathcal{A}_\mathcal{H}$ as $l \notin \mathcal{A}_\mathcal{H}$. From $Q \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} R \setminus \mathcal{A}_\mathcal{H}$ it follows that there exists R' such that $R \setminus \mathcal{A}_\mathcal{H} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow R' \setminus \mathcal{A}_\mathcal{H}$ with $Q' \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} R' \setminus \mathcal{A}_\mathcal{H}$. Since the restriction and hiding operators do not apply to τ and l, we have that $(R \setminus L) \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow (R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ with $((Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L, (R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.

- If $(Q \,/\, \mathcal{A_H}) \backslash L \xrightarrow{\tau}_{\rm a} (Q' \,/\, \mathcal{A_H}) \backslash L$ with $Q \xrightarrow{\tau}_{\rm a} Q'$, then $Q \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} Q' \,/\, \mathcal{A_H}$ as $\tau \notin \mathcal{A_H}$. From $Q \,/\, \mathcal{A_H} \approx_{\rm tw} R \setminus \mathcal{A_H}$ it follows that there exists R' such that $R \setminus \mathcal{A_H} \Longrightarrow R' \setminus \mathcal{A_H}$ with $Q' \,/\, \mathcal{A_H} \approx_{\rm tw} R' \setminus \mathcal{A_H}$. Since the restriction and hiding operators do not apply to τ, we have that $(R \setminus L) \,/\, \mathcal{A_H} \Longrightarrow (R' \setminus L) \,/\, \mathcal{A_H}$ with $((Q' \,/\, \mathcal{A_H}) \setminus L, (R' \setminus L) \,/\, \mathcal{A_H}) \in \mathcal{B}$.
- If $(Q \,/\, \mathcal{A_H}) \backslash L \xrightarrow{\tau}_{\rm a} (Q' \,/\, \mathcal{A_H}) \backslash L$ with $Q \xrightarrow{h}_{\rm a} Q'$, then $Q \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} Q' \,/\, \mathcal{A_H}$ as $h \in \mathcal{A_H}$. The rest of the proof is like the one of the previous case.

In the other three cases, instead, it is $(R \setminus L) \,/\, \mathcal{A_H}$ to move first:

- If $(R \setminus L) \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} (R' \setminus L) \,/\, \mathcal{A_H}$ with $R \xrightarrow{l}_{\rm a} R'$ and $l \notin L$, then $R \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} R' \setminus \mathcal{A_H}$ as $l \notin \mathcal{A_H}$. From $R \setminus \mathcal{A_H} \approx_{\rm tw} Q \,/\, \mathcal{A_H}$ it follows that there exists Q' such that $Q \,/\, \mathcal{A_H} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow Q' \,/\, \mathcal{A_H}$ with $R' \setminus \mathcal{A_H} \approx_{\rm tw} Q' \,/\, \mathcal{A_H}$. Since the restriction operator does not apply to τ and l, we have that $(Q \,/\, ,\mathcal{A_H}) \setminus L \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow (Q' \,/\, \mathcal{A_H}) \setminus L$ with $((R' \setminus L) \,/\, \mathcal{A_H}, (Q' \,/\, \mathcal{A_H}) \setminus L) \in \mathcal{B}$.
- If $(R \backslash L) \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (R' \backslash L) \,/\, \mathcal{A_H}$ with $R \xrightarrow{\tau}_{\rm a} R'$, then $R \backslash \mathcal{A_H} \xrightarrow{\tau}_{\rm a} R' \backslash \mathcal{A_H}$ as $\tau \notin \mathcal{A_H}$. From $R \setminus \mathcal{A_H} \approx_{\rm tw} Q \,/\, \mathcal{A_H}$ it follows that there exists Q' such that $Q \,/\, \mathcal{A_H} \Longrightarrow Q' \,/\, \mathcal{A_H}$ with $R' \backslash \mathcal{A_H} \approx_{\rm tw} Q' \,/\, \mathcal{A_H}$. Since the restriction operator does not apply to τ we have that $(Q \,/\, \mathcal{A_H}) \setminus L \Longrightarrow (Q' \,/\, \mathcal{A_H}) \setminus L$ with $((R' \setminus L) \,/\, \mathcal{A_H}, (Q' \,/\, \mathcal{A_H}) \setminus L) \in \mathcal{B}$.
- If $(R \setminus L) \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (R' \setminus L) \,/\, \mathcal{A_H}$ with $R \xrightarrow{h}_{\rm a} R'$ and $h \notin L$, then $R \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} R' \,/\, \mathcal{A_H}$ as $h \in \mathcal{A_H}$ (note that $R \setminus \mathcal{A_H}$ cannot perform h). From $R \,/\, \mathcal{A_H} \approx_{\rm tw} R \setminus \mathcal{A_H}$ – as $P \in \mathrm{SBSNNI}_{\approx_{\rm tw}}$ and $R \in \mathit{reach}(P)$ – and $R \setminus \mathcal{A_H} \approx_{\rm tw} Q \,/\, \mathcal{A_H}$ it follows that there exists Q' such that $Q \,/\, \mathcal{A_H} \Longrightarrow Q' \,/\, \mathcal{A_H}$ with $R' \,/\, \mathcal{A_H} \approx_{\rm tw} Q' \,/\, \mathcal{A_H}$ and hence $R' \setminus \mathcal{A_H} \approx_{\rm tw} Q' \,/\, \mathcal{A_H}$ – as $R' \,/\, \mathcal{A_H} \approx_{\rm tw} R' \setminus \mathcal{A_H}$ due to $P \in \mathrm{SBSNNI}_{\approx_{\rm tw}}$ and $R' \in \mathit{reach}(P)$. Since the restriction operator does not apply to τ, we have that $(Q \,/\, \mathcal{A_H}) \setminus L \Longrightarrow (Q' \,/\, \mathcal{A_H}) \setminus L$ with $((R' \setminus L) \,/\, \mathcal{A_H}, (Q' \,/\, \mathcal{A_H}) \setminus L) \in \mathcal{B}$.

As for delays, suppose $(Q \,/\, \mathcal{A_H}) \setminus L \ \not\xrightarrow{\tau}_{\rm a}$ so that $Q \,/\, \mathcal{A_H} \ \not\xrightarrow{\tau}_{\rm a}$ then from $Q \,/\, \mathcal{A_H} \approx_{\rm tw} R \setminus \mathcal{A_H} \approx_{\rm tw} R \,/\, \mathcal{A_H}$ – as $P \in \mathrm{SBSNNI}_{\approx_{\rm tw}}$ and $R \in \mathit{reach}(P)$ – it follows that there exists $R \,/\, \mathcal{A_H} \Longrightarrow \bar{R} \,/\, \mathcal{A_H}$ such that $\bar{R} \,/\, \mathcal{A_H} \ \not\xrightarrow{\tau}_{\rm a}$ and $Q \,/\, \mathcal{A_H} \approx_{\rm tw} \bar{R} \,/\, \mathcal{A_H} \approx_{\rm tw} \bar{R} \backslash \mathcal{A_H}$. Since restriction and hiding do not operate on τ we have that $(R \setminus L) \,/\, \mathcal{A_H} \Longrightarrow (\bar{R} \setminus L) \,/\, \mathcal{A_H}$ with $(\bar{R} \setminus L) \,/\, \mathcal{A_H} \ \not\xrightarrow{\tau}_{\rm a}$ and $((Q \,/\, \mathcal{A_H}) \setminus L, (\bar{R} \setminus L) \,/\, \mathcal{A_H}) \in \mathcal{B}$. If $(Q \,/\, \mathcal{A_H}) \setminus L \xrightarrow{t}_{\rm t} (Q' \,/\, \mathcal{A_H}) \setminus L$ then we have that $Q_1 \,/\, \mathcal{A_H} \xrightarrow{t}_{\rm t} Q'_1 \,/\, \mathcal{A_H}$ and hence $\bar{R} \setminus \mathcal{A_H} \stackrel{t}{\Longrightarrow} R' \setminus \mathcal{A_H}$ with $R' \setminus \mathcal{A_H} \approx_{\rm tw} Q' \,/\, \mathcal{A_H}$. Since restriction and hiding do not apply to τ and timed transitions, it follows that $(\bar{R} \setminus L) \,/\, \mathcal{A_H} \stackrel{t}{\Longrightarrow} (R' \setminus L) \,/\, \mathcal{A_H}$ with $((Q' \,/\, \mathcal{A_H}) \setminus L, (R' \setminus L) \,/\, \mathcal{A_H}) \in \mathcal{B}$.

If we suppose that $(R \setminus L) \,/\, \mathcal{A_H} \ \not\xrightarrow{\tau}_{\rm a}$ then the reasoning is similar to the previous case.

5. Given an arbitrary $P \in \mathrm{SBSNNI}_{\approx_{\rm tw}}$ and an arbitrary $L \subseteq \mathcal{A_L}$, for every $P' \in \mathit{reach}(P)$ it holds that $P' \setminus \mathcal{A_H} \approx_{\rm tw} P' \,/\, \mathcal{A_H}$, from which we derive that $(P' \setminus \mathcal{A_H}) \,/\, L \approx_{\rm tw} (P' / \mathcal{A_H}) \,/\, L$ because $\approx_{\rm tw}$ is a congruence with respect to the hiding operator (see the proof of Lemma 1). Since $L \cap \mathcal{A_H} = \emptyset$, we

have that $(P' \setminus \mathcal{A}_{\mathcal{H}}) / L$ is isomorphic to $(P' / L) \setminus \mathcal{A}_{\mathcal{H}}$ and $(P' / \mathcal{A}_{\mathcal{H}}) / L$ is isomorphic to $(P' / L) / \mathcal{A}_{\mathcal{H}}$, hence $(P' / L) \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} (P' / L) / \mathcal{A}_{\mathcal{H}}$, i.e., P' / L is $\mathrm{BSNNI}_{\approx_{\rm tw}}$.

We now prove the results for $\mathrm{SBNDC}_{\approx_{\rm tw}}$:

1. Given an arbitrary $P \in \mathrm{SBNDC}_{\approx_{\rm tw}}$ and an arbitrary $a \in \mathcal{A}_\tau \setminus \mathcal{A}_{\mathcal{H}}$, it trivially holds that $a \,.\, P \in \mathrm{SBNDC}_{\approx_{\rm tw}}$.
2. Given an arbitrary $P \in \mathrm{SBNDC}_{\approx_{\rm tw}}$ and an arbitrary $t \in \mathbb{N}_{>0}$, it trivially holds that $(t) \,.\, P \in \mathrm{SBNDC}_{\approx_{\rm tw}}$.
3. Given two arbitrary $P_1, P_2 \in \mathrm{SBNDC}_{\approx_{\rm tw}}$ and an arbitrary $L \subseteq \mathcal{A}$, the result follows by proving that the symmetric relation $\mathcal{B} = \{((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}, (R_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}}), ((R_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}}, (Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}) \mid Q_1 \parallel_L Q_2, R_1 \parallel_L R_2 \in \mathit{reach}(P_1 \parallel_L P_2) \wedge Q_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} R_1 \setminus \mathcal{A}_{\mathcal{H}} \wedge Q_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} R_2 \setminus \mathcal{A}_{\mathcal{H}}\}$ is a weak timed bisimulation, as can be seen by observing that whenever $P'_1 \parallel_L P'_2 \xrightarrow{h}_{\rm a} P''_1 \parallel_L P''_2$ for $P'_1 \parallel_L P'_2 \in \mathit{reach}(P_1 \parallel_L P_2)$:
   - If $P'_1 \xrightarrow{h}_{\rm a} P''_1$, $P''_2 = P'_2$ (hence $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} P''_2 \setminus \mathcal{A}_{\mathcal{H}}$), and $h \notin L$, then from
     $P_1 \in \mathrm{SBNDC}_{\approx_{\rm tw}}$ it follows that $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} P''_1 \setminus \mathcal{A}_{\mathcal{H}}$, which in turn entails that $(P'_1 \parallel_L P'_2) \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} (P''_1 \parallel_L P''_2) \setminus \mathcal{A}_{\mathcal{H}}$ because $\approx_{\rm tw}$ is a congruence with respect to the parallel composition operator due to Lemma 1(2) and restriction distributes over parallel composition.
   - If $P'_2 \xrightarrow{h}_{\rm a} P''_2$, $P''_1 = P'_1$, and $h \notin L$, then we reason like in the previous case.
   - If $P'_1 \xrightarrow{h}_{\rm a} P''_1$, $P'_2 \xrightarrow{h}_{\rm a} P''_2$, and $h \in L$, then from $P_1, P_2 \in \mathrm{SBNDC}_{\approx_{\rm tw}}$ it follows that $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} P''_1 \setminus \mathcal{A}_{\mathcal{H}}$ and $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} P''_2 \setminus \mathcal{A}_{\mathcal{H}}$, which in turn entail that $(P'_1 \parallel_L P'_2) \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} (P''_1 \parallel_L P''_2) \setminus \mathcal{A}_{\mathcal{H}}$ because $\approx_{\rm tw}$ is a congruence with respect to the parallel composition operator due to Lemma 1(2) and restriction distributes over parallel composition.

   Assuming that $((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}, (R_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$, there are five cases:
   - If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}$ with $Q_1 \xrightarrow{l}_{\rm a} Q'_1$ and $l \notin L$, then $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} Q'_1 \setminus \mathcal{A}_{\mathcal{H}}$ as $l \notin \mathcal{A}_{\mathcal{H}}$. From $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} R_1 \setminus \mathcal{A}_{\mathcal{H}}$ it follows that there exists R'_1 such that $R_1 \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow R'_1 \setminus \mathcal{A}_{\mathcal{H}}$ with $Q'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} R'_1 \setminus \mathcal{A}_{\mathcal{H}}$. Since synchronization does not apply to τ, we have that $(R_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow (R'_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}}$ with and $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}, (R'_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$.
   - If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} (Q_1 \parallel_L Q'_2) \setminus \mathcal{A}_{\mathcal{H}}$ with $Q_2 \xrightarrow{l}_{\rm a} Q'_2$ and $l \notin L$, then the proof is similar to the one of the previous case.
   - If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_{\mathcal{H}}$ with $Q_i \xrightarrow{l}_{\rm a} Q'_i$ for $i \in \{1, 2\}$ and $l \in L$, then $Q_i \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} Q'_i \setminus \mathcal{A}_{\mathcal{H}}$ as $l \notin \mathcal{A}_{\mathcal{H}}$. From $Q_i \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} R_i \setminus \mathcal{A}_{\mathcal{H}}$ it follows that there exists R'_i such that $R_i \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow R'_i \setminus \mathcal{A}_{\mathcal{H}}$ with $Q'_i \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} R'_i \setminus \mathcal{A}_{\mathcal{H}}$. Since synchronization does not apply to τ, we have that $(R_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow \xrightarrow{l}_{\rm a} \Longrightarrow (R'_1 \parallel_L R'_2) \setminus \mathcal{A}_{\mathcal{H}}$ with $((Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_{\mathcal{H}}, (R'_1 \parallel_L R'_2) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$.

- If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}$ with $Q_1 \xrightarrow{\tau}_{\rm a} Q'_1$, then $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} Q'_1 \setminus \mathcal{A}_{\mathcal{H}}$ as $\tau \notin \mathcal{A}_{\mathcal{H}}$. From $Q_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} R_1 \setminus \mathcal{A}_{\mathcal{H}}$ it follows that there exists R'_1 such that $R_1 \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow R'_1 \setminus \mathcal{A}_{\mathcal{H}}$ with $Q'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} R'_1 \setminus \mathcal{A}_{\mathcal{H}}$. Since synchronization does not apply to τ, we have that $(R_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow (R'_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}}$ with $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}, (R'_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$.
- If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} (Q_1 \parallel_L Q'_2) \setminus \mathcal{A}_{\mathcal{H}}$ with $Q_2 \xrightarrow{\tau}_{\rm a} Q'_2$, then the proof is similar to the one of the previous case.

As for delays, suppose that $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \;\not\xrightarrow{\tau}_{\rm a}$ so that $Q_i \setminus \mathcal{A}_{\mathcal{H}} \;\not\xrightarrow{\tau}_{\rm a}$ for $i \in \{1, 2\}$ then from $Q_i \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} R_i \setminus \mathcal{A}_{\mathcal{H}}$ it follows that there exist $R_i \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow \bar{R}_i \setminus \mathcal{A}_{\mathcal{H}}$ with $\bar{R}_i \setminus \mathcal{A}_{\mathcal{H}} \not\xrightarrow{\tau}_{\rm a}$ and $Q_i \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} \bar{R}_i \setminus \mathcal{A}_{\mathcal{H}}$. Since syncrhonization do not apply to τ, we have that $(R_1 \parallel_L R_2) \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow (\bar{R}_1 \parallel_L \bar{R}_2) \setminus \mathcal{A}_{\mathcal{H}}$ with $(\bar{R}_1 \parallel_L \bar{R}_2) \setminus \mathcal{A}_{\mathcal{H}} \not\xrightarrow{\tau}_{\rm a}$ and $((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}, (\bar{R}_1 \parallel_L \bar{R}_2) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$. If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{t}_{\rm t} (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_{\mathcal{H}}$ then $Q_i \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{t}_{\rm t} Q'_i \setminus \mathcal{A}_{\mathcal{H}}$ and we have $\bar{R}_i \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow R'_i \setminus \mathcal{A}_{\mathcal{H}}$ with $Q'_i \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} R'_i \setminus \mathcal{A}_{\mathcal{H}}$. Since the restriction operator does not apply to timed transitions, and since they can be splitted and merged by *TimeSplit* and *TimeMerge* we have that $(\bar{R}_1 \parallel_L \bar{R}_2) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{t}{\Longrightarrow} (R'_1 \parallel_L R'_2) \setminus \mathcal{A}_{\mathcal{H}}$ with $((Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_{\mathcal{H}}, (R'_1 \parallel_L R'_2) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$.

4. Given an arbitrary $P \in \mathrm{SBNDC}_{\approx_{\rm tw}}$ and an arbitrary $L \subseteq \mathcal{A}$, for every $P' \in \mathit{reach}(P)$ and for every P'' such that $P' \xrightarrow{h}_{\rm a} P''$ it holds that $P' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} P'' \setminus \mathcal{A}_{\mathcal{H}}$, from which we derive that $(P' \setminus \mathcal{A}_{\mathcal{H}}) \setminus L \approx_{\rm tw} (P'' \setminus \mathcal{A}_{\mathcal{H}}) \setminus L$ because $\approx_{\rm tw}$ is a congruence with respect to the restriction operator (see the proof of Lemma 1). Since $(P' \setminus \mathcal{A}_{\mathcal{H}}) \setminus L$ is isomorphic to $(P' \setminus L) \setminus \mathcal{A}_{\mathcal{H}}$ and $(P'' \setminus \mathcal{A}_{\mathcal{H}}) \setminus L$ is isomorphic to $(P'' \setminus L) \setminus \mathcal{A}_{\mathcal{H}}$, we have that $(P' \setminus L) \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} (P'' \setminus L) \setminus \mathcal{A}_{\mathcal{H}}$.
5. Given an arbitrary $P \in \mathrm{SBNDC}_{\approx_{\rm tw}}$ and an arbitrary $L \subseteq \mathcal{A}_{\mathcal{L}}$, for every $P' \in \mathit{reach}(P)$ and for every P'' such that $P' \xrightarrow{h}_{\rm a} P''$ it holds that $P' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} P'' \setminus \mathcal{A}_{\mathcal{H}}$, from which we derive that $(P' \setminus \mathcal{A}_{\mathcal{H}}) \,/\, L \approx_{\rm tw} (P'' \setminus \mathcal{A}_{\mathcal{H}}) \,/\, L$ because $\approx_{\rm tw}$ is a congruence with respect to the hiding operator (see the proof of Lemma 1). Since $L \cap \mathcal{A}_{\mathcal{H}} = \emptyset$, we have that $(P' \setminus \mathcal{A}_{\mathcal{H}}) \,/\, L$ is isomorphic to $(P' \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ and $(P'' \setminus \mathcal{A}_{\mathcal{H}}) \,/\, L$ is isomorphic to $(P'' \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$, hence $(P' \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tw} (P'' \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$.

We now prove the same result for the $\approx_{\rm tb}$-based properties. As for the first part of the proof, we first prove the results for $\mathrm{SBSNNI}_{\approx_{\rm tb}}$, and hence for $\mathrm{P\_BNDC}_{\approx_{\rm tb}}$ too by virtue of the forthcoming Theorem 3:

1. Given an arbitrary $P \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$ and an arbitrary $a \in \mathcal{A}_{\mathcal{L}} \cup \{\tau\}$, we proceed as in the $\approx_{\rm tw}$ case.
2. Given an arbitrary $P \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$ and an arbitrary $t \in \mathbb{N}_{>0}$, we proceed as in the $\approx_{\rm tw}$ case.
3. Given two arbitrary $P_1, P_2 \in \mathbb{P}$ such that $Q_1, Q_2 \in \mathit{reach}(P_1)$, $R_1, R_2 \in \mathit{reach}(P_2)$, and arbitrary $L \subseteq \mathcal{A}_{\mathcal{L}}$ the result follows by proving that the symmetric relation $\mathcal{B} = \{((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}, (R_1 \parallel_L R_2) \,/\, \mathcal{A}_{\mathcal{H}}) \mid Q_1 \parallel_L Q_2) \in \mathit{reach}(P_1 \parallel_L P_2) \wedge (R_1 \parallel_L R_2 \in \mathit{reach}(P_1 \parallel_L P_2) \wedge Q_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} R_1 \,/\, \mathcal{A}_{\mathcal{H}} \wedge Q_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} R_2 \,/\, \mathcal{A}_{\mathcal{H}}\}$ by taking Q_1 identical to R_1 and Q_2 identical to R_2. There are thirteen cases for action transitions based on the operational semantic rules in Table 1. In the first five cases, it is $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_{\mathcal{H}}$ to move first:

- If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_1 \xrightarrow{l}_{\rm a} Q'_1$ and $l \notin L$, then $Q_1 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} Q'_1 \setminus \mathcal{A}_\mathcal{H}$ as $l \notin \mathcal{A}_\mathcal{H}$. From $Q_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} R_1 \,/\, \mathcal{A}_\mathcal{H}$ it follows that there exist $\bar{R}_1$ and R'_1 such that $R_1 \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow \bar{R}_1 \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} R'_1 \,/\, \mathcal{A}_\mathcal{H}$ with $Q_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} \bar{R}_1 \,/\, \mathcal{A}_\mathcal{H}$ and $Q'_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} R'_1 \,/\, \mathcal{A}_\mathcal{H}$. Since synchronization does not apply to τ and l, we have that $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow (\bar{R}_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}$ with $((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (\bar{R}_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ and $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (R'_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
- If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (Q_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_2 \xrightarrow{l}_{\rm a} Q'_2$ and $l \notin L$, then the proof is similar to the one of the previous case.
- If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_i \xrightarrow{l}_{\rm a} Q'_i$ for $i \in \{1, 2\}$ and $l \in L$, then $Q_i \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} Q'_i \setminus \mathcal{A}_\mathcal{H}$ as $l \notin \mathcal{A}_\mathcal{H}$. From $Q_i \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} R_i \,/\, \mathcal{A}_\mathcal{H}$ it follows that there exist $\bar{R}_i$ and R'_i such that $R_i \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow \bar{R}_i \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} R'_i \,/\, \mathcal{A}_\mathcal{H}$ with $Q_i \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} \bar{R}_i \,/\, \mathcal{A}_\mathcal{H}$ and $Q'_i \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} R'_i \,/\, \mathcal{A}_\mathcal{H}$. Since synchronization does not apply to τ, we have that $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow (\bar{R}_1 \parallel_L \bar{R}_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (R'_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}$ with $((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (\bar{R}_1 \parallel_L \bar{R}_2) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ and $((Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}, (R'_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
- If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_1 \xrightarrow{\tau}_{\rm a} Q'_1$, then $Q_1 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} Q'_1 \setminus \mathcal{A}_\mathcal{H}$ as $\tau \notin \mathcal{A}_\mathcal{H}$. From $Q_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} R_1 \,/\, \mathcal{A}_\mathcal{H}$ it follows that either $Q'_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} R_1 \,/\, \mathcal{A}_\mathcal{H}$, or there exist $\bar{R}_1$ and R'_1 such that $R_1 \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow \bar{R}_1 \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} R'_1 \,/\, \mathcal{A}_\mathcal{H}$ with $Q_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} \bar{R}_1 \,/\, \mathcal{A}_\mathcal{H}$ and $Q'_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} R'_1 \,/\, \mathcal{A}_\mathcal{H}$. In the former subcase $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}$ is allowed to stay idle with $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$, while in the latter subcase, since synchronization does not apply to τ, we have that $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow (\bar{R}_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}$ with $((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (\bar{R}_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ and $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (R'_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
- If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (Q_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_2 \xrightarrow{\tau}_{\rm a} Q'_2$, then the proof is similar to the one of the previous case.

In the other seven cases, instead, it is $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}$ to move first:

- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H}$ with $R_1 \xrightarrow{l}_{\rm a} R'_1$ and $l \notin L$, then $R_1 \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} R'_1 \,/\, \mathcal{A}_\mathcal{H}$ as $l \notin \mathcal{A}_\mathcal{H}$. From $R_1 \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} Q_1 \setminus \mathcal{A}_\mathcal{H}$ it follows that there exist $\bar{Q}_1$ and Q'_1 such that $Q_1 \setminus \mathcal{A}_\mathcal{H} \Longrightarrow \bar{Q}_1 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} Q'_1 \setminus \mathcal{A}_\mathcal{H}$ with $R_1 \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} \bar{Q}_1 \setminus \mathcal{A}_\mathcal{H}$ and $R'_1 \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} Q'_1 \setminus \mathcal{A}_\mathcal{H}$. Since synchronization does not apply to τ and l, we have that $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \Longrightarrow (\bar{Q}_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}$ with $((R_1 \parallel_L R_2) / \mathcal{A}_\mathcal{H}, (\bar{Q}_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ and $((R'_1 \parallel_L R_2) / \mathcal{A}_\mathcal{H}, (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (R_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}$ with $R_2 \xrightarrow{l}_{\rm a} R'_2$ and $l \notin L$, then the proof is similar to the one of the previous case.
- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (R'_1 \parallel_L R'_2) \,/\, \mathcal{A}_\mathcal{H}$ with $R_i \xrightarrow{l}_{\rm a} R'_i$ for $i \in \{1, 2\}$ and $l \in L$, then $R_i \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} R'_i \,/\, \mathcal{A}_\mathcal{H}$ as $l \notin \mathcal{A}_\mathcal{H}$. From $R_i \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} Q_i \setminus \mathcal{A}_\mathcal{H}$ it follows that there exist $\bar{Q}_i$ and Q'_i such that $Q_i \setminus \mathcal{A}_\mathcal{H} \Longrightarrow \bar{Q}_i \setminus$

$\mathcal{A_H} \xrightarrow{l}_{\rm a} Q'_i \setminus \mathcal{A_H}$ with $R_i \,/\, \mathcal{A_H} \approx_{\rm tb} \bar{Q}_i \setminus \mathcal{A_H}$ and $R'_i \,/\, \mathcal{A_H} \approx_{\rm tb} Q'_i \setminus \mathcal{A_H}$. Since synchronization does not apply to τ, we have that $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \Longrightarrow (\bar{Q}_1 \parallel_L \bar{Q}_2) \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A_H}$ with $((R_1 \parallel_L R_2)/\mathcal{A_H}, (\bar{Q}_1 \parallel_L \bar{Q}_2) \setminus \mathcal{A_H}) \in \mathcal{B}$ and $((R'_1 \parallel_L R'_2) \,/\, \mathcal{A_H}, (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A_H}) \in \mathcal{B}$.
- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}$ with $R_1 \xrightarrow{\tau}_{\rm a} R'_1$, then $R_1 \,/\mathcal{A_H} \xrightarrow{\tau}_{\rm a} R'_1 \,/\mathcal{A_H}$ as $\tau \notin \mathcal{A_H}$. From $R_1 \,/\, \mathcal{A_H} \approx_{\rm tb} Q_1 \setminus \mathcal{A_H}$ it follows that either $R'_1 \,/\, \mathcal{A_H} \approx_{\rm tb} Q_1 \setminus \mathcal{A_H}$, or there exist $\bar{Q}_1$ and Q'_1 such that $Q_1 \setminus \mathcal{A_H} \Longrightarrow \bar{Q}_1 \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} Q'_1 \setminus \mathcal{A_H}$ with $R_1 \,/\, \mathcal{A_H} \approx_{\rm tb} \bar{Q}_1 \setminus \mathcal{A_H}$ and $R'_1 \,/\, \mathcal{A_H} \approx_{\rm tb} Q'_1 \setminus \mathcal{A_H}$. In the former subcase $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H}$ is allowed to stay idle with $((R'_1 \parallel_L R_2)/\mathcal{A_H}, (Q_1 \parallel_L Q_2) \setminus \mathcal{A_H}) \in \mathcal{B}$, while in the latter subcase, since synchronization does not apply to τ, we have that $(Q_1 \parallel_L Q_2) \setminus \mathcal{A_H} \Longrightarrow (\bar{Q}_1 \parallel_L Q_2) \setminus \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}$ with $((R_1 \parallel_L R_2)/\mathcal{A_H}, (\bar{Q}_1 \parallel_L Q_2) \setminus \mathcal{A_H}) \in \mathcal{B}$ and $((R'_1 \parallel_L R_2)/\mathcal{A_H}, (Q'_1 \parallel_L Q_2) \setminus \mathcal{A_H}) \in \mathcal{B}$.
- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (R_1 \parallel_L R'_2) \,/\, \mathcal{A_H}$ with $R_2 \xrightarrow{\tau}_{\rm a} R'_2$, then the proof is similar to the one of the previous case.
- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (R'_1 \parallel_L R_2) \,/\, \mathcal{A_H}$ with $R_1 \xrightarrow{h}_{\rm a} R'_1$ and $h \notin L$, then $R_1 \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} R'_1 \,/\, \mathcal{A_H}$ as $h \in \mathcal{A_H}$. The rest of the proof is like the one of the fourth case.
- If $(R_1 \parallel_L R_2) \,/\, \mathcal{A_H} \xrightarrow{\tau}_{\rm a} (R_1 \parallel_L R'_2) \,/\, \mathcal{A_H}$ with $R_2 \xrightarrow{h}_{\rm a} R'_2$ and $h \notin L$, then the proof is similar to the one of the previous case.

As for delays, then the proof is similar to the one of the $\approx_{\rm tw}$ case.

4. Given an arbitrary $P \in \text{SBSNNI}_{\approx_{\rm tb}}$ and an arbitrary $L \subseteq \mathcal{A}$, the result follows by proving that the symmetric relation $\mathcal{B} = \{((P_1 \,/\, \mathcal{A_H}) \setminus L, (P_2 \setminus L) \,/\, \mathcal{A_H}), ((P_2 \setminus L) \,/\, \mathcal{A_H}, (P_1 \,/\, \mathcal{A_H}) \setminus L) \mid P_1, P_2 \in \mathit{reach}(P) \land P_1 \,/\, \mathcal{A_H} \approx_{\rm tb} P_2 \setminus \mathcal{A_H}\}$ is a timed branching bisimulation, as can be seen by taking P_1 identical to P_2 – which will be denoted by P' – because:
   - $(P' \setminus L) \setminus \mathcal{A_H} \approx_{\rm tb} (P' \setminus \mathcal{A_H}) \setminus L$ as the order in which restriction sets are considered is unimportant.
   - $(P' \setminus \mathcal{A_H}) \setminus L \approx_{\rm tb} (P' \,/\, \mathcal{A_H}) \setminus L$ due to $P' \setminus \mathcal{A_H} \approx_{\rm tb} P' \,/\, \mathcal{A_H}$ – as $P \in \text{SBSNNI}_{\approx_{\rm tb}}$ and $P' \in \mathit{reach}(P)$ – and $\approx_{\rm tb}$ being a congruence with respect to the restriction operator (see the proof of Lemma 1).
   - $(P' \,/\, \mathcal{A_H}) \setminus L \approx_{\rm tb} (P' \setminus L) \,/\, \mathcal{A_H}$ as $((P' \,/\, \mathcal{A_H}) \setminus L, (P' \setminus L) \,/\, \mathcal{A_H}) \in \mathcal{B}$.
   - From the transitivity of $\approx_{\rm tb}$ it follows that $(P' \setminus L) \setminus \mathcal{A_H} \approx_{\rm tb} (P' \setminus L) \,/\, \mathcal{A_H}$.

   Starting from $(Q \,/\, \mathcal{A_H}) \setminus L$ and $(R \setminus L) \,/\, \mathcal{A_H}$ related by $\mathcal{B}$, so that $Q \,/\, \mathcal{A_H} \approx_{\rm tb} R \setminus \mathcal{A_H}$, there are six cases for action transitions based on the operational semantic rules in Table 1. In the first three cases, it is $(Q \,/\, \mathcal{A_H}) \setminus L$ to move first:
   - If $(Q \,/\, \mathcal{A_H}) \setminus L \xrightarrow{l}_{\rm a} (Q' \,/\, \mathcal{A_H}) \setminus L$ with $Q \xrightarrow{l}_{\rm a} Q'$ and $l \notin L$, then $Q \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} Q' \,/\, \mathcal{A_H}$ as $l \notin \mathcal{A_H}$. From $Q \,/\, \mathcal{A_H} \approx_{\rm tb} R \setminus \mathcal{A_H}$ it follows that there exist $\bar{R}$ and R' such that $R \setminus \mathcal{A_H} \Longrightarrow \bar{R} \setminus \mathcal{A_H} \xrightarrow{l}_{\rm a} R' \setminus \mathcal{A_H}$ with $Q \,/\, \mathcal{A_H} \approx_{\rm tb} \bar{R} \setminus \mathcal{A_H}$ and $Q' \,/\, \mathcal{A_H} \approx_{\rm tb} R' \setminus \mathcal{A_H}$. Since the restriction and hiding operators do not apply to τ and l, we have that $(R \setminus L) \,/\, \mathcal{A_H} \Longrightarrow (\bar{R} \setminus L) \,/\, \mathcal{A_H} \xrightarrow{l}_{\rm a} (R' \setminus L) \,/\, \mathcal{A_H}$ with $((Q \,/\, \mathcal{A_H}) \setminus L, (\bar{R} \setminus L) \,/\, \mathcal{A_H}) \in \mathcal{B}$ and $((Q' \,/\, \mathcal{A_H}) \setminus L, (R' \setminus L) \,/\, \mathcal{A_H}) \in \mathcal{B}$.

- If $(Q \,/\, \mathcal{A}_\mathcal{H}) \backslash L \xrightarrow{\tau}_{\rm a} (Q' \,/\, \mathcal{A}_\mathcal{H}) \backslash L$ with $Q \xrightarrow{\tau}_{\rm a} Q'$, then $Q \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} Q' \,/\, \mathcal{A}_\mathcal{H}$ as $\tau \notin \mathcal{A}_\mathcal{H}$. From $Q \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} R \backslash \mathcal{A}_\mathcal{H}$ it follows that either $Q' \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} R \backslash \mathcal{A}_\mathcal{H}$, or there exist $\bar{R}$ and R' such that $R \backslash \mathcal{A}_\mathcal{H} \Longrightarrow \bar{R} \backslash \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} R' \backslash \mathcal{A}_\mathcal{H}$ with $Q \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} \bar{R} \backslash \mathcal{A}_\mathcal{H}$ and $Q' \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} R' \backslash \mathcal{A}_\mathcal{H}$. In the former subcase $(R \backslash L) \,/\, \mathcal{A}_\mathcal{H}$ is allowed to stay idle with $((Q' \,/\, \mathcal{A}_\mathcal{H}) \backslash L, (R \backslash L) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$, while in the latter subcase, since the restriction and hiding operators do not apply to τ, we have that $(R \setminus L) \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow (\bar{R} \setminus L) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ with $((Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L, (\bar{R} \setminus L) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ and $((Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L, (R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
- If $(Q \,/\, \mathcal{A}_\mathcal{H}) \backslash L \xrightarrow{\tau}_{\rm a} (Q' \,/\, \mathcal{A}_\mathcal{H}) \backslash L$ with $Q \xrightarrow{h}_{\rm a} Q'$, then $Q \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} Q' \,/\, \mathcal{A}_\mathcal{H}$ as $h \in \mathcal{A}_\mathcal{H}$. The rest of the proof is like the one of the previous case.

In the other three cases, instead, it is $(R \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ to move first:

- If $(R \setminus L) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} (R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ with $R \xrightarrow{l}_{\rm a} R'$ and $l \notin L$, then $R \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} R' \setminus \mathcal{A}_\mathcal{H}$ as $l \notin \mathcal{A}_\mathcal{H}$. From $R \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} Q \,/\, \mathcal{A}_\mathcal{H}$ it follows that there exist $\bar{Q}$ and Q' such that $Q \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow \bar{Q} \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} Q' \,/\, \mathcal{A}_\mathcal{H}$ with $R \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} \bar{Q} \,/\, \mathcal{A}_\mathcal{H}$ and $R' \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} Q' \,/\, \mathcal{A}_\mathcal{H}$. Since the restriction operator does not apply to τ and l, we have that $(Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L \Longrightarrow (\bar{Q} \,/\, \mathcal{A}_\mathcal{H}) \setminus L \xrightarrow{l}_{\rm a} (Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L$ with $((R \setminus L) \,/\, \mathcal{A}_\mathcal{H}, (\bar{Q} \,/\, \mathcal{A}_\mathcal{H}) \setminus L) \in \mathcal{B}$ and $((R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}, (Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L) \in \mathcal{B}$.
- If $(R \backslash L) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (R' \backslash L) \,/\, \mathcal{A}_\mathcal{H}$ with $R \xrightarrow{\tau}_{\rm a} R'$, then $R \backslash \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} R' \backslash \mathcal{A}_\mathcal{H}$ as $\tau \notin \mathcal{A}_\mathcal{H}$. From $R \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} Q \,/\, \mathcal{A}_\mathcal{H}$ it follows that either $R' \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} Q \,/\, \mathcal{A}_\mathcal{H}$, or there exist $\bar{Q}$ and Q' such that $Q \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow \bar{Q} \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} Q' \,/\, \mathcal{A}_\mathcal{H}$ with $R \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} \bar{Q} \,/\, \mathcal{A}_\mathcal{H}$ and $R' \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} Q' \,/\, \mathcal{A}_\mathcal{H}$. In the former subcase $(Q \,/\, \mathcal{A}_\mathcal{H}) \backslash L$ is allowed to stay idle with $((R' \backslash L) \,/\, \mathcal{A}_\mathcal{H}, (Q \,/\, \mathcal{A}_\mathcal{H}) \backslash L) \in \mathcal{B}$, while in the latter subcase, since the restriction operator does not apply to τ we have that $(Q \,/\, \mathcal{A}_\mathcal{H}) \backslash L \Longrightarrow (\bar{Q} \,/\, \mathcal{A}_\mathcal{H}) \backslash L \xrightarrow{\tau}_{\rm a} (Q' \,/\, \mathcal{A}_\mathcal{H}) \backslash L$ with $((R \backslash L) \,/\, \mathcal{A}_\mathcal{H}, (\bar{Q} \,/\, \mathcal{A}_\mathcal{H}) \backslash L) \in \mathcal{B}$ and $((R' \backslash L) \,/\, \mathcal{A}_\mathcal{H}, (Q' \,/\, \mathcal{A}_\mathcal{H}) \backslash L) \in \mathcal{B}$.
- If $(R \setminus L) \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}$ with $R \xrightarrow{h}_{\rm a} R'$ and $h \notin L$, then $R \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} R' \,/\, \mathcal{A}_\mathcal{H}$ as $h \in \mathcal{A}_\mathcal{H}$ (note that $R \setminus \mathcal{A}_\mathcal{H}$ cannot perform h). From $R \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} R \setminus \mathcal{A}_\mathcal{H}$ – as $P \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$ and $R \in \mathit{reach}(P)$ – and $R \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} Q \,/\, \mathcal{A}_\mathcal{H}$ it follows that either $R' \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} Q \,/\, \mathcal{A}_\mathcal{H}$ and hence $R' \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} Q \,/\, \mathcal{A}_\mathcal{H}$ – as $R' \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} R' \setminus \mathcal{A}_\mathcal{H}$ due to $P \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$ and $R' \in \mathit{reach}(P)$ – or there exist $\bar{Q}$ and Q' such that $Q \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow \bar{Q} \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} Q' \,/\, \mathcal{A}_\mathcal{H}$ with $R \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} \bar{Q} \,/\, \mathcal{A}_\mathcal{H}$ and $R' \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} Q' \,/\, \mathcal{A}_\mathcal{H}$ and hence $R \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} \bar{Q} \,/\, \mathcal{A}_\mathcal{H}$ and $R' \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} Q' \,/\, \mathcal{A}_\mathcal{H}$. In the former subcase $(Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L$ is allowed to stay idle with $((R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}, (Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L) \in \mathcal{B}$, while in the latter subcase, since the restriction operator does not apply to τ transitions, we have that $(Q \,/\, \mathcal{A}_\mathcal{H}) \setminus L \Longrightarrow (\bar{Q} \,/\, \mathcal{A}_\mathcal{H}) \backslash L \xrightarrow{\tau}_{\rm a} (Q' \,/\, \mathcal{A}_\mathcal{H}) \backslash L$ with $((R \backslash L) \,/\, \mathcal{A}_\mathcal{H}, (\bar{Q} \,/\, \mathcal{A}_\mathcal{H}) \backslash L) \in \mathcal{B}$ and $((R' \setminus L) \,/\, \mathcal{A}_\mathcal{H}, (Q' \,/\, \mathcal{A}_\mathcal{H}) \setminus L) \in \mathcal{B}$.

As for delays, we proceed as in the $\approx_{\rm tw}$ case.

5. Given an arbitrary $P \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$ and an arbitrary $L \subseteq \mathcal{A}_\mathcal{L}$ we proceed as in the $\approx_{\rm tw}$ case.

We now prove the result for the $\text{SBNDC}_{\approx_{\text{tb}}}$ properties.

1. Given an arbitrary $P \in \text{SBNDC}_{\approx_{\text{tb}}}$ and an arbitrary $a \in \mathcal{A}_\tau \setminus \mathcal{A}_\mathcal{H}$, it trivially holds that $a \,.\, P \in \text{SBNDC}_{\approx_{\text{tb}}}$.
2. Given an arbitrary $P \in \text{SBNDC}_{\approx_{\text{tb}}}$ and an arbitrary $t \in \mathbb{N}_{>0}$, it trivially holds that $(\lambda) \,.\, P \in \text{SBNDC}_{\approx_{\text{tb}}}$.
3. Assuming that $((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (R_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$, there are five cases:
   - If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\text{a}} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_1 \xrightarrow{l}_{\text{a}} Q'_1$ and $l \notin L$, then $Q_1 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\text{a}} Q'_1 \setminus \mathcal{A}_\mathcal{H}$ as $l \notin \mathcal{A}_\mathcal{H}$. From $Q_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\text{tb}} R_1 \setminus \mathcal{A}_\mathcal{H}$ it follows that there exist $\bar{R}_1$ and R'_1 such that $R_1 \setminus \mathcal{A}_\mathcal{H} \Longrightarrow_{\text{a}} \bar{R}_1 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\text{a}} R'_1 \setminus \mathcal{A}_\mathcal{H}$ with $Q_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\text{tb}} \bar{R}_1 \setminus \mathcal{A}_\mathcal{H}$ and $Q'_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\text{tb}} R'_1 \setminus \mathcal{A}_\mathcal{H}$. Since synchronization does not apply to τ, it follows that $(R_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H} \Longrightarrow_{\text{a}} (\bar{R}_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\text{a}} (R'_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H}$ with $((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (\bar{R}_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ and $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (R'_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
   - If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\text{a}} (Q_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_2 \xrightarrow{l}_{\text{a}} Q'_2$ and $l \notin L$, then the proof is similar to the one of the previous case.
   - If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\text{a}} (Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_i \xrightarrow{l}_{\text{a}} Q'_i$ for $i \in \{1, 2\}$ and $l \in L$, then $Q_i \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\text{a}} Q'_i \setminus \mathcal{A}_\mathcal{H}$ as $l \notin \mathcal{A}_\mathcal{H}$. From $Q_i \setminus \mathcal{A}_\mathcal{H} \approx_{\text{b}} R_i \setminus \mathcal{A}_\mathcal{H}$ it follows that there exist $\bar{R}_i$ and R'_i such that $R_i \setminus \mathcal{A}_\mathcal{H} \Longrightarrow_{\text{a}} \bar{R}_i \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\text{a}} R'_i \setminus \mathcal{A}_\mathcal{H}$ with $Q_i \setminus \mathcal{A}_\mathcal{H} \approx_{\text{tb}} \bar{R}_i \setminus \mathcal{A}_\mathcal{H}$ and $Q'_i \setminus \mathcal{A}_\mathcal{H} \approx_{\text{tb}} R'_i \setminus \mathcal{A}_\mathcal{H}$. Since synchronization does not apply to τ, it follows that $(R_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H} \Longrightarrow_{\text{a}} (\bar{R}_1 \parallel_L \bar{R}_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\text{a}} (R'_1 \parallel_L R'_2) \setminus \mathcal{A}_\mathcal{H}$ with $((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (\bar{R}_1 \parallel_L \bar{R}_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ and $((Q'_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}, (R'_1 \parallel_L R'_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
   - If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\text{a}} (Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_1 \xrightarrow{\tau}_{\text{a}} Q'_1$, then $Q_1 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\text{a}} Q'_1 \setminus \mathcal{A}_\mathcal{H}$. From $Q_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\text{tb}} R_1 \setminus \mathcal{A}_\mathcal{H}$ it follows that either $Q'_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\text{tb}} R_1 \setminus \mathcal{A}_\mathcal{H}$, or there exist $\bar{R}_1$ and R'_1 such that $R_1 \setminus \mathcal{A}_\mathcal{H} \Longrightarrow_{\text{a}} \bar{R}_1 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\text{a}} R'_1 \setminus \mathcal{A}_\mathcal{H}$ with $Q_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\text{tb}} \bar{R}_1 \setminus \mathcal{A}_\mathcal{H}$ and $Q'_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\text{tb}} R'_1 \setminus \mathcal{A}_\mathcal{H}$. In the former subcase $(R_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H}$ is allowed to stay idle with $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (R_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$, while in the latter subcase, since synchronization does not apply to τ, it follows that $(R_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H} \Longrightarrow_{\text{a}} (\bar{R}_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\text{a}} (R'_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H}$ with $((Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (\bar{R}_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ and $((Q'_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H}, (R'_1 \parallel_L R_2) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.
   - If $(Q_1 \parallel_L Q_2) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\text{a}} (Q_1 \parallel_L Q'_2) \setminus \mathcal{A}_\mathcal{H}$ with $Q_2 \xrightarrow{\tau}_{\text{a}} Q'_2$, then the proof is similar to the one of the previous case.

   As for delay, we proceed as in the $\approx_{\text{tw}}$ case.
4. Given an arbitrary $P \in \text{SBNDC}_{\approx_{\text{tb}}}$ and an arbitrary $L \subseteq \mathcal{A}$, we proceed as in the $\approx_{\text{tw}}$ case.
5. Given an arbitrary $P \in \text{SBNDC}_{\approx_{\text{tb}}}$ and an arbitrary $L \subseteq \mathcal{A}_\mathcal{L}$, we proceed as in the $\approx_{\text{tw}}$. ■

Proof of Theorem 3. We first prove the results for the $\approx_{\text{tw}}$-based properties. Let us examine each relationship separately:

- $\text{SBNDC}_{\approx_{\text{tw}}} \subset \text{SBSNNI}_{\approx_{\text{tw}}}$. We need to prove that for a given $P \in \mathbb{P}$, if $P \in$ SBNDC, it follows that for every P' reachable from P, $P' \in \text{BSNNI}_{\approx_{\text{tw}}}$. Since the processes we are considering are not recursive we can treat them as trees, and hence we can proceed by induction on their depth. In this case we will proceed by induction on the depth of P:
  - If the depth of P is 0 then P has no outgoing transitions and it behaves as $\underline{0}$. This obviously entails that $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{tw}} P \,/\, \mathcal{A}_{\mathcal{H}}$.
  - If the depth of P is $n+1$ with $n \in \mathbb{N}$, then take any P' of depth n such that $P \xrightarrow{a}_{\text{a}} P'$ or $P \xrightarrow{t}_{\text{t}} P'$. By hypothesis, $P, P' \in \text{SBNDC}_{\approx_{\text{tw}}}$ and by induction hypothesis $P' \in \text{SBSNNI}_{\approx_{\text{tw}}}$. Hence, we just need to prove that $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{tw}} P \,/\, \mathcal{A}_{\mathcal{H}}$. There are three cases:
    - ∗ If $a \notin \mathcal{A}_{\mathcal{H}}$ then both $P \setminus \mathcal{A}_{\mathcal{H}}$ and $P \,/\, \mathcal{A}_{\mathcal{H}}$ can execute a and reach, respectively, $P' \setminus \mathcal{A}_{\mathcal{H}}$ and $P' \,/\, \mathcal{A}_{\mathcal{H}}$, which are weakly timed bisimilar by induction hypothesis. Thus Definition 3 is respected.
    - ∗ If $a \in \mathcal{A}_{\mathcal{H}}$ we have that $P \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\text{a}} P' \,/\, \mathcal{A}_{\mathcal{H}}$, with $P \xrightarrow{a}_{\text{a}} P'$. By induction hypothesis we have that $P' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{tw}} P' \,/\, \mathcal{A}_{\mathcal{H}}$, and since $a \in \mathcal{A}_{\mathcal{H}}$ and $P \in \text{SBNDC}_{\approx_{\text{tw}}}$ we have $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{tw}} P' \setminus \mathcal{A}_{\mathcal{H}}$. By transitivity it follows that $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{tw}} P' \,/\, \mathcal{A}_{\mathcal{H}}$ which, combined with $P \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\text{a}} P' \,/\, \mathcal{A}_{\mathcal{H}}$, determines the condition required by Definition 3.
    - ∗ If $P \xrightarrow{t}_{\text{t}} P'$ then both $P \setminus \mathcal{A}_{\mathcal{H}}$ and $P \,/\, \mathcal{A}_{\mathcal{H}}$ can perform the same transitions, i.e., $P \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{t}_{\text{t}} P' \setminus \mathcal{A}_{\mathcal{H}}$ and $P \,/\, \mathcal{A}_{\mathcal{H}} \xrightarrow{t}_{\text{t}} P' \,/\, \mathcal{A}_{\mathcal{H}}$, because the hiding and restriction operators do not apply to timed transitions. The processes $P' \,/\, \mathcal{A}_{\mathcal{H}}$ and $P' \setminus \mathcal{A}_{\mathcal{H}}$ are weakly timed bisimilar because of the induction hypothesis.
- $\text{SBSNNI}_{\approx_{\text{tw}}} = \text{P\_BNDC}_{\approx_{\text{tw}}}$. We first prove that $\text{P\_BNDC}_{\approx_{\text{t}}} \subseteq \text{SBSNNI}_{\approx_{\text{tw}}}$. If $P \in \text{P\_BNDC}_{\approx_{\text{tw}}}$, then $P' \in \text{BNDC}_{\approx_{\text{tw}}}$ for every $P' \in \mathit{reach}(P)$. Since $\text{BNDC}_{\approx_{\text{tw}}} \subset \text{BSNNI}_{\approx_{\text{tw}}}$ as will be shown in the last case of the proof of this part of the theorem, $P' \in \text{BSNNI}_{\approx_{\text{tw}}}$ for every $P' \in \mathit{reach}(P)$, i.e., $P \in \text{SBSNNI}_{\approx_{\text{t}}}$.

  The fact that $\text{SBSNNI}_{\approx_{\text{tw}}} \subseteq \text{P\_BNDC}_{\approx_{\text{tw}}}$ will follow by proving that the symmetric relation $\mathcal{B} = \{(P'_1 \setminus \mathcal{A}_{\mathcal{H}}, ((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}), (((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}, P'_1 \setminus \mathcal{A}_{\mathcal{H}}) \mid P'_1 \in \mathit{reach}(P_1) \wedge P'_2 \in \mathit{reach}(P_2) \wedge Q$ executing only actions in $\mathcal{A}_{\mathcal{H}} \wedge L \subseteq \mathcal{A}_{\mathcal{H}} \wedge P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{tw}} P'_2 \,/\, \mathcal{A}_{\mathcal{H}} \wedge P_2 \in \text{SBSNNI}_{\approx_{\text{tw}}}\}$ is a weak timed bisimulation, as can be seen by taking P'_1 identical to P'_2 and both reachable from $P \in \text{SBSNNI}_{\approx_{\text{tw}}}$. Assuming that $P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ and $((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ are related by $\mathcal{B}$ – so that $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{tw}} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ – there are six cases. In the first two cases, it is $P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ to move first:
  - Let $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\text{a}} P''_1 \setminus \mathcal{A}_{\mathcal{H}}$. We observe that from $P'_2 \in \mathit{reach}(P_2)$ and $P_2 \in \text{SBSNNI}_{\approx_{\text{tw}}}$ it follows that $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{tw}} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$, so that $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{tw}} P'_2 \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\text{tw}} P'_2 \setminus \mathcal{A}_{\mathcal{H}}$, i.e., $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{tw}} P'_2 \setminus \mathcal{A}_{\mathcal{H}}$, as $\approx_{\text{tw}}$ is symmetric and transitive. As a consequence, since $l \neq \tau$ there exists $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\Longrightarrow} P''_2 \setminus \mathcal{A}_{\mathcal{H}}$ such that $P''_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\text{tw}} P''_2 \setminus \mathcal{A}_{\mathcal{H}}$. Thus $((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \stackrel{l}{\Longrightarrow} ((P''_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ with $(P''_1 \setminus \mathcal{A}_{\mathcal{H}}, ((P''_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$

because $P''_1 \in \mathit{reach}(P_1)$, $P''_2 \in \mathit{reach}(P_2)$, and $P''_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P''_2 \,/\, \mathcal{A}_\mathcal{H}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tw}}$.

- Let $P'_1 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} P''_1 \setminus \mathcal{A}_\mathcal{H}$. The proof is like the one of the previous case with $\Longrightarrow$ used in place of $\stackrel{l}{\Longrightarrow}$.

In the other four cases, instead, it is $((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H}$ to move first:

- Let $((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} ((P''_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H}$ with $P'_2 \xrightarrow{l}_{\rm a} P''_2$ so that $P'_2 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{l}_{\rm a} P''_2 \setminus \mathcal{A}_\mathcal{H}$ as $l \notin \mathcal{A}_\mathcal{H}$. We observe that from $P'_2 \in \mathit{reach}(P_2)$ and $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tw}}$ it follows that $P'_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P'_2 \,/\, \mathcal{A}_\mathcal{H}$, so that $P'_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P'_2 \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} P'_1 \setminus \mathcal{A}_\mathcal{H}$, i.e., $P'_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P'_1 \setminus \mathcal{A}_\mathcal{H}$, as $\approx_{\rm tw}$ is symmetric and transitive. As a consequence, since $l \neq \tau$ there exists $P'_1 \setminus \mathcal{A}_\mathcal{H} \stackrel{l}{\Longrightarrow} P''_1 \setminus \mathcal{A}_\mathcal{H}$ such that $P''_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P''_1 \setminus \mathcal{A}_\mathcal{H}$. Thus $(((P''_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H}, P''_1 \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ because $P''_1 \in \mathit{reach}(P_1)$, $P''_2 \in \mathit{reach}(P_2)$, and $P''_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P''_2 \,/\, \mathcal{A}_\mathcal{H}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm t}}$.
- Let $((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} ((P''_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H}$ with $P'_2 \xrightarrow{\tau}_{\rm a} P''_2$ so that $P'_2 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} P''_2 \setminus \mathcal{A}_\mathcal{H}$ as $\tau \notin \mathcal{A}_\mathcal{H}$. The proof is like the one of the previous case with $\Longrightarrow$ used in place of $\stackrel{l}{\Longrightarrow}$.
- If $((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} ((P'_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A}_\mathcal{H}$ with $Q \xrightarrow{\tau}_{\rm a} Q'$, then trivially
$(((P'_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A}_\mathcal{H}, P'_1 \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ as $P'_2 \approx_{\rm tw} P'_2$ and hence $P'_2 \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} P'_2 \,/\, \mathcal{A}_\mathcal{H}$ by Lemma 1(4).
- Let $((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} ((P''_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A}_\mathcal{H}$ with $P'_2 \xrightarrow{h}_{\rm a} P''_2$ – so that $P'_2 \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} P''_2 \,/\, \mathcal{A}_\mathcal{H}$ as $h \in \mathcal{A}_\mathcal{H}$ – and $Q \xrightarrow{h}_{\rm a} Q'$ for $h \in L$. We observe that from $P'_2, P''_2 \in \mathit{reach}(P_2)$ and $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tw}}$ it follows that $P'_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P'_2 \,/\, \mathcal{A}_\mathcal{H}$ and $P''_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P''_2 \,/\, \mathcal{A}_\mathcal{H}$, so that $P'_2 \setminus \mathcal{A}_\mathcal{H} \Longrightarrow P''_2 \setminus \mathcal{A}_\mathcal{H}$ as $P'_2 \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} P''_2 \,/\, \mathcal{A}_\mathcal{H}$ and $P'_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P'_2 \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} P'_1 \setminus \mathcal{A}_\mathcal{H}$, i.e., $P'_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P'_1 \setminus \mathcal{A}_\mathcal{H}$, as $\approx_{\rm tw}$ is symmetric and transitive. As a consequence there exists $P'_1 \setminus \mathcal{A}_\mathcal{H} \Longrightarrow P''_1 \setminus \mathcal{A}_\mathcal{H}$ such that $P''_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P''_1 \setminus \mathcal{A}_\mathcal{H}$. Thus $(((P''_2 \parallel_L Q') \,/\, L) \setminus \mathcal{A}_\mathcal{H}, P''_1 \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ because $P''_1 \in \mathit{reach}(P_1)$, $P''_2 \in \mathit{reach}(P_2)$, and $P''_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P''_2 \,/\, \mathcal{A}_\mathcal{H}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tw}}$.

As for delays, suppose $P'_1 \setminus \mathcal{A}_\mathcal{H} \;\; \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ then from $P'_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P'_2 \,/\, \mathcal{A}_\mathcal{H}$ it follows that there exists $P'_2 \,/\, \mathcal{A}_\mathcal{H} \Longrightarrow \bar{P}'_2 \,/\, \mathcal{A}_\mathcal{H}$ with $\bar{P}'_2 \,/\, \mathcal{A}_\mathcal{H} \;\; \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ and $P'_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} \bar{P}'_2 \,/\, \mathcal{A}_\mathcal{H}$. Since parallel composition, hiding and restriction do not operate on τ it follows that $((P'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H} \Longrightarrow ((\bar{P}'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H}$ with $((\bar{P}'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H} \;\stackrel{\tau}{\not\longrightarrow}_{\rm a}$ and $(P'_1 \setminus \mathcal{A}_\mathcal{H}, ((\bar{P}'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$. If $P'_1 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{t}_{\rm t} P''_1 \setminus \mathcal{A}_\mathcal{H}$ then $\bar{P}'_2 \,/\, \mathcal{A}_\mathcal{H} \stackrel{t}{\Longrightarrow} P''_2 \,/\, \mathcal{A}_\mathcal{H}$ with $P''_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tw} P''_2 \,/\, \mathcal{A}_\mathcal{H}$. Since parallel composition, hiding and restriction do not operate on τ and timed transitions, and since we assume that Q can let pass the same time as $\bar{P}'_2$, it follows that $((\bar{P}'_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H} \stackrel{t}{\Longrightarrow} ((P''_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H}$ with $(P''_1 \setminus \mathcal{A}_\mathcal{H}, ((P''_2 \parallel_L Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$.

– $\mathrm{SBSNNI}_{\approx_{\rm tw}} \subset \mathrm{BNDC}_{\approx_{\rm tw}}$. If $P \in \mathrm{SBSNNI}_{\approx_{\rm tw}} = \mathrm{P\_BNDC}_{\approx_{\rm tw}}$, then it immediately follows that $P \in \mathrm{BNDC}_{\approx_{\rm tw}}$.

- $\mathrm{BNDC}_{\approx_{\mathrm{tw}}} \subset \mathrm{BSNNI}_{\approx_{\mathrm{tw}}}$. If $P \in \mathrm{BNDC}_{\approx_{\mathrm{tw}}}$, i.e., $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{tw}} (P \parallel_L Q) / L) \setminus \mathcal{A}_{\mathcal{H}}$ for all $Q \in \mathbb{P}$ such that every $Q' \in \mathit{reach}(Q)$ executes only actions in $\mathcal{A}_{\mathcal{H}}$ and for all $L \subseteq \mathcal{A}_{\mathcal{H}}$, then we can consider in particular $\hat{Q}$ capable of stepwise mimicking the high-level behavior of P, in the sense that $\hat{Q}$ is able to synchronize with all the high-level actions executed by P and its reachable processes, along with $\hat{L} = \mathcal{A}_{\mathcal{H}}$. As a consequence $(P \parallel_{\hat{L}} \hat{Q}) / \hat{L}) \setminus \mathcal{A}_{\mathcal{H}}$ is isomorphic to $P / \mathcal{A}_{\mathcal{H}}$, hence $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{tw}} P / \mathcal{A}_{\mathcal{H}}$, i.e., $P \in \mathrm{BSNNI}_{\approx_{\mathrm{tw}}}$.

We now prove the same results for the $\approx_{\mathrm{tb}}$-based properties. Let us examine each relationship separately:

- $\mathrm{SBNDC}_{\approx_{\mathrm{tb}}} \subset \mathrm{SBSNNI}_{\approx_{\mathrm{tb}}}$. We need to prove that for a given $P \in \mathbb{P}$, if $P \in$ SBNDC, it follows that for every P' reachable from P, $P' \in \mathrm{BSNNI}_{\approx_{\mathrm{tb}}}$. Since the processes we are considering are not recursive we can treat them as trees, and hence we can proceed by induction on their depth. In this case we will proceed by induction on the depth of P:
  - If the depth of P is 0 then P has no outgoing transitions and it behaves as $\underline{0}$. This obviously entails that $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{tb}} P / \mathcal{A}_{\mathcal{H}}$.
  - If the depth of P is $n + 1$ with $n \in \mathbb{N}$, then take any P' of depth n such that $P \xrightarrow{a}_{\mathrm{a}} P'$. By hypothesis, $P, P' \in \mathrm{SBNDC}_{\approx_{\mathrm{tb}}}$ and by induction hypothesis $P' \in \mathrm{SBSNNI}_{\approx_{\mathrm{tb}}}$. Hence, we just need to prove that $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{tb}} P / \mathcal{A}_{\mathcal{H}}$. There are three cases:
    - If $a \notin \mathcal{A}_{\mathcal{H}}$ then both $P \setminus \mathcal{A}_{\mathcal{H}}$ and $P / \mathcal{A}_{\mathcal{H}}$ can execute a and reach, respectively, $P' \setminus \mathcal{A}_{\mathcal{H}}$ and $P' / \mathcal{A}_{\mathcal{H}}$, which are timed branching bisimilar by induction hypothesis. Thus Definition 4 is respected.
    - If $a \in \mathcal{A}_{\mathcal{H}}$ we have that $P / \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} P' / \mathcal{A}_{\mathcal{H}}$, with $P \xrightarrow{a}_{\mathrm{a}} P'$. By induction hypothesis we have that $P' \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{tb}} P' / \mathcal{A}_{\mathcal{H}}$, and since $a \in \mathcal{A}_{\mathcal{H}}$ and $P \in \mathrm{SBNDC}_{\approx_{\mathrm{tb}}}$ we have $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{tb}} P' \setminus \mathcal{A}_{\mathcal{H}}$. By transitivity it follows that $P \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{tb}} P' / \mathcal{A}_{\mathcal{H}}$ which, combined with $P / \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\mathrm{a}} P' / \mathcal{A}_{\mathcal{H}}$, determines the condition required by Definition 4.
    - If $P \xrightarrow{t}_{\mathrm{t}} P'$ then both $P \setminus \mathcal{A}_{\mathcal{H}}$ and $P / \mathcal{A}_{\mathcal{H}}$ can perform the same transitions, i.e., $P \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{t}_{\mathrm{t}} P' \setminus \mathcal{A}_{\mathcal{H}}$ and $P / \mathcal{A}_{\mathcal{H}} \xrightarrow{t}_{\mathrm{t}} P' / \mathcal{A}_{\mathcal{H}}$, because the hiding and restriction operators do not apply to timed transitions. The processes $P' / \mathcal{A}_{\mathcal{H}}$ and $P' \setminus \mathcal{A}_{\mathcal{H}}$ are weakly timed bisimilar because of the induction hypothesis.
- $\mathrm{SBSNNI}_{\approx_{\mathrm{tb}}} = \mathrm{P\_BNDC}_{\approx_{\mathrm{tb}}}$. We first prove that $\mathrm{P\_BNDC}_{\approx_{\mathrm{tb}}} \subseteq \mathrm{SBSNNI}_{\approx_{\mathrm{tb}}}$. If $P \in \mathrm{P\_BNDC}_{\approx_{\mathrm{tb}}}$, then $P' \in \mathrm{BNDC}_{\approx_{\mathrm{tb}}}$ for every $P' \in \mathit{reach}(P)$. Since $\mathrm{BNDC}_{\approx_{\mathrm{tb}}} \subset \mathrm{BSNNI}_{\approx_{\mathrm{tb}}}$ as will be shown in the last case of the proof of this theorem, $P' \in \mathrm{BSNNI}_{\approx_{\mathrm{tb}}}$ for every $P' \in \mathit{reach}(P)$, i.e., $P \in \mathrm{SBSNNI}_{\approx_{\mathrm{tb}}}$.
  The fact that $\mathrm{SBSNNI}_{\approx_{\mathrm{tb}}} \subseteq \mathrm{P\_BNDC}_{\approx_{\mathrm{tb}}}$ will follow by proving that the symmetric relation $\mathcal{B} = \{(P'_1 \setminus \mathcal{A}_{\mathcal{H}}, ((P'_2 \parallel_L Q) / L) \setminus \mathcal{A}_{\mathcal{H}}), (((P'_2 \parallel_L Q) / L) \setminus \mathcal{A}_{\mathcal{H}}, P'_1 \setminus \mathcal{A}_{\mathcal{H}}) \mid P'_1 \in \mathit{reach}(P_1) \wedge P'_2 \in \mathit{reach}(P_2) \wedge Q$ executing only actions in $\mathcal{A}_{\mathcal{H}} \wedge L \subseteq \mathcal{A}_{\mathcal{H}} \wedge P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{tb}} P'_2 / \mathcal{A}_{\mathcal{H}} \wedge P_2 \in \mathrm{SBSNNI}_{\approx_{\mathrm{tb}}}\}$ is a timed branching bisimulation, as can be seen by taking P'_1 identical to P'_2 and both reachable from $P \in \mathrm{SBSNNI}_{\approx_{\mathrm{tb}}}$. Assuming that $P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ and $((P'_2 \parallel_L Q) / L) \setminus \mathcal{A}_{\mathcal{H}}$ are

related by $\mathcal{B}$ – so that $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ – there are six cases. In the first two cases, it is $P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ to move first:

- Let $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} P''_1 \setminus \mathcal{A}_{\mathcal{H}}$. We observe that from $P'_2 \in \mathit{reach}(P_2)$ and $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$ it follows that $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$, so that $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P'_2 \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P'_2 \setminus \mathcal{A}_{\mathcal{H}}$, i.e., $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P'_2 \setminus \mathcal{A}_{\mathcal{H}}$, as $\approx_{\rm tb}$ is symmetric and transitive. As a consequence, since $l \neq \tau$ there exists $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow \bar{P}'_2 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} P''_2 \setminus \mathcal{A}_{\mathcal{H}}$ such that $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} \bar{P}'_2 \setminus \mathcal{A}_{\mathcal{H}}$ and $P''_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P''_2 \setminus \mathcal{A}_{\mathcal{H}}$. Thus $((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow ((\bar{P}'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} ((P''_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ with $(P'_1 \setminus \mathcal{A}_{\mathcal{H}}, ((\bar{P}'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ – because $P'_1 \in \mathit{reach}(P_1)$, $\bar{P}'_2 \in \mathit{reach}(P_2)$, and $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm b} \bar{P}'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$ – and $(P''_1 \setminus \mathcal{A}_{\mathcal{H}}, ((P''_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ – because $P''_1 \in \mathit{reach}(P_1)$, $P''_2 \in \mathit{reach}(P_2)$, and $P''_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P''_2 \,/\, \mathcal{A}_{\mathcal{H}}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$.
- If $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} P''_1 \setminus \mathcal{A}_{\mathcal{H}}$ there are two subcases:
  - ∗ If $P''_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ then $(P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ is allowed to stay idle with
    $(P''_1 \setminus \mathcal{A}_{\mathcal{H}}, ((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ because $P''_1 \in \mathit{reach}(P_1)$ and $P'_2 \in \mathit{reach}(P_2)$.
  - ∗ If $P''_1 \setminus \mathcal{A}_{\mathcal{H}} \not\approx_{\rm tb} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ then the proof is like the one of the previous case with $\xrightarrow{\tau}_{\rm a}$ used in place of $\xrightarrow{l}_{\rm a}$.

In the other four cases, instead, it is $((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ to move first:

- Let $((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} ((P''_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ with $P'_2 \xrightarrow{l}_{\rm a} P''_2$ so that $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} P''_2 \setminus \mathcal{A}_{\mathcal{H}}$ as $l \notin \mathcal{A}_{\mathcal{H}}$. We observe that from $P'_2 \in \mathit{reach}(P_2)$ and $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$ it follows that $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$, so that $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P'_2 \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P'_1 \setminus \mathcal{A}_{\mathcal{H}}$, i.e., $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P'_1 \setminus \mathcal{A}_{\mathcal{H}}$, as $\approx_{\rm tb}$ is symmetric and transitive. As a consequence, since $l \neq \tau$ there exists $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \Longrightarrow \bar{P}'_1 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{l}_{\rm a} P''_1 \setminus \mathcal{A}_{\mathcal{H}}$ such that $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} \bar{P}'_1 \setminus \mathcal{A}_{\mathcal{H}}$ and $P''_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P''_1 \setminus \mathcal{A}_{\mathcal{H}}$. Thus $(((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}, \bar{P}'_1 \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ – because $\bar{P}'_1 \in \mathit{reach}(P_1)$, $P'_2 \in \mathit{reach}(P_2)$, and $\bar{P}'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$ – and $(((P''_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}, P''_1 \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ – because $P''_1 \in \mathit{reach}(P_1)$, $P''_2 \in \mathit{reach}(P_2)$, and $P''_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P''_2 \,/\, \mathcal{A}_{\mathcal{H}}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$.
- If $((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} ((P''_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ with $P'_2 \xrightarrow{\tau}_{\rm a} P''_2$ so that $P'_2 \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} P''_2 \setminus \mathcal{A}_{\mathcal{H}}$ as $\tau \notin \mathcal{A}_{\mathcal{H}}$, there are two subcases:
  - ∗ If $P''_2 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ then $P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ is allowed to stay idle with $(((P''_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}, P'_1 \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ because $P'_1 \in \mathit{reach}(P_1)$, $P''_2 \in \mathit{reach}(P_2)$, and $P'_1 \setminus \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P''_2 \,/\, \mathcal{A}_{\mathcal{H}}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$.
  - ∗ If $P''_2 \setminus \mathcal{A}_{\mathcal{H}} \not\approx_{\rm tb} P'_1 \setminus \mathcal{A}_{\mathcal{H}}$ then the proof is like the one of the previous case with $\xrightarrow{\tau}_{\rm a}$ used in place of $\xrightarrow{l}_{\rm a}$.
- If $((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_{\mathcal{H}} \xrightarrow{\tau}_{\rm a} ((P'_2 \,\|_L\, Q') \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}$ with $Q \xrightarrow{\tau}_{\rm a} Q'$, then trivially $(((P'_2 \,\|_L\, Q') \,/\, L) \setminus \mathcal{A}_{\mathcal{H}}, P'_1 \setminus \mathcal{A}_{\mathcal{H}}) \in \mathcal{B}$ as $P'_2 \approx_{\rm tb} P'_2$ and hence $P'_2 \,/\, \mathcal{A}_{\mathcal{H}} \approx_{\rm tb} P'_2 \,/\, \mathcal{A}_{\mathcal{H}}$ by Lemma 1(4).

- Let $((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} ((P''_2 \,\|_L\, Q') \,/\, L) \setminus \mathcal{A}_\mathcal{H}$ with $P'_2 \xrightarrow{h}_{\rm a} P''_2$ – so that $P'_2 \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} P''_2 \,/\, \mathcal{A}_\mathcal{H}$ as $h \in \mathcal{A}_\mathcal{H}$ – and $Q \xrightarrow{h}_{\rm a} Q'$ for $h \in L$. We observe that from $P'_2, P''_2 \in \mathit{reach}(P_2)$ and $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$ it follows that $P'_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} P'_2 \,/\, \mathcal{A}_\mathcal{H}$ and $P''_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} P''_2 \,/\, \mathcal{A}_\mathcal{H}$, so that $P'_2 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} P''_2 \setminus \mathcal{A}_\mathcal{H}$ and $P'_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} P'_2 \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tb} P'_1 \setminus \mathcal{A}_\mathcal{H}$, i.e., $P'_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} P'_1 \setminus \mathcal{A}_\mathcal{H}$, as $\approx_{\rm tb}$ is symmetric and transitive. There are two subcases:
  - ∗ If $P''_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} P'_1 \setminus \mathcal{A}_\mathcal{H}$ then $P'_1 \setminus \mathcal{A}_\mathcal{H}$ is allowed to stay idle with $(((P''_2 \,\|_L\, Q') \,/\, L) \setminus \mathcal{A}_\mathcal{H}, P'_1 \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ because $P'_1 \in \mathit{reach}(P_1)$, $P''_2 \in \mathit{reach}(P_2)$, and $P'_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} P''_2 \,/\, \mathcal{A}_\mathcal{H}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$.
  - ∗ If $P''_2 \setminus \mathcal{A}_\mathcal{H} \not\approx_{\rm tb} P'_1 \setminus \mathcal{A}_\mathcal{H}$ then there exists $P'_1 \setminus \mathcal{A}_\mathcal{H} \Longrightarrow \bar{P}'_1 \setminus \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} P''_1 \setminus \mathcal{A}_\mathcal{H}$ such that $P'_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} \bar{P}'_1 \setminus \mathcal{A}_\mathcal{H}$ and $P''_2 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} P''_1 \setminus \mathcal{A}_\mathcal{H}$. Thus $(((P'_2 \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H}, \bar{P}'_1 \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ – because $\bar{P}'_1 \in \mathit{reach}(P_1)$, $P'_2 \in \mathit{reach}(P_2)$, and $\bar{P}'_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} P'_2 \,/\, \mathcal{A}_\mathcal{H}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$ – and $(((P''_2 \,\|_L\, Q') \,/\, L) \setminus \mathcal{A}_\mathcal{H}, P''_1 \setminus \mathcal{A}_\mathcal{H}) \in \mathcal{B}$ – because $P''_1 \in \mathit{reach}(P_1)$, $P''_2 \in \mathit{reach}(P_2)$, and $P''_1 \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} P''_2 \,/\, \mathcal{A}_\mathcal{H}$ as $P_2 \in \mathrm{SBSNNI}_{\approx_{\rm tb}}$.

  As for delays, we reason as in the $\approx_{\rm tw}$.
- $\mathrm{SBSNNI}_{\approx_{\rm tb}} \subset \mathrm{BNDC}_{\approx_{\rm tb}}$. If $P \in \mathrm{SBSNNI}_{\approx_{\rm tb}} = \mathrm{P\_BNDC}_{\approx_{\rm tb}}$, then it immediately follows that $P \in \mathrm{BNDC}_{\approx_{\rm tb}}$.
- $\mathrm{BNDC}_{\approx_{\rm tb}} \subset \mathrm{BSNNI}_{\approx_{\rm tb}}$. If $P \in \mathrm{BNDC}_{\approx_{\rm tb}}$, i.e., $P \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} (P \,\|_L\, Q) \,/\, L) \setminus \mathcal{A}_\mathcal{H}$ for all $Q \in \mathbb{P}$ such that every $Q' \in \mathit{reach}(Q)$ executes only actions in $\mathcal{A}_\mathcal{H}$ and for all $L \subseteq \mathcal{A}_\mathcal{H}$, then we can consider in particular $\hat{Q}$ capable of stepwise mimicking the high-level behavior of P, in the sense that $\hat{Q}$ is able to synchronize with all the high-level actions executed by P and its reachable processes, along with $\hat{L} = \mathcal{A}_\mathcal{H}$. As a consequence $(P \,\|_{\hat{L}}\, \hat{Q}) \,/\, \hat{L}) \setminus \mathcal{A}_\mathcal{H}$ is isomorphic to $P \,/\, \mathcal{A}_\mathcal{H}$, hence $P \setminus \mathcal{A}_\mathcal{H} \approx_{\rm tb} P \,/\, \mathcal{A}_\mathcal{H}$, i.e., $P \in \mathrm{BSNNI}_{\approx_{\rm tb}}$. ■

Proof of Theorem 5 Let Q be $P_1 + h \,.\, P_2$ (the proof is similar for Q equal to $P_2 + h \,.\, P_1$) and observe that no high-level actions occur in every process reachable from Q except Q itself:

1. Since the only high-level action occurring in Q is h, in the proof of $Q \in \mathrm{BSNNI}_{\approx_{\rm tw}}$ the only interesting case is the transition $Q \,/\, \mathcal{A}_\mathcal{H} \xrightarrow{\tau}_{\rm a} (P_2) \,/\, \mathcal{A}_\mathcal{H}$, to which $Q \setminus \mathcal{A}_\mathcal{H}$ responds by staying idle because $(P_2) \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} P_2 \approx_{\rm tw} P_2 \approx_{\rm tw} P_1 \approx_{\rm tw} Q \setminus \mathcal{A}_\mathcal{H}$, i.e., $(P_2) \,/\, \mathcal{A}_\mathcal{H} \approx_{\rm tw} Q \setminus \mathcal{A}_\mathcal{H}$ as $\approx_{\rm tw}$ is symmetric and transitive.
   On the other hand, $Q \notin \mathrm{BSNNI}_{\approx_{\rm tb}}$ because $P_2 \not\approx_{\rm tb} P_1$ in the same situation as before.
2. Since $Q \in \mathrm{BSNNI}_{\approx_{\rm tw}}$ by the previous result and no high-level actions occur in every process reachable from Q, it holds that $Q \in \mathrm{SBSNNI}_{\approx_{\rm tw}}$ and hence $Q \in \mathrm{BNDC}_{\approx_{\rm tw}}$ by virtue of Theorem 3.
   On the other hand, from $Q \notin \mathrm{BSNNI}_{\approx_{\rm tb}}$ by the previous result it follows that $Q \notin \mathrm{BNDC}_{\approx_{\rm tb}}$ by virtue of Theorem 3.
3. We already know from the proof of the previous result that $Q \in \mathrm{SBSNNI}_{\approx_{\rm tw}}$.
   On the other hand, from $Q \notin \mathrm{BSNNI}_{\approx_{\rm tb}}$ by the first result it follows that $Q \notin \mathrm{SBSNNI}_{\approx_{\rm tb}}$ by virtue of Theorem 3.

4. An immediate consequence of $\mathrm{P\_BNDC}_{\approx_{\mathrm{tw}}} = \mathrm{SBSNNI}_{\approx_{\mathrm{tw}}}$ and $\mathrm{P\_BNDC}_{\approx_{\mathrm{tb}}} = \mathrm{SBSNNI}_{\approx_{\mathrm{tb}}}$ as established by Theorem 3.
5. Since the only high-level action occurring in Q is h, in the proof of $Q \in \mathrm{SBNDC}_{\approx_{\mathrm{tw}}}$ the only interesting case is the transition $Q \xrightarrow{h}_{\mathrm{a}} P_2$, for which it holds that $Q \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{tw}} P_1 \approx_{\mathrm{tw}} P_2 \approx_{\mathrm{tw}} P_2 \approx_{\mathrm{tw}} (P_2) \setminus \mathcal{A}_{\mathcal{H}}$, i.e., $Q \setminus \mathcal{A}_{\mathcal{H}} \approx_{\mathrm{tw}} (P_2) \setminus \mathcal{A}_{\mathcal{H}}$ as $\approx_{\mathrm{tw}}$ is transitive.
On the other hand, $Q \notin \mathrm{SBNDC}_{\approx_{\mathrm{tb}}}$ because $P_1 \not\approx_{\mathrm{tb}} P_2$ in the same situation as before. ■

Proof of Proposition 2

1. We need to prove that the symmetric relation $\mathcal{B} = \{(nd(P_1), nd(P_2)) \mid P_1 \approx_{\mathrm{tw}} P_2\}$ is a weak bisimulation. We start by observing that from $P_1 \approx_{\mathrm{tw}} P_2$ it follows that for each $P_1 \xrightarrow{a}_{\mathrm{a}} P_1'$ there exists $P_2 \stackrel{\hat{a}}{\Longrightarrow} P_2'$ such that $P_1' \approx_{\mathrm{tw}} P_2'$. Since $nd(P_1)$ and $nd(P_2)$ are obtained by only replacing each timed transition with a τ-transition, we have that for each $nd(P_1) \xrightarrow{a}_{\mathrm{a}} nd(P_1')$ there exists $nd(P_2) \stackrel{\hat{a}}{\Longrightarrow}_{\mathrm{a}} nd(P_2')$ such that $(nd(P_1'), nd(P_2')) \in \mathcal{B}$. Similarly, if $nd(P_1) \xrightarrow{\tau}_{\mathrm{a}} nd(P_1')$ with $P_1 \xrightarrow{t}_{\mathrm{t}} P_1'$ then we have that $P_2 \Longrightarrow \bar{P}_1 \stackrel{t}{\Longrightarrow} P_2'$ with $P_1 \approx_{\mathrm{tw}} \bar{P}_2$ and $P_1' \approx_{\mathrm{tw}} P_2'$. Since those timed transitions are turned into τ transitions in $nd(P_1)$ and $nd(P_2)$ it follows that there exists $nd(P_2) \Longrightarrow nd(P_2')$ with $(nd(P_1'), nd(P_2')) \in \mathcal{B}$.
2. We need to prove that the symmetric relation $\mathcal{B} = \{nd(P_1), nd(P_2)) \mid P_1 \approx_{\mathrm{tb}} P_2\}$ is a timed branching bisimulation. We start by observing that from $P_1 \approx_{\mathrm{tb}} P_2$ it follows that for each $P_1 \xrightarrow{a}_{\mathrm{a}} P_1'$ either $a = \tau$ and $P_1' \approx_{\mathrm{tb}} P_2$, or there exists $P_2 \Longrightarrow \bar{P}_2 \xrightarrow{a}_{\mathrm{a}} P_2'$ such that $P_1 \approx_{\mathrm{tb}} \bar{P}_2$ and $P_1' \approx_{\mathrm{tb}} P_2'$. Since $nd(P_1)$ and $nd(P_2)$ enable the same τ transitions of P_1 and P_2 for each $nd(P_1) \xrightarrow{a}_{\mathrm{a}} nd(P_1')$ either $a = \tau$ and $(nd(P_1'), nd(P_2)) \in \mathcal{B}$, or there exists $nd(P_2) \stackrel{\tau^*}{\Longrightarrow}_{\mathrm{a}} nd(\bar{P}_2) \xrightarrow{a}_{\mathrm{a}} nd(P_2')$ such that $(nd(P_1), nd(\bar{P}_2)) \in \mathcal{B}$ and $(nd(P_1'), nd(P_2')) \in \mathcal{B}$. If $nd(P_1) \xrightarrow{\tau}_{\mathrm{a}} nd(P_1')$ with $P_1 \xrightarrow{t}_{\mathrm{t}} P_1'$ then we proceed like the $\approx_{\mathrm{tw}}$ case. ■

Proof of Corollary 1 The result follows directly from Proposition 1. ■

Proof of Lemma 2 Given $s_1, s_2 \in \mathcal{S}$ with $s_1 \approx_{\mathrm{tbf}} s_2$, consider the transitive closure $\mathcal{B}^+$ of the reflexive and symmetric relation $\mathcal{B} = \approx_{\mathrm{tbf}} \cup \{(\rho_1'', \rho_2''), (\rho_2'', \rho_1'') \in (run(s_1) \times run(s_2)) \cup (run(s_2) \times run(s_1)) \mid \exists \rho_1' \in run(s_1), \rho_2' \in run(s_2).\, \rho_1' \Longrightarrow \rho_1'' \wedge \rho_2' \Longrightarrow \rho_2'' \wedge \rho_1' \approx_{\mathrm{tbf}} \rho_2'' \wedge \rho_1'' \approx_{\mathrm{tbf}} \rho_2'\}$. The result will follow by proving that $\mathcal{B}^+$ is a weak timed back-and-forth bisimulation, because this implies that $\rho_1'' \approx_{\mathrm{tbf}} \rho_2''$ for every additional pair – i.e., $\mathcal{B}^+$ satisfies the cross property – as well as $\mathcal{B}^+ = \approx_{\mathrm{tbf}}$ – hence $\approx_{\mathrm{tbf}}$ satisfies the cross property too.
Let $(\rho_1'', \rho_2'') \in \mathcal{B} \setminus \approx_{\mathrm{tbf}}$ to avoid trivial cases. Then there exist $\rho_1' \in run(s_1)$ and $\rho_2' \in run(s_2)$ such that $\rho_1' \Longrightarrow \rho_1''$, $\rho_2' \Longrightarrow \rho_2''$, $\rho_1' \approx_{\mathrm{tbf}} \rho_2''$, and $\rho_1'' \approx_{\mathrm{tbf}} \rho_2'$. There are two cases for action transitions:

- Assume that $\rho_1'' \xrightarrow{a}_{\mathrm{a}} \rho_1'''$, from which it follows that $\rho_1' \Longrightarrow \rho_1'' \xrightarrow{a}_{\mathrm{a}} \rho_1'''$. From $\rho_1' \approx_{\mathrm{tbf}} \rho_2''$ we get $\rho_2'' \Longrightarrow \xrightarrow{a}_{\mathrm{a}} \Longrightarrow \rho_2'''$, or $\rho_2'' \Longrightarrow \rho_2'''$ when $a = \tau$, with

$\rho_1''' \approx_{\rm tbf} \rho_2'''$ and hence $(\rho_1''', \rho_2''') \in \mathcal{B}$.
When starting from $\rho_2'' \xrightarrow{a}_{\rm a} \rho_2'''$, we exploit $\rho_2' \Longrightarrow \rho_2''$ and $\rho_1'' \approx_{\rm tbf} \rho_2'$ instead.

- Assume that $\rho_1''' \xrightarrow{a}_{\rm a} \rho_1''$. From $\rho_1'' \approx_{\rm tbf} \rho_2'$ we get $\rho_2''' \Longrightarrow \xrightarrow{a}_{\rm a} \Longrightarrow \rho_2'$, so that $\rho_2''' \Longrightarrow \xrightarrow{a}_{\rm a} \Longrightarrow \rho_2''$, or $\rho_2''' \Longrightarrow \rho_2'$ when $a = \tau$, so that $\rho_2''' \Longrightarrow \rho_2''$, with $\rho_1''' \approx_{\rm tbf} \rho_2'''$ and hence $(\rho_1''', \rho_2''') \in \mathcal{B}$.
  When starting from $\rho_2''' \xrightarrow{a}_{\rm a} \rho_2''$, we exploit $\rho_1' \approx_{\rm tbf} \rho_2''$ and $\rho_1' \Longrightarrow \rho_1''$ instead.

Moreover, there are two further cases for timed transitions:

- Assume that $\rho_1'' \Longrightarrow \bar{\rho}_1''$ with $\bar{\rho}_1'' \not\xrightarrow{\tau}_{\rm a}$, from which follows that $\rho_1' \Longrightarrow \bar{\rho}_1''$. From $\rho_1' \approx_{\rm tbf} \rho_2''$ we get $\rho_2'' \Longrightarrow \bar{\rho}_2''$ with $\bar{\rho}_2'' \not\xrightarrow{\tau}_{\rm a}$, $\bar{\rho}_1'' \approx_{\rm tbf} \bar{\rho}_2''$, and hence $(\bar{\rho}_1'', \bar{\rho}_2'') \in \mathcal{B}$. If $\bar{\rho}_1'' \xrightarrow{t}_{\rm t} \rho_1'''$ then $\bar{\rho}_2'' \stackrel{t}{\Longrightarrow} \rho_2'''$ with $\rho_1''' \approx_{\rm tbf} \rho_2'''$ and hence $(\rho_1''', \rho_2''') \in \mathcal{B}$.
  When starting from $\rho_2'' \xrightarrow{a}_{\rm a} \rho_2'''$, we exploit $\rho_2' \Longrightarrow \rho_2''$ and $\rho_1'' \approx_{\rm tbf} \rho_2'$ instead.
- Assume that $\rho_1''' \xrightarrow{t}_{\rm t} \rho_1''$. From $\rho_1'' \approx_{\rm tbf} \rho_2'$ we get $\rho_2''' \stackrel{t}{\Longrightarrow} \rho_2'$, so that $\rho_2''' \stackrel{t}{\Longrightarrow} \rho_2''$ with $\rho_1''' \approx_{\rm tbf} \rho_2'''$ and hence $(\rho_1''', \rho_2''') \in \mathcal{B}$.
  When starting from $\rho_2''' \xrightarrow{a}_{\rm a} \rho_2''$, we exploit $\rho_1' \approx_{\rm tbf} \rho_2''$ and $\rho_1' \Longrightarrow \rho_1''$ instead.

■

Proof of Theorem 6. The proof is divided into two parts:

- Suppose that $s_1 \approx_{\rm tbf} s_2$ and let $\mathcal{B}$ be a weak timed back-and-forth bisimulation over $\mathcal{U}$ such that $((s_1, \varepsilon), (s_2, \varepsilon)) \in \mathcal{B}$. Assume that $\mathcal{B}$ only contains all the pairs of $\approx_{\rm tbf}$-equivalent runs from s_1 and s_2, so that Lemma 2 is applicable to $\mathcal{B}$. We show that $\mathcal{B}' = \{(last(\rho_1), last(\rho_2)) \mid (\rho_1, \rho_2) \in \mathcal{B}\}$ is a timed branching bisimulation over the states in $\mathcal{S}$ reachable from s_1 and s_2, from which $s_1 \approx_{\rm tb} s_2$ will follow. Note that $\mathcal{B}'$ is an equivalence relation because so is $\mathcal{B}$.
  Given $(last(\rho_1), last(\rho_2)) \in \mathcal{B}'$, by definition of $\mathcal{B}'$ we have that $(\rho_1, \rho_2) \in \mathcal{B}$. Let $r_k = last(\rho_k)$ for $k \in \{1, 2\}$, so that $(r_1, r_2) \in \mathcal{B}'$. Suppose that $r_1 \xrightarrow{a}_{\rm a} r_1'$, i.e., $\rho_1 \xrightarrow{a}_{\rm a} \rho_1'$ where $last(\rho_1') = r_1'$. There are two cases:
  - If $a = \tau$ then from $(\rho_1, \rho_2) \in \mathcal{B}$ it follows that $\rho_2 \Longrightarrow \rho_2'$ with $(\rho_1', \rho_2') \in \mathcal{B}$. This means that we have a sequence of $n \geq 0$ transitions of the form $\rho_{2,i} \xrightarrow{\tau}_{\rm a} \rho_{2,i+1}$ for all $0 \leq i \leq n-1$ where $\rho_{2,0}$ is ρ_2 while $\rho_{2,n}$ is ρ_2' so that $(\rho_1', \rho_{2,n}) \in \mathcal{B}$ as $(\rho_1', \rho_2') \in \mathcal{B}$.
    If $n = 0$ then we are done because ρ_2' is ρ_2 and hence $(\rho_1', \rho_2) \in \mathcal{B}$ as $(\rho_1', \rho_2') \in \mathcal{B}$ – thus $(r_1', r_2) \in \mathcal{B}'$ – otherwise from $\rho_{2,n}$ we go back to $\rho_{2,n-1}$ via $\rho_{2,n-1} \xrightarrow{\tau}_{\rm a} \rho_{2,n}$. Recalling that $(\rho_1', \rho_{2,n}) \in \mathcal{B}$ then ρ_1' can respond by staying idle so that $(\rho_1', \rho_{2,n-1}) \in \mathcal{B}$ and we are done because $\rho_{2,n-1}$ is ρ_2 and hence $(\rho_1', \rho_2) \in \mathcal{B}$ as $(\rho_1', \rho_{2,n-1}) \in \mathcal{B}$ – thus $(r_1', r_2) \in \mathcal{B}'$ – otherwise we go further back to $\rho_{2,n-2}$ via $\rho_{2,n-2} \xrightarrow{\tau}_{\rm a} \rho_{2,n-1}$. Then ρ_1' can respond by staying idle, so that $(\rho_1', \rho_{2,n-2}) \in \mathcal{B}$ then we are done because $\rho_{2,n-2}$ is ρ_2 and hence $(\rho_1', \rho_2) \in \mathcal{B}$ as $(\rho_1', \rho_{2,n-2}) \in \mathcal{B}$ – thus $(r_1', r_2) \in \mathcal{B}'$ – otherwise we keep going backward.

By repeating this procedure, since $(\rho'_1, \rho_{2,n}) \in \mathcal{B}$ either we get to $(\rho'_1, \rho_{2,n-n}) \in \mathcal{B}$ and we are done because this implies that $(\rho'_1, \rho_2) \in \mathcal{B}$ – thus $(r'_1, r_2) \in \mathcal{B}'$ – or for some $0 < m \leq n$ such that $(\rho'_1, \rho_{2,m}) \in \mathcal{B}$ the incoming transition $\rho_{2,m-1} \xrightarrow{\tau}_{\rm a} \rho_{2,m}$ is matched by $\bar{\rho}_1 \Longrightarrow \rho_1 \xrightarrow{\tau}_{\rm a} \rho'_1$ with $(\bar{\rho}_1, \rho_{2,m-1}) \in \mathcal{B}$. In the latter case, since $\mathit{last}(\rho_1), \mathit{last}(\rho_{2,m-1}) \in \mathcal{S}_{\rm n}$, $\bar{\rho}_1 \Longrightarrow \rho_1$, $\rho_2 \Longrightarrow \rho_{2,m-1}$, $(\bar{\rho}_1, \rho_{2,m-1}) \in \mathcal{B}$, and $(\rho_1, \rho_2) \in \mathcal{B}$, from Lemma 2 it follows that $(\rho_1, \rho_{2,m-1}) \in \mathcal{B}$. Consequently $\rho_2 \Longrightarrow \rho_{2,m-1} \xrightarrow{\tau}_{\rm a} \rho_{2,m}$ with $(\rho_1, \rho_{2,m-1}) \in \mathcal{B}$ and $(\rho'_1, \rho_{2,m}) \in \mathcal{B}$ – thus $r_2 \Longrightarrow \mathit{last}(\rho_{2,m-1}) \xrightarrow{\tau}_{\rm a} \mathit{last}(\rho_{2,m})$ with $(r_1, \mathit{last}(\rho_{2,m-1})) \in \mathcal{B}'$ and $(r'_1, \mathit{last}(\rho_{2,m})) \in \mathcal{B}'$.

- If $a \neq \tau$ then from $(\rho_1, \rho_2) \in \mathcal{B}$ it follows that $\rho_2 \Longrightarrow \bar{\rho}_2 \xrightarrow{a}_{\rm a} \bar{\rho}'_2 \Longrightarrow \rho'_2$ with $(\rho'_1, \rho'_2) \in \mathcal{B}$.
  From $(\rho'_1, \rho'_2) \in \mathcal{B}$ and $\bar{\rho}'_2 \Longrightarrow \rho'_2$ it follows that $\bar{\rho}'_1 \Longrightarrow \rho'_1$ with $(\bar{\rho}'_1, \bar{\rho}'_2) \in \mathcal{B}$. Since $\rho_1 \xrightarrow{a}_{\rm a} \rho'_1$ and hence the last transition in ρ'_1 is labeled with a, we derive that $\bar{\rho}'_1$ is ρ'_1 and hence $(\rho'_1, \bar{\rho}'_2) \in \mathcal{B}$.
  From $(\rho'_1, \bar{\rho}'_2) \in \mathcal{B}$ and $\bar{\rho}_2 \xrightarrow{a}_{\rm a} \bar{\rho}'_2$ it follows that $\bar{\rho}_1 \Longrightarrow \rho_1 \xrightarrow{a}_{\rm a} \rho'_1$ with $(\bar{\rho}_1, \bar{\rho}_2) \in \mathcal{B}$.
  Since $\mathit{last}(\rho_1), \mathit{last}(\bar{\rho}_2) \in \mathcal{S}_{\rm n}$, $\bar{\rho}_1 \Longrightarrow \rho_1$, $\rho_2 \Longrightarrow \bar{\rho}_2$, $(\bar{\rho}_1, \bar{\rho}_2) \in \mathcal{B}$, and $(\rho_1, \rho_2) \in \mathcal{B}$, from Lemma 2 it follows that $(\rho_1, \bar{\rho}_2) \in \mathcal{B}$.
  Consequently $\rho_2 \Longrightarrow \bar{\rho}_2 \xrightarrow{a}_{\rm a} \bar{\rho}'_2$ with $(\rho_1, \bar{\rho}_2) \in \mathcal{B}$ and $(\rho'_1, \bar{\rho}'_2) \in \mathcal{B}$ – thus $r_2 \Longrightarrow \mathit{last}(\bar{\rho}_2) \xrightarrow{a}_{\rm a} \mathit{last}(\bar{\rho}'_2)$ with $(r_1, \mathit{last}(\bar{\rho}_2)) \in \mathcal{B}'$ and $(r'_1, \mathit{last}(\bar{\rho}'_2)) \in \mathcal{B}'$.

As for delays, suppose $r_1 \Longrightarrow \bar{r}_1$ with $\bar{r}_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, i.e., $\rho_1 \Longrightarrow \bar{\rho}_1$ with $\mathit{last}(\bar{\rho}_1) = \bar{r}_1$ and $\bar{\rho}_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ then from $(\rho_1, \rho_2) \in \mathcal{B}$ we get $\rho_2 \Longrightarrow \bar{\rho}_2$ with $\bar{\rho}_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ and $(\bar{\rho}_1, \bar{\rho}_2) \in \mathcal{B}$, i.e., $r_2 \Longrightarrow \bar{r}_2$ with $\mathit{last}(\bar{\rho}_2) = \bar{r}_2$ and $\bar{r}_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ and $(\bar{r}_1, \bar{r}_2) \in \mathcal{B}'$. Therefore if $\bar{r}_1 \xrightarrow{t}_{\rm t} r'_1$, i.e., $\bar{\rho}_1 \xrightarrow{t}_{\rm t} \rho'_1$ with $\mathit{last}(\rho'_1) = r'_1$ we get $\bar{\rho}_2 \stackrel{t}{\Longrightarrow} \rho'_2$ with $(\rho'_1, \rho'_2) \in \mathcal{B}$, i.e., $\bar{r}_2 \stackrel{t}{\Longrightarrow} r'_2$ with $\mathit{last}(\rho'_2) = r'_2$ and hence $(r'_1, r'_2) \in \mathcal{B}'$.

– Suppose that $s_1 \approx_{\rm tb} s_2$ and let $\mathcal{B}$ be a timed branching bisimulation over $\mathcal{S}$ such that $(s_1, s_2) \in \mathcal{B}$. Assume that $\mathcal{B}$ only contains all the pairs of $\approx_{\rm tb}$-equivalent states reachable from s_1 and s_2. We show that the reflexive and transitive closure $\mathcal{B}'^*$ of $\mathcal{B}' = \{(\rho_1, \rho_2), (\rho_2, \rho_1) \in (\mathit{run}(s_1) \times \mathit{run}(s_2)) \cup (\mathit{run}(s_2) \times \mathit{run}(s_1)) \mid (\mathit{last}(\rho_1), \mathit{last}(\rho_2)) \in \mathcal{B}\}$ is a weak timed back-and-forth bisimulation over the runs in $\mathcal{U}$ from s_1 and s_2, from which $(s_1, \varepsilon) \approx_{\rm tbf} (s_2, \varepsilon)$, i.e., $s_1 \approx_{\rm tbf} s_2$, will follow.
  Given $(\rho_1, \rho_2) \in \mathcal{B}'$, by definition of $\mathcal{B}'$ we have that $(\mathit{last}(\rho_1), \mathit{last}(\rho_2)) \in \mathcal{B}$. Let $r_k = \mathit{last}(\rho_k)$ for $k \in \{1, 2\}$, so that $(r_1, r_2) \in \mathcal{B}$. For action transitions there are two cases:
  - If $\rho_1 \xrightarrow{a}_{\rm a} \rho'_1$, i.e., $r_1 \xrightarrow{a}_{\rm a} r'_1$ where $r'_1 = \mathit{last}(\rho'_1)$, then either $a = \tau$ and $(r'_1, r'_2) \in \mathcal{B}$ where $r'_2 = r_2$, or $r_2 \Longrightarrow \bar{r}_2 \xrightarrow{a}_{\rm a} r'_2$ with $(r_1, \bar{r}_2) \in \mathcal{B}$ and $(r'_1, r'_2) \in \mathcal{B}$. In both cases $\rho_2 \stackrel{\hat{a}}{\Longrightarrow} \rho'_2$ where $\mathit{last}(\rho'_2) = r'_2$, so that $(\rho'_1, \rho'_2) \in \mathcal{B}'$.
  - If $\rho'_1 \xrightarrow{a}_{\rm a} \rho_1$, i.e., $r'_1 \xrightarrow{a}_{\rm a} r_1$ where $r'_1 = \mathit{last}(\rho'_1)$, there are two subcases:
    * If ρ'_1 is (s_1, ε), i.e., $r'_1 \xrightarrow{a}_{\rm a} r_1$ is $s_1 \xrightarrow{a}_{\rm a} r_1$ and $\mathit{last}(\rho'_1) = s_1$, then from $(s_1, s_2) \in \mathcal{B}$ it follows that either $a = \tau$ and $(r_1, r_2) \in \mathcal{B}$ where

$r_2 = s_2$, or $s_2 \Longrightarrow \bar{r}_2 \xrightarrow{a}_{\rm a} r_2$ with $(s_1, \bar{r}_2) \in \mathcal{B}$ and $(r_1, r_2) \in \mathcal{B}$. In both cases $\rho'_2 \stackrel{\hat{a}}{\Longrightarrow} \rho_2$ where $last(\rho'_2) = s_2$, so that $(\rho'_1, \rho'_2) \in \mathcal{B}'$.

  * If ρ'_1 is not (s_1, ε) then from $(s_1, s_2) \in \mathcal{B}$ it follows that s_1 reaches r'_1 with a sequence of moves that are $\mathcal{B}$-compatible with those with which s_2 reaches some r'_2 such that $(r'_1, r'_2) \in \mathcal{B}$ as $\mathcal{B}$ only contains all the states reachable from s_1 and s_2. Therefore either $a = \tau$ and $(r_1, r'_2) \in \mathcal{B}$ where $r'_2 = r_2$, or $r'_2 \Longrightarrow \bar{r}_2 \xrightarrow{a}_{\rm a} r_2$ with $(r'_1, \bar{r}_2) \in \mathcal{B}$ and $(r_1, r_2) \in \mathcal{B}$. In both cases $\rho'_2 \stackrel{\hat{a}}{\Longrightarrow} \rho_2$ where $last(\rho'_2) = r'_2$, so that $(\rho'_1, \rho'_2) \in \mathcal{B}'$.

Moreover, there are two further cases for timed transition:

- If $\rho_1 \Longrightarrow \bar{\rho}_1$ with $\bar{\rho}_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$, i.e., $r_1 \Longrightarrow \bar{r}_1$ with $\bar{r}_1 = last(\bar{\rho}_1)$ and $\bar{r}_1 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ then from $(r_1, r_2) \in \mathcal{B}$ it follows that $r_2 \Longrightarrow \bar{r}_2$ with $\bar{r}_2 \stackrel{\tau}{\not\longrightarrow}_{\rm a}$ and $(\bar{r}_1, \bar{r}_2) \in \mathcal{B}$ and hence $\rho_2 \Longrightarrow \bar{\rho}_2$ with $last(\bar{\rho}_2) = \bar{r}_2$ and $(\bar{\rho}_1, \bar{\rho}_2) \in \mathcal{B}'$. Therefore, if $\bar{\rho}_1 \xrightarrow{t}_{\rm t} \rho'_1$, i.e., $\bar{r}_1 \xrightarrow{t}_{\rm t} r'_1$ with $r'_1 = last(\rho'_1)$ we get $\bar{r}_2 \stackrel{t}{\Longrightarrow} r'_2$ with $(r'_1, r'_2) \in \mathcal{B}$ and hence $\bar{\rho}_2 \stackrel{t}{\Longrightarrow} \rho'_2$ with $last(\rho'_2) = r'_2$ and $(\rho'_1, \rho'_2) \in \mathcal{B}'$.
- If $\rho'_1 \xrightarrow{t}_{\rm t} \rho_1$, i.e., $r'_1 \xrightarrow{t}_{\rm t} r_1$ where $r'_1 = last(\rho'_1)$, there are two cases:
  * If ρ'_1 is (s_1, ε), i.e., $r'_1 \xrightarrow{t}_{\rm t} r_1$ is $s_1 \xrightarrow{t}_{\rm t} r_1$ and $last(\rho'_1) = s_1$, then from $(s_1, s_2) \in \mathcal{B}$ it follows that $s_2 \stackrel{t}{\Longrightarrow} r_2$ with $(r_1, r_2) \in \mathcal{B}$. Hence, $\rho'_2 \stackrel{t}{\Longrightarrow} \rho_2$ where $last(\rho'_2) = s_2$, so that $(\rho'_1, \rho'_2) \in \mathcal{B}'$.
  * If ρ'_1 is not (s_1, ε) then from $(s_1, s_2) \in \mathcal{B}$ it follows that s_1 reaches r'_1 with a sequence of moves that are $\mathcal{B}$-compatible with those with which s_2 reaches some r'_2 such that $(r'_1, r'_2) \in \mathcal{B}$ as $\mathcal{B}$ only contains all the states reachable from s_1 and s_2. Therefore $r'_2 \stackrel{t}{\Longrightarrow} r_2$ with $(r_1, r_2) \in \mathcal{B}$. Hence, $\rho'_2 \stackrel{t}{\Longrightarrow} \rho_2$ where $last(\rho'_2) = r'_2$, so that $(\rho'_1, \rho'_2) \in \mathcal{B}'$. ■