Genesis and Evolution of ULTraS: Metamodel, Metaequivalences, Metaresults

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Abstract. We discuss the genesis of the ULTraS metamodel and summarize its evolution arising from the introduction of coherent resolutions of nondeterminism and reachability-consistent semirings.

1 The ULTraS Metamodel

In 2009, within the Italian project PaCo – *Performability-Aware Computing: Logics, Models, and Languages*, I started working with Rocco De Nicola and Michele Loreti on the definition of a general, state-transition behavioral model, hopefully paving the way to the development of a *unifying theory* as well as *reuse facilities* in the field of concurrency, without resorting to abstract representations such as the categorical ones based on coalgebras and bialgebras.

Together with Diego Latella and Mieke Massink, Rocco and Michele had already done much work in that framework, aiming at providing a uniform definition of the structural operational semantics for various stochastic process calculi. To this purpose, they developed *rate-based transition systems* [14], which then evolved into the semiring-based metamodel known as FuTS – *state-to-function labeled transition system* [15,35].

Rocco, Michele, and I wanted to explore a different direction, not related to languages and their semantics. Our first objective was to define a metamodel general enough to encompass specific behavioral models widely used in the concurrency literature, featuring nondeterminism, probabilities, deterministic/stochastic time, or a combination of them. We thus came up in [4] with a metamodel that we called ULTraS – *uniform labeled transition system* (then exemplified in [5] as an extension of rate-based transition systems to formalize process semantics), which was fully elaborated in [6] and further fine-tuned in [3].

ULTraS is a *discrete-state metamodel* parameterized with respect to a set D, where D-values are interpreted as different degrees of *one-step reachability.* These values are assumed to be ordered according to a reflexive and transitive relation $\sqsubseteq_D$, which is equipped with minimum $\perp_D$ expressing *unreachability.* Let us denote by $(S \rightarrow D)$ the set of functions from a set S to D. When S is a set of states, every element Δ of $(S \rightarrow D)$ can be interpreted as a function that *distributes reachability* over all possible next states. The set of states $supp(\Delta) = \{s \in S \mid \Delta(s) \neq \perp_D\}$ that are reachable according to Δ is called the *support* of Δ.

The set $(S \rightarrow D)_{\mathrm{nefs}}$ of D-distributions Δ over S is considered, which satisfies the constraint $0 < |supp(\Delta)| < \omega$. The first part of the constraint establishes that the target distribution of each transition has a *nonempty support*, so to avoid distributions always returning $\bot_D$ and hence transitions leading to nowhere. The second part of the constraint ensures that the same distribution has a *finite support*, a fact that will enable a correct definition of behavioral metaequivalences.

Definition 1. *Let* $(D, \sqsubseteq_D, \bot_D)$ *be a preordered set equipped with minimum. A* uniform labeled transition system *on it, or* D-ULTraS *for short, is a triple* $\mathcal{U} = (S, A, \longrightarrow)$ *where:*

- $S \neq \emptyset$ *is an at most countable set of states.*
- $A \neq \emptyset$ *is a countable set of transition-labeling actions.*
- $\longrightarrow \subseteq S \times A \times (S \to D)_{\mathrm{nefs}}$ *is a transition relation.* ■

Every transition (s, a, Δ) of $\mathcal{U}$ is written $s \xrightarrow{a} \Delta$, where $\Delta(s')$ is a D-value quantifying the degree of reachability of s' from s via that a-transition, with $\Delta(s') = \bot_D$ meaning that s' is not reachable with that transition. In the directed graph description of $\mathcal{U}$ (see the forthcoming Figs. 1, 2, 3, 4, 5, 6), vertices represent states and action-labeled edges represent action-labeled transitions. Given a transition $s \xrightarrow{a} \Delta$, the corresponding a-labeled edge goes from the vertex representing state s to a set of vertices linked by a dashed line, each of which represents a state $s' \in supp(\Delta)$ and is labeled with $\Delta(s')$.

In [6,9] we showed what follows about the choice of D:

- $\mathbb{B} = \{\bot, \top\}$, with $\bot \sqsubseteq_{\mathbb{B}} \top$, captures nondeterministic models such as:
  - labeled transition systems (LTS) [30], i.e., fully nondeterministic processes;
  - timed automata (TA) [1] – provided that S and A are allowed to be uncountable – where time is deterministic.
- $\mathbb{R}_{[0,1]}$, with the usual $\leq$, captures probabilistic models such as:
  - action-labeled discrete-time Markov chains (ADTMC) [48], i.e., fully probabilistic processes;
  - Markov decision processes (MDP) [17]/Rabin probabilistic automata [41], i.e., reactive probabilistic processes according to the terminology of [21];
  - Segala probabilistic automata (PA) [42], i.e., nondeterministic and probabilistic processes;
  - probabilistic timed automata (PTA) [33] – provided that S and A are allowed to be uncountable – where time is deterministic.
  - Markov automata (MA) [18], where time is stochastic.
- $\mathbb{R}_{\geq 0}$, with the usual $\leq$, captures stochastic models such as:
  - action-labeled continuous-time Markov chains (ACTMC) [48], i.e., fully stochastic processes;
  - continuous-time Markov decision processes (CTMDP) [40] / Knast probabilistic automata [31], i.e., reactive stochastic processes;
  - nondeterministic and stochastic processes intended as extensions of PA.

The definition of the ULTraS metamodel is extremely parsimonious, in the sense that it does not require any algebraic structure, really necessary only for behavioral relations and language semantics. It simply relies on a *preordered set equipped with minimum*, because this is sufficient to express reachability degrees for the various states when performing a transition, as well as unreachability.

2 Behavioral Metaequivalences on ULTraS

The second objective of Rocco, Michele, and myself was to define, on ULTraS, *behavioral metaequivalences* general enough to encompass equivalences for specific classes of processes appeared in the literature. In [6] we focused on three approaches – bisimulation [38,36], testing [13], and trace [26] – so to cover to some extent the linear-time/branching-time spectrum [20]. We showed that:

- Bisimulation metaequivalence can be instantiated to bisimilarities for:
  - fully nondeterministic processes [23];
  - fully probabilistic processes [19];
  - reactive probabilistic processes [34];
  - fully stochastic processes [25,24];
  - reactive stochastic processes [37].
- Trace metaequivalence can be instantiated to trace equivalences for:
  - fully nondeterministic processes [10];
  - fully probabilistic processes [29];
  - reactive probabilistic processes [46];
  - fully stochastic processes [50,2].
- Testing metaequivalence can be instantiated to testing equivalences for:
  - fully nondeterministic processes [13];
  - fully probabilistic processes [11,12];
  - reactive probabilistic processes [32];
  - fully stochastic processes [2].

Surprisingly enough, it turned out that our behavioral metaequivalences, as defined in [6], were *not* able to capture the following well known equivalences for nondeterministic and probabilistic processes:

- The bisimulation equivalences of [22,45] are finer than the one derivable from our bisimulation metaequivalence. The latter, studied in [8] and akin to the ones in [49,47], contains the former as coarsest congruence with respect to parallel composition, and has the nice property of being characterized by (a minor variant of) the probabilistic modal logic PML [34] like in the case of fully/reactive probabilistic processes [34] and alternating PA [39].
- The trace equivalence of [43] is finer than the one derivable from our trace metaequivalence. The latter, studied in [7], has the nice property of being a congruence with respect to parallel composition.
- The testing equivalences of [51,28,44,16] are finer than the one derivable from our testing metaequivalence. The latter, studied in [7], has the nice property of being backward compatible with testing equivalences for fully nondeterministic, fully probabilistic, and reactive probabilistic processes without imposing any restriction on the set of tests.

In order to retrieve also the aforementioned equivalences, in [3] I introduced the notion of *resolution of nondeterminism* in the ULTraS framework – with a formalization inspired by testing theories for nondeterministic and probabilistic processes – and, similar to what we did with Rocco and Michele in [8,7], I played with the order of certain universal quantifiers in the definition of the metaequivalences thereby obtaining *pre-* and *post-metaequivalences*.

2.1 Resolutions of Nondeterminism

When several transitions depart from the same state, they describe a choice among different behaviors, but the presence of these choices may hamper the calculations that will be required by behavioral metaequivalences. A *resolution* of a state s of a D-ULTRAS $\mathcal{U}$ is the result of a possible way of resolving choices starting from s, as if a *scheduler* were applied that, at the current state, selects one of its outgoing transitions or no transitions at all.

Following [27], in [3] I formalized a resolution as a D-ULTRAS $\mathcal{Z}$ with a tree-like structure – whose branching points correspond to target distributions of transitions – obtained by unfolding from s the graph structure of $\mathcal{U}$ and by selecting at each reached state at most one of its outgoing transitions. Since $\mathcal{U}$ can be cyclic, I made use of a *correspondence function* from the acyclic state space of $\mathcal{Z}$ to the original state space of $\mathcal{U}$. This function must be bijective[1] between the support of the target distribution of each transition in $\mathcal{Z}$ and the support of the target distribution of the corresponding transition in $\mathcal{U}$.

Definition 2. *Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS and $s \in S$. A D-ULTRAS $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}})$, with no cycles and Z disjoint from S, is a* resolution *of s, written $\mathcal{Z} \in Res(s)$, iff there exists a correspondence function $corr_{\mathcal{Z}} : Z \to S$ such that $s = corr_{\mathcal{Z}}(z_s)$, for some $z_s \in Z$, and for all $z \in Z$ it holds that:*

- *If $z \xrightarrow{a}_{\mathcal{Z}} \Delta$ then $corr_{\mathcal{Z}}(z) \xrightarrow{a} \Delta'$, with $corr_{\mathcal{Z}}$ being bijective between $supp(\Delta)$ and $supp(\Delta')$ and $\Delta(z') = \Delta'(corr_{\mathcal{Z}}(z'))$ for all $z' \in supp(\Delta)$.*
- *At most one transition departs from z.* ■

For bisimulation semantics, choices need to be resolved only at the first step or, more generally, only at each of the first k steps in case of a multistep definition of bisimilarity. A notion of *partial resolution* is thus introduced. It has the same characteristics as a resolution in its initial part – i.e., states not in S for ensuring the absence of cycles and choices – but, after the first k steps, its states and transitions are identical to the original ones.

Definition 3. *Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS, $s \in S$, and $k \in \mathbb{N}_{\geq 1}$. A D-ULTRAS $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}})$ is a* k-resolution *of s, written $\mathcal{Z} \in k\text{-}Res(s)$, iff there exists a correspondence function $corr_{\mathcal{Z}} : Z \to S$ such that $s = corr_{\mathcal{Z}}(z_s)$, for some $z_s \in Z$, and for all $z \in Z$ it holds that:*

- *If $z \xrightarrow{a}_{\mathcal{Z}} \Delta$ then $corr_{\mathcal{Z}}(z) \xrightarrow{a} \Delta'$, with $corr_{\mathcal{Z}}$ being bijective between $supp(\Delta)$ and $supp(\Delta')$ and $\Delta(z') = \Delta'(corr_{\mathcal{Z}}(z'))$ for all $z' \in supp(\Delta)$.*
- *If z is reachable from z_s with a sequence of less than k transitions, then:*
  - *$z \notin S$;*
  - *z cannot be part of a cycle;*
  - *z has at most one outgoing transition;*

  otherwise z is equal to $corr_{\mathcal{Z}}(z) \in S$ and has the same outgoing transitions that it has in $\mathcal{U}$. ■

[1] Requiring only injectivity as in [3] is not enough because it does not ensure that the former distribution preserves the overall reachability mass of the latter distribution (unlike the probabilistic case, in general there is no predefined reachability mass).

2.2 Reachability-Consistent Semirings

To express the calculations needed by behavioral metaequivalences, in [3] I further assumed that D has a *commutative semiring* structure – thereby reconciling ULTraS with FuTS to a large extent – i.e., that D is equipped with two binary operations denoted by $\oplus$ and $\otimes$, with the latter distributing over the former, which satisfy the following properties:

- $\otimes$ is associative and commutative and admits neutral element 1_D and absorbing element 0_D. This multiplicative operation enables the calculation of multistep reachability from values of consecutive single-step reachability along the same trajectory.
- $\oplus$ is associative and commutative and admits neutral element 0_D. This additive operation is useful for aggregating values of multistep reachability along different trajectories starting from the same state, as well as for shorthands of the form $\Delta(S') = \bigoplus_{s' \in S'} \Delta(s')$ given a transition $s \xrightarrow{a} \Delta$.

In [3] I also assumed that these two binary operations are *reachability consistent*, in the sense that they satisfy the following additional properties in accordance with the intuition behind the concept of reachability:

- $0_D = \bot_D$ (i.e., the zero of the semiring denotes unreachability).
- $d_1 \otimes d_2 \neq 0_D$ whenever $d_1 \neq 0_D \neq d_2$ (hence two consecutive steps cannot result in unreachability).
- The sum via $\oplus$ of finitely many values 1_D is always different from 0_D (known as *characteristic zero*; it ensures that two nonzero values sum up to zero only if they are one the inverse of the other w.r.t. $\oplus$, thus avoiding inappropriate zero results when aggregating values of trajectories from the same state).

For example, the following reachability-consistent semirings can be used:

- $(\mathbb{B}, \vee, \wedge, \bot, \top)$ for nondeterministic models;
- $(\mathbb{R}_{\geq 0}, +, \times, 0, 1)$ for probabilistic and stochastic models;

while characteristic zero rules out all semirings $(\mathbb{Z}_n, +_n, \times_n, 0, 1)$ of the classes of integer numbers that are congruent modulo $n \in \mathbb{N}_{\geq 2}$.

2.3 Measure Schemata for Multistep Reachability

The definition of behavioral metaequivalences requires the capability of measuring the degree of reachability of a given set of states from a given state when executing a sequence of transitions labeled with a certain sequence of actions. On the basis of [6], in [3] I provided a notion of *measure schema* for a D-ULTraS $\mathcal{U}$ as a set of homogeneously defined *measure functions*, one for each resolution $\mathcal{Z}$ of $\mathcal{U}$. In the following, A^* denotes the set of traces over an action set A, ε the empty trace, and $|\alpha|$ the length of a trace $\alpha \in A^*$.

Definition 4. *Let* $(D, \oplus, \otimes, 0_D, 1_D)$ *be a reachability-consistent semiring and* $\mathcal{U} = (S, A, \longrightarrow)$ *be a* D-ULTRAS*. A* D-measure schema $\mathcal{M}$ *for* $\mathcal{U}$ *is a set of* measure functions *of the form* $\mathcal{M}_{\mathcal{Z}} : Z \times A^* \times 2^Z \rightarrow D$*, one for each* $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in Res(s)$ *and* $s \in S$*, which are inductively defined on the length of their second argument by letting* $\mathcal{M}_{\mathcal{Z}}(z, \alpha, Z')$ *be equal to:*

$$\begin{cases} f_{\mathcal{Z}}(\bigoplus\limits_{z' \in supp(\Delta)} (\Delta(z') \otimes \mathcal{M}_{\mathcal{Z}}(z', \alpha', Z')), z, a, \Delta) & \textit{if } \alpha = a\,\alpha' \textit{ and } z \xrightarrow{a}_{\mathcal{Z}} \Delta \\ 1_D & \textit{if } \alpha = \varepsilon \textit{ and } z \in Z' \\ 0_D & \textit{otherwise} \end{cases}$$

where $f_{\mathcal{Z}} : D \times Z \times A \times (Z \rightarrow D)_{\text{nefs}} \rightarrow D$. ■

In the first clause, the value of $\mathcal{M}_{\mathcal{Z}}(z, \alpha, Z')$ is built around a sum of products of D-values, with the summation being well defined because $supp(\Delta)$ is finite as established in Def. 1. The definition above applies to $\mathcal{Z} \in k\text{-}Res(s)$ by restricting to traces $\alpha \in A^*$ such that $|\alpha| \leq k$ (note that $Z' \subseteq S$ when $|\alpha| = k$). For simplicity, $\mathcal{M}$ will often indicate both the measure schema and any of its measure functions $\mathcal{M}_{\mathcal{Z}}$, with $\mathcal{M}_{\text{nd}}$ being used when the semiring is $(\mathbb{B}, \vee, \wedge, \bot, \top)$ and $\mathcal{M}_{\text{pb}}$ when it is $(\mathbb{R}_{\geq 0}, +, \times, 0, 1)$.

To provide some degree of flexibility, further parameters, internal or external to $\mathcal{U}$, may be taken into account. On the one hand, the auxiliary function $f_{\mathcal{Z}}$ returns its first argument unless otherwise stated, but can also exploit information related to the source state z, the action label a, or the target distribution Δ of the transition elicited in the first clause. On the other hand, when necessary $\mathcal{M}_{\mathcal{Z}}$ is allowed to depend on arguments external to $\mathcal{U}$, such as time [3], which are consistently inherited by $f_{\mathcal{Z}}$ (the codomain of both functions remains D).

2.4 Bisimulation and Trace Pre-/Post-Metaequivalences: Coherency

In [3] I focused on the two endpoints of the linear-time/branching-time spectrum and redefined bisimulation and trace semantics for ULTRAS with respect to [6] on the basis of the newly introduced concepts: resolutions of nondeterminism, reachability-consistent semirings, measure schemata. This allowed me to capture *also* the equivalences for nondeterministic and probabilistic processes.

For bisimulation semantics there are two different metaequivalences, $\sim_{\text{B}}^{\text{pre}}$ and $\sim_{\text{B}}^{\text{post}}$. Both are defined in the style of [34], which requires bisimulations to be equivalence relations, but deal with *sets of equivalence classes*, rather than only with individual equivalence classes, to avoid an undesirable decrease of the discriminating power in certain circumstances. The difference between the two metaequivalences lies in the position – underlined in the definition below – of the universal quantification over sets of equivalence classes.

In the first case, inspired by [49,47,8] and referred to as *pre*-bisimulation, the quantification occurs *before* the transition of the challenger and the transition of the defender. In the second case, which is the widely accepted approach of [45] referred to as *post*-bisimulation, the quantification occurs *after* those two transitions. Given an equivalence relation $\mathcal{B}$ over a state space S together with a set of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$, $\bigcup \mathcal{G} \subseteq S$ denotes the union of all the equivalence classes in $\mathcal{G}$. The two considered transitions are represented via 1-resolutions.

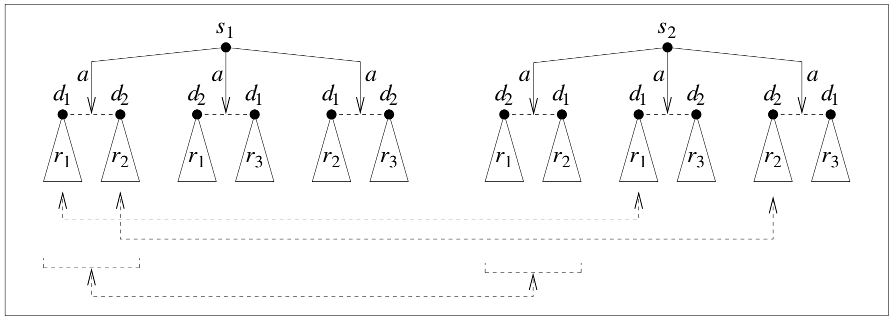

Fig. 1. Difference between bisimulation metaequivalences: $s_1 \not\sim^{\rm post}_{\rm B,\mathcal{M}} s_2$, $s_1 \sim^{\rm pre}_{\rm B,\mathcal{M}} s_2$

Definition 5. *Let* $(D, \oplus, \otimes, 0_D, 1_D)$ *be a reachability-consistent semiring,* $\mathcal{U} = (S, A, \longrightarrow)$ *be a* D-ULTRAS*,* $\mathcal{M}$ *be a* D*-measure schema for* $\mathcal{U}$*, and* $s_1, s_2 \in S$*:*

- $s_1 \sim^{\rm pre}_{\rm B,\mathcal{M}} s_2$ *iff there exists an* $\mathcal{M}$*-pre-bisimulation* $\mathcal{B}$ *over* S *such that* $(s_1, s_2) \in \mathcal{B}$*. An equivalence relation* $\mathcal{B}$ *over* S *is an* $\mathcal{M}$-pre-bisimulation *iff, whenever* $(s_1, s_2) \in \mathcal{B}$*, then for all* $a \in A$ *and* $\underline{\textit{for all } \mathcal{G} \in 2^{S/\mathcal{B}}}$ *it holds that for each* $\mathcal{Z}_1 \in 1\text{-}Res(s_1)$ *there exists* $\mathcal{Z}_2 \in 1\text{-}Res(s_2)$ *such that:*
$$\mathcal{M}(z_{s_1}, a, \bigcup \mathcal{G}) \; = \; \mathcal{M}(z_{s_2}, a, \bigcup \mathcal{G})$$
- $s_1 \sim^{\rm post}_{\rm B,\mathcal{M}} s_2$ *iff there exists an* $\mathcal{M}$*-post-bisimulation* $\mathcal{B}$ *over* S *such that* $(s_1, s_2) \in \mathcal{B}$*. An equivalence relation* $\mathcal{B}$ *over* S *is an* $\mathcal{M}$-post-bisimulation *iff, whenever* $(s_1, s_2) \in \mathcal{B}$*, then for all* $a \in A$ *it holds that for each* $\mathcal{Z}_1 \in 1\text{-}Res(s_1)$ *there exists* $\mathcal{Z}_2 \in 1\text{-}Res(s_2)$ *such that* $\underline{\textit{for all } \mathcal{G} \in 2^{S/\mathcal{B}}}$*:*
$$\mathcal{M}(z_{s_1}, a, \bigcup \mathcal{G}) \; = \; \mathcal{M}(z_{s_2}, a, \bigcup \mathcal{G}) \qquad \blacksquare$$

To understand the difference between the two bisimulation metaequivalences, consider the two D-ULTRAS models in Fig. 1. Both models feature *internal nondeterminism* (due to the three a-transitions departing from s_1 and s_2), the same distinct D-values d_1 and d_2, and the same inequivalent continuations given by the D-ULTRAS submodels with initial states r_1, r_2, r_3. Notice that both the D-values and the continuations are *shuffled within* each model, while only the D-values are *shuffled across* the two models too. It holds that $s_1 \not\sim^{\rm post}_{\rm B,\mathcal{M}} s_2$ because, e.g., the leftmost a-transition of s_1 is not matched by any of the three a-transitions of s_2. In contrast, $s_1 \sim^{\rm pre}_{\rm B,\mathcal{M}} s_2$. For instance, the leftmost a-transition of s_1 is matched by the central (resp. rightmost) a-transition of s_2 with respect to the equivalence class of r_1 (resp. r_2) and by the leftmost a-transition of s_2 with respect to the union of the equivalence classes of r_1 and r_2 (see the dashed arrow-headed lines in Fig. 1).

Also for trace semantics there are two different metaequivalences, $\sim^{\rm pre}_{\rm T}$ and $\sim^{\rm post}_{\rm T}$, which differ for the position of the universal quantification over traces. In the first case, inspired by [7], the quantification occurs *before* the computation of the challenger and the computation of the defender, so that superscript *pre* is used. In the second case, which is the widely accepted approach of [43], the quantification occurs *after* those two computations, hence superscript *post*.

In the definition of trace semantics, the considered computations are represented through resolutions. The ULTRAS submodels rooted in the support of the target distribution of a transition are not necessarily distinct and can have several outgoing transitions. Therefore, on the resolution side, the scheduler has the freedom of making *different* decisions in different occurrences of the *same* submodel within a target distribution. Unfortunately, this results in overdiscriminating trace metaequivalences.

Unlike [3], in this paper I limit the excessive power of schedulers by restricting myself to *coherent resolutions*, i.e., resolutions in which the decisions made in different occurrences of the same submodel are coherent with each other. This can be expressed by reasoning on suitable sets of traces, each extended with its *degree of executability* in a given resolution.

Given $a \in A$, $d \in D \setminus \{0_D\}$, and $T, T_1, T_2 \subseteq A^* \times (D \setminus \{0_D\})$, let:

$$\begin{aligned}
a \,.\, T &= \{(a\,\alpha, d') \mid (\alpha, d') \in T\} \\
d \otimes T &= \{(\alpha, d \otimes d') \mid (\alpha, d') \in T\} \\
tr(T) &= \{\alpha \in A^* \mid (\alpha, d') \in T \text{ for some } d' \in D \setminus \{0_D\}\} \\
T_1 \oplus T_2 &= \{(\alpha, d_1 \oplus d_2) \mid (\alpha, d_1) \in T_1 \wedge (\alpha, d_2) \in T_2\} \\
&\quad \cup \{(\alpha, d_1) \in T_1 \mid \text{there exists no } (\alpha, d_2) \in T_2 \text{ or there exists} \\
&\qquad\qquad \alpha' \neq \alpha \text{ in either } tr(T_1) \text{ or } tr(T_2) \text{ such that } |\alpha'| \leq |\alpha|\} \\
&\quad \cup \{(\alpha, d_2) \in T_2 \mid \text{there exists no } (\alpha, d_1) \in T_1 \text{ or there exists} \\
&\qquad\qquad \alpha' \neq \alpha \text{ in either } tr(T_1) \text{ or } tr(T_2) \text{ such that } |\alpha'| \leq |\alpha|\}
\end{aligned}$$

The set of *coherent D-traces* of a state s of a D-ULTRAS is defined as follows:

$$\mathcal{T}^{\mathrm{c}}_{D}(s) \;=\; \bigcup_{n \in \mathbb{N}} \mathcal{T}^{\mathrm{c}}_{D,n}(s)$$

where $\mathcal{T}^{\mathrm{c}}_{D,n}(s)$ is the set of coherent D-traces of s having length at most n:

$$\begin{aligned}
\mathcal{T}^{\mathrm{c}}_{D,0}(s) \;&=\; \{(\varepsilon, 1_D)\} \\
\mathcal{T}^{\mathrm{c}}_{D,n+1}(s) \;&=\; \{(\varepsilon, 1_D)\} \cup \bigcup_{s \xrightarrow{a} \Delta} a \,.\, \left(\bigoplus_{\Theta \subseteq A^*} \; \bigoplus_{s' \in supp(\Delta)}^{tr(\mathcal{T}^{\mathrm{c}}_{D,n}(s')) = \Theta} (\Delta(s') \otimes \mathcal{T}^{\mathrm{c}}_{D,n}(s')) \right)
\end{aligned}$$

Definition 6. *Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS, $s \in S$, $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in Res(s)$ with correspondence function $corr_{\mathcal{Z}} : Z \rightarrow S$. $\mathcal{Z}$ is said to be a* coherent resolution *of s, written $\mathcal{Z} \in Res^{\mathrm{c}}(s)$, iff for all $z \in Z$, whenever $z \xrightarrow{a}_{\mathcal{Z}} \Delta$, then for all $z', z'' \in supp(\Delta)$ and $n \in \mathbb{N}$:*

$$tr(\mathcal{T}^{\mathrm{c}}_{D,n}(corr_{\mathcal{Z}}(z'))) = tr(\mathcal{T}^{\mathrm{c}}_{D,n}(corr_{\mathcal{Z}}(z''))) \implies tr(\mathcal{T}^{\mathrm{c}}_{D,n}(z')) = tr(\mathcal{T}^{\mathrm{c}}_{D,n}(z'')) \;\blacksquare$$

Definition 7. *Let $(D, \oplus, \otimes, 0_D, 1_D)$ be a reachability-consistent semiring, $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, and $s_1, s_2 \in S$:*

- *$s_1 \sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}} s_2$ iff <u>for all $\alpha \in A^*$</u> it holds that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res^{\mathrm{c}}(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res^{\mathrm{c}}(s_2)$ such that:*
$$\mathcal{M}(z_{s_1}, \alpha, \mathcal{Z}_1) \;=\; \mathcal{M}(z_{s_2}, \alpha, \mathcal{Z}_2)$$
and also the condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied.
- *$s_1 \sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}} s_2$ iff it holds that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res^{\mathrm{c}}(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res^{\mathrm{c}}(s_2)$ such that <u>for all $\alpha \in A^*$</u>:*
$$\mathcal{M}(z_{s_1}, \alpha, \mathcal{Z}_1) \;=\; \mathcal{M}(z_{s_2}, \alpha, \mathcal{Z}_2)$$
and also the condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied. $\blacksquare$

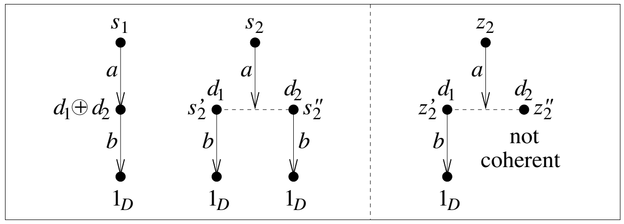

Fig. 2. Validity of Prop. 1(3) thanks to resolution coherency: $s_1 \sim^{\text{post}}_{\text{B},\mathcal{M}} s_2$, $s_1 \sim^{\text{post}}_{\text{T},\mathcal{M}} s_2$

2.5 Comparing Bisimulation and Trace Metaequivalences

The outcome of the comparison of the discriminating power of the four behavioral metaequivalences is recalled below from [3].

Proposition 1. *Let* $(D, \oplus, \otimes, 0_D, 1_D)$ *be a reachability-consistent semiring,* $\mathcal{U} = (S, A, \longrightarrow)$ *be a* D*-ULTRAS, and* $\mathcal{M}$ *be a* D*-measure schema for* $\mathcal{U}$*. Then:*

1. $\sim^{\text{post}}_{\text{B},\mathcal{M}} \subseteq \sim^{\text{pre}}_{\text{B},\mathcal{M}}$*, with* $\sim^{\text{post}}_{\text{B},\mathcal{M}} = \sim^{\text{pre}}_{\text{B},\mathcal{M}}$ *if* $\mathcal{U}$ *has no internal nondeterminism.*
2. $\sim^{\text{post}}_{\text{T},\mathcal{M}} \subseteq \sim^{\text{pre}}_{\text{T},\mathcal{M}}$.
3. $\sim^{\text{post}}_{\text{B},\mathcal{M}} \subseteq \sim^{\text{post}}_{\text{T},\mathcal{M}}$. ■

The validity of the third property[2] above is ensured by the coherency of the resolutions used in the definition of the trace metaequivalences. Consider for instance the two D-ULTRAS models in the leftmost part of Fig. 2, where $s_1 \sim^{\text{post}}_{\text{B},\mathcal{M}} s_2$ and $s_1 \sim^{\text{post}}_{\text{T},\mathcal{M}} s_2$. The latter identification is made possible by resolution coherency in $\sim^{\text{post}}_{\text{T},\mathcal{M}}$. Indeed, $\mathcal{T}^{\text{c}}_D(s'_2) = \{(\varepsilon, 1_D), (b, 1_D)\} = \mathcal{T}^{\text{c}}_D(s''_2)$. Therefore, the resolution of s_2 coinciding with the entire second model is coherent, while the one in the rightmost part of Fig. 2 is not, because $\mathcal{T}^{\text{c}}_D(z'_2) = \{(\varepsilon, 1_D), (b, 1_D)\} \neq \{(\varepsilon, 1_D)\} = \mathcal{T}^{\text{c}}_D(z''_2)$, and would lead to $s_1 \not\sim^{\text{post}}_{\text{T},\mathcal{M}} s_2$ if it were admitted.

As far as the strictness of the inclusions in Prop. 1 and the incomparability of certain metaequivalences are concerned, consider the three $\mathbb{B}$-ULTRAS models in the upper part of Fig. 3 – where only the second one features internal nondeterminism and $b_1 \neq b_2$ – together with their maximal resolutions in the lower part of Fig. 3 ($\top$ is omitted in the case of target distributions with singleton support). It turns out what follows:

- $s_1 \sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{nd}}} s_2$ but $s_1 \not\sim^{\text{post}}_{\text{B},\mathcal{M}_{\text{nd}}} s_2$ because the only a-transition of s_1 cannot be matched, in the $\mathcal{M}_{\text{nd}}$-post-bisimulation game, by any of the two a-transitions of s_2, as the transition of s_1 can reach two different equivalence classes, while each transition of s_2 can reach only one class.

[2] The proof is the same as the third property of Prop. 3.5 of [3], which is now correct in its inductive part ($|\alpha| = n+1$, $a' = a$, "either α' ...") due to resolution coherency.

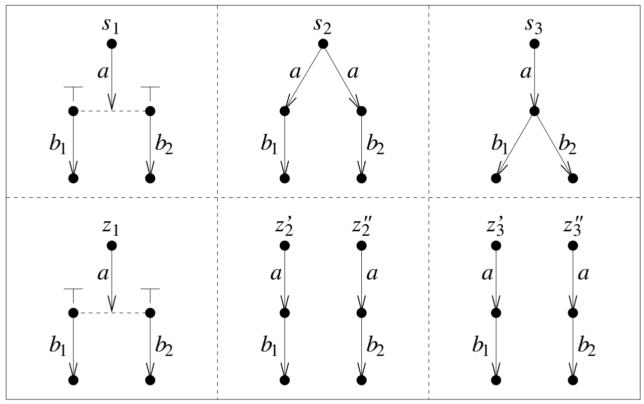

Fig. 3. Strictness of inclusions in Prop. 1 and incomparability of metaequivalences

- $s_1 \not\sim^{\rm pre}_{\rm B,\mathcal{M}_{nd}} s_3$, and hence $s_1 \not\sim^{\rm post}_{\rm B,\mathcal{M}_{nd}} s_3$, because the state reached by the a-transition of s_3 enables two actions and, as a consequence, cannot be equivalent to any of the two states reached by the a-transition of s_1. Indeed, although s_2 and s_3 have the same resolutions, their maximal 1-resolutions are different; for s_2 they coincide with the two maximal resolutions, while for s_3 the only maximal 1-resolution coincides with the original model.
- $s_1 \sim^{\rm pre}_{\rm T,\mathcal{M}_{nd}} s_2$ but $s_1 \not\sim^{\rm post}_{\rm T,\mathcal{M}_{nd}} s_2$ because the only maximal resolution of s_1 cannot be matched, in the case of $\sim^{\rm post}_{\rm T,\mathcal{M}_{nd}}$, by any of the two maximal resolutions of s_2, as the maximal resolution of s_1 has two different maximal traces, while each maximal resolution of s_2 has only one maximal trace.
- $s_1 \sim^{\rm pre}_{\rm T,\mathcal{M}_{nd}} s_3$ but $s_1 \not\sim^{\rm post}_{\rm T,\mathcal{M}_{nd}} s_3$ because s_3 has the same resolutions as s_2. This shows that, unlike bisimulation semantics, $\sim^{\rm pre}_{\rm T,\mathcal{M}}$ and $\sim^{\rm post}_{\rm T,\mathcal{M}}$ do not coincide even in the absence of internal nondeterminism, unless excluding $\mathbb{B}$-ULTraS models such as the first one that cannot be considered the canonical representation of any labeled transition system and, more generally, all semirings with a set $D \neq \mathbb{B}$ containing a value $d \neq 0_D$ such that $d \oplus d = d$ (so that trace a would distinguish s_1 from s_3 – and also s_2 – w.r.t. $\sim^{\rm pre}_{\rm T,\mathcal{M}}$).
- $s_2 \sim^{\rm post}_{\rm T,\mathcal{M}_{nd}} s_3$ as they have the same resolutions, but $s_2 \not\sim^{\rm post}_{\rm B,\mathcal{M}_{nd}} s_3$.
- $\sim^{\rm pre}_{\rm B,\mathcal{M}}$ is generally incomparable with $\sim^{\rm post}_{\rm T,\mathcal{M}}$ and $\sim^{\rm pre}_{\rm T,\mathcal{M}}$. On the one hand, $s_2 \not\sim^{\rm pre}_{\rm B,\mathcal{M}_{nd}} s_3$ while $s_2 \sim^{\rm post}_{\rm T,\mathcal{M}_{nd}} s_3$ and $s_2 \sim^{\rm pre}_{\rm T,\mathcal{M}_{nd}} s_3$. On the other hand, in Fig. 1 it holds that $s_1 \sim^{\rm pre}_{\rm B,\mathcal{M}} s_2$ while $s_1 \not\sim^{\rm post}_{\rm T,\mathcal{M}} s_2$; moreover $s_1 \not\sim^{\rm pre}_{\rm T,\mathcal{M}} s_2$ if r_1 (resp. r_2) has a b-transition that reaches with degree d'_b (resp. d''_b) a terminal state, whenever degrees $(d_1 \otimes d'_b) \oplus (d_2 \otimes d''_b)$ and $(d_2 \otimes d'_b) \oplus (d_1 \otimes d''_b)$ associated with trace $a\,b$ – which is assumed not to be executable via r_3 – are different from each other and from $d_1 \otimes d'_b$ and $d_2 \otimes d''_b$.

2.6 Alternative Characterizations of Trace Metaequivalences

On the basis of [7], in [3] I provided an alternative characterization of $\sim^{\rm pre}_{\rm T,\mathcal{M}}$, which is slightly revised here. Since this metaequivalence treats traces individually regardless of the resolutions in which they can be executed, two states turn out to be equivalent according to $\sim^{\rm pre}_{\rm T,\mathcal{M}}$ iff they have the same set of D-traces.

The validity of the lemma below relies on the use of coherent resolutions, together with the fact that the definition of $T_1 \oplus T_2$ before Def. 6 also includes (α, d_1) taken from T_1 and (α, d_2) taken from T_2 without summing up their degrees, provided that there exists another trace α' not longer than α in only one of T_1 and T_2 – meaning that T_1 and T_2 stem from two inequivalent states.[3]

Lemma 1. *Let* $(D, \oplus, \otimes, 0_D, 1_D)$ *be a reachability-consistent semiring,* $\mathcal{U} = (S, A, \longrightarrow)$ *be a* D*-ULTraS,* $\mathcal{M}$ *be a* D*-measure schema for* $\mathcal{U}$*. Let* $(\alpha, d) \in A^* \times (D \setminus \{0_D\})$ *and* $s \in S$*. Then* $(\alpha, d) \in T^{\rm c}_D(s)$ *iff there exists* $(Z, A, \longrightarrow_{\mathcal{Z}}) \in Res^{\rm c}(s)$ *such that* $\mathcal{M}(z_s, \alpha, Z) = d$*.* ■

Theorem 1. *Let* $s_1, s_2 \in S$*. Then* $s_1 \sim^{\rm pre}_{\rm T,\mathcal{M}} s_2$ *iff* $T^{\rm c}_D(s_1) = T^{\rm c}_D(s_2)$*.* ■

An analogous characterization can be provided for $\sim^{\rm post}_{\rm T,\mathcal{M}}$ by reasoning in terms of *coherent* D*-trace distributions*, so to bind extended D-traces to the resolutions in which they can be executed. For a state s, what is obtained is a family of sets of extended D-traces instead of a flat set:

$$TD^{\rm c}(s) = \{T^{\rm c}(z_s) \mid \text{there exists } \mathcal{Z} \in Res^{\rm c}(s) \text{ whose initial state is } z_s\}$$

from which the result below immediately follows.

Theorem 2. *Let* $s_1, s_2 \in S$*. Then* $s_1 \sim^{\rm post}_{\rm T,\mathcal{M}} s_2$ *iff* $TD^{\rm c}_D(s_1) = TD^{\rm c}_D(s_2)$*.* ■

3 Metaresults for Behavioral Metaequivalences

After the identification of models and equivalences captured or generated by the ULTraS framework, the ongoing research is aimed at investigating the properties of behavioral metaequivalences. The objective of this activity is to produce *metaresults*, in the sense that the obtained results should be valid *regardless of specific classes of processes*, thereby leading to a *unifying process theory*.

The compositionality metaresults established in [3] for bisimulation and trace pre-/post-metaequivalences are now discussed, by rephrasing them in the setting of a general process calculus relying on the same underpinnings as ULTraS. The definition of the semantics for this language makes use of the two binary operations provided by the underlying reachability-consistent semiring.

[3] The definition of $T_1 \oplus T_2$ before Lemma 4.11 of [3] should be rectified by removing the two instances of "α occurring only in ..." as resolutions are not coherent there (otherwise the if part of Lemma 4.11(2) would not hold).

$$\frac{\mathcal{D} \longmapsto \Delta}{a\,.\,\mathcal{D} \xrightarrow{a} \Delta}$$

$$\frac{P_1 \xrightarrow{a} \Delta}{P_1 + P_2 \xrightarrow{a} \Delta} \qquad \frac{P_2 \xrightarrow{a} \Delta}{P_1 + P_2 \xrightarrow{a} \Delta}$$

$$\frac{P_1 \xrightarrow{a} \Delta_1 \quad a \notin L}{P_1 \parallel_L P_2 \xrightarrow{a} \Delta_1 \otimes \delta_{P_2}} \qquad \frac{P_2 \xrightarrow{a} \Delta_2 \quad a \notin L}{P_1 \parallel_L P_2 \xrightarrow{a} \delta_{P_1} \otimes \Delta_2}$$

$$\frac{P_1 \xrightarrow{a} \Delta_1 \quad P_2 \xrightarrow{a} \Delta_2 \quad a \in L}{P_1 \parallel_L P_2 \xrightarrow{a} \Delta_1 \otimes \Delta_2}$$

$$d \triangleright P \longmapsto \{(P, d)\} \qquad \frac{\mathcal{D}_1 \longmapsto \Delta_1 \quad \mathcal{D}_2 \longmapsto \Delta_2}{\mathcal{D}_1 \mathbin{\Diamond\!\!\!\!+} \mathcal{D}_2 \longmapsto \Delta_1 \oplus \Delta_2}$$

Table 1. Structural operational semantic rules for UProC

3.1 A Process Algebraic View of ULTraS

Given a preordered set D equipped with minimum that yields a reachability-consistent semiring $(D, \oplus, \otimes, 0_D, 1_D)$, together with a countable set A of actions, the syntax for UProC – *uniform process calculus* features two levels, one for the set $\mathbb{P}$ of processes and one for the set $\mathbb{D}$ of reachability distributions:

$$\begin{array}{lcl} P & ::= & \underline{0} \mid a\,.\,\mathcal{D} \mid P + P \mid P \parallel_L P \\ \mathcal{D} & ::= & d \triangleright P \mid \mathcal{D} \mathbin{\Diamond\!\!\!\!+} \mathcal{D} \end{array}$$

where $a \in A$, $L \subseteq A$, $d \in D \setminus \{0_D\}$.

The structural operational semantic rules in Table 1 generate a D-ULTraS $(\mathbb{P}, A, \longrightarrow)$ by exploiting the semiring operations. The primary transition relation $\longrightarrow$ is defined as the smallest subset of $\mathbb{P} \times A \times (\mathbb{P} \to D)_{\text{nefs}}$ satisfying the rules in the upper part, where $\otimes$ is lifted to reachability distributions over the parallel composition of processes by letting $(\Delta_1 \otimes \Delta_2)(P_1 \parallel_L P_2) = \Delta_1(P_1) \otimes \Delta_2(P_2)$, while δ_P is the reachability distribution identically equal to 0_D except in P where its value is 1_D. The secondary transition relation $\longmapsto$ is the smallest subset of $\mathbb{D} \times (\mathbb{P} \to D)_{\text{nefs}}$ satisfying the rules in the lower part, with $\{(P, d)\}$ being a shorthand for the reachability distribution identically equal to 0_D except in P where its value is d; furthermore, $\oplus$ is lifted to reachability distributions by letting $(\Delta_1 \oplus \Delta_2)(P) = \Delta_1(P) \oplus \Delta_2(P)$. Let $\mathit{supp}(\mathcal{D}) = \mathit{supp}(\Delta)$ if $\mathcal{D} \longmapsto \Delta$.

As far as the process operator $+$ is concerned, it expresses a generic choice to be interpreted on the basis of D. For example, if $D = \mathbb{B}$ then the choice is nondeterministic. If instead $D = \mathbb{R}_{\geq 0}$, in the presence of alternative identical actions – corresponding to identically labeled transitions departing from the same state – the choice is nondeterministic and a (variant of) probabilistic automata can be derived; otherwise, the choice may be regarded as probabilistic, in the sense that a Markov chain or a Markov decision process may be obtained. Moreover, note that a probabilistic process term like $P_1 \, {}_p\!+ P_2$, where $p \in \mathbb{R}_{]0,1[}$, can be rendered as $\tau \,.\, (p \triangleright P_1 \oplus (1-p) \triangleright P_2)$ in UProC, where τ is the invisible action.

3.2 Congruence with Respect to Distribution/Dynamic Operators

Let us investigate the compositionality properties of the four behavioral metaequivalences with respect to the operators of UProC. Due to the two-level format of the syntax, as a preliminary step the metaequivalences are lifted from processes to reachability distributions over processes. Extending [34], this can be done by considering $\mathcal{D}_1, \mathcal{D}_2 \in \mathbb{D}$ related by an equivalence relation $\sim$ over $\mathbb{P}$ when $\mathcal{D}_1$ and $\mathcal{D}_2$ assign the same reachability degree to the same equivalence class, i.e., $\Delta_1(C) = \Delta_2(C)$ for all $C \in \mathbb{P}/\sim$ with $\mathcal{D}_1 \longmapsto \Delta_1$ and $\mathcal{D}_2 \longmapsto \Delta_2$. Note that, given $\mathcal{D} \longmapsto \Delta$, it holds that $\Delta \in (\mathbb{P} \to D)_{\text{nefs}}$ and hence $\Delta(C)$, i.e., $\bigoplus_{P \in C} \Delta(P)$, can only have finitely many summands different from 0_D.

Compositionality with respect to the distribution operators $\triangleright$ and $\oplus$ can be established in a way that abstracts from the specific behavioral metaequivalence.

Theorem 3. *Let* $\sim_{\mathcal{M}} \in \{\sim^{\text{pre}}_{\text{B},\mathcal{M}}, \sim^{\text{post}}_{\text{B},\mathcal{M}}, \sim^{\text{pre}}_{\text{T},\mathcal{M}}, \sim^{\text{post}}_{\text{T},\mathcal{M}}\}$ *for a measure schema* $\mathcal{M}$ *over the* D-ULTraS *semantics of* UProC. *Let* $P_1, P_2 \in \mathbb{P}$ *and* $\mathcal{D}_1, \mathcal{D}_2 \in \mathbb{D}$. *If* $P_1 \sim_{\mathcal{M}} P_2$ *and* $\mathcal{D}_1 \sim_{\mathcal{M}} \mathcal{D}_2$, *then:*

1. $d \triangleright P_1 \sim_{\mathcal{M}} d \triangleright P_2$ *for all* $d \in D \setminus \{0_D\}$.
2. $\mathcal{D}_1 \oplus \mathcal{D} \sim_{\mathcal{M}} \mathcal{D}_2 \oplus \mathcal{D}$ *and* $\mathcal{D} \oplus \mathcal{D}_1 \sim_{\mathcal{M}} \mathcal{D} \oplus \mathcal{D}_2$ *for all* $\mathcal{D} \in \mathbb{D}$. ■

As far as the two dynamic process operators are concerned, there are different proofs for bisimulation and trace semantics, which are reworkings of those in [3].

Theorem 4. *Let* $\sim_{\mathcal{M}} \in \{\sim^{\text{pre}}_{\text{B},\mathcal{M}}, \sim^{\text{post}}_{\text{B},\mathcal{M}}, \sim^{\text{pre}}_{\text{T},\mathcal{M}}, \sim^{\text{post}}_{\text{T},\mathcal{M}}\}$ *for a measure schema* $\mathcal{M}$ *over the* D-ULTraS *semantics of* UProC. *Let* $P_1, P_2 \in \mathbb{P}$ *and* $\mathcal{D}_1, \mathcal{D}_2 \in \mathbb{D}$. *If* $P_1 \sim_{\mathcal{M}} P_2$ *and* $\mathcal{D}_1 \sim_{\mathcal{M}} \mathcal{D}_2$, *then:*

1. $a \,.\, \mathcal{D}_1 \sim_{\mathcal{M}} a \,.\, \mathcal{D}_2$ *for all* $a \in A$.
2. $P_1 + P \sim_{\mathcal{M}} P_2 + P$ *and* $P + P_1 \sim_{\mathcal{M}} P + P_2$ *for all* $P \in \mathbb{P}$. ■

Unlike Thm. 4.2 of [3], trace metaequivalences are *full congruences* with respect to *action prefix*. If ordinary resolutions were considered instead of coherent ones, a lack of compositionality would arise in the general setting of ULTraS because the continuation after an action is *not a single process*, but a reachability distribution over processes.

This can be illustrated through the following UProC terms P_1 and P_2:

$$\begin{aligned} P_1 &= a \,.\, (d_1 \triangleright Q_1 \oplus d_2 \triangleright Q_2) \\ P_2 &= a \,.\, (d_1 \triangleright Q_2 \oplus d_2 \triangleright Q_2) \\ Q_1 &= a' \,.\, b \,.\, \underline{0} + a' \,.\, c \,.\, \underline{0} \\ Q_2 &= a' \,.\, (b \,.\, \underline{0} + c \,.\, \underline{0}) \end{aligned}$$

where a sequence of action prefixes like $a' \,.\, b \,.\, \underline{0}$ is a shorthand for $a' \,.\, (d \triangleright b \,.\, (d \triangleright \underline{0}))$ for some $d \in D \setminus \{0_D\}$. Their underlying D-ULTraS models are shown in the

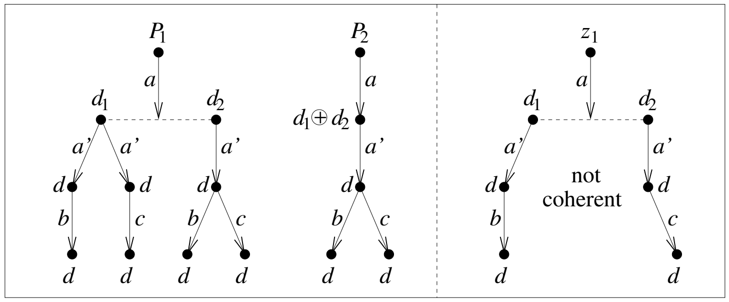

Fig. 4. Full compositionality w.r.t. action prefix thanks to resolution coherency

leftmost part of Fig. 4. It is easy to see Q_1 and Q_2 are trace equivalent, hence the two distributions describing the a-continuations of P_1 and P_2 are trace equivalent too. However, if one considers the trace $\alpha = a\,a'b$ and the resolution of P_1 shown in the rightmost part of Fig. 4 – in which α is executable with degree $d_1 \otimes d \otimes d$ – then no resolution of P_2 is capable of matching it – as the executability degree would be $(d_1 \oplus d_2) \otimes d \otimes d$ or 0_D – unless $D = \mathbb{B}$ in which case $d_1 = d_2 = \top$ and $d_1 \oplus d_2 = \top \vee \top = \top$. As can be noted, that resolution of P_1 is not coherent, as the scheduler makes different decisions in the two trace equivalent submodels respectively rooted at Q_1 and Q_2, thereby producing two resolutions of those two submodels that are no longer trace equivalent.

3.3 Congruence with Respect to Parallel Composition

Addressing parallel composition is much more involved. Following [3], the first metaresult states that $\sim^{\text{post}}_{\text{B},\mathcal{M}}$ is a congruence with respect to parallel composition always, i.e., for every possible ULTRAS. As a consequence of Prop. 1, this is the case also for $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ in the absence of internal nondeterminism.

Theorem 5. *Let $\mathcal{M}$ be a measure schema for the D-ULTRAS semantics of UProC. Let $P_1, P_2 \in \mathbb{P}$. If $P_1 \sim^{\text{post}}_{\text{B},\mathcal{M}} P_2$, then $P_1 \parallel_L P \sim^{\text{post}}_{\text{B},\mathcal{M}} P_2 \parallel_L P$ and $P \parallel_L P_1 \sim^{\text{post}}_{\text{B},\mathcal{M}} P \parallel_L P_2$ for all $L \subseteq A$ and $P \in \mathbb{P}$.* ■

Corollary 1. *Let $\mathcal{M}$ be a measure schema for the D-ULTRAS semantics of UProC. Let $P_1, P_2 \in \mathbb{P}$ have no internal nondeterminism. If $P_1 \sim^{\text{pre}}_{\text{B},\mathcal{M}} P_2$, then $P_1 \parallel_L P \sim^{\text{pre}}_{\text{B},\mathcal{M}} P_2 \parallel_L P$ and $P \parallel_L P_1 \sim^{\text{pre}}_{\text{B},\mathcal{M}} P \parallel_L P_2$ for all $L \subseteq A$ and $P \in \mathbb{P}$.* ■

As for the compositionality of $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ in the presence of internal nondeterminism, let us consider the case $|D| = 2$, i.e., the simplest reachability-consistent semiring $(\mathbb{B}, \vee, \wedge, \bot, \top)$ together with the corresponding measure schema $\mathcal{M}_{\text{nd}}$. In this specific case, $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ turns out to be a congruence with respect to parallel composition. Intuitively, in addition to the coinductive nature of bisimulation, the reason is that, starting from transitions whose target distributions can only contain $\top$ and $\bot$ as values, their parallel composition cannot generate, for the target distributions of the resulting transitions, values different from $\top$ and $\bot$.

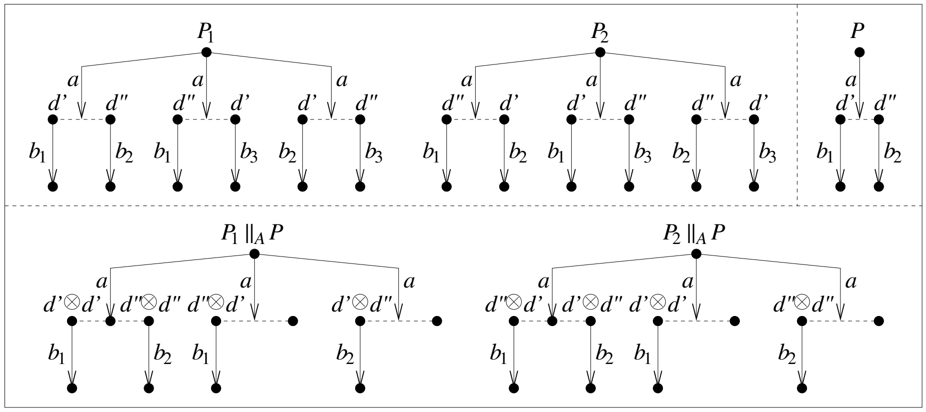

Fig. 5. $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ is not compositional when $|D| \geq 3$ and there is internal nondeterminism

Theorem 6. *Let $\mathcal{M}_{\text{nd}}$ be the measure schema for the* $\mathbb{B}$-ULTRAS *semantics of* UPROC. *Let $P_1, P_2 \in \mathbb{P}$. If $P_1 \sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{nd}}} P_2$, then $P_1 \parallel_L P \sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{nd}}} P_2 \parallel_L P$ and $P \parallel_L P_1 \sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{nd}}} P \parallel_L P_2$ for all $L \subseteq A$ and $P \in \mathbb{P}$.* ■

In all the other cases, i.e., when $|D| \geq 3$ and internal nondeterminism is present, the relation $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ is no longer guaranteed to be a congruence with respect to parallel composition.

Consider for instance the first two D-ULTRAS models in the upper part of Fig. 5 (D-values of terminal states are omitted), where $d', d'' \in D$ satisfy $d' \neq d''$ and $d' \neq 0_D \neq d''$ (these values exist because $|D| \geq 3$). Given a D-measure schema $\mathcal{M}$, it holds that $P_1 \sim^{\text{pre}}_{\text{B},\mathcal{M}} P_2$. However, if the last D-ULTRAS in the upper part is taken into account, the two D-ULTRAS models in the lower part of Fig. 5 are obtained, which satisfy $P_1 \parallel_A P \not\sim^{\text{pre}}_{\text{B},\mathcal{M}} P_2 \parallel_A P$. The reason is that, when examining the set of equivalence classes whose states can perform b_1 or b_2, the leftmost a-transition of $P_1 \parallel_A P$ is not matched by any a-transition of $P_2 \parallel_A P$ whenever $(d' \otimes d') \oplus (d'' \otimes d'') \notin \{(d'' \otimes d') \oplus (d' \otimes d''),\, d' \otimes d',\, d'' \otimes d''\}$.

A *coarsest congruence metaresult* relating $\sim^{\text{post}}_{\text{B},\mathcal{M}}$ and $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ for $|D| \geq 3$ can be established whenever the reachability-consistent semiring $(D, \oplus, \otimes, 0_D, 1_D)$ is a *field* – like, e.g., $(\mathbb{Q}, +, \times, 0, 1)$, $(\mathbb{R}, +, \times, 0, 1)$, and $(\mathbb{C}, +, \times, 0, 1)$ – which means that the inverse operations with respect to $\oplus$ and $\otimes$ exist:

- $d \ominus d = d \oplus inv_{\oplus}(d) = inv_{\oplus}(d) \oplus d = 0_D$ for all $d \in D$.
- $d \oslash d = d \otimes inv_{\otimes}(d) = inv_{\otimes}(d) \otimes d = 1_D$ for all $d \in D \setminus \{0_D\}$.

Such a metaresult holds under *image finiteness* – i.e., when the number of identically labeled transitions departing from any state is finite – and relies on the fact that transitions have target distributions with *finite support*. The proof exploits the algebraic and topological properties of the *vector spaces* that can be built on top of the field, as well as *characteristic zero*, which guarantees that the field and hence the vector spaces on it are infinite.

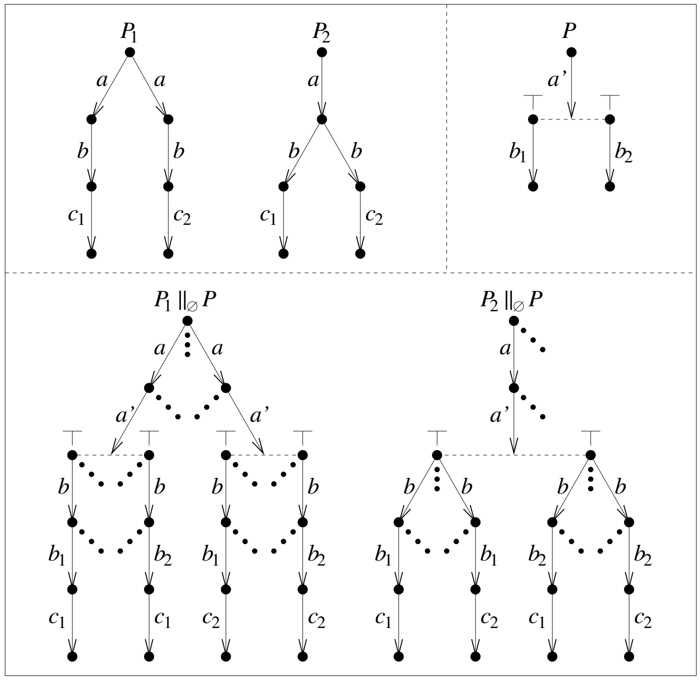

Fig. 6. $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}}$ is not compositional

Theorem 7. *Let* $(D, \oplus, \ominus, \otimes, \oslash, 0_D, 1_D)$ *be a reachability-consistent field and* $\mathcal{M}$ *be a measure schema for the* D-ULTraS *semantics of* UProC. *Let* $P_1, P_2 \in \mathbb{P}$ *be image finite. Then* $P_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} P_2$ *iff* $P_1 \parallel_L P \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}} P_2 \parallel_L P$ *for all* $L \subseteq A$ *and* $P \in \mathbb{P}$. ■

In the case of trace semantics, it is $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}}$ that, for every possible ULTraS, is a congruence with respect to parallel composition, hence no compositionality connection can be established with $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}}$ as the latter is finer than the former. The proof of this congruence metaresult for $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}}$ exploits the alternative characterization of Thm. 1.

Theorem 8. *Let* $\mathcal{M}$ *be a measure schema for the* D-ULTraS *semantics of* UProC. *Let* $P_1, P_2 \in \mathbb{P}$. *If* $P_1 \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} P_2$, *then* $P_1 \parallel_L P \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} P_2 \parallel_L P$ *and* $P \parallel_L P_1 \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} P \parallel_L P_2$ *for all* $L \subseteq A$ *and* $P \in \mathbb{P}$. ■

As for the compositionality of $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}}$, even under the simplest reachability-consistent semiring $(\mathbb{B}, \vee, \wedge, \perp, \top)$ and the corresponding measure schema $\mathcal{M}_{\mathrm{nd}}$ the relation is not a congruence with respect to parallel composition, unless excluding $\mathbb{B}$-ULTraS models that cannot be regarded as the canonical representation of any labeled transition system (for a congruence counterexample based on $\mathcal{M}_{\mathrm{pb}}$, see Fig. 3 of [43]).

Consider for instance the first two $\mathbb{B}$-ULTraS models in the upper part of Fig. 6 ($\top$ is omitted in the case of target distributions with singleton support), which satisfy $P_1 \sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}_{\mathrm{nd}}} P_2$. If the last $\mathbb{B}$-ULTraS in the upper part is taken into account, the two $\mathbb{B}$-ULTraS models in the lower part of Fig. 6 are obtained (dots stands for transitions that are not shown), which satisfy $P_1 \parallel_{\emptyset} P \not\sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}_{\mathrm{nd}}} P_2 \parallel_{\emptyset} P$. This is witnessed by the maximal resolutions of $P_1 \parallel_{\emptyset} P$ and $P_2 \parallel_{\emptyset} P$ that start with trace $a\, a'$ and continue with one of the traces in $\{b\, b_1\, c_1, b\, b_1\, c_2, b\, b_2\, c_1, b\, b_2\, c_2\}$. As an example, the maximal resolution of $P_2 \parallel_{\emptyset} P$ whose associated set of maximal traces is $\{a\, a'\, b\, b_1\, c_1, a\, a'\, b\, b_2\, c_2\}$ is not matched under $\sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}_{\mathrm{nd}}}$ by any maximal resolution of $P_1 \parallel_{\emptyset} P$.

3.4 Final Remarks

In conclusion, the metaresults of [3] – which have been reformulated here in a process algebraic setting – confirm a foundational difference between bisimulation and trace semantics. This difference refers to compositionality with respect to parallel combinators and shows up under internal nondeterminism, as had emerged in the specific case of nondeterministic and probabilistic processes [8,7].

A question naturally arises: *is there a semantics for which both pre- and post-metaequivalences are always congruences with respect to parallel composition?*

4 Future Directions

I plan to keep putting ULTraS at work on behavioral metaequivalences to further extend the resulting unifying process theory. In particular, I would like to investigate:

- Equational characterization metaresults.
- Logical characterization metaresults.
- Metaresults for other bisimulation-/trace-based metaequivalences.
- Metaresults for testing metaequivalences.
- The spectrum of metaequivalences on ULTraS.

As far as behavioral metarelations are concerned, it is also worth studying:

- Behavioral metapreorders.
- Weak variants of behavioral metarelations.
- Approximate variants of behavioral metarelations.

Finally, on the metamodel side, it would be interesting to capture also:

- Interleaving models with continuous state spaces.
- Truly concurrent models such as Petri nets and event structures.

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