A Process Algebraic Theory of Reversible Concurrent Systems

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Concurrency: Nondeterminism vs. Irreversibility

- Systems composed of many interconnected computing parts that communicate by exchanging information or simply synchronizing.
- Models: shared memory, message passing, web services, . . .
- Types: centralized/distributed/decentralized, static/dynamic/mobile.
- Aspects: functionality, security, reliability, performance, . . .
- Nondeterminism: the input does not uniquely define the output.
- Due to different speeds, interaction scheme, scheduling policies, . . .
- Does the output uniquely define the input? What if it is not the case?
- Irreversibility: typical of functions that are *not invertible*.
- Example: conjunctions/disjunctions computed inside circuits.

Reversible Computing

- What does (ir)reversibility mean in computing?
- Well established concept in mathematics, physics, chemistry, biology: inverse function, operation, element, reaction, ...
- Much more recent in informatics: seminal papers by Landauer in 1961 and Bennett in 1973 on IBM Journal of Research and Development.
- Landauer principle states that any manipulation of information that is *irreversible* – i.e., causes information loss – such as:
  - erasure/overwriting of bits
  - merging of computation paths

  must be accompanied by a corresponding *entropy increase*.
- Minimal *heat generation* due to *extra work* for standardizing signals and making them independent of their history, so that it becomes *impossible to determine the input from the output*.

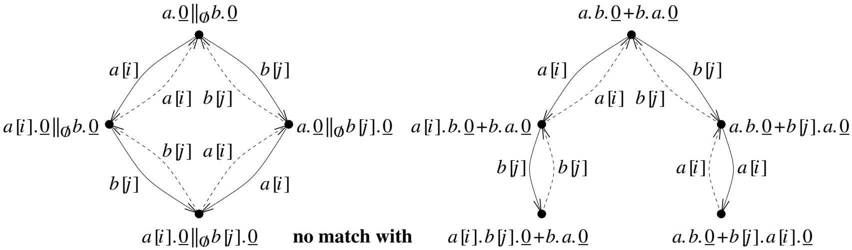
- Due to Landauer principle, the logical irreversibility of a function implies the physical irreversibility of computing that function and the consequent dissipative effects.
- Experimentally verified by Bérut et al in 2012 and revisited in terms of its physical foundations by Frank in 2018.
- Every reversible computation, where no information is lost instead, may be potentially carried out without dissipating further heat.
- Lower energy consumption could therefore be achieved by resorting to reversible computing.
- There are many other applications of reversible computing:
  - Biochemical reaction modeling (nature).
  - Parallel discrete-event simulation (speedup).
  - Fault tolerant computing systems (rollback).
  - Robotics and control theory (backtrack).
  - Concurrent program debugging (reproducibility).

- Two directions of computation in a reversible system:
  - Forward: coincides with the normal way of computing.
  - Backward: the effects of the forward one are undone (when needed).
- How to proceed backward? Same path as the forward direction?
- Not necessarily, especially in the case of a concurrent system, where causally independent paths should be deemed equivalent.
- Different notions of reversibility developed in different settings:
  - Causal reversibility is the capability of going back to a past state in a way that is *consistent with the computational history* of the system (easy for sequential systems, hard for concurrent and distributed ones).
  - Time reversibility refers to the conditions under which the stochastic behavior remains the same when the *direction of time* is reversed (quantitative system models, efficient performance evaluation).
  - The former implies the latter in models based on Markov chains.

Reversibility in Process Algebra

- There are no inverse process algebraic operators!
- The *dynamic* approach of [DanosKrivine04] yielding RCCS uses explicit stack-based memories attached to processes to record all the actions executed by those processes.
- A single transition relation is defined, while actions are divided into forward and backward resulting in forward and backward transitions.
- The *static* approach of [PhillipsUlidowski07] yielding CCSK is a method to reverse calculi by retaining within process syntax:
  - all executed actions, which are suitably decorated;
  - all dynamic operators, which are therefore treated as static.
- A forward transition relation and a backward transition relation are separately defined, labeled with communication keys so as to know who synchronized with whom when building backward transitions.

- In [PU07] forward-reverse bisimilarity has been introduced too, which is truly concurrent as it does not satisfy the expansion law of parallel composition into a choice among all possible action sequencings:



- With back-and-forth bisimilarity [DeNicolaMontanariVaandrager90] the interleaving view can be restored as this bisimilarity is defined on computations instead of states to preserve both causality and history (one transition relation, viewed as bidirectional, outgoing/incoming).

- What are the properties of bisimilarity over reversible processes?
- Minimal process calculus tailored for reversible processes to *comparatively* study congruence, axioms, and logics for:
  - Forward-reverse bisimilarity.
  - Forward-only bisimilarity.
  - Reverse-only bisimilarity.
- Two different kinds of bisimilarities:
  - Strong bisimilarities (all actions are treated in the same way).
  - Weak bisimilarities (abstraction from unobservable actions).
- Initially only sequential processes (i.e., no parallel composition) to be neutral with respect to interleaving view vs. true concurrency.
- Then add parallel composition and investigate expansion laws.

Reversible Nondeterministic Sequential Processes

- We usually describe only the future behavior of processes.
- [PU07] encodes information about the past behavior in the syntax:

$$P \; ::= \; \underline{0} \mid a \,.\, P \mid a^{\dagger} .\, P \mid P + P$$

- Countable set A of actions, including the unobservable action τ.
- $a^{\dagger} .\, P$ executed action a, its forward continuation is inside P, and can undo a after all executed actions within P have been undone.
- Uniform action decorations like in [BoudolCastellani94] instead of communication keys [PU07].
- Consequence of a single transition relation [DMV90].
- No need to distinguish between forward and backward actions [DK04].
- Outgoing vs. incoming transitions in the bisimulation game [DMV90].

- Initial processes: all the actions are unexecuted (they coincide with standard, forward-only processes).
- Final processes: all the actions along a path have been executed (several paths in the presence of $+$, only one is chosen – †-marked).
- Work with the set $\mathbb{P}$ of reachable processes:

$$
\begin{array}{rcl}
\mathit{reachable}(\underline{0}) & & \\
\mathit{reachable}(a\,.\,P) & \Longleftarrow & \mathit{initial}(P) \\
\mathit{reachable}(a^{\dagger}\,.\,P) & \Longleftarrow & \mathit{reachable}(P) \\
\mathit{reachable}(P_1 + P_2) & \Longleftarrow & (\mathit{reachable}(P_1) \wedge \mathit{initial}(P_2)) \vee \\
 & & (\mathit{initial}(P_1) \wedge \mathit{reachable}(P_2))
\end{array}
$$

- In $P_1 + P_2$ both subprocesses can be initial (at least one must be).
- Every initial or final process is reachable too ($\underline{0}$ is both).
- $\mathbb{P}$ also contains processes that are neither initial nor final: $a^{\dagger}\,.\,b\,.\,\underline{0}$.
- Past actions can never follow future actions: $b\,.\,a^{\dagger}\,.\,\underline{0} \notin \mathbb{P}$.

- Since all information needed to enable reversibility is in the syntax, action prefix and choice are made static by the semantics [PU07].
- Semantics defined according to the structural operational approach: labeled transition system $(\mathbb{P}, A, \longrightarrow)$ where $\longrightarrow \subseteq \mathbb{P} \times A \times \mathbb{P}$.
- Single transition relation viewed as symmetric to meet loop property: *executed actions can be undone and undone actions can be redone* (necessary condition for any reasonable notion of reversibility).
- Outgoing/incoming transitions for forward/backward bisimilarity like in [DMV90].
- Transition $P \xrightarrow{a} P'$ goes:
  - forward if it is viewed as an outgoing transition of P, in which case action a is done.
  - backward if it is viewed as an incoming transition of P', in which case action a is undone.

- Semantic rules for action prefix:

$$\frac{\mathit{initial}(P)}{a\,.\,P \xrightarrow{a} a^{\dagger}\,.\,P} \qquad \frac{P \xrightarrow{b} P'}{a^{\dagger}\,.\,P \xrightarrow{b} a^{\dagger}\,.\,P'}$$

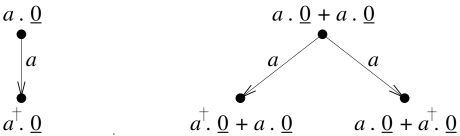
- The prefix related to the executed action is *not discarded*.
- It becomes a †-decorated part of the target process, necessary to offer again that action after rolling back.
- Additional rule for performing unexecuted actions that are preceded by already executed actions (direct consequence of making prefix static).
- This rule propagates actions executed by initial subprocesses.
- Can we view $a^{\dagger}\,.\,\_$ as the inverse operator of $a\,.\,\_$?

- Semantic rules for alternative composition:

$$\frac{P_1 \xrightarrow{a} P_1' \quad \textit{initial}(P_2)}{P_1 + P_2 \xrightarrow{a} P_1' + P_2} \qquad \frac{P_2 \xrightarrow{a} P_2' \quad \textit{initial}(P_1)}{P_1 + P_2 \xrightarrow{a} P_1 + P_2'}$$

- The subprocess not involved in the executed action is *not discarded* but cannot proceed further (only the non-initial subprocess can).
- It becomes part of the target process, which is necessary for offering again the original choice after undoing all the executed actions.
- If both subprocesses are initial, both rules apply (nondet. choice).
- If not, should operator $+$ become something like $+^{\dagger}$?
  Not needed due to action decorations within either subprocess.

- The labeled transition system underlying an initial process is a *tree*, whose branching points correspond to occurrences of $+$:
  - Every non-final process has at least one outgoing transition.
  - Every non-initial process has exactly one incoming transition due to decorations associated with executed actions.
- Consider the two initial processes $a \,.\, \underline{0}$ and $a \,.\, \underline{0} + a \,.\, \underline{0}$:



- Single a-transition on the right in a forward-only process calculus.
- These two distinct processes should be considered equivalent though.

Bisimilarities for Reversible Nondeterministic Processes

- Bisimulation game: *outgoing* transitions for forward direction and *incoming* transitions for backward direction [DMV90].
- A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a:
  - Forward bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
    - for each $P_1 \xrightarrow{a} P_1'$ there exists $P_2 \xrightarrow{a} P_2'$ such that $(P_1', P_2') \in \mathcal{B}$.
  - Reverse bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
    - for each $P_1' \xrightarrow{a} P_1$ there exists $P_2' \xrightarrow{a} P_2$ such that $(P_1', P_2') \in \mathcal{B}$.
  - Forward-reverse bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
    - for each $P_1 \xrightarrow{a} P_1'$ there exists $P_2 \xrightarrow{a} P_2'$ such that $(P_1', P_2') \in \mathcal{B}$;
    - for each $P_1' \xrightarrow{a} P_1$ there exists $P_2' \xrightarrow{a} P_2$ such that $(P_1', P_2') \in \mathcal{B}$.
- Largest such relations: $\sim_{\mathrm{FB}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$.
- In order for $P_1, P_2 \in \mathbb{P}$ to be identified by $\sim_{\mathrm{FB}}/\sim_{\mathrm{RB}}$, the sets of actions labeling their outgoing/incoming transitions must coincide (*forward/backward ready set*).

Discriminating Power

- $\sim_{\rm FRB} \subsetneq \sim_{\rm FB} \cap \sim_{\rm RB}$:
  - The inclusion is strict because the final processes $a^{\dagger} . \underline{0}$ and $a^{\dagger} . \underline{0} + c . \underline{0}$ are identified by $\sim_{\rm FB}$ and $\sim_{\rm RB}$, but distinguished by $\sim_{\rm FRB}$.
  - $\sim_{\rm FB}$ and $\sim_{\rm RB}$ are incomparable because $a^{\dagger} . \underline{0} \sim_{\rm FB} \underline{0}$ but $a^{\dagger} . \underline{0} \not\sim_{\rm RB} \underline{0}$ while $a . \underline{0} \sim_{\rm RB} \underline{0}$ but $a . \underline{0} \not\sim_{\rm FB} \underline{0}$.
- First comparative remark ($\sim_{\rm FB}$ vs. $\sim_{\rm RB}$):
  - $\sim_{\rm FRB} = \sim_{\rm FB}$ over initial processes, with $\sim_{\rm RB}$ strictly coarser.
  - $\sim_{\rm FRB} \neq \sim_{\rm RB}$ over final processes because, after going backward, discarded subprocesses come into play again for $\sim_{\rm FRB}$.
- $a . \underline{0}$ and $a . \underline{0} + a . \underline{0}$ are identified by all three bisimilarities as witnessed by any bisimulation containing the pairs $(a . \underline{0}, a . \underline{0} + a . \underline{0})$, $(a^{\dagger} . \underline{0}, a^{\dagger} . \underline{0} + a . \underline{0})$, $(a^{\dagger} . \underline{0}, a . \underline{0} + a^{\dagger} . \underline{0})$.

Compositionality Properties

- $\sim_{\rm FB}$ equates processes with different past: $a_1^{\dagger} \,.\, \underline{0} \sim_{\rm FB} a_2^{\dagger} \,.\, \underline{0} \sim_{\rm FB} \underline{0}$.
- $\sim_{\rm RB}$ equates processes with different future: $a_1 \,.\, \underline{0} \sim_{\rm RB} a_2 \,.\, \underline{0} \sim_{\rm RB} \underline{0}$.
- Second comparative remark:
  - $a^{\dagger} \,.\, b \,.\, \underline{0} \sim_{\rm FB} b \,.\, \underline{0}$ but $a^{\dagger} \,.\, b \,.\, \underline{0} + c \,.\, \underline{0} \not\sim_{\rm FB} b \,.\, \underline{0} + c \,.\, \underline{0}$.
  - $a^{\dagger} \,.\, b \,.\, \underline{0} \not\sim_{\rm RB} b \,.\, \underline{0}$ hence no such compositionality violation for $\sim_{\rm RB}$.
- $\sim_{\rm RB}$ and $\sim_{\rm FRB}$ never identify an initial process with a non-initial one, hence $\sim_{\rm FB}$ has to be made sensitive to the *presence of the past*.
- A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a past-sensitive forward bisimulation iff it is a forward bisimulation in which *initial*$(P_1) \Longleftrightarrow$ *initial*(P_2) for all $(P_1, P_2) \in \mathcal{B}$. Largest such relation: $\sim_{\rm FB:ps}$.
- $a_1^{\dagger} \,.\, \underline{0} \sim_{\rm FB:ps} a_2^{\dagger} \,.\, \underline{0}$, but $a^{\dagger} \,.\, \underline{0} \not\sim_{\rm FB:ps} \underline{0}$ and $a^{\dagger} \,.\, b \,.\, \underline{0} \not\sim_{\rm FB:ps} b \,.\, \underline{0}$.

- Let $P_1, P_2 \in \mathbb{P}$ be s.t. $P_1 \sim P_2$ and take arbitrary $a \in A$ and $P \in \mathbb{P}$.
- All the considered bisimilarities are congruences w.r.t. action prefix:
  - $a \, . \, P_1 \sim a \, . \, P_2$ provided that *initial*$(P_1) \wedge$ *initial*(P_2).
  - $a^{\dagger} . \, P_1 \sim a^{\dagger} . \, P_2$.
- $\sim_{\mathrm{FB:ps}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$ are congruences w.r.t. alternative composition:
  - $P_1 + P \sim P_2 + P$ and $P + P_1 \sim P + P_2$
    provided that *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.
- $\sim_{\mathrm{FB:ps}}$ is the coarsest congruence w.r.t. $+$ contained in $\sim_{\mathrm{FB}}$:
  - $P_1 \sim_{\mathrm{FB:ps}} P_2$ iff $P_1 + P \sim_{\mathrm{FB}} P_2 + P$
    for all $P \in \mathbb{P}$ s.t. *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.

Equational Characterizations

- Deduction system based on these axioms and inference rules on $\mathbb{P}$:
  - Reflexivity: $P = P$.
  - Symmetry: $\dfrac{P_1 = P_2}{P_2 = P_1}$.
  - Transitivity: $\dfrac{P_1 = P_2 \quad P_2 = P_3}{P_1 = P_3}$.
  - .-Substitutivity: $\dfrac{P_1 = P_2 \quad \textit{initial}(P_1) \wedge \textit{initial}(P_2)}{a \,.\, P_1 = a \,.\, P_2}$, $\dfrac{P_1 = P_2}{a^{\dagger} \,.\, P_1 = a^{\dagger} \,.\, P_2}$.
  - $+$-Substitutivity: $\dfrac{P_1 = P_2 \quad \textit{initial}(P) \vee (\textit{initial}(P_1) \wedge \textit{initial}(P_2))}{P_1 + P = P_2 + P \quad P + P_1 = P + P_2}$.
- Correspond to $\sim_{\mathrm{FB:ps}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$ being equivalence relations as well as congruences w.r.t. action prefix and alternative composition.

- Axioms:

$(\mathcal{A}_1)$		$(P + Q) + R$	$=$	$P + (Q + R)$	
$(\mathcal{A}_2)$		$P + Q$	$=$	$Q + P$	
$(\mathcal{A}_3)$		$P + \underline{0}$	$=$	P	
$(\mathcal{A}_4)$	$[\sim_{\text{FB:ps}}]$	$a^{\dagger} . P$	$=$	P	if $\neg \mathit{initial}(P)$
$(\mathcal{A}_5)$	$[\sim_{\text{FB:ps}}]$	$a^{\dagger} . P$	$=$	$b^{\dagger} . P$	if $\mathit{initial}(P)$
$(\mathcal{A}_6)$	$[\sim_{\text{FB:ps}}]$	$P + Q$	$=$	P	if $\neg \mathit{initial}(P)$, where $\mathit{initial}(Q)$
$(\mathcal{A}_7)$	$[\sim_{\text{RB}}]$	$a . P$	$=$	P	where $\mathit{initial}(P)$
$(\mathcal{A}_8)$	$[\sim_{\text{RB}}]$	$P + Q$	$=$	P	if $\mathit{initial}(Q)$
$(\mathcal{A}_9)$	$[\sim_{\text{FB:ps}}]$	$P + P$	$=$	P	where $\mathit{initial}(P)$
$(\mathcal{A}_{10})$	$[\sim_{\text{FRB}}]$	$P + Q$	$=$	P	if $\mathit{initial}(Q) \wedge \mathit{to\_initial}(P) = Q$

- $\mathcal{A}_8$ subsumes $\mathcal{A}_3$ (with $Q = \underline{0}$) and $\mathcal{A}_9$ (with $Q = P$).
- $\mathcal{A}_9$ and $\mathcal{A}_6$ apply in two different cases (P initial or not).
- $\mathcal{A}_{10}$ appeared for the first time in [LanesePhillips21].
- $\vdash^{1,2,3}_{4,5,6,9}$ / $\vdash^{1,2}_{7,8}$ / $\vdash^{1,2,3}_{10}$ sound and complete for $\sim_{\text{FB:ps}}$ / $\sim_{\text{RB}}$ / $\sim_{\text{FRB}}$.
- Third comparative remark: explicit vs. implicit idempotency.

Modal Logic Characterizations

- Hennessy-Milner logic extended with a backward modality (and init) from which suitable fragments are taken.
- Syntax:

$$\phi ::= \text{true} \mid \text{init} \mid \neg\phi \mid \phi \wedge \phi \mid \langle a \rangle \phi \mid \langle a^\dagger \rangle \phi$$

- Semantics:

$$\begin{array}{lcll} P & \models & \text{true} & \text{for all } P \in \mathbb{P} \\ P & \models & \text{init} & \text{iff } \textit{initial}(P) \\ P & \models & \neg\phi & \text{iff } P \not\models \phi \\ P & \models & \phi_1 \wedge \phi_2 & \text{iff } P \models \phi_1 \text{ and } P \models \phi_2 \\ P & \models & \langle a \rangle \phi & \text{iff there is } P' \in \mathbb{P} \text{ such that } P \xrightarrow{a} P' \text{ and } P' \models \phi \\ P & \models & \langle a^\dagger \rangle \phi & \text{iff there is } P' \in \mathbb{P} \text{ such that } P' \xrightarrow{a} P \text{ and } P' \models \phi \end{array}$$

- Fragments characterizing the four strong bisimilarities:

	true	init	$\neg$	$\wedge$	$\langle a \rangle$	$\langle a^\dagger \rangle$
$\mathcal{L}_{\mathrm{FB}}$	✓		✓	✓	✓	
$\mathcal{L}_{\mathrm{FB:ps}}$	✓	✓	✓	✓	✓	
$\mathcal{L}_{\mathrm{RB}}$	✓					✓
$\mathcal{L}_{\mathrm{FRB}}$	✓		✓	✓	✓	✓

- $\mathcal{L}_{\mathrm{FB}}$ / $\mathcal{L}_{\mathrm{FB:ps}}$ / $\mathcal{L}_{\mathrm{RB}}$ / $\mathcal{L}_{\mathrm{FRB}}$ characterizes $\sim_{\mathrm{FB}}$ / $\sim_{\mathrm{FB:ps}}$ / $\sim_{\mathrm{RB}}$ / $\sim_{\mathrm{FRB}}$:
  $P_1 \sim_B P_2$ iff $\forall \phi \in \mathcal{L}_B .\, P_1 \models \phi \Longleftrightarrow P_2 \models \phi$.
- $\sim_{\mathrm{RB}}$ boils down to reverse trace equivalence!
- Every process has at most one incoming transition.

Weak Bisimilarities

- Abstracting from τ-actions: $P \xRightarrow{\tau^*} P'$, $P \xRightarrow{\tau^*} \xrightarrow{a} \xRightarrow{\tau^*} P'$.
- A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a ($a \in A \setminus \{\tau\}$):
  - Weak forward bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$:
    - for each $P_1 \xrightarrow{\tau} P_1'$ there is $P_2 \xRightarrow{\tau^*} P_2'$ s.t. $(P_1', P_2') \in \mathcal{B}$;
    - for each $P_1 \xrightarrow{a} P_1'$ there is $P_2 \xRightarrow{\tau^*} \xrightarrow{a} \xRightarrow{\tau^*} P_2'$ s.t. $(P_1', P_2') \in \mathcal{B}$.
  - Weak reverse bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
    - for each $P_1' \xrightarrow{\tau} P_1$ there is $P_2' \xRightarrow{\tau^*} P_2$ s.t. $(P_1', P_2') \in \mathcal{B}$;
    - for each $P_1' \xrightarrow{a} P_1$ there is $P_2' \xRightarrow{\tau^*} \xrightarrow{a} \xRightarrow{\tau^*} P_2$ s.t. $(P_1', P_2') \in \mathcal{B}$.
  - Weak forward-reverse bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
    - for each $P_1 \xrightarrow{\tau} P_1'$ there is $P_2 \xRightarrow{\tau^*} P_2'$ s.t. $(P_1', P_2') \in \mathcal{B}$;
    - for each $P_1 \xrightarrow{a} P_1'$ there is $P_2 \xRightarrow{\tau^*} \xrightarrow{a} \xRightarrow{\tau^*} P_2'$ s.t. $(P_1', P_2') \in \mathcal{B}$;
    - for each $P_1' \xrightarrow{\tau} P_1$ there is $P_2' \xRightarrow{\tau^*} P_2$ s.t. $(P_1', P_2') \in \mathcal{B}$;
    - for each $P_1' \xrightarrow{a} P_1$ there is $P_2' \xRightarrow{\tau^*} \xrightarrow{a} \xRightarrow{\tau^*} P_2$ s.t. $(P_1', P_2') \in \mathcal{B}$.
- Largest such relations: $\approx_{\mathrm{FB}}$, $\approx_{\mathrm{RB}}$, $\approx_{\mathrm{FRB}}$.

- Each weak bisimilarity is strictly coarser than its strong counterpart.
- $\approx_{\mathrm{FRB}} \subsetneq \approx_{\mathrm{FB}} \cap \approx_{\mathrm{RB}}$ with $\approx_{\mathrm{FB}}$ and $\approx_{\mathrm{RB}}$ being incomparable.
- $\approx_{\mathrm{FRB}} \neq \approx_{\mathrm{FB}}$ over initial processes:
  - $\tau\,.\,a\,.\,\underline{0} + a\,.\,\underline{0} + b\,.\,\underline{0}$ and $\tau\,.\,a\,.\,\underline{0} + b\,.\,\underline{0}$ are identified by $\approx_{\mathrm{FB}}$ but told apart by $\approx_{\mathrm{FRB}}$
    - Doing a on the left is matched by doing τ and then a on the right.
    - Undoing a on the right cannot be matched on the left.
  - $c\,.\,(\tau\,.\,a\,.\,\underline{0} + a\,.\,\underline{0} + b\,.\,\underline{0})$ and $c\,.\,(\tau\,.\,a\,.\,\underline{0} + b\,.\,\underline{0})$ is an analogous counterexample with non-initial τ-actions:
    - Doing c on one side is matched by doing c on the other side.
    - Doing a on the left is matched by doing τ and then a on the right.
    - Undoing a on the right cannot be matched on the left.

- Neither $\approx_{\mathrm{FB}}$ nor $\approx_{\mathrm{FRB}}$ is compositional:
  - $a^{\dagger} . b . \underline{0} \approx_{\mathrm{FB}} b . \underline{0}$ but $a^{\dagger} . b . \underline{0} + c . \underline{0} \not\approx_{\mathrm{FB}} b . \underline{0} + c . \underline{0}$ (same as $\sim_{\mathrm{FB}}$).
  - $\tau . a . \underline{0} \approx_{\mathrm{FB}} a . \underline{0}$ but $\tau . a . \underline{0} + b . \underline{0} \not\approx_{\mathrm{FB}} a . \underline{0} + b . \underline{0}$.
  - $\tau . a . \underline{0} \approx_{\mathrm{FRB}} a . \underline{0}$ but $\tau . a . \underline{0} + b . \underline{0} \not\approx_{\mathrm{FRB}} a . \underline{0} + b . \underline{0}$.
- Weak congruence construction à la Milner does not work here.
- A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a weak past-sensitive forward bisim. iff it is a weak forward bisim. in which $\mathit{initial}(P_1) \Longleftrightarrow \mathit{initial}(P_2)$ for all $(P_1, P_2) \in \mathcal{B}$. Largest such relation: $\approx_{\mathrm{FB:ps}}$.
- A symm. rel. $\mathcal{B}$ over $\mathbb{P}$ is a weak past-sensitive forward-reverse bisim. iff it is a weak forward-reverse bisim. s.t. $\mathit{initial}(P_1) \Longleftrightarrow \mathit{initial}(P_2)$ for all $(P_1, P_2) \in \mathcal{B}$. Largest such relation: $\approx_{\mathrm{FRB:ps}}$.
- $\sim_{\mathrm{FRB}} \subsetneq \approx_{\mathrm{FRB:ps}}$ as the former satisfies the initiality condition.

- Let $P_1, P_2 \in \mathbb{P}$ be s.t. $P_1 \approx P_2$ and take arbitrary $a \in A$ and $P \in \mathbb{P}$.
- All the considered bisimilarities are congruences w.r.t. action prefix:
  - $a \,.\, P_1 \approx a \,.\, P_2$ provided that *initial*$(P_1) \wedge$ *initial*(P_2).
  - $a^{\dagger} \,.\, P_1 \approx a^{\dagger} \,.\, P_2$.
- $\approx_{\mathrm{FB:ps}}$, $\approx_{\mathrm{RB}}$, $\approx_{\mathrm{FRB:ps}}$ are congruences w.r.t. alternative composition:
  - $P_1 + P \approx P_2 + P$ and $P + P_1 \approx P + P_2$
    provided that *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.
- $\approx_{\mathrm{FB:ps}}$ is the coarsest congruence w.r.t. $+$ contained in $\approx_{\mathrm{FB}}$:
  - $P_1 \approx_{\mathrm{FB:ps}} P_2$ iff $P_1 + P \approx_{\mathrm{FB}} P_2 + P$
    for all $P \in \mathbb{P}$ s.t. *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.
- $\approx_{\mathrm{FRB:ps}}$ is the coarsest congruence w.r.t. $+$ contained in $\approx_{\mathrm{FRB}}$:
  - $P_1 \approx_{\mathrm{FRB:ps}} P_2$ iff $P_1 + P \approx_{\mathrm{FRB}} P_2 + P$
    for all $P \in \mathbb{P}$ s.t. *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.

- Additional axioms (τ-laws):

$(\mathcal{A}_1^\tau)$	$[\approx_{\mathrm{FB:ps}}]$	$a\,.\,\tau\,.\,P$	$=$	$a\,.\,P$	where $initial(P)$
$(\mathcal{A}_2^\tau)$	$[\approx_{\mathrm{FB:ps}}]$	$P + \tau\,.\,P$	$=$	$\tau\,.\,P$	where $initial(P)$
$(\mathcal{A}_3^\tau)$	$[\approx_{\mathrm{FB:ps}}]$	$a\,.\,(P + \tau\,.\,Q) + a\,.\,Q$	$=$	$a\,.\,(P + \tau\,.\,Q)$	where P, Q initial
$(\mathcal{A}_4^\tau)$	$[\approx_{\mathrm{FB:ps}}]$	$a^\dagger.\,\tau\,.\,P$	$=$	$a^\dagger.\,P$	where $initial(P)$
$(\mathcal{A}_5^\tau)$	$[\approx_{\mathrm{RB}}]$	$\tau^\dagger.\,P$	$=$	P	
$(\mathcal{A}_6^\tau)$	$[\approx_{\mathrm{FRB:ps}}]$	$a\,.\,(\tau\,.\,(P + Q) + P)$	$=$	$a\,.\,(P + Q)$	where P, Q initial
$(\mathcal{A}_7^\tau)$	$[\approx_{\mathrm{FRB:ps}}]$	$a^\dagger.\,(\tau\,.\,(P + Q) + P')$	$=$	$a^\dagger.\,(P' + Q)$	if $to\_initial(P') = P$, where P, Q initial
$(\mathcal{A}_8^\tau)$	$[\approx_{\mathrm{FRB:ps}}]$	$a^\dagger.\,(\tau^\dagger.\,(P' + Q) + P)$	$=$	$a^\dagger.\,(P' + Q)$	if $to\_initial(P') = P$, where $initial(P)$

- $\mathcal{A}_1^\tau$, $\mathcal{A}_2^\tau$, $\mathcal{A}_3^\tau$ are Milner τ-laws, $\mathcal{A}_4^\tau$ is needed for completeness.
- $\mathcal{A}_5^\tau$ is a variant of $\tau\,.\,P = P$ (not valid for weak bisim. congruence).
- $\mathcal{A}_6^\tau$ is Van Glabbeek – Weijland τ-law, $\mathcal{A}_7^\tau$ and $\mathcal{A}_8^\tau$ needed for complet.
- $\vdash_{1,2,3,4}^{1,2,3,4,5,6,9}$ / $\vdash_{5}^{1,2,7,8}$ / $\vdash_{6,7,8}^{1,2,3,10}$ sound and complete for $\approx_{\mathrm{FB:ps}}$ / $\approx_{\mathrm{RB}}$ / $\approx_{\mathrm{FRB:ps}}$.
- $\approx_{\mathrm{FRB}}$ is branching bisimilarity over initial processes!

- Modal logic with weak forward/backward modalities ($a \in A \setminus \{\tau\}$):

$$\phi \ ::= \ \text{true} \mid \text{init} \mid \neg\phi \mid \phi \wedge \phi \mid \langle\langle\tau\rangle\rangle\phi \mid \langle\langle a\rangle\rangle\phi \mid \langle\langle\tau^\dagger\rangle\rangle\phi \mid \langle\langle a^\dagger\rangle\rangle\phi$$

- Semantics:

$$
\begin{array}{lcll}
P & \models & \text{true} & \text{for all } P \in \mathbb{P} \\
P & \models & \text{init} & \text{iff } \textit{initial}(P) \\
P & \models & \neg\phi & \text{iff } P \not\models \phi \\
P & \models & \phi_1 \wedge \phi_2 & \text{iff } P \models \phi_1 \text{ and } P \models \phi_2 \\
P & \models & \langle\langle\tau\rangle\rangle\phi & \text{iff there is } P' \in \mathbb{P} \text{ such that } P \stackrel{\tau^*}{\Longrightarrow} P' \text{ and } P' \models \phi \\
P & \models & \langle\langle a\rangle\rangle\phi & \text{iff there is } P' \in \mathbb{P} \text{ such that } P \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P' \text{ and } P' \models \phi \\
P & \models & \langle\langle\tau^\dagger\rangle\rangle\phi & \text{iff there is } P' \in \mathbb{P} \text{ such that } P' \stackrel{\tau^*}{\Longrightarrow} P \text{ and } P' \models \phi \\
P & \models & \langle\langle a^\dagger\rangle\rangle\phi & \text{iff there is } P' \in \mathbb{P} \text{ such that } P' \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P \text{ and } P' \models \phi
\end{array}
$$

- Fragments characterizing the five weak bisimilarities:

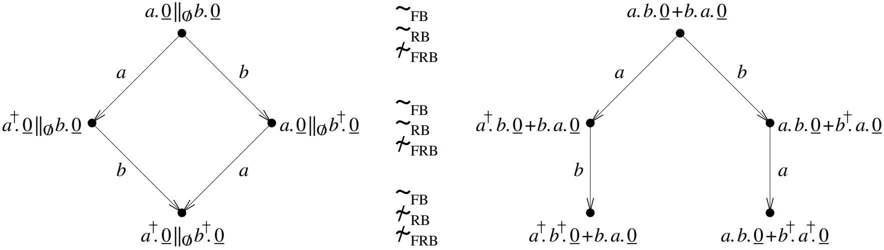
	true	init	$\neg$	$\wedge$	$\langle\langle\tau\rangle\rangle$	$\langle\langle a\rangle\rangle$	$\langle\langle\tau^\dagger\rangle\rangle$	$\langle\langle a^\dagger\rangle\rangle$
$\mathcal{L}^\tau_{\mathrm{FB}}$	✓		✓	✓	✓	✓		
$\mathcal{L}^\tau_{\mathrm{FB:ps}}$	✓	✓	✓	✓	✓	✓		
$\mathcal{L}^\tau_{\mathrm{RB}}$	✓						✓	✓
$\mathcal{L}^\tau_{\mathrm{FRB}}$	✓		✓	✓	✓	✓	✓	✓
$\mathcal{L}^\tau_{\mathrm{FRB:ps}}$	✓	✓	✓	✓	✓	✓	✓	✓

- $\mathcal{L}^\tau_{\mathrm{FB}}$ / $\mathcal{L}^\tau_{\mathrm{FB:ps}}$ / $\mathcal{L}^\tau_{\mathrm{RB}}$ / $\mathcal{L}^\tau_{\mathrm{FRB}}$ / $\mathcal{L}^\tau_{\mathrm{FRB:ps}}$ characterizes $\approx_{\mathrm{FB}}$ / $\approx_{\mathrm{FB:ps}}$ / $\approx_{\mathrm{RB}}$ / $\approx_{\mathrm{FRB}}$ / $\approx_{\mathrm{FRB:ps}}$:
  $P_1 \approx_B P_2$ iff $\forall \phi \in \mathcal{L}^\tau_B.\, P_1 \models \phi \iff P_2 \models \phi$.

- We now include parallel composition in the syntax:

$$P \ ::= \ \underline{0} \mid a\,.\,P \mid a^{\dagger}.\,P \mid P + P \mid P \,\|_L\, P$$

- Then for $a \neq b$:



- $\sim_{\mathrm{FB}}$ is interleaving, while $\sim_{\mathrm{RB}}$ and $\sim_{\mathrm{FRB}}$ are truly concurrent.
- What are the expansion laws for $\sim_{\mathrm{FB}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$?

- Expansion laws for forward-only calculi in the interleaving setting identify $a\,.\,\underline{0} \parallel_{\emptyset} b\,.\,\underline{0}$ and $a\,.\,b\,.\,\underline{0} + b\,.\,a\,.\,\underline{0}$ [HennessyMilner85].
- Used also in truly concurrent semantics to distinguish those processes by adding suitable discriminating information within action prefixes:
  - Causal bisimilarity [DarondeauDegano90] (corresponding to history-preserving bisimilarity): every action is enriched with the set of its causing actions expressed as backward pointers, hence $<a, \emptyset>\,.\,<b, \emptyset>\,.\,\underline{0} + <b, \emptyset>\,.\,<a, \emptyset>\,.\,\underline{0}$ and $<a, \emptyset>\,.\,<b, \{1\}>\,.\,\underline{0} + <b, \emptyset>\,.\,<a, \{1\}>\,.\,\underline{0}$.
  - Location bisimilarity [BoudolCastellaniHennessyKiehn94]: every action is enriched with the name of the location in which it is executed, hence $<a, l_a>\,.\,<b, l_b>\,.\,\underline{0} + <b, l_b>\,.\,<a, l_a>\,.\,\underline{0}$ and $<a, l_a>\,.\,<b, l_a l_b>\,.\,\underline{0} + <b, l_b>\,.\,<a, l_b l_a>\,.\,\underline{0}$.
  - Pomset bisimilarity [BoudolCastellani88]: a prefix may contain the combination of actions that are independent of each other, hence $a\,.\,b\,.\,\underline{0} + b\,.\,a\,.\,\underline{0} + (a \parallel b)\,.\,\underline{0}$.

- How to uniformly derive expansion laws for $\sim_{\mathrm{FB}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$?
- Proved trees approach of [DeganoPriami92].
- Label every transition with a proof term [BoudolCastellani88], which is an action preceded by the operators in the scope of which it occurs:

$$\theta \ ::= \ a \mid .\theta \mid \dot{+}\theta \mid +\!\!.\,\theta \mid \rfloor\!\rfloor\theta \mid \lfloor\!\lfloor\theta \mid \langle\theta,\theta\rangle$$

- The equivalence of interest then drives an observation function that maps proof terms to the required observations.
- Interleaving: proof terms are reduced to the actions they contain.
- True concurrency: they are transformed into actions extended with suitable discriminating information (encoding processes accordingly).
- Information not necessarily available in the operational semantics, as is the case with $\sim_{\mathrm{RB}}$ and $\sim_{\mathrm{FRB}}$.

- Proved operational semantic rules:

$$\frac{\mathit{initial}(P)}{a\,.\,P \xrightarrow{a} a^{\dagger}\,.\,P} \qquad \frac{P \xrightarrow{\theta} P'}{a^{\dagger}\,.\,P \xrightarrow{.\theta} a^{\dagger}\,.\,P'}$$

$$\frac{P_1 \xrightarrow{\theta} P_1' \quad \mathit{initial}(P_2)}{P_1 + P_2 \xrightarrow{+\theta} P_1' + P_2} \qquad \frac{P_2 \xrightarrow{\theta} P_2' \quad \mathit{initial}(P_1)}{P_1 + P_2 \xrightarrow{+\theta} P_1 + P_2'}$$

$$\frac{P_1 \xrightarrow{\theta} P_1' \quad \mathit{act}(\theta) \notin L}{P_1 \parallel_L P_2 \xrightarrow{\parallel\theta} P_1' \parallel_L P_2} \qquad \frac{P_2 \xrightarrow{\theta} P_2' \quad \mathit{act}(\theta) \notin L}{P_1 \parallel_L P_2 \xrightarrow{\parallel\theta} P_1 \parallel_L P_2'}$$

$$\frac{P_1 \xrightarrow{\theta_1} P_1' \quad P_2 \xrightarrow{\theta_2} P_2' \quad \mathit{act}(\theta_1) = \mathit{act}(\theta_2) \in L}{P_1 \parallel_L P_2 \xrightarrow{\langle\theta_1,\theta_2\rangle} P_1' \parallel_L P_2'}$$

- Forward clause rephrased:
  - For each $P_1 \xrightarrow{\theta_1} P'_1$ there exists $P_2 \xrightarrow{\theta_2} P'_2$ such that $\mathit{act}(\theta_1) = \mathit{act}(\theta_2)$ and $(P'_1, P'_2) \in \mathcal{B}$.
- Backward clause rephrased:
  - For each $P'_1 \xrightarrow{\theta_1} P_1$ there exists $P'_2 \xrightarrow{\theta_2} P_2$ such that $\mathit{act}(\theta_1) = \mathit{act}(\theta_2)$ and $(P'_1, P'_2) \in \mathcal{B}$.
- Observation function ℓ applied to proof terms labeling transitions, so that $\ell(\theta_1)$ and $\ell(\theta_2)$ are considered in the bisimulation game.
- May depend on other possible parameters that are present in the proved labeled transition system.
- Preserves actions: $\ell(\theta_1) = \ell(\theta_2)$ implies $\mathit{act}(\theta_1) = \mathit{act}(\theta_2)$.
- $\sim_{\mathrm{FB:ps:}\ell_{\mathrm{F}}}$, $\sim_{\mathrm{RB:}\ell_{\mathrm{R}}}$, $\sim_{\mathrm{FRB:}\ell_{\mathrm{FR}}}$ are the three resulting equivalences.
- When do they coincide with $\sim_{\mathrm{FB:ps}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$?
- What is the discriminating information needed by $\sim_{\mathrm{RB}}$ and $\sim_{\mathrm{FRB}}$?

- $\sim_{\mathrm{FB:ps:}\ell_{\mathrm{F}}} = \sim_{\mathrm{FB:ps}}$ when $\ell_{\mathrm{F}}(\theta) = \mathit{act}(\theta)$.
- Axiomatization of $\sim_{\mathrm{FB:ps}}$ over reversible concurrent processes:

$$
\begin{array}{llcll}
(\mathcal{A}_{\mathrm{F},1}) & (P + Q) + R & = & P + (Q + R) & \\
(\mathcal{A}_{\mathrm{F},2}) & P + Q & = & Q + P & \\
(\mathcal{A}_{\mathrm{F},3}) & P + \underline{0} & = & P & \\
(\mathcal{A}_{\mathrm{F},4}) & P + P & = & P & \text{where } \mathit{initial}(P) \\
(\mathcal{A}_{\mathrm{F},5}) & a^{\dagger} \,.\, P & = & P & \text{if } \neg\mathit{initial}(P) \\
(\mathcal{A}_{\mathrm{F},6}) & a^{\dagger} \,.\, P & = & b^{\dagger} \,.\, P & \text{if } \mathit{initial}(P) \\
(\mathcal{A}_{\mathrm{F},7}) & P + Q & = & P & \text{if } \neg\mathit{initial}(P), \text{ where } \mathit{initial}(Q) \\
(\mathcal{A}_{\mathrm{F},8}) & P_1 \parallel_L P_2 & = & \multicolumn{2}{l}{[a^{\dagger}.] \Big(\sum\limits_{i \in I_1, a_{1,i} \notin L} a_{1,i} \,.\, (P_{1,i} \parallel_L P'_2) + \sum\limits_{i \in I_2, a_{2,i} \notin L} a_{2,i} \,.\, (P'_1 \parallel_L P_{2,i}) + \sum\limits_{i \in I_1, a_{1,i} \in L} \; \sum\limits_{j \in I_2, a_{2,j} = a_{1,i}} a_{1,i} \,.\, (P_{1,i} \parallel_L P_{2,j}) \Big)}
\end{array}
$$

- $P_k = [a_k^{\dagger}.]P'_k$ with $P'_k = \sum\limits_{i \in I_k} a_{k,i} \,.\, P_{k,i}$ for $k \in \{1, 2\}$.
- $[a^{\dagger}.]$ stands for an optional executed action prefix.

- $\sim_{\mathrm{RB}:\ell_{\mathrm{R}}} = \sim_{\mathrm{RB}}$ and $\sim_{\mathrm{FRB}:\ell_{\mathrm{FR}}} = \sim_{\mathrm{FRB}}$ when $\ell_{\mathrm{R}}(\theta)_{P'} = \ell_{\mathrm{FR}}(\theta)_{P'} \triangleq \ell_{\mathrm{brs}}(\theta)_{P'} = {<}\mathit{act}(\theta), \mathit{brs}(P'){>}$ for every proved transition $P \xrightarrow{\theta} P'$.
- $\mathit{brs}(P')$ is the backward ready set of P', the set of actions labeling the incoming transitions of P'.
- Then $a\,.\,\underline{0} \parallel_{\emptyset} b\,.\,\underline{0}$ is encoded as
  ${<}a, \{a\}{>}\,.\,{<}b, \{a, b\}{>}\,.\,\underline{0} + {<}b, \{b\}{>}\,.\,{<}a, \{a, b\}{>}\,.\,\underline{0}$
  while $a\,.\,b\,.\,\underline{0} + b\,.\,a\,.\,\underline{0}$ is encoded as
  ${<}a, \{a\}{>}\,.\,{<}b, \{b\}{>}\,.\,\underline{0} + {<}b, \{b\}{>}\,.\,{<}a, \{a\}{>}\,.\,\underline{0}$.
- The encoding of $a^{\dagger}\,.\,\underline{0} \parallel_{\emptyset} b^{\dagger}\,.\,\underline{0}$ is
  either ${<}a^{\dagger}, \{a\}{>}\,.\,{<}b^{\dagger}, \{a, b\}{>}\,.\,\underline{0} + {<}b, \{b\}{>}\,.\,{<}a, \{a, b\}{>}\,.\,\underline{0}$
  or ${<}a, \{a\}{>}\,.\,{<}b, \{a, b\}{>}\,.\,\underline{0} + {<}b^{\dagger}, \{b\}{>}\,.\,{<}a^{\dagger}, \{a, b\}{>}\,.\,\underline{0}$.
- Depends on the trace of actions executed so far.
- *It cannot be*
  ${<}a^{\dagger}, \{a\}{>}\,.\,{<}b^{\dagger}, \{a, b\}{>}\,.\,\underline{0} + {<}b^{\dagger}, \{b\}{>}\,.\,{<}a^{\dagger}, \{a, b\}{>}\,.\,\underline{0}$.

- Axiomatization of $\sim_{\mathrm{RB}}$ over reversible concurrent processes:

$$
\begin{array}{lrcll}
(\mathcal{A}_{\mathrm{R},1}) & \widetilde{(P+Q)+R} & = & \widetilde{P+(Q+R)} & \\
(\mathcal{A}_{\mathrm{R},2}) & \widetilde{P+Q} & = & \widetilde{Q+P} & \\
(\mathcal{A}_{\mathrm{R},3}) & \widetilde{a\,.\,P} & = & \widetilde{P} & \text{where } \mathit{initial}(P) \\
(\mathcal{A}_{\mathrm{R},4}) & \widetilde{P+Q} & = & \widetilde{P} & \text{if } \mathit{initial}(Q) \\
(\mathcal{A}_{\mathrm{R},5}) & \widetilde{P_1 \parallel_L P_2} & = & e\ell^{\varepsilon}_{\mathrm{brs}}(\widetilde{P_1}, \widetilde{P_2}, L)_{P_1 \parallel_L P_2} &
\end{array}
$$

- $P_k = \underline{0}$ or $P_k = a^{\dagger}\,.\,P'_k$ for $k \in \{1, 2\}$.
- Axiomatization of $\sim_{\mathrm{FRB}}$ over reversible concurrent processes:

$$
\begin{array}{lrcll}
(\mathcal{A}_{\mathrm{FR},1}) & \widetilde{(P+Q)+R} & = & \widetilde{P+(Q+R)} & \\
(\mathcal{A}_{\mathrm{FR},2}) & \widetilde{P+Q} & = & \widetilde{Q+P} & \\
(\mathcal{A}_{\mathrm{FR},3}) & \widetilde{P+\underline{0}} & = & \widetilde{P} & \\
(\mathcal{A}_{\mathrm{FR},4}) & \widetilde{P+Q} & = & \widetilde{P} & \text{if } \mathit{initial}(Q) \wedge \mathit{to\_initial}(P) = Q \\
(\mathcal{A}_{\mathrm{FR},5}) & \widetilde{P_1 \parallel_L P_2} & = & e\ell^{\varepsilon}_{\mathrm{brs}}(\widetilde{P_1}, \widetilde{P_2}, L)_{P_1 \parallel_L P_2} &
\end{array}
$$

- $P_k = [a^{\dagger}\,.\,P'_k\, +]\sum_{i \in I_k} a_{k,i}\,.\,P_{k,i}$ for $k \in \{1, 2\}$.

Hereditary History-Preserving Bisimilarity

- For $a = b$ the two encodings $<a, \{a\}> . <b, \{a, b\}> . \underline{0} + <b, \{b\}> . <a, \{a, b\}> . \underline{0}$ and $<a, \{a\}> . <b, \{b\}> . \underline{0} + <b, \{b\}> . <a, \{a\}> . \underline{0}$ coincide.
- Then $a . \underline{0} \parallel_{\emptyset} a . \underline{0} \ \sim_{\rm FRB} \ a . a . \underline{0} + a . a . \underline{0} \ \sim_{\rm FRB} \ a . a . \underline{0}$.
- But $a . \underline{0} \parallel_{\emptyset} a . \underline{0} \not\sim_{\rm HHPB} a . a . \underline{0}$.
- Backward ready multisets distinguish them again and this yields the same power as hereditary history-preserving bisimilarity.
- $\sim_{\rm FRB:brm}$ provides an operational view of $\sim_{\rm HHPB}$.
- No need of identifying identically labeled events, just count them.
- The axiomatization of $\sim_{\rm HHPB}$ is a variant of the one of $\sim_{\rm FRB}$.

Concluding Remarks and Future Work

- Process algebraic theory encompassing most of concurrency theory:
  - Forward bisimilarity is the usual bisimilarity.
  - Reverse bisimilarity boils down to reverse trace equivalence.
  - Weak forward-reverse bisimilarity is branching bisimilarity.
  - Expansion laws addressing interleaving and true concurrency.
- Applied to noninterference analysis.
- Theory extended to Markovian sequential processes in the strong case, link with ordinary/exact/strict lumpability and time reversibility.
- Reversibility of deterministic timed processes (time additivity).
- Reversibility of probabilistic processes (alternating model)?
- Markovian sequential processes in the weak case (W-lumpability)?
- What changes when admitting irreversible actions (commit)?

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