Uniform Labeled Transition Systems for Nondeterministic, Probabilistic, and Stochastic Processes

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Abstract. Rate transition systems (RTS) are a special kind of transition systems introduced for defining the stochastic behavior of processes and for associating continuous-time Markov chains with process terms. The transition relation assigns to each process, for each action, the set of possible futures paired with a measure indicating the rates at which they are reached. RTS have been shown to be a uniform model for providing an operational semantics to many stochastic process algebras. In this paper, we define Uniform Labeled TRAnsition Systems (ULTraS) as a generalization of RTS that can be exploited to uniformly describe also nondeterministic and probabilistic variants of process algebras. We then present a general notion of behavioral relation for ULTraS that can be instantiated to capture bisimulation and trace equivalences for fully nondeterministic, fully probabilistic, and fully stochastic processes.

1 Introduction

Process algebras have been successfully used in the last thirty years to model the behavior and prove properties of concurrent systems. The basic ingredients of these formalisms, apart from specific syntactic operators used to define the term algebra, are labeled transition systems (LTS) and behavioral relations in the form of equivalences or preorders. By exploiting the so-called structural operational semantics, a LTS is "compositionally" associated with each term. LTS possibly corresponding to terms describing systems at different levels of abstraction are then compared according to one of the many behavioral relations that have been proposed in the literature.

Initially, the behavioral relations were mainly designed to assess whether two systems have comparable functional (extensional) behavior, i.e., whether they could perform similar actions. However, soon after witnessing the success of the process algebraic approach, it was noticed that other aspects of concurrent systems are at least as important as the functional ones. Thus, many variants of process algebras have been introduced to take into account quantitative aspects of concurrent systems and we have seen proposals of *(deterministically) timed* process algebra, *probabilistic* process algebras, and *stochastic(ally timed)* process algebras. Their semantics has then been rendered in terms of richer LTS quotiented with new behavioral relations and we have read of (deterministically) timed, probabilistic, and stochastic(ally timed) relations.

The line of research targeted to stochastic variants of process algebras has been particularly productive due to the importance of shared-resource systems. The main aim has been the integration of qualitative descriptions with quantitative (especially performance) ones in a single mathematical framework by building on the combination of LTS and continuous-time Markov chains (CTMC), one of the most successful approaches to modeling and analyzing the performance of computer systems and networks. The common feature of the most prominent stochastic process algebra proposals is that the actions used to label transitions are enriched with rates of exponentially distributed *random variables* characterizing their duration. Although the same class of random variables is assumed in many languages, the underlying models and notions are significantly different, in particular with respect to the issue of the correct representation of the *race condition* principle when modeling the choice operator (see, e.g., [7]).

In [5], two of the authors of the present paper, together with D. Latella and M. Massink, proposed a variant of LTS, namely *rate transition systems* (RTS), as a tool for providing semantics to some of the most representative stochastic process languages. Within LTS, the transition relation describes the evolution of a system from one state to another as determined by the execution of specific actions, thus it is a set of triples (*state*, *action*, *state*). Within RTS, the transition relation $\rightarrowtail$ associates with a given state P and a given transition label (action) a a function, say $\mathscr{P}$, mapping each term into a nonnegative real number. The transition $P \overset{a}{\rightarrowtail} \mathscr{P}$ has the following meaning: if $\mathscr{P}(Q) = v$ with $v \neq 0$, then Q is reachable from P by executing a, the duration of such an execution being exponentially distributed with rate v; if $\mathscr{P}(Q) = 0$, then Q is not reachable from P via a.

RTS have been used for providing a uniform semantic framework for modeling many of the different stochastic process languages. This facilitates reasoning about them and throwing light on their similarities as well as on their differences. In [4], we considered a limited number of significant stochastic process calculi. We provided the RTS semantics for TIPP [6], EMPA [3], PEPA [9], and IML [8] as representatives of the class of stochastic languages based on the CSP-like, multipart interaction paradigm. Moreover, we also considered stochastic CCS and stochastic π-calculus [14] as examples of languages based on the two-way interaction paradigm.

In this paper, we aim at performing a step further in the direction of providing a uniform characterization of the semantics of different process calculi. We propose a framework more general than RTS, which can be instantiated to model both classical process algebras usually handled via LTS and process algebras with quantitative information like probability and time. We will introduce ULTraS – *Uniform Labeled TRAnsition Systems* – as a generalization of RTS and show that they can be used to uniformly describe the nondeterministic, probabilistic, and stochastic variants of process algebras. We will then introduce a general notion of equivalence that can be instantiated to capture the nondeterministic, probabilistic, and stochastic versions of trace and bisimulation equivalence.

Within ULTRAS, the transition relation associates with a state and a given transition label a function mapping each (next) state into an element of a domain D. In order to be uniform with classical nondeterministic calculi, we do encode quantitative information inside the next-state function. More precisely, rather than having transition leading to a next state, we do work with a notion of next-state distribution, meaning that we quantify the possibility of having every process term as the next state after executing a certain action.

By appropriately changing the domain D, we can capture different models of concurrent systems. For example, we will see that if D is the Boolean algebra $\mathbb{B}$ consisting of the two values $\top$ and $\bot$ we can capture classical LTS, while if D is the set $\mathbb{R}_{[0,1]}$ we do capture fully probabilistic models, and when D is the set $\mathbb{R}_{\geq 0}$ we do capture fully stochastic models.

The advantage of the proposed uniform modeling is twofold. On the one hand, we show that the way the semantics for calculi with quantitative information has been defined so far is indeed the natural extension of the definition of the semantics for calculi with only qualitative information. On the other hand, we make calculi with quantitative information more understandable for those people with a process algebraic background who are not familiar with probability/time.

Of course, modeling state transitions and their annotations is one of the key ingredients; however, we need also to worry about how they are combined to obtain computations and how we do deem that from two states we can obtain "equivalent" computation trees. In order to do that, we introduce the notion of *trace* and *measured trace*. Based on them, we define trace equivalence and bisimulation equivalence over ULTRAS and study their relationships with the corresponding equivalences in the literature once we "appropriately" instantiate the domain D to capture well-studied models.

One of the key ingredients of the equivalence definition is a *measure function* that associates a suitable value with every triple composed of a state s, a trace α, and a state subset S'. To capture classical equivalences over nondeterministic systems, the measure of a computation labeled with α from state s to a state in S' yields $\top$ if the computation does exist and $\bot$ otherwise. To capture probabilistic equivalences, the measure yields a value in $\mathbb{R}_{[0,1]}$ that represents the probability of the set of computations labeled with trace α that reach a state in S' from state s. For stochastic equivalences, we consider two cases: the *end-to-end* delay and the *step-by-step* delay of traces. In the first case, the measure function yields the probability that the set of computations labeled with trace α lead to a state in S' from state s within t time units. In the second case, the measure function considers, instead, the probability of the set of computations labeled with α that go from s to S' within a certain number of time units for each single step.

The rest of the paper is organized as follows. In Sect. 2, we introduce ULTRAS and bisimulation and trace equivalences over them. In Sect. 3, we instantiate ULTRAS to obtain, in a row, *fully nondeterministic* processes (i.e., classical LTS), *fully probabilistic* processes (i.e., classical *action-labeled discrete-*

time Markov chains – ADTMC), and *fully stochastic* processes (i.e., classical *action-labeled continuous-time Markov chains* – ACTMC). In Sects. 4, 5, and 6, we prove that bisimulation and trace equivalences for the various instantiations of ULTraS coincide with the corresponding equivalences defined in the literature for LTS, ADTMC, and ACTMC, respectively. Finally, Sect. 7 concludes the paper and outlines future work.

2 Uniform Labeled Transition Systems

LTS consist of a set of states, a set of transition labels, and a transition relation. States correspond to the configurations processes can reach. Labels describe the actions processes can perform internally or that are used to interact with the environment. The transition relation describes process evolution as determined by the execution of specific actions.

In this section, we introduce a generalization of LTS that aims at providing a uniform framework that can be used for defining the behavior of different kinds of process. In the new model, named ULTraS from Uniform Labeled TRAnsition Systems, the transition relation associates with any source state and any transition label a function mapping each possible target state into an element of a domain D. The definition of ULTraS is provided in Sect. 2.1, while in Sect. 2.2 we show how to define behavioral equivalences on ULTraS.

2.1 Definition of the Uniform Model

In the following, we assume that D is a complete partial order with $\bot$ being its least element. We also denote by $[S \to D]$ the set of functions from S to D, which is ranged over by $\mathcal{D}$.

Definition 1. *A uniform labeled transition system on D (D-ULTraS for short) is a triple:*

$$\mathcal{U} \;=\; (S, A, \longrightarrow)$$

where:

- *S is a countable set of states.*
- *A is a countable set of transition-labeling actions.*
- *$\longrightarrow\; \subseteq S \times A \times [S \to D]$ is a transition relation.*

We say that the D-ULTraS $\mathcal{U}$ is functional iff $\longrightarrow$ is a function from $S \times A$ to $[S \to D]$. ■

Every transition $(s, a, \mathcal{D})$ is written $s \xrightarrow{a} \mathcal{D}$, with $\mathcal{D}(s')$ being a D-value quantifying the reachability of s' from s via the execution of a. In order to avoid ambiguity, when considering functional ULTraS we will often write $\mathcal{D}_{s,a}(s')$ to denote the same value.

2.2 Behavioral Equivalences on the Uniform Model

We now show how two behavioral equivalences lying at the opposite end points of the linear-time/branching-time spectrum [16] like trace equivalence and bisimilarity can be defined on ULTRAS. Later on, we will see that they coincide with their classical definitions in the case of fully nondeterministic, fully probabilistic, and fully stochastic processes when mapping these processes into ULTRAS.

In order to define the two equivalences on ULTRAS, first of all we have to introduce traces and measure functions. The former are sequences of actions and identify possible observable computations in ULTRAS. The latter give measures of reachability of elements in a set of states $S' \subseteq S$ starting from a state $s \in S$ via a fixed trace $\alpha \in A^*$.

Definition 2. *Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS. A trace α for $\mathcal{U}$ is a finite sequence of transition labels in A^*, where $\alpha = \varepsilon$ denotes the empty sequence while operation "$\_ \circ \_$" denotes sequence concatenation.* ■

Definition 3. *Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS and M be a lattice. A measure function for $\mathcal{U}$ is a function $\mathcal{M}_M : S \times A^* \times 2^S \to M$.* ■

In the setting of ULTRAS, both trace equivalence and bisimilarity are parameterized with respect to a measure function. Indeed, different measure functions can induce different equivalences on the same D-ULTRAS depending on the support set and the operations of M. Although D and M may share the same support set, this is not necessarily the case as we will see when addressing fully stochastic processes. In fact, while D-values encode one-step reachability, M-values are measures computed (on the basis of D-values) along computations.

Trace equivalence is straightforward: two states are trace equivalent if every trace has the same measure with respect to the entire set of states S when starting from the two considered states.

Definition 4. *Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS and $\mathcal{M}_M$ be a measure function for $\mathcal{U}$. We say that $s_1, s_2 \in S$ are $\mathcal{M}_M$-trace equivalent, written $s_1 \sim_{\mathrm{Tr},\mathcal{M}_M} s_2$, iff for all traces $\alpha \in A^*$:*

$$\mathcal{M}_M(s_1, \alpha, S) \; = \; \mathcal{M}_M(s_2, \alpha, S)$$

■

While trace equivalence simply compares any two states without taking into account the states reached at the end of the trace, a bisimulation relation also poses constraints on the final states.

Definition 5. *Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS and $\mathcal{M}_M$ be a measure function for $\mathcal{U}$. An equivalence relation $\mathcal{B}$ over S is an $\mathcal{M}_M$-bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all traces $\alpha \in A^*$ and equivalence classes $C \in S/\mathcal{B}$:*

$$\mathcal{M}_M(s_1, \alpha, C) \; = \; \mathcal{M}_M(s_2, \alpha, C)$$

We say that $s_1, s_2 \in S$ are $\mathcal{M}_M$-bisimilar, written $s_1 \sim_{\mathrm{B},\mathcal{M}_M} s_2$, iff there exists an $\mathcal{M}_M$-bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. ■

3 Mapping Classical Models into the Uniform Model

In this section, we show how classical models used for describing fully nondeterministic, fully probabilistic, and fully stochastic processes can be defined in terms of ULTRAS. In particular, we consider labeled transition systems in Sect. 3.1, action-labeled discrete-time Markov chains in Sect. 3.2, and action-labeled continuous-time Markov chains in Sect. 3.3.

3.1 A Fully Nondeterministic Specialization: LTS

Fully nondeterministic processes are traditionally represented through state-transition graphs in which every transition is labeled with the action determining the corresponding state change. In these graphs, there is no information about how to choose among the various transitions departing from a state.

Definition 6. *A labeled transition system (LTS for short) is a triple* $(S, A, \longrightarrow)$ *where:*

- S *is a countable set of states.*
- A *is a countable set of transition-labeling actions.*
- $\longrightarrow \subseteq S \times A \times S$ *is a transition relation.* ■

Every transition (s, a, s') is written $s \xrightarrow{a} s'$ and means that it is possible to reach s' from s by executing a.

It is straightforward to see that a LTS is a functional $\mathbb{B}$-ULTRAS – where $\mathbb{B} = \{\bot, \top\}$ is the Boolean algebra – in which, given a transition $s \xrightarrow{a} \mathcal{D}$, $\mathcal{D}(s') = \bot$ means that it is not possible to reach s' from s by executing a, whereas $\mathcal{D}(s') = \top$ means that it is possible.

3.2 A Fully Probabilistic Specialization: ADTMC

Fully probabilistic processes, also called generative probabilistic processes according to the terminology of [17], can be represented through state-transition graphs in which every transition is labeled with both the action and the probability of the corresponding state change. In other words, each such process can be represented as a discrete-time Markov chain [15] whose transitions are additionally labeled with actions. [1]

In the following, we use $\{\!|$ and $|\!\}$ to delimit multisets. We also assume that the summation over an empty multiset of numbers is zero.

Definition 7. *An action-labeled discrete-time Markov chain (ADTMC for short) is a triple* $(S, A, \longrightarrow)$ *where:*

[1] The name discrete-time Markov chain is used here for historical reasons. Since time does not come into play, a name like time-abstract Markov chain would be better. A discrete-time interpretation is appropriate only when all state changes occur at equidistant time points.

– *S is a countable set of states.*
– *A is a countable set of transition-labeling actions.*
– *$\longrightarrow \subseteq S \times A \times \mathbb{R}_{(0,1]} \times S$ is a transition relation.*
– *For all $s, s' \in S$ and $a \in A$, whenever $(s, a, p_1, s'), (s, a, p_2, s') \in \longrightarrow$, then $p_1 = p_2$.*
– *For all $s \in S$, it holds that $\sum\{\!| p \in \mathbb{R}_{(0,1]} \mid \exists a \in A, s' \in S. (s, a, p, s') \in \longrightarrow |\!\} \in \{0, 1\}$.* ■

Every transition (s, a, p, s') is written $s \xrightarrow{a,p} s'$, with p being the probability with which s' is reached from s by executing a.

It is straightforward to see that an ADTMC is a functional $\mathbb{R}_{[0,1]}$-ULTraS in which $\sum_{a \in A} \sum_{s' \in S} \mathcal{D}_{s,a}(s') \in \{0, 1\}$ for all $s \in S$. Given a transition $s \xrightarrow{a} \mathcal{D}$, $\mathcal{D}(s') = 0$ means that it is not possible to reach s' from s by executing a, whereas $\mathcal{D}(s') \in \mathbb{R}_{(0,1]}$ means that it is possible with probability $\mathcal{D}(s')$.

3.3 A Fully Stochastic Specialization: ACTMC

Fully stochastic processes in which the notion of time is formalized by means of exponentially distributed durations, also called Markovian processes, can be represented through state-transition graphs in which every transition is labeled with both the action and the rate of the corresponding state change. In other words, each such process can be represented as a continuous-time Markov chain [15] whose transitions are additionally labeled with actions.

This Markov chain can be viewed as an ADTMC in which every state s has associated with it an exponentially distributed sojourn time, which is uniquely identified by a positive real number $\mathrm{E}(s)$ called rate, whose reciprocal coincides with the average sojourn time in s. Assuming that transition firing is governed by a race policy, this is equivalent to replacing the probability labeling each transition departing from s with a rate given by $\mathrm{E}(s)$ multiplied by the transition probability. Consistent with the fact that the minimum of a set of exponentially distributed random variables is exponentially distributed with rate equal to the sum of the original rates, the sum of the transition rates is equal to $\mathrm{E}(s)$.

Definition 8. *An action-labeled continuous-time Markov chain (ACTMC for short) is a triple $(S, A, \longrightarrow)$ where:*

– *S is a countable set of states.*
– *A is a countable set of transition-labeling actions.*
– *$\longrightarrow \subseteq S \times A \times \mathbb{R}_{>0} \times S$ is a transition relation.*
– *For all $s, s' \in S$ and $a \in A$, whenever $(s, a, \lambda_1, s'), (s, a, \lambda_2, s') \in \longrightarrow$, then $\lambda_1 = \lambda_2$.* ■

Every transition (s, a, λ, s') is written $s \xrightarrow{a,\lambda} s'$, with λ being the rate at which s' is reached from s by executing a.

It is straightforward to see that an ACTMC is a functional $\mathbb{R}_{\geq 0}$-ULTraS. Given a transition $s \xrightarrow{a} \mathcal{D}$, $\mathcal{D}(s') = 0$ means that it is not possible to reach

s' from s by executing a, whereas $\mathcal{D}(s') \in \mathbb{R}_{>0}$ means that it is possible at rate $\mathcal{D}(s')$.

4 Equivalences for Fully Nondeterministic Processes

In this section, we instantiate the two behavioral equivalences of Sect. 2.2 – i.e., bisimilarity and trace equivalence – for fully nondeterministic processes represented as functional $\mathbb{B}$-ULTRAS. This is accomplished by introducing a measure function $\mathcal{M}_{\mathbb{B}}$ that associates a suitable constant $\mathbb{B}$-valued function with every triple composed of a state, a trace, and a state subset.

Definition 9. *Let $\mathcal{U} = (S, A, \longrightarrow)$ be a functional $\mathbb{B}$-ULTRAS. The measure function $\mathcal{M}_{\mathbb{B}} : S \times A^* \times 2^S \to \mathbb{B}$ for $\mathcal{U}$ is inductively defined as follows:*

$$\mathcal{M}_{\mathbb{B}}(s, \alpha, S') = \begin{cases} \bigvee\limits_{s' \in S} \mathcal{D}_{s,a}(s') \wedge \mathcal{M}_{\mathbb{B}}(s', \alpha', S') & \text{if } \alpha = a \circ \alpha' \\ \top & \text{if } \alpha = \varepsilon \text{ and } s \in S' \\ \bot & \text{if } \alpha = \varepsilon \text{ and } s \notin S' \end{cases}$$ ■

The value $\mathcal{M}_{\mathbb{B}}(s, \alpha, S')$ establishes whether there exists a computation labeled with trace α that leads to a state in S' from state s. If such a computation exists, then $\mathcal{M}_{\mathbb{B}}(s, \alpha, S') = \top$, otherwise $\mathcal{M}_{\mathbb{B}}(s, \alpha, S') = \bot$.

We now show that each of the two resulting behavioral equivalences $\sim_{\mathrm{B},\mathcal{M}_{\mathbb{B}}}$ and $\sim_{\mathrm{Tr},\mathcal{M}_{\mathbb{B}}}$ on functional $\mathbb{B}$-ULTRAS coincides with the corresponding behavioral equivalence defined in the literature on LTS.

Given two LTS $(S_i, A_i, \longrightarrow_i)$, $i = 1, 2$, with $S_1 \cap S_2 = \emptyset$, consider the LTS $(S, A, \longrightarrow)$ where $S = S_1 \cup S_2$, $A = A_1 \cup A_2$, and $\longrightarrow \; = \; \longrightarrow_1 \cup \longrightarrow_2$.

Bisimilarity for LTS [13] captures the ability of two states of mimicking each other's behavior step by step.

Definition 10. *A binary relation $\mathcal{B}$ over S is a bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$:*

- *Whenever $s_1 \xrightarrow{a} s'_1$, then $s_2 \xrightarrow{a} s'_2$ with $(s'_1, s'_2) \in \mathcal{B}$.*
- *Whenever $s_2 \xrightarrow{a} s'_2$, then $s_1 \xrightarrow{a} s'_1$ with $(s'_1, s'_2) \in \mathcal{B}$.*

We say that $s_1, s_2 \in S$ are bisimilar, written $s_1 \sim_{\mathrm{B}} s_2$, iff there exists a bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. ■

Theorem 1. *For all $s_1, s_2 \in S$:*

$$s_1 \sim_{\mathrm{B}} s_2 \iff s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathbb{B}}} s_2$$

Proof. The proof is divided into two parts:

- *Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{B}} s_2$. From $s_1 \sim_{\mathrm{B}} s_2$, it follows that there exists a bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. Observing that the reflexive and transitive closure $\mathcal{B}'$ of $\mathcal{B}$ is still a bisimulation over S such that $(s_1, s_2) \in \mathcal{B}'$, it turns out that $\mathcal{B}'$ is an $\mathcal{M}_{\mathbb{B}}$-bisimulation and hence*

$s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathbb{B}}} s_2$. In fact, for all $s'_1, s'_2 \in S$, whenever $(s'_1, s'_2) \in \mathcal{B}'$, then for all $\alpha \in A^*$ and $C \in S/\mathcal{B}'$:

$$\mathcal{M}_{\mathbb{B}}(s'_1, \alpha, C) \;=\; \mathcal{M}_{\mathbb{B}}(s'_2, \alpha, C)$$

as we now prove by proceeding by induction on $|\alpha|$:

- *If $|\alpha| = 0$, then for all $C \in S/\mathcal{B}'$ it holds that:*
  $$\mathcal{M}_{\mathbb{B}}(s'_1, \alpha, C) \;=\; \top \;=\; \mathcal{M}_{\mathbb{B}}(s'_2, \alpha, C)$$
  whenever $s'_1, s'_2 \in C$ and:
  $$\mathcal{M}_{\mathbb{B}}(s'_1, \alpha, C) \;=\; \bot \;=\; \mathcal{M}_{\mathbb{B}}(s'_2, \alpha, C)$$
  whenever $s'_1, s'_2 \notin C$.
- *Let $|\alpha| = n \in \mathbb{N}_{>0}$ and assume that the result holds for all traces of length $n - 1$. Supposing $\alpha = a \circ \alpha'$, we note that for all $s \in S$ and $C \in S/\mathcal{B}'$ it holds that:*
  $$\begin{aligned}\mathcal{M}_{\mathbb{B}}(s, \alpha, C) &= \bigvee_{s' \in S} \mathcal{D}_{s,a}(s') \wedge \mathcal{M}_{\mathbb{B}}(s', \alpha', C)\\ &= \bigvee_{C' \in S/\mathcal{B}} \bigvee_{s' \in C'} \mathcal{D}_{s,a}(s') \wedge \mathcal{M}_{\mathbb{B}}(s', \alpha', C)\\ &= \bigvee_{C' \in S/\mathcal{B}} \mathcal{M}_{\mathbb{B}}(s_{C'}, \alpha', C) \wedge \bigvee_{s' \in C'} \mathcal{D}_{s,a}(s')\\ &= \bigvee_{C' \in S/\mathcal{B}} \mathcal{M}_{\mathbb{B}}(s_{C'}, \alpha', C) \wedge (\exists s' \in C'.\, s \xrightarrow{a} s')\end{aligned}$$
  where $s_{C'} \in C'$ and the factorization of $\mathcal{M}_{\mathbb{B}}(s_{C'}, \alpha', C)$ stems from the application of the induction hypothesis on α' to all states of each equivalence class C'. Since $\exists s' \in C'.\, s'_1 \xrightarrow{a} s'$ iff $\exists s' \in C'.\, s'_2 \xrightarrow{a} s'$ by virtue of $(s'_1, s'_2) \in \mathcal{B}'$, we derive that for all $C \in S/\mathcal{B}'$:
  $$\mathcal{M}_{\mathbb{B}}(s'_1, \alpha, C) \;=\; \mathcal{M}_{\mathbb{B}}(s'_2, \alpha, C)$$

– *Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathbb{B}}} s_2$. From $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathbb{B}}} s_2$, it follows that there exists an $\mathcal{M}_{\mathbb{B}}$-bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. It turns out that $\mathcal{B}$ is also a bisimulation and hence $s_1 \sim_{\mathrm{B}} s_2$. In fact, for all $s'_1, s'_2 \in S$ such that $(s'_1, s'_2) \in \mathcal{B}$, from the definition of $\mathcal{M}_{\mathbb{B}}$-bisimulation it follows in particular that for all $a \in A$ and $C \in S/\mathcal{B}$:*

$$\mathcal{M}_{\mathbb{B}}(s'_1, a, C) \;=\; \mathcal{M}_{\mathbb{B}}(s'_2, a, C)$$

Since for all $s \in S$, $a \in A$, and $C \in S/\mathcal{B}$ it holds that:

$$\mathcal{M}_{\mathbb{B}}(s, a, C) \;=\; \bigvee_{s' \in C} \mathcal{D}_{s,a}(s') \;=\; (\exists s' \in C.\, s \xrightarrow{a} s')$$

we immediately derive that for all $a \in A$:

- *Whenever $s'_1 \xrightarrow{a} s''_1$, then $s'_2 \xrightarrow{a} s''_2$ with $(s''_1, s''_2) \in \mathcal{B}$.*
- *Whenever $s'_2 \xrightarrow{a} s''_2$, then $s'_1 \xrightarrow{a} s''_1$ with $(s''_1, s''_2) \in \mathcal{B}$.* ■

Trace equivalence for LTS [10] compares the traces labeling the computations executable from two states. We lift the transition relation $\longrightarrow$ from actions to action sequences by letting $s \xRightarrow{\varepsilon} s$ and $s \xRightarrow{a_1 \ldots a_n} s' \equiv s \xrightarrow{a_1} s_1 \ldots s_{n-1} \xrightarrow{a_n} s'$ for $n \in \mathbb{N}_{>0}$. Given $s \in S$ and $\alpha \in A^*$, we also write $s \xRightarrow{\alpha}$ to denote the existence of $s' \in S$ such that $s \xRightarrow{\alpha} s'$.

Definition 11. *We say that $s_1, s_2 \in S$ are trace equivalent, written $s_1 \sim_{\mathrm{Tr}} s_2$, iff for all traces $\alpha \in A^*$:*

$$s_1 \xRightarrow{\alpha} \text{ iff } s_2 \xRightarrow{\alpha}$$ ■

Theorem 2. *For all* $s_1, s_2 \in S$*:*

$$s_1 \sim_{\mathrm{Tr}} s_2 \iff s_1 \sim_{\mathrm{Tr},\mathcal{M}_{\mathbb{B}}} s_2$$

Proof. Let $s_1, s_2 \in S$ *be such that* $s_1 \sim_{\mathrm{Tr}} s_2$*. The fact that* $s_1 \sim_{\mathrm{Tr}} s_2$ *is equivalent by definition to the fact that for all* $\alpha \in A^*$*:*

$$s_1 \xrightarrow{\alpha} \text{ iff } s_2 \xrightarrow{\alpha}$$

Since for all $s \in S$ *it holds that:*

$$(s \xrightarrow{\alpha}) \; = \; \begin{cases} \bigvee\limits_{s' \in S} \mathcal{D}_{s,a}(s') \wedge (s' \xrightarrow{\alpha'}) & \text{if } \alpha = a \circ \alpha' \\ \top & \text{if } \alpha = \varepsilon \end{cases}$$

and hence:

$$(s \xrightarrow{\alpha}) \; = \; \mathcal{M}_{\mathbb{B}}(s, \alpha, S)$$

we immediately derive that the fact that for all $\alpha \in A^*$ $s_1 \xrightarrow{\alpha}$ *iff* $s_2 \xrightarrow{\alpha}$ *is equivalent to to the fact that for all* $\alpha \in A^*$*:*

$$\mathcal{M}_{\mathbb{B}}(s_1, \alpha, S) \; = \; \mathcal{M}_{\mathbb{B}}(s_2, \alpha, S)$$

which in turn is equivalent by definition to $s_1 \sim_{\mathrm{Tr},\mathcal{M}_{\mathbb{B}}} s_2$*.* ■

5 Equivalences for Fully Probabilistic Processes

In this section, we extend the work in the previous section by additionally taking into account the execution probability of transitions. More precisely, we instantiate the two behavioral equivalences of Sect. 2.2 for fully probabilistic processes represented as functional $\mathbb{R}_{[0,1]}$-ULTraS. This is accomplished by introducing a measure function that associates a suitable constant $\mathbb{R}_{[0,1]}$-valued function with every triple composed of a state, a trace, and a state subset.

Definition 12. *Let* $\mathcal{U} = (S, A, \longrightarrow)$ *be a functional* $\mathbb{R}_{[0,1]}$-ULTraS*. The measure function* $\mathcal{M}_{\mathbb{R}_{[0,1]}} : S \times A^* \times 2^S \to \mathbb{R}_{[0,1]}$ *for* $\mathcal{U}$ *is inductively defined as follows:*

$$\mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, S') \; = \; \begin{cases} \sum\limits_{s' \in S} \mathcal{D}_{s,a}(s') \cdot \mathcal{M}_{\mathbb{R}_{[0,1]}}(s', \alpha', S') & \text{if } \alpha = a \circ \alpha' \\ 1 & \text{if } \alpha = \varepsilon \text{ and } s \in S' \\ 0 & \text{if } \alpha = \varepsilon \text{ and } s \notin S' \end{cases}$$ ■

The value $\mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, S')$ is the probability of the set of computations labeled with trace α that lead to a state in S' from state s. If there are no such computations, then $\mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, S') = 0$, otherwise $\mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, S') \in \mathbb{R}_{(0,1]}$.

We now show that each of the two resulting behavioral equivalences $\sim_{\mathrm{B},\mathcal{M}_{\mathbb{R}_{[0,1]}}}$ and $\sim_{\mathrm{Tr},\mathcal{M}_{\mathbb{R}_{[0,1]}}}$ on functional $\mathbb{R}_{[0,1]}$-ULTraS coincides with the corresponding behavioral equivalence defined in the literature on ADTMC.

Given two ADTMC $(S_i, A_i, \longrightarrow_i)$, $i = 1, 2$, with $S_1 \cap S_2 = \emptyset$, consider the ADTMC $(S, A, \longrightarrow)$ where $S = S_1 \cup S_2$, $A = A_1 \cup A_2$, and $\longrightarrow \; = \; \longrightarrow_1 \cup \longrightarrow_2$.

Bisimilarity for ADTMC [12] relies on the comparison of state exit probabilities. [2] The exit probability of a state $s \in S$ is the probability with which s can

[2] To be precise, probabilistic bisimilarity was defined in [12] for reactive probabilistic processes, but the same definition applies to fully probabilistic processes too.

execute transitions labeled with a certain action $a \in A$ that lead to a certain destination $S' \subseteq S$: $prob_{\mathrm{e}}(s, a, S') = \sum \{\!| p \in \mathbb{R}_{(0,1]} \mid \exists s' \in S'. s \xrightarrow{a,p} s' |\!\}$.

Definition 13. *An equivalence relation $\mathcal{B}$ over S is a probabilistic bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and equivalence classes $C \in S/\mathcal{B}$:*

$$prob_{\mathrm{e}}(s_1, a, C) \ = \ prob_{\mathrm{e}}(s_2, a, C)$$

We say that $s_1, s_2 \in S$ are probabilistic bisimilar, written $s_1 \sim_{\mathrm{PB}} s_2$, iff there exists a probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. ■

Theorem 3. *For all $s_1, s_2 \in S$:*

$$s_1 \sim_{\mathrm{PB}} s_2 \iff s_1 \sim_{\mathrm{B}, \mathcal{M}_{\mathbb{R}_{[0,1]}}} s_2$$

Proof. *The proof is divided into two parts:*

- *Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{PB}} s_2$. From $s_1 \sim_{\mathrm{PB}} s_2$, it follows that there exists a probabilistic bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. It turns out that $\mathcal{B}$ is also an $\mathcal{M}_{\mathbb{R}_{[0,1]}}$-bisimulation and hence $s_1 \sim_{\mathrm{B}, \mathcal{M}_{\mathbb{R}_{[0,1]}}} s_2$. In fact, for all $s'_1, s'_2 \in S$, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $\alpha \in A^*$ and $C \in S/\mathcal{B}$:*

$$\mathcal{M}_{\mathbb{R}_{[0,1]}}(s'_1, \alpha, C) \ = \ \mathcal{M}_{\mathbb{R}_{[0,1]}}(s'_2, \alpha, C)$$

  as we now prove by proceeding by induction on $|\alpha|$:
  - *If $|\alpha| = 0$, then for all $C \in S/\mathcal{B}$ it holds that:*

$$\mathcal{M}_{\mathbb{R}_{[0,1]}}(s'_1, \alpha, C) \ = \ 1 \ = \ \mathcal{M}_{\mathbb{R}_{[0,1]}}(s'_2, \alpha, C)$$

    whenever $s'_1, s'_2 \in C$ and:

$$\mathcal{M}_{\mathbb{R}_{[0,1]}}(s'_1, \alpha, C) \ = \ 0 \ = \ \mathcal{M}_{\mathbb{R}_{[0,1]}}(s'_2, \alpha, C)$$

    whenever $s'_1, s'_2 \notin C$.
  - *Let $|\alpha| = n \in \mathbb{N}_{>0}$ and assume that the result holds for all traces of length $n - 1$. Supposing $\alpha = a \circ \alpha'$, we note that for all $s \in S$ and $C \in S/\mathcal{B}$ it holds that:*

$$\begin{aligned} \mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, C) &= \sum_{s' \in S} \mathcal{D}_{s,a}(s') \cdot \mathcal{M}_{\mathbb{R}_{[0,1]}}(s', \alpha', C) \\ &= \sum_{C' \in S/\mathcal{B}} \sum_{s' \in C'} \mathcal{D}_{s,a}(s') \cdot \mathcal{M}_{\mathbb{R}_{[0,1]}}(s', \alpha', C) \\ &= \sum_{C' \in S/\mathcal{B}} \mathcal{M}_{\mathbb{R}_{[0,1]}}(s_{C'}, \alpha', C) \cdot \sum_{s' \in C'} \mathcal{D}_{s,a}(s') \\ &= \sum_{C' \in S/\mathcal{B}} \mathcal{M}_{\mathbb{R}_{[0,1]}}(s_{C'}, \alpha', C) \cdot prob_{\mathrm{e}}(s, a, C') \end{aligned}$$

    where $s_{C'} \in C'$ and the factorization of $\mathcal{M}_{\mathbb{R}_{[0,1]}}(s_{C'}, \alpha', C)$ stems from the application of the induction hypothesis on α' to all states of each equivalence class C'. Since $prob_{\mathrm{e}}(s'_1, a, C') = prob_{\mathrm{e}}(s'_2, a, C')$ by virtue of $(s'_1, s'_2) \in \mathcal{B}$, we derive that for all $C \in S/\mathcal{B}$:

$$\mathcal{M}_{\mathbb{R}_{[0,1]}}(s'_1, \alpha, C) \ = \ \mathcal{M}_{\mathbb{R}_{[0,1]}}(s'_2, \alpha, C)$$

- *Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{B}, \mathcal{M}_{\mathbb{R}_{[0,1]}}} s_2$. From $s_1 \sim_{\mathrm{B}, \mathcal{M}_{\mathbb{R}_{[0,1]}}} s_2$, it follows that there exists an $\mathcal{M}_{\mathbb{R}_{[0,1]}}$-bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. It turns out that $\mathcal{B}$ is also a probabilistic bisimulation and hence $s_1 \sim_{\mathrm{PB}} s_2$. In fact, for all $s'_1, s'_2 \in S$ such that $(s'_1, s'_2) \in \mathcal{B}$, from the definition of $\mathcal{M}_{\mathbb{R}_{[0,1]}}$-bisimulation it follows in particular that for all $a \in A$ and $C \in S/\mathcal{B}$:*

$$\mathcal{M}_{\mathbb{R}_{[0,1]}}(s_1, a, C) \;=\; \mathcal{M}_{\mathbb{R}_{[0,1]}}(s_2, a, C)$$

Since for all $s \in S$, $a \in A$, and $C \in S/\mathcal{B}$ it holds that:

$$\mathcal{M}_{\mathbb{R}_{[0,1]}}(s, a, C) \;=\; \sum_{s' \in C} \mathcal{D}_{s,a}(s') \;=\; \mathit{prob}_{\mathrm{e}}(s, a, C)$$

we immediately derive that for all $a \in A$ and $C \in S/\mathcal{B}$:

$$\mathit{prob}_{\mathrm{e}}(s'_1, a, C) \;=\; \mathit{prob}_{\mathrm{e}}(s'_2, a, C)$$

■

Trace equivalence for ADTMC [11] is based on the comparison of the execution probabilities of analogous computations. Given $s \in S$, we denote by $\mathcal{C}_{\mathrm{f}}(s)$ the set of finite-length computations of s and by $|c|$ the length of any $c \in \mathcal{C}_{\mathrm{f}}(s)$. We say that two distinct computations in $\mathcal{C}_{\mathrm{f}}(s)$ are independent of each other iff neither is a proper prefix of the other one. The probability of executing $c \in \mathcal{C}_{\mathrm{f}}(s)$ is the product of the execution probabilities of the transitions of c:

$$\mathit{prob}(c) \;=\; \begin{cases} 1 & \text{if } |c| = 0 \\ p \cdot \mathit{prob}(c') & \text{if } c \equiv s \xrightarrow{a,p} c' \end{cases}$$

which is lifted to $C \subseteq \mathcal{C}_{\mathrm{f}}(s)$ as follows:

$$\mathit{prob}(C) \;=\; \sum_{c \in C} \mathit{prob}(c)$$

whenever C is finite and all of its computations are independent of each other. Indicating with $\mathit{trace}(c)$ the sequence of actions labeling the transitions of $c \in \mathcal{C}_{\mathrm{f}}(s)$, we say that c is compatible with $\alpha \in A^*$ iff $\mathit{trace}(c) = \alpha$ and we denote by $\mathcal{CC}(s, \alpha)$ the set of computations in $\mathcal{C}_{\mathrm{f}}(s)$ that are compatible with α.

Definition 14. *We say that $s_1, s_2 \in S$ are probabilistic trace equivalent, written $s_1 \sim_{\mathrm{PTr}} s_2$, iff for all traces $\alpha \in A^*$:*

$$\mathit{prob}(\mathcal{CC}(s_1, \alpha)) \;=\; \mathit{prob}(\mathcal{CC}(s_2, \alpha))$$

■

Theorem 4. *For all $s_1, s_2 \in S$:*

$$s_1 \sim_{\mathrm{PTr}} s_2 \iff s_1 \sim_{\mathrm{Tr}, \mathcal{M}_{\mathbb{R}_{[0,1]}}} s_2$$

Proof. *Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{PTr}} s_2$. The fact that $s_1 \sim_{\mathrm{PTr}} s_2$ is equivalent by definition to the fact that for all $\alpha \in A^*$:*

$$\mathit{prob}(\mathcal{CC}(s_1, \alpha)) \;=\; \mathit{prob}(\mathcal{CC}(s_2, \alpha))$$

Since for all $s \in S$ it holds that:

$$\mathit{prob}(\mathcal{CC}(s, \alpha)) \;=\; \begin{cases} \sum_{s' \in S} \mathcal{D}_{s,a}(s') \cdot \mathit{prob}(\mathcal{CC}(s', \alpha')) & \text{if } \alpha = a \circ \alpha' \\ 1 & \text{if } \alpha = \varepsilon \end{cases}$$

and hence:

$$\mathit{prob}(\mathcal{CC}(s, \alpha)) \;=\; \mathcal{M}_{\mathbb{R}_{[0,1]}}(s, \alpha, S)$$

we immediately derive that the fact that for all $\alpha \in A^*$ $\mathit{prob}(\mathcal{CC}(s_1, \alpha)) = \mathit{prob}(\mathcal{CC}(s_2, \alpha))$ is equivalent to the fact that for all $\alpha \in A^*$:

$$\mathcal{M}_{\mathbb{R}_{[0,1]}}(s_1, \alpha, S) \;=\; \mathcal{M}_{\mathbb{R}_{[0,1]}}(s_2, \alpha, S)$$

which in turn is equivalent by definition to $s_1 \sim_{\mathrm{Tr}, \mathcal{M}_{\mathbb{R}_{[0,1]}}} s_2$. ■

6 Equivalences for Fully Stochastic Processes

In this section, we further extend the work in the previous two sections by additionally taking into account a notion of time formalized by means of the exponentially distributed durations of transitions. More precisely, we instantiate the two behavioral equivalences of Sect. 2.2 for fully stochastic processes involving only exponential distributions – i.e., fully Markovian processes – represented as functional $\mathbb{R}_{\geq 0}$-ULTRAS. Unlike the previous two sections, when defining the measure function we distinguish between two cases. In Sect. 6.1 we take into account the end-to-end delay of traces, whereas in Sect. 6.2 we consider the step-by-step delay of traces.

6.1 The End-To-End Case

The measure function for the end-to-end case associates a suitable $\mathbb{R}_{[0,1]}$-valued function with every triple composed of a state, a trace, and a state subset, which is parameterized with respect to the end-to-end delay $t \in \mathbb{R}_{\geq 0}$ of the trace.

Definition 15. *Let $\mathcal{U} = (S, A, \longrightarrow)$ be a functional $\mathbb{R}_{\geq 0}$-ULTRAS. The end-to-end measure function $\mathcal{M}_{\mathrm{ete}} : S \times A^* \times 2^S \rightarrow [\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{[0,1]}]$ for $\mathcal{U}$ is inductively defined as follows:*

$$\mathcal{M}_{\mathrm{ete}}(s, \alpha, S')(t) = \begin{cases} \int_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot \sum\limits_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \cdot \mathcal{M}_{\mathrm{ete}}(s', \alpha', S')(t - x)\, dx & \text{if } \alpha = a \circ \alpha' \text{ and } \mathrm{E}(s) > 0 \\ 1 & \text{if } \alpha = \varepsilon \text{ and } s \in S' \\ 0 & \text{if } \alpha = \varepsilon \text{ and } s \notin S' \text{ or } \alpha \neq \varepsilon \text{ and } \mathrm{E}(s) = 0 \end{cases}$$

■

Note that subscript "ete" is a symbolic shorthand for $[\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{[0,1]}]$. The value $\mathcal{M}_{\mathrm{ete}}(s, \alpha, S')(t)$ is the probability of the set of computations labeled with trace α that lead to a state in S' from state s within t time units. If there are no such computations, then $\mathcal{M}_{\mathrm{ete}}(s, \alpha, S')(t) = 0$, otherwise $\mathcal{M}_{\mathrm{ete}}(s, \alpha, S')(t) \in \mathbb{R}_{(0,1]}$. In the case $\alpha = a \circ \alpha'$ and $\mathrm{E}(s) > 0$, this value is computed as the convolution of probability distributions. Assuming to spend $x \in \mathbb{R}_{[0,t]}$ time units in state s, the first operand of the convolution is the exponentially distributed density function quantifying the sojourn time in s, i.e., the derivative with respect to t of $1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}$ evaluated in x. For each state s' reachable from s by executing a, the first operand is multiplied by the probability of the set of computations labeled with the remaining trace α' that lead to a state in S' from state s' within the remaining $t - x$ time units.

We now show that each of the two resulting behavioral equivalences $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}}$ and $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete}}}$ on functional $\mathbb{R}_{\geq 0}$-ULTRAS coincides with the corresponding behavioral equivalence defined in the literature on ACTMC.

Given two ACTMC $(S_i, A_i, \longrightarrow_i)$, $i = 1, 2$, with $S_1 \cap S_2 = \emptyset$, consider the ACTMC $(S, A, \longrightarrow)$ where $S = S_1 \cup S_2$, $A = A_1 \cup A_2$, and $\longrightarrow \; = \; \longrightarrow_1 \cup \longrightarrow_2$.

Bisimilarity for ACTMC [9] relies on the comparison of state exit rates. The exit rate of a state $s \in S$ is the rate at which s can execute transitions labeled with a certain action $a \in A$ that lead to a certain destination $S' \subseteq S$, which is the sum of the rates of those transitions due to the fact that transition firing is governed by the race policy: $\mathit{rate}_{\mathrm{e}}(s, a, S') = \sum\{\!|\, \lambda \in \mathbb{R}_{>0} \mid \exists s' \in S'.\, s \xrightarrow{a,\lambda} s' \,|\!\}$.

Definition 16. *An equivalence relation $\mathcal{B}$ over S is a Markovian bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and equivalence classes $C \in S/\mathcal{B}$:*

$$\mathit{rate}_{\mathrm{e}}(s_1, a, C) \;=\; \mathit{rate}_{\mathrm{e}}(s_2, a, C)$$

We say that $s_1, s_2 \in S$ are Markovian bisimilar, written $s_1 \sim_{\mathrm{MB}} s_2$, iff there exists a Markovian bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. ■

Lemma 1. *For all $s_1, s_2 \in S$:*

$$s_1 \sim_{\mathrm{MB}} s_2 \implies \mathrm{E}(s_1) = \mathrm{E}(s_2)$$

Proof. It stems from the fact that for all $s \in S$:

$$\mathrm{E}(s) \;=\; \sum_{a \in A} \mathit{rate}_{\mathrm{e}}(s, a, S) \;=\; \sum_{a \in A} \sum_{C \in S/\sim_{\mathrm{MB}}} \mathit{rate}_{\mathrm{e}}(s, a, C)$$

■

Lemma 2. *For all $s_1, s_2 \in S$:*

$$s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2 \implies \mathrm{E}(s_1) = \mathrm{E}(s_2)$$

Proof. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2$ and assume $\mathrm{E}(s_1) > 0$ and $\mathrm{E}(s_2) > 0$ in order to avoid trivial cases. Since for all $s \in S$ and $a \in A$ it holds that:

$$\mathcal{M}_{\mathrm{ete}}(s, a, S) \;=\; \sum_{C \in S/\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}}} \mathcal{M}_{\mathrm{ete}}(s, a, C)$$

from $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2$ it follows that:

$$\sum_{a \in A} \mathcal{M}_{\mathrm{ete}}(s_1, a, S) \;=\; \sum_{a \in A} \mathcal{M}_{\mathrm{ete}}(s_2, a, S)$$

Since for all $s \in S$ such that $\mathrm{E}(s) > 0$ and $t \in \mathbb{R}_{\geq 0}$ it holds that:

$$\begin{aligned}
\sum_{a \in A} \mathcal{M}_{\mathrm{ete}}(s, a, S)(t) \;&=\; \sum_{a \in A} \int_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot \sum_{s' \in S} \tfrac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \,\mathrm{d}x \\
&=\; \sum_{a \in A} \sum_{s' \in S} \tfrac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \cdot \int_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \,\mathrm{d}x \\
&=\; \tfrac{1}{\mathrm{E}(s)} \cdot \sum_{a \in A} \sum_{s' \in S} \mathcal{D}_{s,a}(s') \cdot (1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}) \\
&=\; 1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}
\end{aligned}$$

we derive:

$$1 - \mathrm{e}^{-\mathrm{E}(s_1) \cdot t} \;=\; 1 - \mathrm{e}^{-\mathrm{E}(s_2) \cdot t}$$

and hence:

$$\mathrm{E}(s_1) \;=\; \mathrm{E}(s_2)$$

■

Lemma 3. *Let $s_1, s_2 \in S$. Whenever $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2$, then for all $a \in A$ and $C \in S/\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}}$:*

$$\sum_{s' \in C} \mathcal{D}_{s_1,a}(s') \;=\; \sum_{s' \in C} \mathcal{D}_{s_2,a}(s')$$

Proof. Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2$ and assume $\mathrm{E}(s_1) > 0$ and $\mathrm{E}(s_2) > 0$ in order to avoid trivial cases. From $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2$, it follows in *particular that for all $a \in A$ and $C \in S/\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}}$:*

$$\mathcal{M}_{\mathrm{ete}}(s_1, a, C) \; = \; \mathcal{M}_{\mathrm{ete}}(s_2, a, C)$$

Since for all $s \in S$ such that $\mathrm{E}(s) > 0$ and $t \in \mathbb{R}_{\geq 0}$ it holds that:

$$\begin{aligned}\mathcal{M}_{\mathrm{ete}}(s, a, C)(t) \; &= \; \int_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot \sum_{s' \in C} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \, \mathrm{d}x \\ &= \; (\int_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \, \mathrm{d}x) \cdot \frac{1}{\mathrm{E}(s)} \cdot \sum_{s' \in C} \mathcal{D}_{s,a}(s') \\ &= \; \frac{1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}}{\mathrm{E}(s)} \cdot \sum_{s' \in C} \mathcal{D}_{s,a}(s')\end{aligned}$$

we derive:

$$\frac{1 - \mathrm{e}^{-\mathrm{E}(s_1) \cdot t}}{\mathrm{E}(s_1)} \cdot \sum_{s' \in C} \mathcal{D}_{s_1,a}(s') \; = \; \frac{1 - \mathrm{e}^{-\mathrm{E}(s_2) \cdot t}}{\mathrm{E}(s_2)} \cdot \sum_{s' \in C} \mathcal{D}_{s_2,a}(s')$$

and hence:

$$\sum_{s' \in C} \mathcal{D}_{s_1,a}(s') \; = \; \sum_{s' \in C} \mathcal{D}_{s_2,a}(s')$$

by virtue of Lemma 2. ■

Theorem 5. *For all $s_1, s_2 \in S$:*

$$s_1 \sim_{\mathrm{MB}} s_2 \iff s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2$$

Proof. The proof is divided into two parts:

- *Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{MB}} s_2$ and assume $\mathrm{E}(s_1) > 0$ and $\mathrm{E}(s_2) > 0$ in order to avoid trivial cases. From $s_1 \sim_{\mathrm{MB}} s_2$, it follows that there exists a Markovian bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. It turns out that $\mathcal{B}$ is also an $\mathcal{M}_{\mathrm{ete}}$-bisimulation and hence $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2$. In fact, for all $s'_1, s'_2 \in S$, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $\alpha \in A^*$ and $C \in S/\mathcal{B}$:*

  $$\mathcal{M}_{\mathrm{ete}}(s'_1, \alpha, C) \; = \; \mathcal{M}_{\mathrm{ete}}(s'_2, \alpha, C)$$

  as we now prove by proceeding by induction on $|\alpha|$:

  - *If $|\alpha| = 0$, then for all $C \in S/\mathcal{B}$ and $t \in \mathbb{R}_{\geq 0}$ it holds that:*

    $$\mathcal{M}_{\mathrm{ete}}(s'_1, \alpha, C)(t) \; = \; 1 \; = \; \mathcal{M}_{\mathrm{ete}}(s'_2, \alpha, C)(t)$$

    whenever $s'_1, s'_2 \in C$ and:

    $$\mathcal{M}_{\mathrm{ete}}(s'_1, \alpha, C)(t) \; = \; 0 \; = \; \mathcal{M}_{\mathrm{ete}}(s'_2, \alpha, C)(t)$$

    whenever $s'_1, s'_2 \notin C$.
  - *Let $|\alpha| = n \in \mathbb{N}_{>0}$ and assume that the result holds for all traces of length $n - 1$. Supposing $\alpha = a \circ \alpha'$, we note that for all $s \in S$ such that $\mathrm{E}(s) > 0$, $C \in S/\mathcal{B}$, and $t \in \mathbb{R}_{\geq 0}$ it holds that $\mathcal{M}_{\mathrm{ete}}(s, \alpha, C)(t)$ is equal to:*

    $$\begin{aligned}&\int_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot \sum_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \cdot \mathcal{M}_{\mathrm{ete}}(s', \alpha', C)(t - x) \, dx \\ &= \int_0^t \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot \sum_{C' \in S/\mathcal{B}} \sum_{s' \in C'} \mathcal{D}_{s,a}(s') \cdot \mathcal{M}_{\mathrm{ete}}(s', \alpha', C)(t - x) \, dx \\ &= \int_0^t \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot \sum_{C' \in S/\mathcal{B}} \mathcal{M}_{\mathrm{ete}}(s_{C'}, \alpha', C)(t - x) \cdot \sum_{s' \in C'} \mathcal{D}_{s,a}(s') \, dx \\ &= \int_0^t \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot \sum_{C' \in S/\mathcal{B}} \mathcal{M}_{\mathrm{ete}}(s_{C'}, \alpha', C)(t - x) \cdot rate_{\mathrm{e}}(s, a, C') \, dx\end{aligned}$$

    where $s_{C'} \in C'$ and the factorization of $\mathcal{M}_{\mathrm{ete}}(s_{C'}, \alpha', C)(t - x)$ stems *from the application of the induction hypothesis on α' to all states of each equivalence class C'. Since $\mathrm{E}(s'_1) = \mathrm{E}(s'_2)$ by virtue of $(s'_1, s'_2) \in \mathcal{B}$ and Lemma 1 and $rate_{\mathrm{e}}(s'_1, a, C') = rate_{\mathrm{e}}(s'_2, a, C')$ by virtue of $(s'_1, s'_2) \in \mathcal{B}$, we derive that for all $C \in S/\mathcal{B}$ and $t \in \mathbb{R}_{\geq 0}$:*

$$\mathcal{M}_{\mathrm{sbs}}(s'_1, \alpha, C)(t) \; = \; \mathcal{M}_{\mathrm{sbs}}(s'_2, \alpha, C)(t)$$

– *Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2$. From $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}} s_2$, it follows that there exists an $\mathcal{M}_{\mathrm{ete}}$-bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. It turns out that $\mathcal{B}$ is also a Markovian bisimulation and hence $s_1 \sim_{\mathrm{MB}} s_2$. In fact, for all $s'_1, s'_2 \in S$ such that $(s'_1, s'_2) \in \mathcal{B}$, from Lemma 3 it follows that for all $a \in A$ and $C \in S/\mathcal{B}$:*

$$\sum_{s' \in C} \mathcal{D}_{s'_1,a}(s') \; = \; \sum_{s' \in C} \mathcal{D}_{s'_2,a}(s')$$

Since for all $s \in S$, $a \in A$, and $C \in S/\mathcal{B}$ it holds that:

$$\sum_{s' \in C} \mathcal{D}_{s,a}(s') \; = \; rate_{\mathrm{e}}(s, a, C)$$

we immediately derive that for all $a \in A$ and $C \in S/\mathcal{B}$:

$$rate_{\mathrm{e}}(s'_1, a, C) \; = \; rate_{\mathrm{e}}(s'_2, a, C)$$ ■

Trace equivalence for ACTMC is based on the comparison of the execution probabilities and the average durations of analogous computations. Here, by average duration of a computation we intend its end-to-end average duration [2]. Given $s \in S$, the probability of executing $c \in \mathcal{C}_{\mathrm{f}}(s)$ is the product of the rate-based execution probabilities of the transitions of c: [3]

$$prob(c) \; = \; \begin{cases} 1 & \text{if } |c| = 0 \\ \frac{\lambda}{\mathrm{E}(s)} \cdot prob(c') & \text{if } c \equiv s \xrightarrow{a,\lambda} c' \end{cases}$$

The end-to-end average duration of c is the sum of the average sojourn times in the states traversed by c:

$$time_{\mathrm{a,ete}}(c) \; = \; \begin{cases} 0 & \text{if } |c| = 0 \\ \frac{1}{\mathrm{E}(s)} + time_{\mathrm{a,ete}}(c') & \text{if } c \equiv s \xrightarrow{a,\lambda} c' \end{cases}$$

and we denote by $C_{\leq t}$ the set of computations in $C \subseteq \mathcal{C}_{\mathrm{f}}(s)$ whose end-to-end average duration is not greater than $t \in \mathbb{R}_{\geq 0}$.

Definition 17. *We say that $s_1, s_2 \in S$ are end-to-end Markovian trace equivalent, written $s_1 \sim_{\mathrm{MTr,ete}} s_2$, iff for all traces $\alpha \in A^*$ and amounts of time $t \in \mathbb{R}_{\geq 0}$:*

$$prob(\mathcal{CC}_{\leq t}(s_1, \alpha)) \; = \; prob(\mathcal{CC}_{\leq t}(s_2, \alpha))$$ ■

Theorem 6. *For all $s_1, s_2 \in S$:*

$$s_1 \sim_{\mathrm{MTr,ete}} s_2 \iff s_1 \sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete}}} s_2$$

Proof. *Given $s \in S$, we define the end-to-end duration of $c \in \mathcal{C}_{\mathrm{f}}(s)$ as the sum of the random variables quantifying the average sojourn times in the states traversed by c:*

[3] With abuse of notation, we use the same name *prob* employed in the ADTMC case.

$$time_{\mathrm{d,ete}}(c) \; = \; \begin{cases} Det_0 & \text{if } |c| = 0 \\ Exp_{\mathrm{E}(s)} + time_{\mathrm{d,ete}}(c') & \text{if } c \equiv s \xrightarrow{a,\lambda} c' \end{cases}$$

where Det_0 is the random variable equal to 0 with probability 1, while $Exp_{\mathrm{E}(s)}$ is the exponentially distributed random variable with rate $\mathrm{E}(s)$. Moreover, we define the probability distribution of executing a computation in $C \subseteq \mathcal{C}_{\mathrm{f}}(s)$ within $t \in \mathbb{R}_{\geq 0}$ time units by letting:

$$prob_{\mathrm{d,ete}}(C, t) \; = \; \sum_{c \in C} prob(c) \cdot \Pr\{time_{\mathrm{d,ete}}(c) \leq t\}$$

whenever C is finite and all of its computations are independent of each other.

Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{MTr,ete}} s_2$ and assume $\mathrm{E}(s_1) > 0$ and $\mathrm{E}(s_2) > 0$ in order to avoid trivial cases. The fact that $s_1 \sim_{\mathrm{MTr,ete}} s_2$ is equivalent by definition to the fact that for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

$$prob(\mathcal{CC}_{\leq t}(s_1, \alpha)) \; = \; prob(\mathcal{CC}_{\leq t}(s_2, \alpha))$$

which in turn is equivalent by virtue of [2] to the fact that for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$:

$$prob_{\mathrm{d,ete}}(\mathcal{CC}(s_1, \alpha), t) \; = \; prob_{\mathrm{d,ete}}(\mathcal{CC}(s_2, \alpha), t)$$

Since for all $s \in S$ such that $\mathrm{E}(s) > 0$, $\alpha \in A^*$, and $t \in \mathbb{R}_{\geq 0}$ it holds that:

$$prob_{\mathrm{d,ete}}(\mathcal{CC}(s, \alpha), t) \; = \; \begin{cases} \sum\limits_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \cdot \int\limits_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot & \\ \quad \cdot \, prob_{\mathrm{d,ete}}(\mathcal{CC}(s', \alpha'), t - x) \, \mathrm{d}x & \text{if } \alpha = a \circ \alpha' \\ 1 & \text{if } \alpha = \varepsilon \end{cases}$$

and hence:

$$prob_{\mathrm{d,ete}}(\mathcal{CC}(s, \alpha), t) \; = \; \mathcal{M}_{\mathrm{ete}}(s, \alpha, S)(t)$$

we immediately derive that the fact that for all $\alpha \in A^*$ and $t \in \mathbb{R}_{\geq 0}$ $prob_{\mathrm{d,ete}}(\mathcal{CC}(s_1, \alpha), t) = prob_{\mathrm{d,ete}}(\mathcal{CC}(s_2, \alpha), t)$ is equivalent to the fact that for all $\alpha \in A^*$:

$$\mathcal{M}_{\mathrm{ete}}(s_1, \alpha, S) \; = \; \mathcal{M}_{\mathrm{ete}}(s_2, \alpha, S)$$

which in turn is equivalent by definition to $s_1 \sim_{\mathrm{Tr}, \mathcal{M}_{\mathrm{ete}}} s_2$. ■

6.2 The Step-By-Step Case

The measure function for the step-by-step case associates a suitable $\mathbb{R}_{[0,1]}$-valued function with every triple composed of a state, a trace, and a state subset, which is parameterized with respect to the step-by-step delay $\theta \in (\mathbb{R}_{\geq 0})^*$ of the trace.

Definition 18. *Let $\mathcal{U} = (S, A, \longrightarrow)$ be a functional $\mathbb{R}_{\geq 0}$-ULTRAS. The step-by-step measure function $\mathcal{M}_{\mathrm{sbs}} : S \times A^* \times 2^S \rightarrow [(\mathbb{R}_{\geq 0})^* \rightarrow \mathbb{R}_{[0,1]}]$ for $\mathcal{U}$ is inductively defined as follows:*

$$\mathcal{M}_{\mathrm{sbs}}(s, \alpha, S')(\theta) \; = \; \begin{cases} (1 - e^{-\mathrm{E}(s) \cdot t}) \cdot \sum\limits_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \cdot \mathcal{M}_{\mathrm{sbs}}(s', \alpha', S')(\theta') & \\ \qquad\qquad \text{if } \alpha = a \circ \alpha' \text{ and } \theta = t \circ \theta' \text{ and } \mathrm{E}(s) > 0 & \\ 1 \qquad\qquad \text{if } \alpha = \varepsilon \text{ and } s \in S' & \\ 0 \qquad\qquad \text{if } \alpha = \varepsilon \text{ and } s \notin S' \text{ or} & \\ \qquad\qquad\quad \alpha \neq \varepsilon \text{ and } \theta = \varepsilon \text{ or} & \\ \qquad\qquad\quad \alpha \neq \varepsilon \text{ and } \theta \neq \varepsilon \text{ and } \mathrm{E}(s) = 0 & \end{cases}$$

■

Note that subscript "sbs" is a symbolic shorthand for $[(\mathbb{R}_{\geq 0})^* \rightarrow \mathbb{R}_{[0,1]}]$. The value $\mathcal{M}_{\rm sbs}(s, \alpha, S')(\theta)$ is the probability of the set of computations labeled with trace α that lead to a state in S' from state s, such that the delay of the i-th transition of any computation is not greater than $\theta[i]$ for each i ranging from 1 to the length of the computation. If there are no such computations, then $\mathcal{M}_{\rm sbs}(s, \alpha, S')(\theta) = 0$, otherwise $\mathcal{M}_{\rm sbs}(s, \alpha, S')(\theta) \in \mathbb{R}_{(0,1]}$. In the case $\alpha = a \circ \alpha'$ and $\theta = t \circ \theta'$ and $\mathrm{E}(s) > 0$, this value is computed on the basis of the probability of leaving state s within t time units, i.e., $1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}$. For each state s' reachable from s by executing a, this probability is multiplied by the probability of the set of computations labeled with the remaining trace α' that lead to a state in S' from state s' within the remaining time steps θ'.

We now show that each of the two resulting behavioral equivalences $\sim_{\rm B, \mathcal{M}_{sbs}}$ and $\sim_{\rm Tr, \mathcal{M}_{sbs}}$ on functional $\mathbb{R}_{\geq 0}$-ULTRAS coincides with the corresponding behavioral equivalence defined in the literature on ACTMC. In the case of bisimilarity, we consider the same equivalence $\sim_{\rm MB}$ as Sect. 6.1.

Lemma 4. *For all $s_1, s_2 \in S$:*

$$s_1 \sim_{\rm B, \mathcal{M}_{sbs}} s_2 \implies \mathrm{E}(s_1) = \mathrm{E}(s_2)$$

Proof. Similar to the proof of Lemma 2, with the following different calculation for all $s \in S$ such that $\mathrm{E}(s) > 0$ and $\theta = t \circ \theta' \in (\mathbb{R}_{\geq 0})^*$:

$$\begin{aligned}\sum_{a \in A} \mathcal{M}_{\rm sbs}(s, a, S)(\theta) &= \sum_{a \in A} (1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}) \cdot \sum_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \\ &= (1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}) \cdot \frac{1}{\mathrm{E}(s)} \cdot \sum_{a \in A} \sum_{s' \in S} \mathcal{D}_{s,a}(s') \\ &= 1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}\end{aligned}$$

■

Lemma 5. *Let $s_1, s_2 \in S$. Whenever $s_1 \sim_{\rm B, \mathcal{M}_{sbs}} s_2$, then for all $a \in A$ and $C \in S/\sim_{\rm B, \mathcal{M}_{sbs}}$:*

$$\sum_{s' \in C} \mathcal{D}_{s_1,a}(s') = \sum_{s' \in C} \mathcal{D}_{s_2,a}(s')$$

Proof. Similar to the proof of Lemma 3, with the following different calculation for all $s \in S$ such that $\mathrm{E}(s) > 0$ and $\theta = t \circ \theta' \in (\mathbb{R}_{\geq 0})^*$:

$$\begin{aligned}\mathcal{M}_{\rm sbs}(s, a, C)(\theta) &= (1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}) \cdot \sum_{s' \in C} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \\ &= \frac{1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}}{\mathrm{E}(s)} \cdot \sum_{s' \in C} \mathcal{D}_{s,a}(s')\end{aligned}$$

followed by the exploitation of Lemma 4. ■

Theorem 7. *For all $s_1, s_2 \in S$:*

$$s_1 \sim_{\rm B, \mathcal{M}_{ete}} s_2 \iff s_1 \sim_{\rm B, \mathcal{M}_{sbs}} s_2$$

Proof. The proof is divided into two parts:

- *Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\rm B, \mathcal{M}_{ete}} s_2$ and assume $\mathrm{E}(s_1) > 0$ and $\mathrm{E}(s_2) > 0$ in order to avoid trivial cases. From $s_1 \sim_{\rm B, \mathcal{M}_{ete}} s_2$, it follows that there exists an $\mathcal{M}_{\rm ete}$-bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. It turns out that $\mathcal{B}$ is also an $\mathcal{M}_{\rm sbs}$-bisimulation and hence $s_1 \sim_{\rm B, \mathcal{M}_{sbs}} s_2$. In* *fact, for all* $s'_1, s'_2 \in S$*, whenever* $(s'_1, s'_2) \in \mathcal{B}$*, then for all* $\alpha \in A^*$ *and* $C \in S/\mathcal{B}$*:*

$$\mathcal{M}_{\text{sbs}}(s'_1, \alpha, C) \; = \; \mathcal{M}_{\text{sbs}}(s'_2, \alpha, C)$$

as we now prove by proceeding by induction on $|\alpha|$*:*

- *If* $|\alpha| = 0$*, then for all* $C \in S/\mathcal{B}$ *and* $\theta \in (\mathbb{R}_{\geq 0})^*$ *it holds that:*
  $$\mathcal{M}_{\text{sbs}}(s'_1, \alpha, C)(\theta) \; = \; 1 \; = \; \mathcal{M}_{\text{sbs}}(s'_2, \alpha, C)(\theta)$$
  whenever $s'_1, s'_2 \in C$ *and:*
  $$\mathcal{M}_{\text{sbs}}(s'_1, \alpha, C)(\theta) \; = \; 0 \; = \; \mathcal{M}_{\text{sbs}}(s'_2, \alpha, C)(\theta)$$
  whenever $s'_1, s'_2 \notin C$*.*
- *Let* $|\alpha| = n \in \mathbb{N}_{>0}$ *and assume that the result holds for all traces of length* $n - 1$*. Supposing* $\alpha = a \circ \alpha'$*, there are two cases for* $\theta \in (\mathbb{R}_{\geq 0})^*$*:*
  * *If* $\theta = \varepsilon$*, then for all* $C \in S/\mathcal{B}$ *it holds that:*
    $$\mathcal{M}_{\text{sbs}}(s'_1, \alpha, C)(\theta) \; = \; 0 \; = \; \mathcal{M}_{\text{sbs}}(s'_2, \alpha, C)(\theta)$$
  * *Let* $\theta = t \circ \theta'$*. For all* $s \in S$ *such that* $\mathrm{E}(s) > 0$ *and* $C \in S/\mathcal{B}$ *it holds that* $\mathcal{M}_{\text{sbs}}(s, \alpha, C)(\theta)$ *is equal to:*
    $$\begin{aligned}
    &(1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}) \cdot \sum_{s' \in S} \tfrac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \cdot \mathcal{M}_{\text{sbs}}(s', \alpha', C)(\theta') \\
    &= \tfrac{1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}}{\mathrm{E}(s)} \cdot \sum_{C' \in S/\mathcal{B}} \sum_{s' \in C'} \mathcal{D}_{s,a}(s') \cdot \mathcal{M}_{\text{sbs}}(s', \alpha', C)(\theta') \\
    &= \tfrac{1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}}{\mathrm{E}(s)} \cdot \sum_{C' \in S/\mathcal{B}} \mathcal{M}_{\text{sbs}}(s_{C'}, \alpha', C)(\theta') \cdot \sum_{s' \in C'} \mathcal{D}_{s,a}(s')
    \end{aligned}$$
    where $s_{C'} \in C'$ *and the factorization of* $\mathcal{M}_{\text{sbs}}(s_{C'}, \alpha', C)(\theta')$ *stems from the application of the induction hypothesis on* α' *to all states of each equivalence class* C'*. Since* $\mathrm{E}(s'_1) = \mathrm{E}(s'_2)$ *by virtue of* $(s'_1, s'_2) \in \mathcal{B}$ *and Lemma 2 and* $\sum_{s' \in C'} \mathcal{D}_{s'_1,a}(s') = \sum_{s' \in C'} \mathcal{D}_{s'_2,a}(s')$ *by virtue of* $(s'_1, s'_2) \in \mathcal{B}$ *and Lemma 3, we derive that for all* $C \in S/\mathcal{B}$*:*
    $$\mathcal{M}_{\text{sbs}}(s'_1, \alpha, C)(\theta) \; = \; \mathcal{M}_{\text{sbs}}(s'_2, \alpha, C)(\theta)$$

– *The proof of the second part is similar to the proof of the first part, with the following calculation of* $\mathcal{M}_{\text{ete}}(s, \alpha, C)(t)$ *in the induction case for all* $s \in S$ *such that* $\mathrm{E}(s) > 0$*,* $C \in S/\mathcal{B}$*, and* $t \in \mathbb{R}_{\geq 0}$*:*

$$\begin{aligned}
&\int_0^t \mathrm{E}(s) \cdot \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot \sum_{s' \in S} \tfrac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \cdot \mathcal{M}_{\text{ete}}(s', \alpha', C)(t - x) \, \mathrm{d}x \\
&= \int_0^t \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot \sum_{C' \in S/\mathcal{B}} \sum_{s' \in C'} \mathcal{D}_{s,a}(s') \cdot \mathcal{M}_{\text{ete}}(s', \alpha', C)(t - x) \, \mathrm{d}x \\
&= \int_0^t \mathrm{e}^{-\mathrm{E}(s) \cdot x} \cdot \sum_{C' \in S/\mathcal{B}} \mathcal{M}_{\text{ete}}(s_{C'}, \alpha', C)(t - x) \cdot \sum_{s' \in C'} \mathcal{D}_{s,a}(s') \, \mathrm{d}x
\end{aligned}$$

followed by the exploitation of Lemmas 4 and 5. ■

Corollary 1. *For all* $s_1, s_2 \in S$*:*

$$s_1 \sim_{\text{MB}} s_2 \iff s_1 \sim_{\text{B},\mathcal{M}_{\text{ete}}} s_2 \iff s_1 \sim_{\text{B},\mathcal{M}_{\text{sbs}}} s_2 \qquad ■$$

With regard to trace equivalence for ACTMC, unlike Sect. 6.1 here the average duration of a computation is intended as its step-by-step average duration [18]. Given $s \in S$, the step-by-step average duration of $c \in \mathcal{C}_{\mathrm{f}}(s)$ is the sequence of the average sojourn times in the states traversed by c:

$$time_{\mathrm{a,sbs}}(c) \;=\; \begin{cases} \varepsilon & \text{if } |c| = 0 \\ \frac{1}{\mathrm{E}(s)} \circ time_{\mathrm{a,sbs}}(c') & \text{if } c \equiv s \xrightarrow{a,\lambda} c' \end{cases}$$

and we denote by $C_{\leq\theta}$ the set of computations in $C \subseteq \mathcal{C}_{\mathrm{f}}(s)$ whose step-by-step average duration is not greater than $\theta \in (\mathbb{R}_{\geq 0})^*$, i.e., $C_{\leq\theta} = \{c \in C \mid |c| \leq |\theta| \wedge \forall i = 1, \ldots, |c|.\, time_{\mathrm{a,sbs}}(c)[i] \leq \theta[i]\}$.

Definition 19. *We say that $s_1, s_2 \in S$ are step-by-step Markovian trace equivalent, written $s_1 \sim_{\mathrm{MTr,sbs}} s_2$, iff for all traces $\alpha \in A^*$ and sequences of amounts of time $\theta \in (\mathbb{R}_{\geq 0})^*$:*

$$prob(\mathcal{CC}_{\leq\theta}(s_1, \alpha)) \;=\; prob(\mathcal{CC}_{\leq\theta}(s_2, \alpha))$$

■

Theorem 8. *For all $s_1, s_2 \in S$:*

$$s_1 \sim_{\mathrm{MTr,sbs}} s_2 \iff s_1 \sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs}}} s_2$$

Proof. *Given $s \in S$, we define the step-by-step duration of $c \in \mathcal{C}_{\mathrm{f}}(s)$ as the sequence of the random variables quantifying the average sojourn times in the states traversed by c:*

$$time_{\mathrm{d,sbs}}(c) \;=\; \begin{cases} Det_0 & \text{if } |c| = 0 \\ Exp_{\mathrm{E}(s)} \circ time_{\mathrm{d,sbs}}(c') & \text{if } c \equiv s \xrightarrow{a,\lambda} c' \end{cases}$$

where Det_0 and $Exp_{\mathrm{E}(s)}$ are the same as those in the proof of Thm. 6. Moreover, we define the probability distribution of executing a computation in $C \subseteq \mathcal{C}_{\mathrm{f}}(s)$ within a sequence $\theta \in (\mathbb{R}_{\geq 0})^*$ of time units by letting:

$$\begin{aligned} prob_{\mathrm{d,sbs}}(C, \theta) \;&=\; \sum_{c \in C}^{|c| \leq |\theta|} prob(c) \cdot \prod_{i=1}^{|c|} \Pr\{time_{\mathrm{d,sbs}}(c)[i] \leq \theta[i]\} \\ &=\; \sum_{c \in C}^{|c| \leq |\theta|} prob(c) \cdot \prod_{i=1}^{|c|} (1 - \mathrm{e}^{-\theta[i] / time_{\mathrm{a,sbs}}(c)[i]}) \end{aligned}$$

whenever C is finite and all of its computations are independent of each other.

Let $s_1, s_2 \in S$ be such that $s_1 \sim_{\mathrm{MTr,sbs}} s_2$ and assume $\mathrm{E}(s_1) > 0$ and $\mathrm{E}(s_2) > 0$ in order to avoid trivial cases. The fact that $s_1 \sim_{\mathrm{MTr,sbs}} s_2$ is equivalent by definition to the fact that for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$prob(\mathcal{CC}_{\leq\theta}(s_1, \alpha)) \;=\; prob(\mathcal{CC}_{\leq\theta}(s_2, \alpha))$$

which in turn is equivalent by virtue of [1] to the fact that for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$:

$$prob_{\mathrm{d,sbs}}(\mathcal{CC}(s_1, \alpha), \theta) \;=\; prob_{\mathrm{d,sbs}}(\mathcal{CC}(s_2, \alpha), \theta)$$

Since for all $s \in S$ such that $\mathrm{E}(s) > 0$, $\alpha \in A^*$, and $\theta \in (\mathbb{R}_{\geq 0})^*$ it holds that:

$$prob_{\mathrm{d,sbs}}(\mathcal{CC}(s, \alpha), \theta) \;=\; \begin{cases} \sum_{s' \in S} \frac{\mathcal{D}_{s,a}(s')}{\mathrm{E}(s)} \cdot (1 - \mathrm{e}^{-\mathrm{E}(s) \cdot t}) \cdot prob_{\mathrm{d,sbs}}(\mathcal{CC}(s', \alpha'), \theta') & \text{if } \alpha = a \circ \alpha' \text{ and } \theta = t \circ \theta' \\ 1 & \text{if } \alpha = \varepsilon \\ 0 & \text{if } \alpha \neq \varepsilon \text{ and } \theta = \varepsilon \end{cases}$$

and hence:

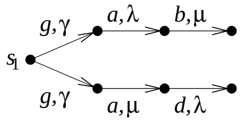
$$prob_{\mathrm{d,sbs}}(\mathcal{CC}(s, \alpha), \theta) \;=\; \mathcal{M}_{\mathrm{sbs}}(s, \alpha, S)(\theta)$$

we immediately derive that the fact that for all $\alpha \in A^*$ and $\theta \in (\mathbb{R}_{\geq 0})^*$ $prob_{\mathrm{d,sbs}}(\mathcal{CC}(s_1, \alpha), \theta) = prob_{\mathrm{d,sbs}}(\mathcal{CC}(s_2, \alpha), \theta)$ is equivalent to the fact that for all $\alpha \in A^*$:

$$\mathcal{M}_{\text{sbs}}(s_1, \alpha, S) \;=\; \mathcal{M}_{\text{sbs}}(s_2, \alpha, S)$$

which in turn is equivalent by definition to $s_1 \sim_{\text{Tr},\mathcal{M}_{\text{sbs}}} s_2$. ■

It is worth observing that $\sim_{\text{MTr,ete}}$ and $\sim_{\text{MTr,sbs}}$ do not coincide. In fact, the latter is finer than the former, because it is somehow able to keep track of the time instants at which the various actions of a trace start/complete their execution. As an example, if we consider the following two ACTMC taken from [1]:



where $\lambda < \mu$ and $b \neq d$, it turns out that $s_1 \sim_{\text{MTr,ete}} s_2$ while $s_1 \not\sim_{\text{MTr,sbs}} s_2$ because $prob(\mathcal{CC}_{\leq\theta}(s_1, \alpha)) = \frac{1}{2} \neq 0 = prob(\mathcal{CC}_{\leq\theta}(s_2, \alpha))$ when $\alpha = g \circ a \circ b$ and $\theta = \frac{1}{2\cdot\gamma} \circ \frac{1}{\lambda} \circ \frac{1}{\mu}$. Therefore, $\sim_{\text{Tr},\mathcal{M}_{\text{ete}}}$ and $\sim_{\text{Tr},\mathcal{M}_{\text{sbs}}}$ do not coincide either.

7 Conclusions and Future Work

In this paper, we have introduced ULTraS as a general framework to uniformly describe the operational semantics of fully nondeterministic, fully probabilistic, and fully stochastic variants of process algebras. Within ULTraS, the transition relation associates with a state and a given transition label a function mapping each state into an element of a domain D. Elements in D are used to associate a weight with each transition. By appropriately changing the domain D, different models of concurrent systems can be represented.

We have then defined two of the most classical notions of behavioral equivalences, namely bisimulation and trace equivalences, over ULTraS and have studied their impact on the characterization as ULTraS of three classical process models: LTS, ADTMC, and ACTMC. In particular, we have shown that the bisimulation and trace equivalences on the models obtained via the ULTraS characterization of LTS, ADTMC, and ACTMC are in full agreement with those specifically considered in the literature for the three different models. We consider this general characterization and the proof of correspondence of the equivalences as a vindication for the originally proposed models.

In the near future, we plan to investigate the applicability of the ULTraS framework to further classes of processes – like deterministically timed processes and processes where nondeterminism and probability or nondeterminism and stochasticity are intertwined – as well as other behavioral equivalences in the linear-time/branching-time spectrum – especially testing equivalences. Moreover, we plan to use ULTraS for describing the operational semantics of a few of the many process description languages that have been proposed in the literature, in order to assess their relative expressiveness of specific operators and establish general properties for the different languages.

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