ULTraS: *A Uniform Framework for Nondeterministic, Probabilistic, and Timed Process Models and Behavioral Equivalences*

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Behavioral Models and Equivalences

- Behavioral models of complex computing systems are mostly based on labeled transition systems [Keller, 1976].
- State-transition graphs in which every transition is labeled with the action/event determining the state change.
- Focus on interaction/communication, as opposed to Kripke structures.
- The next transition to be executed is selected nondeterministically: implementation freedom, lack of information.
- Behavioral equivalences studied in the 1980's to establish a connection between different LTS models that exhibit the same behavior.
- Support top-down design and compositional state space minimization before applying verification techniques such as model checking.

A Unifying View

- Several generalizations to deal with probabilistic and/or timed systems since the late 1980's, yielding different models and equivalences.
- Possibly combining nondeterminism and quantitative aspects.
- Can we provide a unifying definition of the various models/equivalences?
- Do new models/equivalences emerge, which have interesting properties?
- Taking inspiration from two extensions of the LTS model:
  - *Simple probabilistic automata* [Segala, 1995].
  - *Rate transition systems* [De Nicola-Latella-Loreti-Massink, 2009].
- Transition format: next state distribution vs. single next state.

The ULTRAS Model

- $(D, \sqsubseteq_D, \perp_D)$: preordered set equipped with a minimum denoted by $\perp_D$, with each value representing a degree of one-step reachability.
- A uniform labeled transition system on $(D, \sqsubseteq_D, \perp_D)$, or D-ULTRAS, is a triple $\mathcal{U} = (S, A, \longrightarrow)$ where:
  - S is an at most countable set of states.
  - A is a countable set of transition-labeling actions.
  - $\longrightarrow \subseteq S \times A \times [S \to D]$ is a transition relation.
- $\mathcal{U}$ is *functional* iff $\longrightarrow$ is a function from $S \times A$ to $[S \to D]$.
- Given a transition $s \xrightarrow{a} \Delta$, function Δ represents the distribution of reachability of all possible states from s via that transition.
- If $\Delta(s') = \perp_D$, then s' is not reachable from s via that transition.

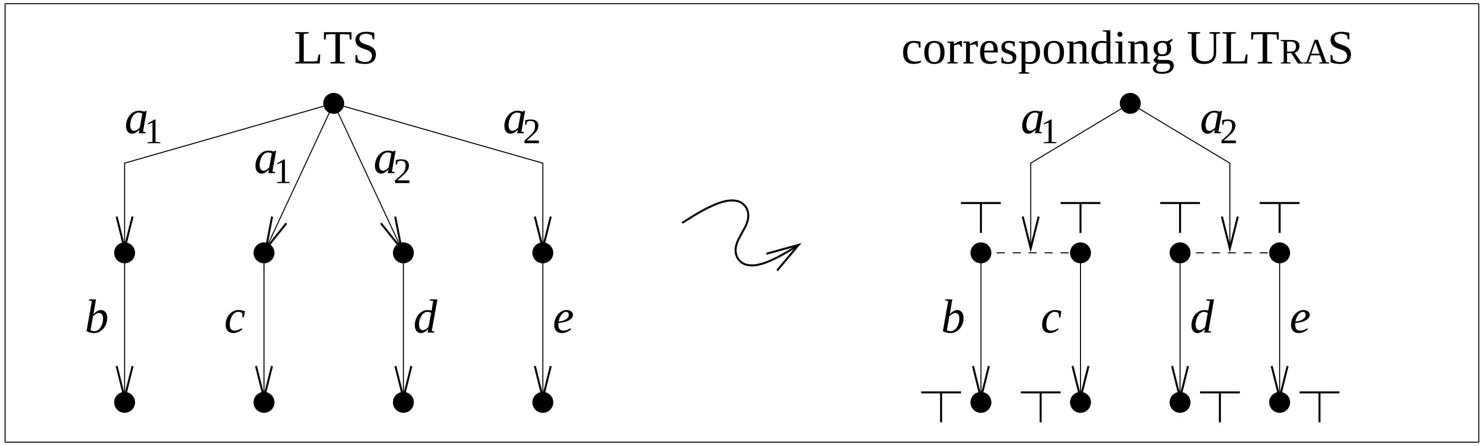
Encoding Nondeterministic Models as ULTRAS

- An LTS can be encoded as a functional $\mathbb{B}$-ULTRAS, where $\mathbb{B} = \{\bot, \top\}$ is the support set of the Boolean algebra with $\bot \sqsubseteq_{\mathbb{B}} \top$.

- An LTS is a triple $(S, A, \longrightarrow)$ where:
  - S is an at most countable set of states.
  - A is a countable set of transition-labeling actions.
  - $\longrightarrow \subseteq S \times A \times S$ is a transition relation.

- Corresponding functional $\mathbb{B}$-ULTRAS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$:
  - $s \xrightarrow{a}_{\mathcal{U}} \Delta_{s,a}$ for all $s \in S$ and $a \in A$.
  - $\Delta_{s,a}(s') = \begin{cases} \top & \text{if } s \xrightarrow{a} s' \\ \bot & \text{if } (s, a, s') \notin \longrightarrow \end{cases}$ for all $s' \in S$.

- External and internal forms of nondeterminism are encoded differently:



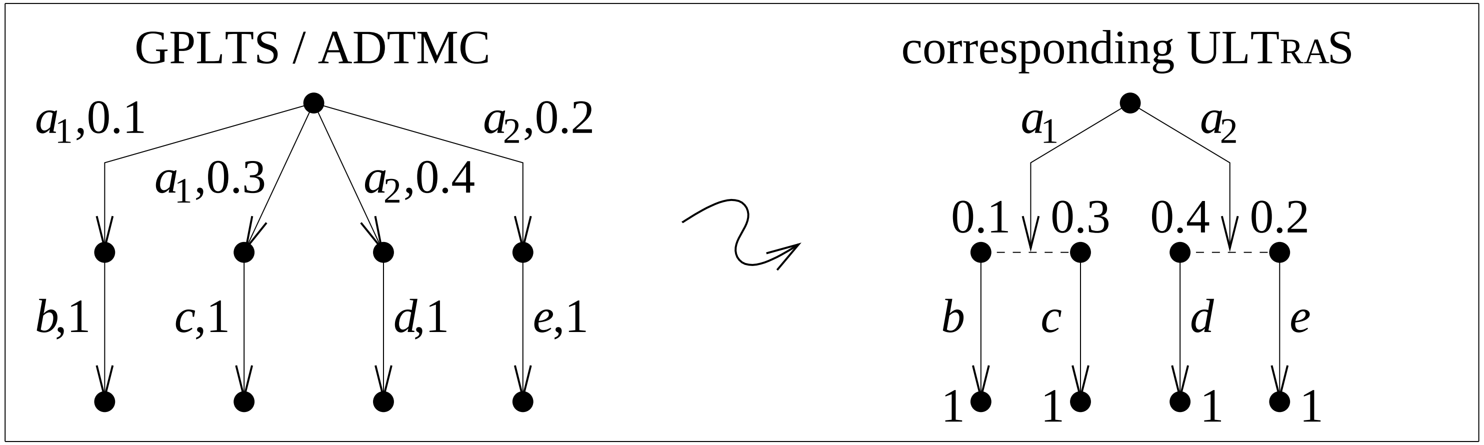
- The resulting functional $\mathbb{B}$-ULTraS models can be viewed as *alternating automata* in which, however, every transition is existential: all states labeled with $\top$ are alternative to each other.

Encoding Probabilistic Models as ULTRAS

- Models featuring probabilities and different levels of nondeterminism.
- A GPLTS (or action-labeled discrete-time Markov chain – ADTMC) can be encoded as a functional $\mathbb{R}_{[0,1]}$-ULTRAS with the usual $\leq$.
- An RPLTS (or discrete-time Markov decision process – MDP) can be encoded as a functional $\mathbb{R}_{[0,1]}$-ULTRAS with the usual $\leq$.
- An NPLTS (which is an MDP with internal nondeterminism) can be encoded as an $\mathbb{R}_{[0,1]}$-ULTRAS with the usual $\leq$.

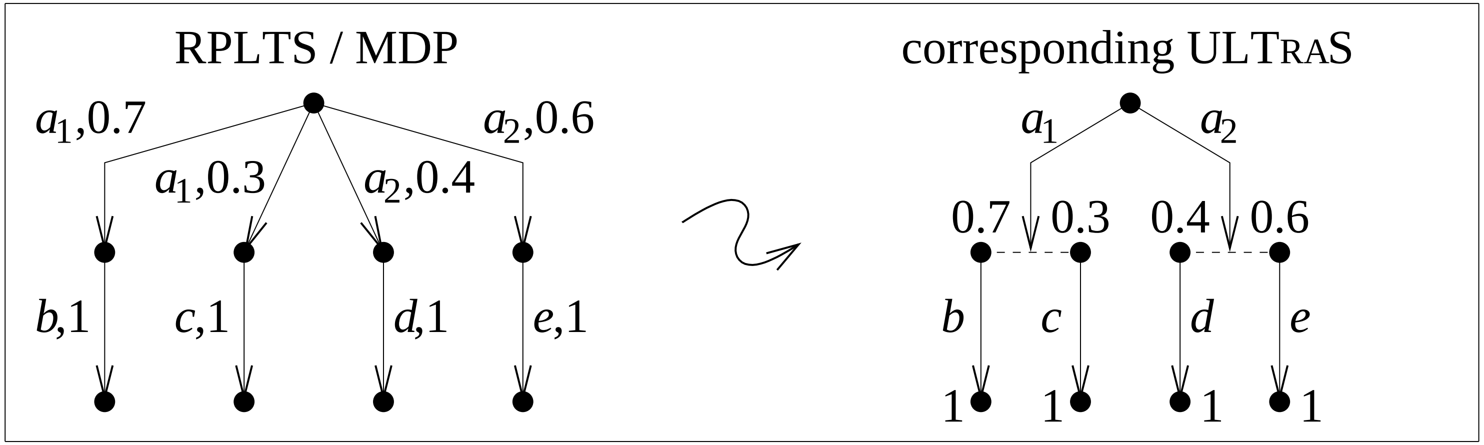
- A generative probabilistic LTS is a triple $(S, A, \longrightarrow)$ where:
  - S is an at most countable set of states.
  - A is a countable set of transition-labeling actions.
  - $\longrightarrow \subseteq S \times A \times \mathbb{R}_{(0,1]} \times S$ is a transition relation.
  - Whenever $s \xrightarrow{a,p_1} s'$ and $s \xrightarrow{a,p_2} s'$, then $p_1 = p_2$.
  - $\sum \{\!| \, p \in \mathbb{R}_{(0,1]} \mid \exists a \in A. \exists s' \in S. \, s \xrightarrow{a,p} s' \, |\!\} \in \{0, 1\}$ for all $s \in S$.
- Corresponding functional $\mathbb{R}_{[0,1]}$-ULTRAS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$:
  - $s \xrightarrow{a}_{\mathcal{U}} \Delta_{s,a}$ for all $s \in S$ and $a \in A$.
  - $\Delta_{s,a}(s') = \begin{cases} p & \text{if } s \xrightarrow{a,p} s' \\ 0 & \text{if } \nexists p \in \mathbb{R}_{(0,1]}. \, s \xrightarrow{a,p} s' \end{cases}$ for all $s' \in S$.

- External and internal probabilistic choices, probability *sub*distributions:



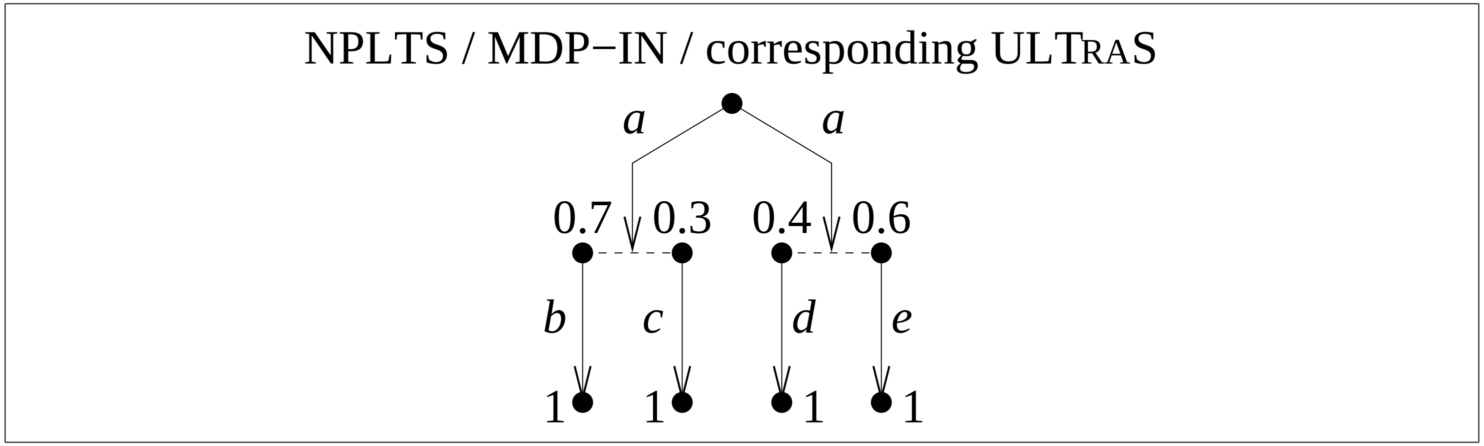
- A reactive probabilistic LTS is a triple $(S, A, \longrightarrow)$ where:
  - S is an at most countable set of states.
  - A is a countable set of transition-labeling actions.
  - $\longrightarrow \subseteq S \times A \times \mathbb{R}_{(0,1]} \times S$ is a transition relation.
  - Whenever $s \xrightarrow{a,p_1} s'$ and $s \xrightarrow{a,p_2} s'$, then $p_1 = p_2$.
  - $\sum\{\!| p \in \mathbb{R}_{(0,1]} \mid \exists s' \in S.\, s \xrightarrow{a,p} s' |\!\} \in \{0, 1\}$ for all $s \in S$ <u>and $a \in A$</u>.
- Corresponding functional $\mathbb{R}_{[0,1]}$-ULTraS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$:
  - $s \xrightarrow{a}_{\mathcal{U}} \Delta_{s,a}$ for all $s \in S$ and $a \in A$.
  - $\Delta_{s,a}(s') = \begin{cases} p & \text{if } s \xrightarrow{a,p} s' \\ 0 & \text{if } \nexists p \in \mathbb{R}_{(0,1]}.\, s \xrightarrow{a,p} s' \end{cases}$ for all $s' \in S$.

- External nondeterminism & internal probabilistic choices:



- A nondeterministic and probabilistic LTS is a triple $(S, A, \longrightarrow)$ where:
  - S is an at most countable set of states.
  - A is a countable set of transition-labeling actions.
  - $\longrightarrow \subseteq S \times A \times [S \to \mathbb{R}_{[0,1]}]$ is a transition relation.
  - $\sum_{s' \in S} \Delta(s') = 1$ for all $s \xrightarrow{a} \Delta$.
- The corresponding $\mathbb{R}_{[0,1]}$-ULTRAS is $(S, A, \longrightarrow)$ itself.
- Not functional due to the coexistence of internal nondeterminism and probabilistic choices.

- External/internal nondeterminism & internal probabilistic choices:

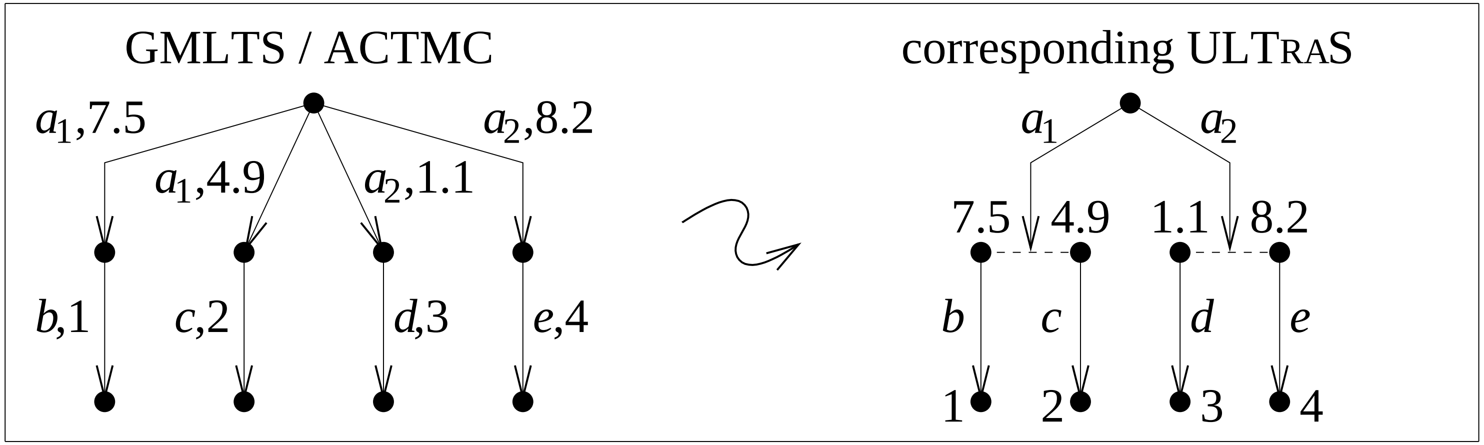


Encoding Stochastic Models as ULTRAS

- Models featuring rates and different levels of nondeterminism.
- Rates encompass both probabilistic and timing aspects.
- A GMLTS (or action-labeled continuous-time Markov chain – ACTMC) can be encoded as a functional $\mathbb{R}_{\geq 0}$-ULTRAS with the usual $\leq$.
- An RMLTS (or continuous-time Markov decision process – CTMDP) can be encoded as a functional $\mathbb{R}_{\geq 0}$-ULTRAS with the usual $\leq$.
- An NMLTS (which is a CTMDP with internal nondeterminism) can be encoded as an $\mathbb{R}_{\geq 0}$-ULTRAS with the usual $\leq$.

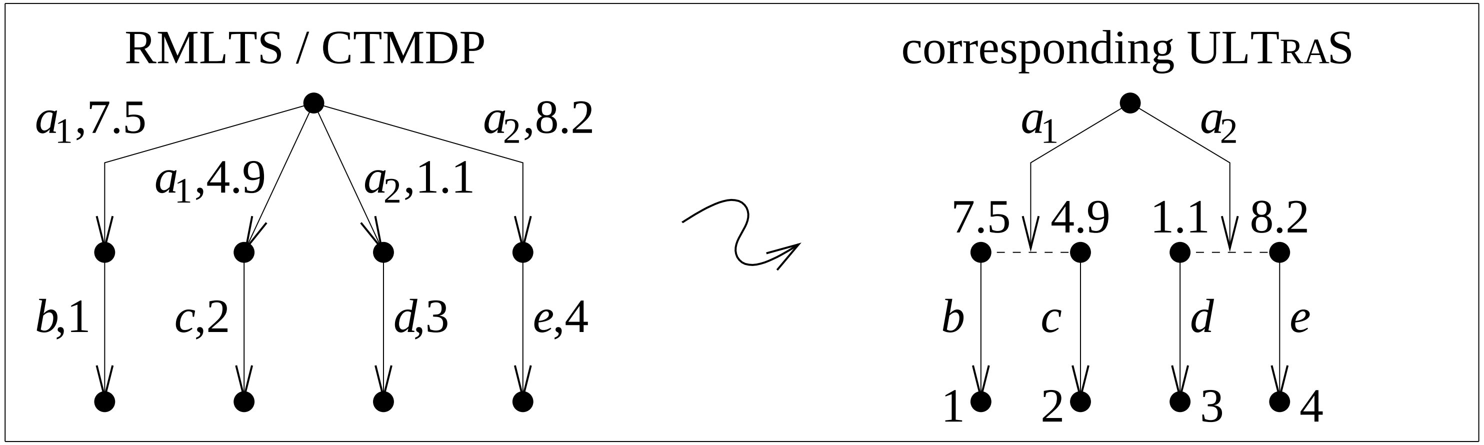
- A generative Markovian LTS is a triple $(S, A, \longrightarrow)$ where:
  - S is an at most countable set of states.
  - A is a countable set of transition-labeling actions.
  - $\longrightarrow \subseteq S \times A \times \mathbb{R}_{>0} \times S$ is a transition relation.
  - Whenever $s \xrightarrow{a,\lambda_1} s'$ and $s \xrightarrow{a,\lambda_2} s'$, then $\lambda_1 = \lambda_2$.
- Corresponding functional $\mathbb{R}_{\geq 0}$-ULTRAS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$:
  - $s \xrightarrow{a}_{\mathcal{U}} \Delta_{s,a}$ for all $s \in S$ and $a \in A$.
  - $\Delta_{s,a}(s') = \begin{cases} \lambda & \text{if } s \xrightarrow{a,\lambda} s' \\ 0 & \text{if } \nexists \lambda \in \mathbb{R}_{>0}.\, s \xrightarrow{a,\lambda} s' \end{cases}$ for all $s' \in S$.

- External and internal rate-based probabilistic choices:



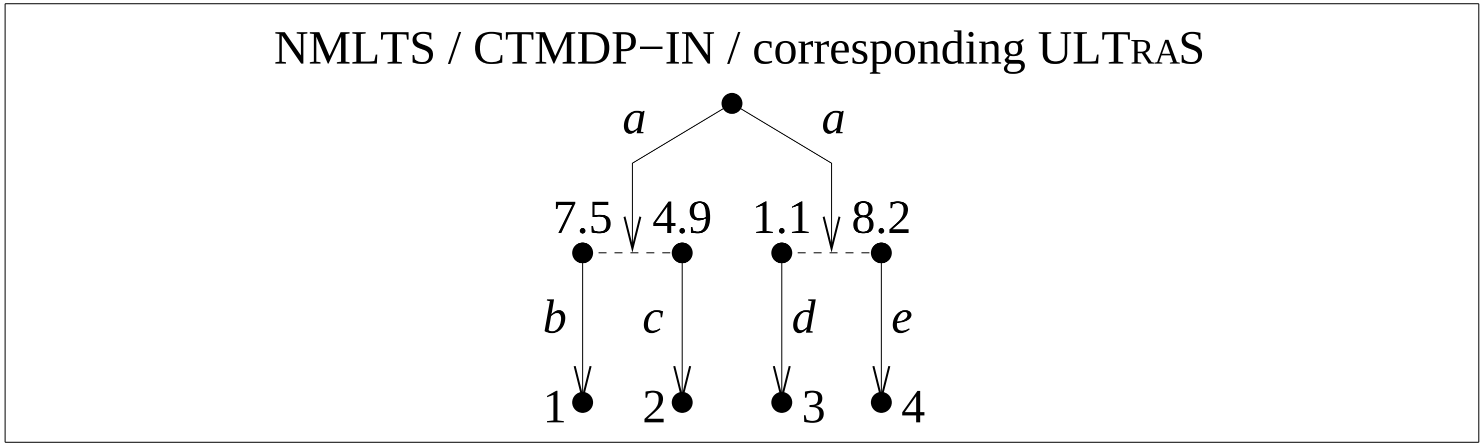
- A reactive Markovian LTS is a triple $(S, A, \longrightarrow)$ where:
  - S is an at most countable set of states.
  - A is a countable set of transition-labeling actions.
  - $\longrightarrow \subseteq S \times A \times \mathbb{R}_{>0} \times S$ is a transition relation.
  - Whenever $s \xrightarrow{a,\lambda_1} s'$ and $s \xrightarrow{a,\lambda_2} s'$, then $\lambda_1 = \lambda_2$.
- Corresponding functional $\mathbb{R}_{\geq 0}$-ULTRAS $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$:
  - $s \xrightarrow{a}_{\mathcal{U}} \Delta_{s,a}$ for all $s \in S$ and $a \in A$.
  - $\Delta_{s,a}(s') = \begin{cases} \lambda & \text{if } s \xrightarrow{a,\lambda} s' \\ 0 & \text{if } \nexists \lambda \in \mathbb{R}_{>0}.\, s \xrightarrow{a,\lambda} s' \end{cases}$ for all $s' \in S$.

- External nondeterminism & internal rate-based probabilistic choices:



- A nondeterministic and Markovian LTS is a triple $(S, A, \longrightarrow)$ where:
  - S is an at most countable set of states.
  - A is a countable set of transition-labeling actions.
  - $\longrightarrow \subseteq S \times A \times [S \rightarrow \mathbb{R}_{\geq 0}]$ is a transition relation.
  - $\sum_{s' \in S} \Delta(s') > 0$ for all $s \xrightarrow{a} \Delta$.
- The corresponding $\mathbb{R}_{\geq 0}$-ULTRAS is $(S, A, \longrightarrow)$ itself.
- Not functional due to the coexistence of internal nondeterminism and rate-based probabilistic choices.

- Ext./int. nondeterminism & internal rate-based probabilistic choices:



Encoding Timed Models as ULTRAS

- Timed automata (TA) are automata with clock variables that measure the passage of time within states, while transitions are instantaneous, may be subject to clock-based guards, and may reset some clocks.
- A TA/TLTS can be encoded as a functional $\mathbb{B}$-ULTRAS with $\bot \sqsubseteq_{\mathbb{B}} \top$.
- Probabilistic timed automata (PTA) are TA where the destination of every transition is a function associating with each state the probability of being the target state.
- A PTA/PTLTS can be encoded as an $\mathbb{R}_{[0,1]}$-ULTRAS with the usual $\leq$.
- Markov automata (MA) retain the probabilistic flavor of transitions of PTA, while temporal aspects are described through exponentially distributed random variables rather than deterministic quantities.
- An MA can be encoded as an $\mathbb{R}_{[0,1]}$-ULTRAS with the usual $\leq$.

- Due to the memoryless property of exponential distributions, clocks are not needed for MA.

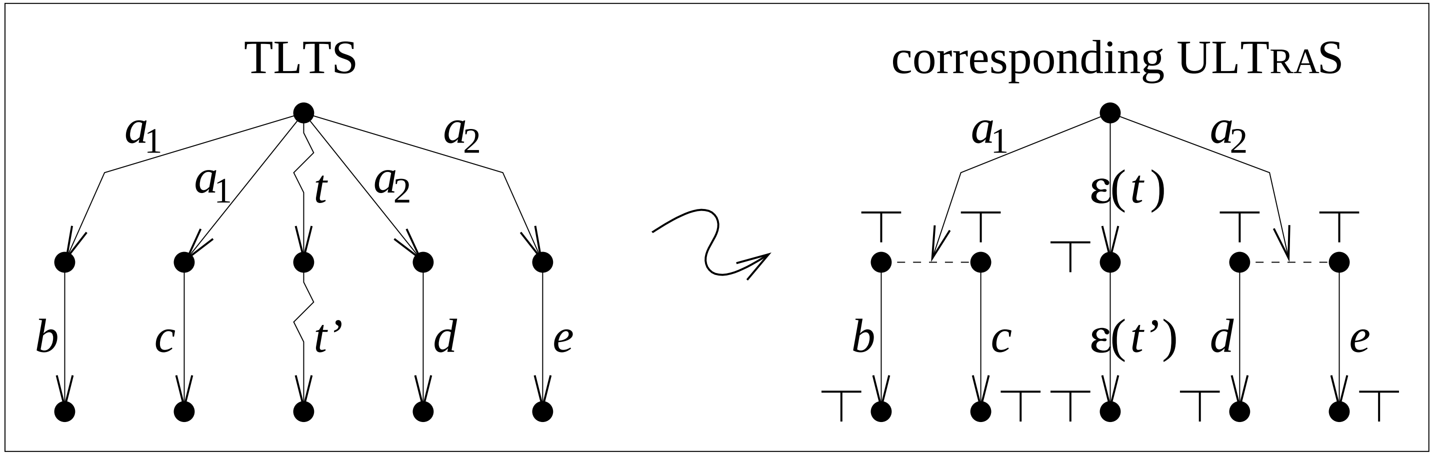
- The presence of $\mathbb{R}_{\geq 0}$-valued clocks causes the LTS-based semantics of TA/PTA to have uncountably many states, each corresponding to a pair formed by a vector of location states and a vector of clock values.

- Need to extend the definition of D-ULTRAS by allowing:
  - The set of states S to be possibly uncountable.
  - The set of transition-labeling actions A to be possibly uncountable.
  - The transition relation $\longrightarrow$ to be $\subseteq S \times A \times [S \rightarrow D]_{\mathrm{cs}}$.

- $[S \rightarrow D]_{\mathrm{cs}}$ is the set of reachability distributions $\Delta : S \rightarrow D$ such that their support $supp(\Delta) = \{s \in S \mid \Delta(s) \neq \bot_D\}$ is at most countable.

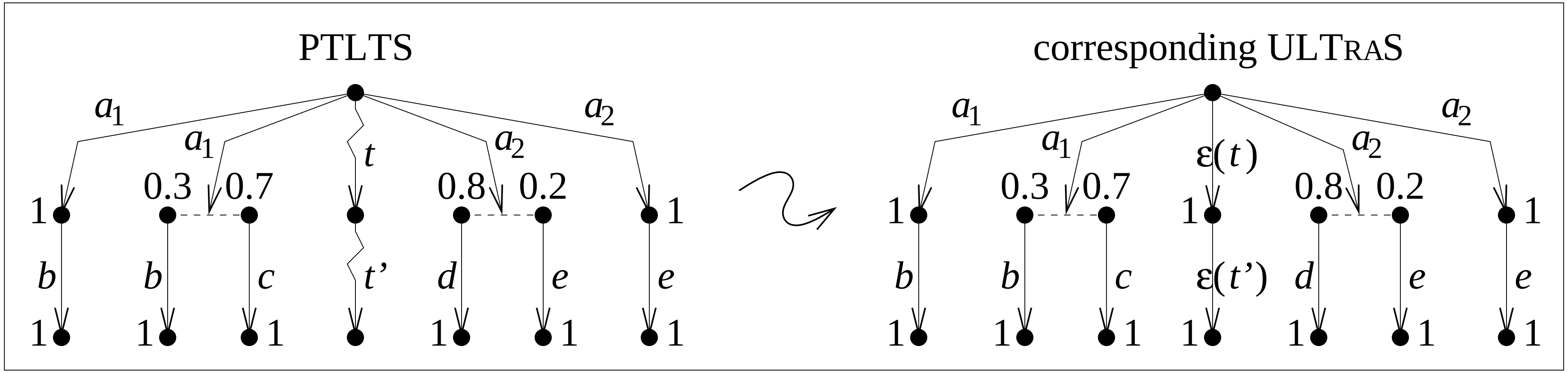
- A timed LTS (TLTS) is a quadruple $(S, A, \longrightarrow, \leadsto)$ where:
  - S is a possibly uncountable set of states.
  - A is a possibly uncountable set of transition-labeling actions.
  - $\longrightarrow \subseteq S \times A \times S$ is an action-transition relation such that the set $\{s' \in S \mid s \xrightarrow{a} s'\}$ is at most countable for all $s \in S$ and $a \in A$.
  - $\leadsto \subseteq S \times \mathbb{R}_{\geq 0} \times S$ is a time-transition relation satisfying:
    * If $s \overset{0}{\leadsto} s'$, then $s' = s$ *[zero delay]*.
    * If $s \overset{t}{\leadsto} s'_1$ and $s \overset{t}{\leadsto} s'_2$, then $s'_1 = s'_2$ *[time determinism]*.
    * $s \overset{t_1+t_2}{\leadsto} s''$ iff $s \overset{t_1}{\leadsto} s'$ and $s' \overset{t_2}{\leadsto} s''$ *[time additivity]*.

- We can merge the two transition relations into a single one by adding a special time-elapsing action $\epsilon(t)$ for every $t \in \mathbb{R}_{\geq 0}$.
- A TLTS can be encoded as a functional $\mathbb{B}$-ULTRAS:
  - External nondeterminism is preserved.
  - Internal nondeterminism is represented within the countable-support reachability distributions constituting the target of the transitions.
- Extends to TA.



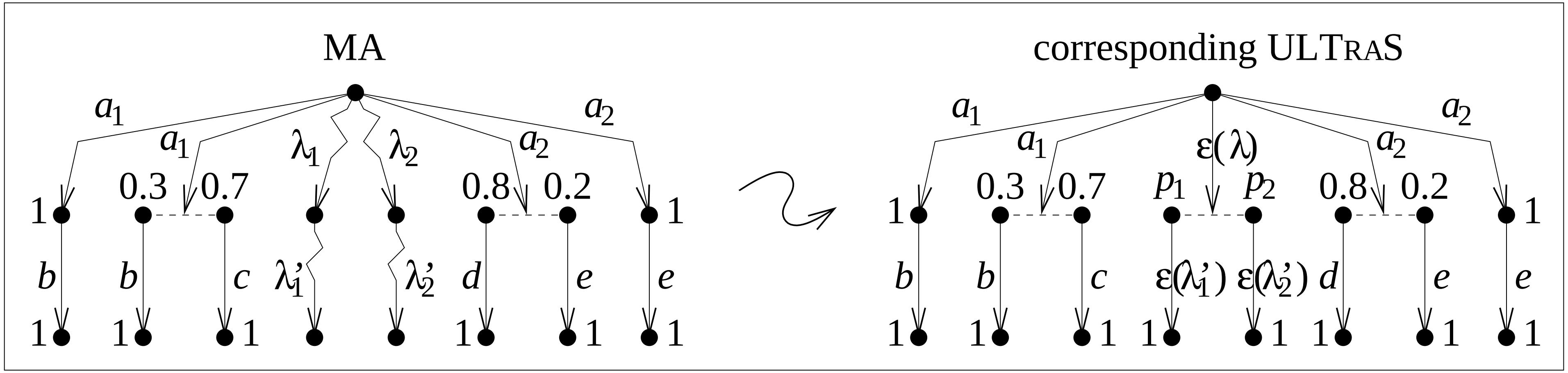
- A probabilistic timed LTS (PTLTS) is a quadruple $(S, A, \longrightarrow, \leadsto)$ where:
  - S is a possibly uncountable set of states.
  - A is a possibly uncountable set of transition-labeling actions.
  - $\longrightarrow \subseteq S \times A \times \mathit{Distr}_{\mathrm{cs}}(S)$ is a probabilistic action-transition relation.
  - $\leadsto \subseteq S \times \mathbb{R}_{\geq 0} \times S$ is a time-transition relation satisfying:
    - If $s \overset{0}{\leadsto} s'$, then $s' = s$ *[zero delay]*.
    - If $s \overset{t}{\leadsto} s'_1$ and $s \overset{t}{\leadsto} s'_2$, then $s'_1 = s'_2$ *[time determinism]*.
    - $s \overset{t_1+t_2}{\leadsto} s''$ iff $s \overset{t_1}{\leadsto} s'$ and $s' \overset{t_2}{\leadsto} s''$ *[time additivity]*.

- The target of any transition labeled with $\epsilon(t)$ is a Dirac distribution.
- A PTLTS can be encoded as an $\mathbb{R}_{[0,1]}$-ULTRAS:
  - External nondeterminism is preserved as for TLTS.
  - Internal nondeterminism is preserved as well.
  - Not functional due to the coexistence of probability and internal nondeterminism.
- Extends to PTA.



- A Markov automaton (MA) is a quadruple $(S, A, \longrightarrow, \dashrightarrow)$ where:
  - S is an at most countable set of states.
  - A is a countable set of transition-labeling actions.
  - $\longrightarrow \subseteq S \times A \times \mathit{Distr}(S)$ is a probabilistic action-transition relation.
  - $\rightsquigarrow \subseteq S \times \mathbb{R}_{\geq 0} \times S$ is a time-transition relation satisfying:
    * If $s \stackrel{0}{\rightsquigarrow} s'$, then $s' = s$ *[zero speed]*.
    * $\sum_{(s,\lambda,s') \in \rightsquigarrow} \lambda < \infty$ for all $s \in S$ *[speed boundedness]*.
- The execution probability of a time transition is proportional to its rate.
- Race policy: the time transition sampling the least duration is the one that is actually executed.
- The sojourn time in a state is exponentially distributed, with rate given by the sum of the rates of the outgoing time transitions.

- An MA can be encoded as an $\mathbb{R}_{[0,1]}$-ULTRAS:
  - External/internal nondeterminism are preserved as for PTLTS.
  - The race policy is preserved by generating for each state a single transition labeled with $\epsilon(\lambda)$, where λ is the sum of the rates, with the target distribution assigning probabilities proportional to rates.
  - Not functional due to the coexistence of probability and internal nondeterminism.



ULTRAS as a Metamodel

- ULTRAS has the potential to provide:
  - A unifying mathematical theory for many models.
  - General results that can be readily instantiated.
  - A comparison and cross-fertilizing framework.
- More general than LTS models weighted over monoids [Klin, 2009].
- Akin to LTS models weighted over semirings.
- Formalizable as specific coalgebras [Miculan-Peressotti, 2014].

Behavioral Equivalences for the ULTRAS Model

- $(M, \sqsubseteq_M, \bot_M)$: preordered set equipped with a minimum denoted by $\bot_M$, with each value representing a degree of multi-step reachability.

- A measure function on $(M, \sqsubseteq_M, \bot_M)$ for $\mathcal{U} = (S, A, \longrightarrow)$, or M-measure function for $\mathcal{U}$, is a function $\mathcal{M}_M : S \times A^* \times 2^S \to M$ such that the value of $\mathcal{M}_M(s, \alpha, S')$ is defined by induction on $|\alpha|$ and depends only on the reachability of a state in S' from state s through computations labeled with trace α.

- A measure function somehow subsumes the existence of two operators:
  - A *multiplicative operator* $\otimes$ that combines into an M-value the D-values corresponding to each individual step along a single computation labeled with trace α that goes from s to S'.
  - An *additive operator* $\oplus$ that combines the M-values computed for each considered computation with the previous operator.

- D and M are not necessarily the same set.
- A D-value $\Delta(s')$ is related to one-step reachability.
- An M-value $\mathcal{M}_M(s, \alpha, S')$ is related to multi-step reachability.
- Testing equivalence for LTS models: the M-value will be a pair of $\mathbb{B}$-values – *rather than a single* $\mathbb{B}$*-value* – to take into account the possibility and the necessity of reaching S' from s after α.
- Equivalences for NPLTS models: the M-value will be a nonemtpy set of $\mathbb{R}_{[0,1]}$-values – *rather than a single* $\mathbb{R}_{[0,1]}$*-value* – to take into account all possible ways of resolving internal nondeterminism.
- Equivalences for stochastic models: the M-value will be an $\mathbb{R}_{[0,1]}$-valued function – *rather than a single* $\mathbb{R}_{\geq 0}$*-value* – representing for each possible end-to-end/step-by-step deadline the probability (or set of probabilities) of reaching S' from s via α within the considered deadline.

- Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS.

- Let $\mathcal{M}_M$ be an M-measure function for $\mathcal{U}$.

- Focus on strong equivalences: no abstraction from invisible actions.

- The simplest equivalence to define is trace equivalence.

- We say that $s_1, s_2 \in S$ are $\mathcal{M}_M$-trace equivalent, written $s_1 \sim_{\mathrm{Tr}, \mathcal{M}_M} s_2$, iff for all traces $\alpha \in A^*$:

$$\mathcal{M}_M(s_1, \alpha, S) = \mathcal{M}_M(s_2, \alpha, S)$$

- Using the entire S as set of destination states means that destination states are not important; what matters is the capability of executing α.

- The definition of bisimulation equivalence concentrates on traces of length 1 and does take into account destination states.
- An equivalence relation $\mathcal{B}$ over S is an $\mathcal{M}_M$-bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in A$ and groups of equivalence classes $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\mathcal{M}_M(s_1, a, \bigcup \mathcal{G}) = \mathcal{M}_M(s_2, a, \bigcup \mathcal{G})$$

- Considering groups of equivalence classes, instead of single equivalence classes, is a useful generalization when dealing with:
  - continuous state spaces;
  - simulation preorders/equivalences.
- We say that $s_1, s_2 \in S$ are $\mathcal{M}_M$-bisimilar, written $s_1 \sim_{\mathrm{B},\mathcal{M}_M} s_2$, iff there exists an $\mathcal{M}_M$-bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$.

- Testing equivalence requires some preliminary definitions.

- A D-observation system is a D-ULTRAS $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ where O contains a distinguished success state denoted by ω such that, whenever $\omega \xrightarrow{a} \Delta$, then $\Delta(o) = \bot_D$ for all $o \in O$.

- Need D-valued function δ for the interaction system $\mathcal{I}(\mathcal{U}, \mathcal{O})$ to combine the target distributions of the synchronizing transitions of $\mathcal{U}$ and $\mathcal{O}$, which preserves $\bot_D$ and is injective.

- States are configurations (s, o) that are successful when $o = \omega$: $\mathcal{S}^{\delta}(\mathcal{U}, \mathcal{O})$.

- We say that $s_1, s_2 \in S$ are $\mathcal{M}^{\delta}_{M}$-testing equivalent, written $s_1 \sim_{\mathrm{Te}, \mathcal{M}^{\delta}_{M}} s_2$, iff for every D-observation system $\mathcal{O} = (O, A, \longrightarrow_{\mathcal{O}})$ with initial state $o \in O$ and for all traces $\alpha \in A^*$:

$$\mathcal{M}^{\delta, \mathcal{O}}_{M}((s_1, o), \alpha, \mathcal{S}^{\delta}(\mathcal{U}, \mathcal{O})) \; = \; \mathcal{M}^{\delta, \mathcal{O}}_{M}((s_2, o), \alpha, \mathcal{S}^{\delta}(\mathcal{U}, \mathcal{O}))$$

Retrieving Existing Behavioral Equivalences

- Most of the bisimulation, trace, and testing equivalences appeared in the literature since the 1980's are captured by our general framework ...
- ... except when internal nondeterminism and probability/stochasticity coexist in the considered model.
- For NPLTS models, we have obtained equivalences different from those appeared in the literature, which possess interesting properties.
- For NMLTS models, there are no equivalences defined in the literature, hence we have provided them for the first time.

- Nondeterministic behavioral equivalences:

LTS	$\sim_{\mathrm{B}}$ [Park, 1981][Milner, 1984] $\sim_{\mathrm{Tr}}$ [Brookes-Hoare-Roscoe, 1984] $\sim_{\mathrm{Te}}$ [De Nicola-Hennessy, 1984]	$\sim_{\mathrm{B},\mathcal{M}_{\mathbb{B},\vee}}$ $\sim_{\mathrm{Tr},\mathcal{M}_{\mathbb{B},\vee}}$ $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{LC}}_{\mathbb{B}\times\mathbb{B}}}$	functional $\mathbb{B}$-ULTRAS

- Nondeterministic measure functions:

$$\mathcal{M}_{\mathbb{B},\vee}(s,\alpha,S') = \begin{cases} \bigvee_{s' \in S \text{ s.t. } \Delta_{s,a}(s') \neq \bot} \mathcal{M}_{\mathbb{B},\vee}(s',\alpha',S') & \alpha = a \circ \alpha' \\ \top; \bot & \alpha = \varepsilon,\ s \in S'? \end{cases}$$

$$\mathcal{M}_{\mathbb{B},\wedge}(s,\alpha,S') = \begin{cases} \bigwedge_{s' \in S \text{ s.t. } \Delta_{s,a}(s') \neq \bot} \mathcal{M}_{\mathbb{B},\wedge}(s',\alpha',S') & \alpha = a \circ \alpha' \\ \top; \bot & \alpha = \varepsilon,\ s \in S'? \end{cases}$$

$$\mathcal{M}_{\mathbb{B}\times\mathbb{B}}(s,\alpha,S') = (\mathcal{M}_{\mathbb{B},\vee}(s,\alpha,S'), \mathcal{M}_{\mathbb{B},\wedge}(s,\alpha,S'))$$

- Probabilistic behavioral equivalences:

GPLTS	$\sim_{\rm PB}$ $\sim_{\rm PTr}$ $\sim_{\rm PTe}$	$\sim_{\rm B,\mathcal{M}_{\mathbb{R}_{[0,1]}}}$ $\sim_{\rm Tr,\mathcal{M}_{\mathbb{R}_{[0,1]}}}$ $\sim_{\rm Te,\mathcal{M}^{\rm NPM}_{\mathbb{R}_{[0,1]}}}$	functional $\mathbb{R}_{[0,1]}$-ULTraS such that for all $s \in S$ $\sum_{a \in A} \sum_{s' \in S} \Delta_{s,a}(s') \in \{0, 1\}$
RPLTS	$\sim_{\rm PB}$ [Larsen-Skou, 1991] $\sim_{\rm PTr}$ $\sim_{\rm PTe}$	$\sim_{\rm B,\mathcal{M}_{\mathbb{R}_{[0,1]}}}$ $\sim_{\rm Tr,\mathcal{M}_{\mathbb{R}_{[0,1]}}}$ $\sim_{\rm Te,\mathcal{M}^{\rm PM}_{\mathbb{R}_{[0,1]}}}$	functional $\mathbb{R}_{[0,1]}$-ULTraS such that for all $s \in S$ and $a \in A$ $\sum_{s' \in S} \Delta_{s,a}(s') \in \{0, 1\}$
NPLTS	$\sim_{\rm PB,N}$ $\sim_{\rm PTr,N}$ $\sim_{\rm PTe,N}$	$\sim_{\rm B,\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}}$ $\sim_{\rm Tr,\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}}}$ $\sim_{\rm Te,\mathcal{M}^{\rm PM}_{2^{\mathbb{R}_{[0,1]}}}}$	$\mathbb{R}_{[0,1]}$-ULTraS such that for all $s \xrightarrow{a} \Delta$ $\sum_{s' \in S} \Delta(s') = 1$

- Probabilistic measure functions:

$\mathcal{M}_{\mathbb{R}_{[0,1]}}(s,\alpha,S')$	$=$	$\begin{cases} \sum_{s' \in S} \Delta_{s,a}(s') \cdot \mathcal{M}_{\mathbb{R}_{[0,1]}}(s',\alpha',S') & \alpha = a \circ \alpha' \\ 1;0 & \alpha = \varepsilon,\ s \in S'? \end{cases}$
$\mathcal{M}_{2^{\mathbb{R}_{[0,1]}}_{\cdot}}(s,\alpha,S')$	$=$	$\begin{cases} \bigcup_{s \xrightarrow{a} \Delta} \{ \sum_{s' \in S} \Delta(s') \cdot p_{s'} \mid p_{s'} \in \mathcal{M}_{2^{\mathbb{R}_{[0,1]}}_{\cdot}}(s',\alpha',S') \} & \alpha = a \circ \alpha' \\ \{1\};\{0\} & \alpha = \varepsilon,\ s \in S'? \end{cases}$

- Stochastic behavioral equivalences:

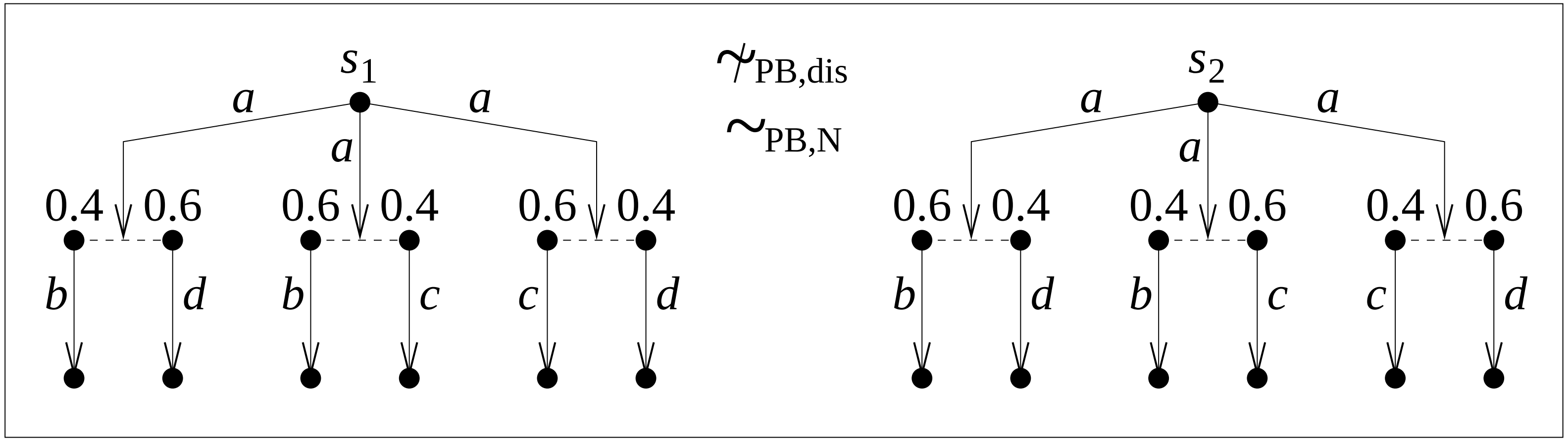
GMLTS	$\sim_{\mathrm{MB}}$ [Hillston, 1996] $\sim_{\mathrm{MTr,ete}}$ $\sim_{\mathrm{MTr,sbs}}$ $\sim_{\mathrm{MTe,ete}}$ $\sim_{\mathrm{MTe,sbs}}$	$\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete}}}$ $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs}}}$ $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete}}}$ $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs}}}$ $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{ete}}}$ $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{sbs}}}$	functional $\mathbb{R}_{\geq 0}$-ULTraS
RMLTS	$\sim_{\mathrm{MB}}$ $\sim_{\mathrm{MTr,ete,R}}$ $\sim_{\mathrm{MTr,sbs,R}}$ $\sim_{\mathrm{MTe,ete,R}}$ $\sim_{\mathrm{MTe,sbs,R}}$	$\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete,R}}}$ $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs,R}}}$ $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete,R}}}$ $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs,R}}}$ $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{ete,R}}}$ $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{sbs,R}}}$	functional $\mathbb{R}_{\geq 0}$-ULTraS
NMLTS	$\sim_{\mathrm{MB,N}}$ $\sim_{\mathrm{MTr,ete,N}}$ $\sim_{\mathrm{MTr,sbs,N}}$ $\sim_{\mathrm{MTe,ete,N}}$ $\sim_{\mathrm{MTe,sbs,N}}$	$\sim_{\mathrm{B},\mathcal{M}_{\mathrm{ete,N}}}$ $\sim_{\mathrm{B},\mathcal{M}_{\mathrm{sbs,N}}}$ $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{ete,N}}}$ $\sim_{\mathrm{Tr},\mathcal{M}_{\mathrm{sbs,N}}}$ $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{ete,N}}}$ $\sim_{\mathrm{Te},\mathcal{M}^{\mathrm{RM}}_{\mathrm{sbs,N}}}$	$\mathbb{R}_{\geq 0}$-ULTraS such that for all $s \xrightarrow{a} \Delta$ $\sum_{s' \in S} \Delta(s') > 0$

- Stochastic measure functions:

$\mathcal{M}_{\text{ete},\text{R}}(s,\alpha,S')(t)$	$= \begin{cases} \int\limits_0^t \mathrm{E}_a(s)\cdot \mathrm{e}^{-\mathrm{E}_a(s)\cdot x}\cdot \sum\limits_{s'\in S}\frac{\Delta_{s,a}(s')}{\mathrm{E}_a(s)}\cdot \mathcal{M}_{\text{ete},\text{R}}(s',\alpha',S')(t-x)\,\mathrm{d}x & \alpha = a\circ\alpha',\ \mathrm{E}_a(s)>0 \\ 1;0 & \alpha=\varepsilon,\ s\in S'? \end{cases}$
$\mathcal{M}_{\text{sbs},\text{R}}(s,\alpha,S')(\theta)$	$= \begin{cases} (1-\mathrm{e}^{-\mathrm{E}_a(s)\cdot t})\cdot \sum\limits_{s'\in S}\frac{\Delta_{s,a}(s')}{\mathrm{E}_a(s)}\cdot \mathcal{M}_{\text{sbs},\text{R}}(s',\alpha',S')(\theta') & \alpha=a\circ\alpha',\ \theta=t\circ\theta',\ \mathrm{E}_a(s)>0 \\ 1;0 & \alpha=\varepsilon,\ s\in S'? \end{cases}$
$\mathcal{M}_{\text{ete},\text{N}}(s,\alpha,S')(t)$	$= \begin{cases} \bigcup\limits_{s\xrightarrow{a}\Delta}\{\int\limits_0^t \Delta(S)\cdot \mathrm{e}^{-\Delta(S)\cdot x}\cdot \sum\limits_{s'\in S}\frac{\Delta(s')}{\Delta(S)}\cdot p_{s'}\,\mathrm{d}x \mid p_{s'}\in \mathcal{M}_{\text{ete},\text{N}}(s',\alpha',S')(t-x)\} & \alpha=a\circ\alpha' \\ \{1\};\{0\} & \alpha=\varepsilon,\ s\in S'? \end{cases}$
$\mathcal{M}_{\text{sbs},\text{N}}(s,\alpha,S')(\theta)$	$= \begin{cases} \bigcup\limits_{s\xrightarrow{a}\Delta}\{(1-\mathrm{e}^{-\Delta(S)\cdot t})\cdot \sum\limits_{s'\in S}\frac{\Delta(s')}{\Delta(S)}\cdot p_{s'} \mid p_{s'}\in \mathcal{M}_{\text{sbs},\text{N}}(s',\alpha',S')(\theta')\} & \alpha=a\circ\alpha',\ \theta=t\circ\theta' \\ \{1\};\{0\} & \alpha=\varepsilon,\ s\in S'? \end{cases}$

New Behavioral Equivalences for NPLTS Models

- Bisimilarity $\sim_{\mathrm{PB,dis}}$ introduced in [Segala-Lynch, 1994].
- An equivalence relation $\mathcal{B}$ over S is a probabilistic group-distribution bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for each $s_1 \xrightarrow{a} \Delta_1$ there exists $s_2 \xrightarrow{a} \Delta_2$ such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $\Delta_1(\bigcup \mathcal{G}) = \Delta_2(\bigcup \mathcal{G})$.
- Very discriminating, not characterized by PML/PCTL.
- Coarsest congruence contained in our $\sim_{\mathrm{PB,N}}$, characterized by PML!
- Obtained by simply anticipating the quantification over $\mathcal{G}$.
- An equivalence relation $\mathcal{B}$ over S is a probabilistic bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that for each $s_1 \xrightarrow{a} \Delta_1$ there exists $s_2 \xrightarrow{a} \Delta_2$ such that $\Delta_1(\bigcup \mathcal{G}) = \Delta_2(\bigcup \mathcal{G})$.


s_1
$\not\sim_{\text{PB,dis}}$
$\sim_{\text{PB,N}}$
s_2
a
a
a
0.4
0.6
0.6
0.4
0.6
0.4
b
d
b
c
c
d
a
a
a
0.6
0.4
0.4
0.6
0.4
0.6
b
d
b
c
c
d

- Trace equivalence $\sim_{\mathrm{PTr,dis}}$ introduced in [Segala, 1995].

- $s_1 \sim_{\mathrm{PTr,dis}} s_2$ iff for each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$ such that for all $\alpha \in A^*$:

$$prob(\mathcal{CC}(z_{s_1}, \alpha)) \; = \; prob(\mathcal{CC}(z_{s_2}, \alpha))$$

  and symmetrically for each $\mathcal{Z}_2 \in Res(s_2)$.

- Very discriminating, not a congruence w.r.t. parallel composition.

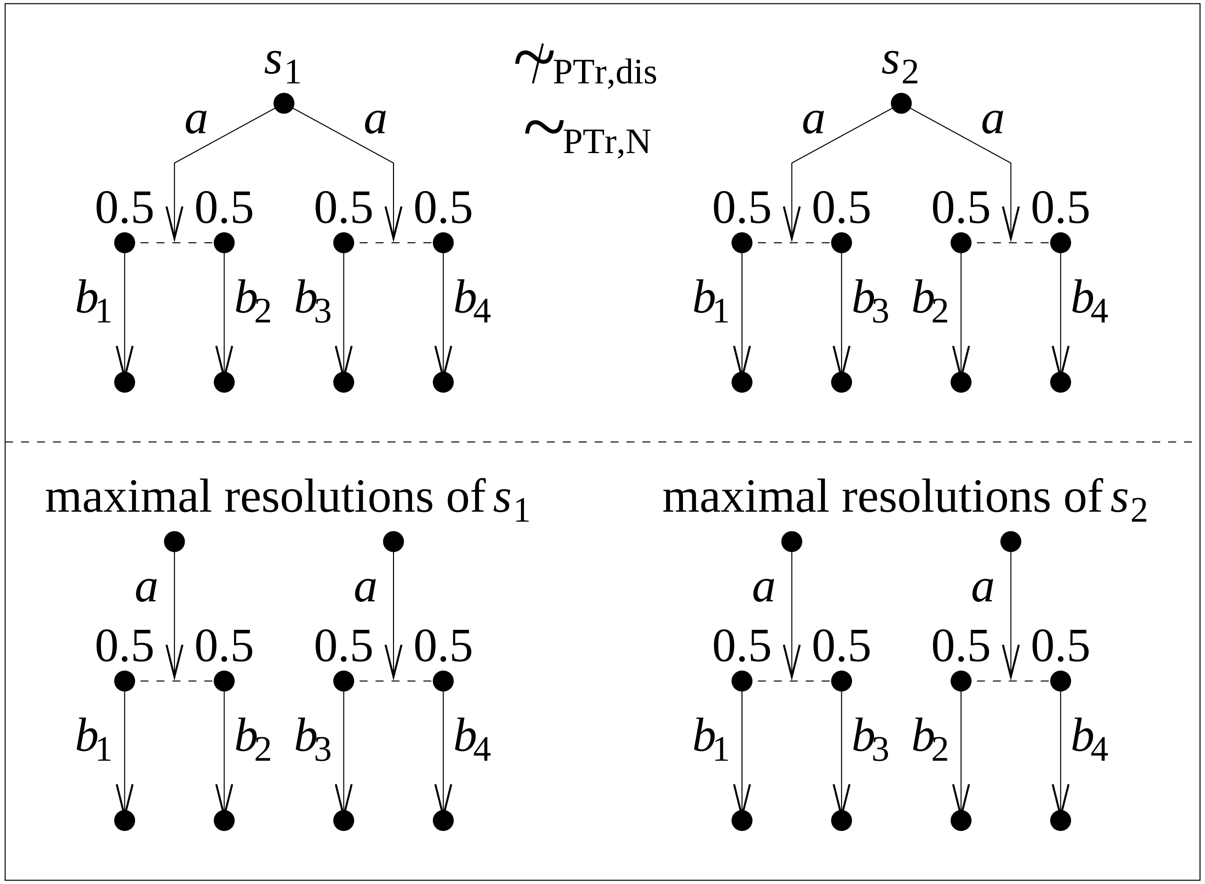
- Our trace equivalence $\sim_{\mathrm{PTr,N}}$ is coarser and compositional!

- Obtained by simply anticipating the quantification over α.

- $s_1 \sim_{\mathrm{PTr,N}} s_2$ iff for all $\alpha \in A^*$ it holds that for each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$ such that:

$$prob(\mathcal{CC}(z_{s_1}, \alpha)) \; = \; prob(\mathcal{CC}(z_{s_2}, \alpha))$$

  and symmetrically for each $\mathcal{Z}_2 \in Res(s_2)$.



s1
≁PTr,dis
∼PTr,N
s2
a
a
a
a
0.5 0.5 0.5 0.5
0.5 0.5 0.5 0.5
b1 b2 b3 b4
b1 b3 b2 b4
maximal resolutions of s1
maximal resolutions of s2

- Testing equivalence $\sim_{\text{PTe-}\sqcup\sqcap}$ of [Yi-Larsen, 1992; Jonsson-Yi, 1995] then revisited in [Segala, 1996; Deng-Van Glabbeek-Hennessy-Morgan, 2008].

- $s_1 \sim_{\text{PTe-}\sqcup\sqcap} s_2$ iff for every $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$:

$$\bigsqcup_{\mathcal{Z}_1 \in Res_{\max}(s_1, o)} prob(\mathcal{SC}(z_{s_1,o})) = \bigsqcup_{\mathcal{Z}_2 \in Res_{\max}(s_2, o)} prob(\mathcal{SC}(z_{s_2,o}))$$

$$\bigsqcap_{\mathcal{Z}_1 \in Res_{\max}(s_1, o)} prob(\mathcal{SC}(z_{s_1,o})) = \bigsqcap_{\mathcal{Z}_2 \in Res_{\max}(s_2, o)} prob(\mathcal{SC}(z_{s_2,o}))$$

- Very discriminating, not fully compatible with the classical one.

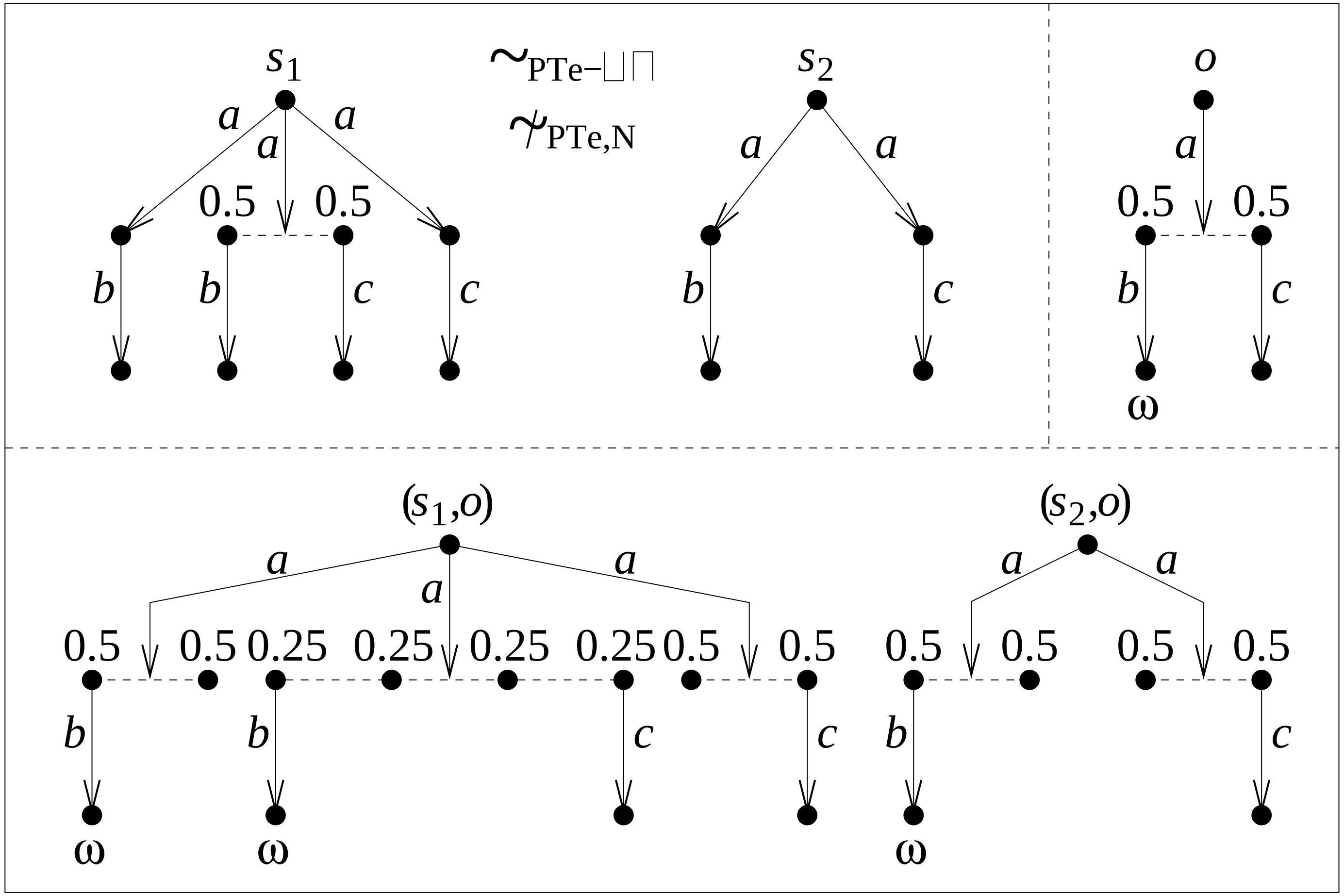
- Our testing equiv. $\sim_{\text{PTe,N}}$ is fully compatible with the classical one!

- Considering success probabilities in a trace-by-trace fashion.

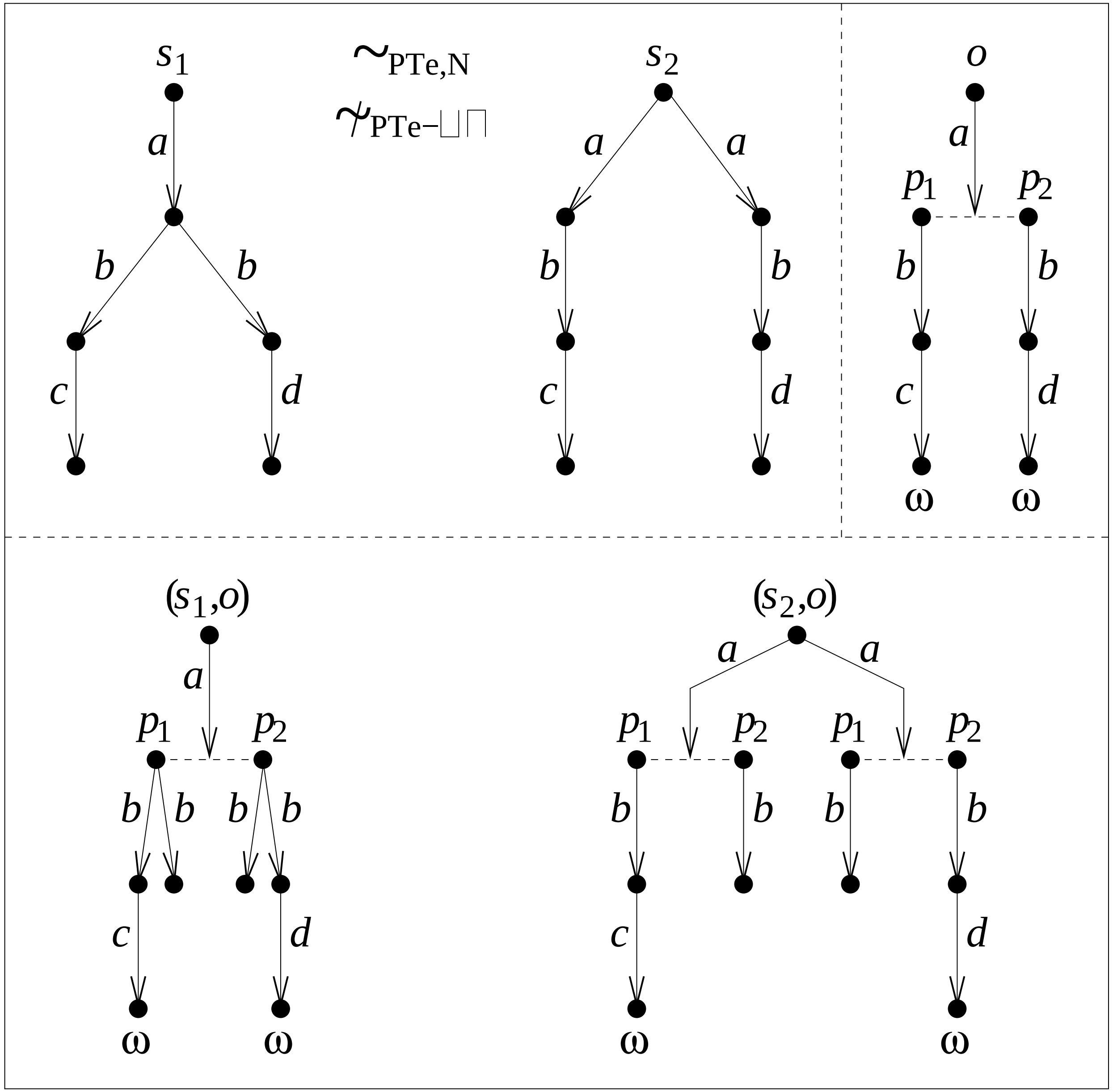
- $s_1 \sim_{\text{PTe,N}} s_2$ iff for every $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$ and for all $\alpha \in A^*$ it holds that for each $\mathcal{Z}_1 \in Res_{\max,\mathcal{C},\alpha}(s_1, o)$ there exists $\mathcal{Z}_2 \in Res_{\max,\mathcal{C},\alpha}(s_2, o)$ such that:

$$prob(\mathcal{SCC}(z_{s_1,o}, \alpha)) = prob(\mathcal{SCC}(z_{s_2,o}, \alpha))$$

  and symmetrically for each $\mathcal{Z}_2 \in Res_{\max,\mathcal{C},\alpha}(s_2, o)$.



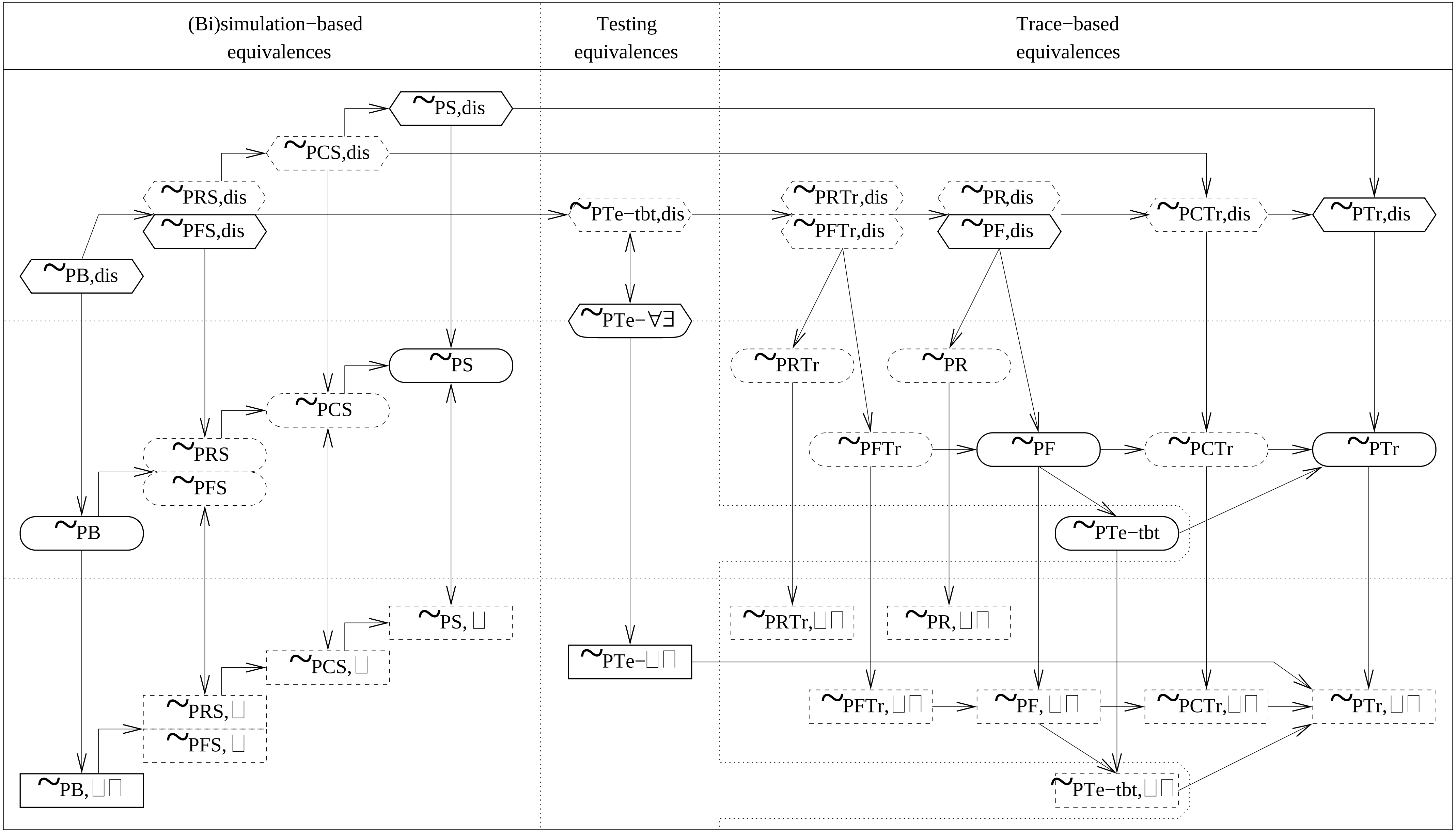
s1
∼PTe-⊔⊓
≁PTe,N
s2
o
a
0.5
b
c
ω
(s1,o)
(s2,o)
0.25



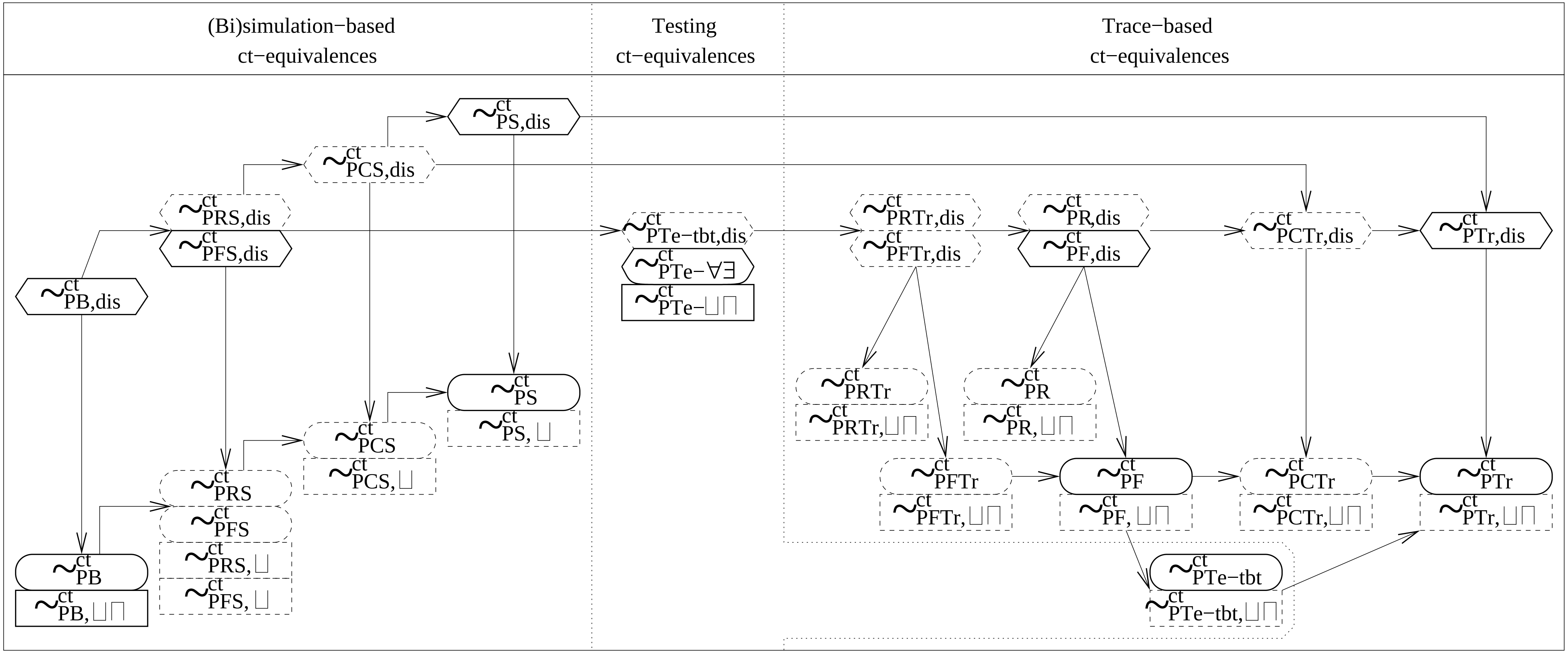
s_1
$\sim_{\mathrm{PTe,N}}$
$\not\sim_{\mathrm{PTe}-\sqcup\sqcap}$
s_2
o
a
b
c
d
p_1
p_2
ω
(s_1,o)
(s_2,o)

Spectrum of NPLTS Behavioral Equivalences

- Spectrum of LTS behavioral equivalences in [Van Glabbeek, 1990] and of GPLTS behavioral equivalences in [Jou-Smolka, 1990]: one fragment.
- Three different fragments in the spectrum for NPLTS models:
  - $\sim_{\rm PB,dis}$ and $\sim_{\rm PTr,dis}$ require fully matching resolutions: for every trace, the probability of performing that trace must be the same in both resolutions, which thus possess the same trace distribution.
  - $\sim_{\rm PB}$ ($\sim_{\rm PB,N}$) and $\sim_{\rm PTr}$ ($\sim_{\rm PTr,N}$) use partially matching resolutions: a resolution on one side is allowed to be matched by different resolutions on the other side with respect to different traces.
  - $\sim_{\rm PTe\text{-}\sqcup\sqcap}$ considers only extremal probabilities over all resolutions.
- Deterministic schedulers vs. randomized schedulers.



(Bi)simulation−based equivalences
Testing equivalences
Trace−based equivalences
~PS,dis
~PCS,dis
~PRS,dis
~PFS,dis
~PB,dis
~PTe−tbt,dis
~PRTr,dis
~PFTr,dis
~PR,dis
~PF,dis
~PCTr,dis
~PTr,dis
~PTe−∀∃
~PS
~PCS
~PRS
~PFS
~PB
~PRTr
~PR
~PFTr
~PF
~PCTr
~PTr
~PTe−tbt
~PS,⊔
~PCS,⊔
~PRS,⊔
~PFS,⊔
~PB,⊔⊓
~PTe−⊔⊓
~PRTr,⊔⊓
~PR,⊔⊓
~PFTr,⊔⊓
~PF,⊔⊓
~PCTr,⊔⊓
~PTr,⊔⊓
~PTe−tbt,⊔⊓


(Bi)simulation−based ct−equivalences
Testing ct−equivalences
Trace−based ct−equivalences
$\sim^{ct}_{PS,dis}$
$\sim^{ct}_{PCS,dis}$
$\sim^{ct}_{PRS,dis}$
$\sim^{ct}_{PFS,dis}$
$\sim^{ct}_{PB,dis}$
$\sim^{ct}_{PS}$
$\sim^{ct}_{PS,\sqcup}$
$\sim^{ct}_{PCS}$
$\sim^{ct}_{PCS,\sqcup}$
$\sim^{ct}_{PRS}$
$\sim^{ct}_{PFS}$
$\sim^{ct}_{PRS,\sqcup}$
$\sim^{ct}_{PFS,\sqcup}$
$\sim^{ct}_{PB}$
$\sim^{ct}_{PB,\sqcup\sqcap}$
$\sim^{ct}_{PTe-tbt,dis}$
$\sim^{ct}_{PTe-\forall\exists}$
$\sim^{ct}_{PTe-\sqcup\sqcap}$
$\sim^{ct}_{PRTr,dis}$
$\sim^{ct}_{PFTr,dis}$
$\sim^{ct}_{PR,dis}$
$\sim^{ct}_{PF,dis}$
$\sim^{ct}_{PCTr,dis}$
$\sim^{ct}_{PTr,dis}$
$\sim^{ct}_{PRTr}$
$\sim^{ct}_{PRTr,\sqcup\sqcap}$
$\sim^{ct}_{PR}$
$\sim^{ct}_{PR,\sqcup\sqcap}$
$\sim^{ct}_{PFTr}$
$\sim^{ct}_{PFTr,\sqcup\sqcap}$
$\sim^{ct}_{PF}$
$\sim^{ct}_{PF,\sqcup\sqcap}$
$\sim^{ct}_{PCTr}$
$\sim^{ct}_{PCTr,\sqcup\sqcap}$
$\sim^{ct}_{PTr}$
$\sim^{ct}_{PTr,\sqcup\sqcap}$
$\sim^{ct}_{PTe-tbt}$
$\sim^{ct}_{PTe-tbt,\sqcup\sqcap}$

Future Work

- Defining an ULTRAS-based operational semantics of process calculi of nondeterministic, probabilistic, stochastic, timed, or mixed nature, for investigating their relative expressiveness.

- Studying a generic process algebra together with uniform results for congruence properties & equational/logical characterizations, as well as uniform algorithms for equivalence checking and model checking.

- Providing uniform definitions of weak behavioral equivalences.

- Extension of ULTRAS with transitions of the form $\Delta \xrightarrow{a} \Delta'$:
  - State distributions describing alternatives among global states: Kleisli lifting of state-to-state-distribution reachability relations.
  - State distributions describing combinations of local states: Petri nets as $\mathbb{N}$-ULTRAS models in which states are Petri net places and transitions are Petri net transitions.