# Towards General Axiomatizations for Bisimilarity and Trace Semantics 

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#### Abstract

We study general equational characterizations for bisimulation and trace semantics via the respective post-/pre-metaequivalences defined on the ULTraS metamodel. This yields axiomatizations encompassing those appeared in the literature, as well as new ones, for bisimulation and trace equivalences when applied to specific classes of processes. The equational laws are developed incrementally, by starting with some core axioms and then singling out additional axioms for bisimulation post-/pre-metaequivalences on the one hand, and different additional axioms for trace post-/pre-metaequivalences on the other hand. The axiomatizations highlight the fundamental differences in the discriminating power between bisimulation semantics and trace semantics, regardless of specific classes of processes. Moreover, they generalize idempotency laws of bisimilarity and choice-deferring laws of trace semantics, in addition to formalizing shuffling laws for pre-metaequivalences.


## 1 Introduction

Process calculi [6] constitute a foundational algebraic tool for the specification and verification of concurrent, distributed, and mobile systems. Their syntax includes operators for expressing concepts such as sequential/alternative/parallel composition, action hiding/restriction/renaming, and recursion. Their semantics is typically formalized via structural operational rules associating a labeled transition system with each process term. Many behavioral equivalences have been proposed to identify syntactically different process terms on the basis of observational criteria. Sound and complete axiomatizations have been developed to emphasize the equational laws on which the equivalences rely, which can then be exploited for the algebraic manipulation of process terms.

These axiomatizations are usually provided for a specific class of processes (e.g., nondeterministic, probabilistic, or timed), while we are interested in investigating general equational laws that are valid for multiple classes of processes. This can be accomplished by working with behavioral metamodels. On the one hand, they act as unifying theories by underpinning a deeper understanding of specific models through a uniform view of the models themselves. On the other hand, they support the study of metaresults, i.e., results that are valid for all the specific models that are embodied. Frameworks like operational semantic rule formats [2]19], Segala probabilistic automata [40, and weighted automata [23] can be viewed to some extent as behavioral metamodels, even
though their emphasis is more on ensuring certain properties in a general setting or achieving a higher expressivity. More recently, behavioral metamodels such as WLTS - weighted labeled transition systems [31], FUTS - state-to-function labeled transition systems [20|34], and ULTraS - uniform labeled transition systems 101517 , have been developed with the explicit purpose of paving the way to unifying theories.

In this paper, we investigate equational characterization metaresults for the two endpoints of the branching-time - linear-time spectrum [48, i.e., bisimulation semantics and trace semantics, through the corresponding behavioral post-/pre-metaequivalences defined on ULTRAS. In this metamodel - which encompasses a wide gamut of behavioral models ranging from nondeterministic transition systems to action-labeled Markov chains and several variants of automata with probability and time - every action-labeled transition goes from a state to a reachability distribution over states. We make use of this metamodel because from its inception it has been equipped with several behavioral relations. In particular, it has inspired some new relations in the probabilistic setting, whose properties have been analyzed in [13|11] , and has been used in [7] to study compositionality metaresults for bisimulation and trace semantics.

The equational characterization metaresults are developed incrementally on an ULTraS-based process calculus named UProC, which contains only dynamic process operators such as action prefix and alternative composition as well as operators on state reachability distributions. This calculus provides the minimum set of operators that are necessary to highlight the fundamental differences in the discriminating power of the considered metaequivalences.

We start with some core axioms establishing associativity and commutativity of alternative composition and distribution composition, together with the existence of a neutral element for alternative composition. Then we single out additional axioms for the various semantics, by considering for each of them a post-metaequivalence and a pre-metaequivalence differring for the way in which resolutions of nondeterminism have to match each other, for a total of two bisimulation metaequivalences $\sim_{B}^{\text {post }}$ and $\sim_{B}^{\text {pre }}$ plus two trace metaequivalences $\sim_{T}^{\text {post }}$ and $\sim_{T}^{\text {pre }}$. Here is a summary of our contributions and their relationships with previous axiomatizations:

- The equational characterization of $\sim_{\mathrm{B}}^{\text {post }}$ relying on idempotency axioms for alternative and reachability distribution compositions is the expected one, generalizes the well known ones of $36|29| 4|27| 26$ related to various classes of specific processes, and is in agreement with the coalgebraic one of 43].
- The equational characterization of $\sim_{\mathrm{B}}^{\text {pre }}$ relying on a $B$-shuffing axiom is new and yields the first axiomatization for the bisimilarities over nondeterministic and probabilistic processes of [13|45] characterized by the probabilistic modal and temporal logics of $33 \mid 25$.
- The equational characterization of $\sim_{\mathrm{T}}^{\text {post }}$ relying on choice-deferring axioms generalizes the well known ones of [18/39] for nondeterministic processes, is in agreement with some of the axioms of the coalgebraic ones of [17|44], and provides the first axiomatization for the probabilistic trace equivalences
of 2941] given that the axiomatization in [37] holds for the simulation-like coarsest congruence [35] contained in the equivalence of 41].
- The equational characterization of $\sim_{\mathrm{T}}^{\text {pre }}$ relying on a T-shuffing axiom is new and opens the way to the first axiomatization of the compositional trace semantics over nondeterministic and probabilistic processes of [11].

The proof of completeness of the equational characterizations of $\sim_{B}^{\text {post }}$ and $\sim_{B}^{\text {pre }}$ is based on a preliminary reduction of process terms into sum normal form [36]. In contrast, for $\sim_{T}^{\text {post }}$ the proof employs the technique of [5/48] by using the choice-deferring axioms as graph rewriting rules to transform the completeness problem for $\sim_{T}^{\text {post }}$ over arbitrary process terms into the completeness problem for $\sim_{B}^{\text {post }}$ over process terms in a sum normal form specific to $\sim_{T}^{\text {post }}$. The completeness problem for $\sim_{\mathrm{T}}^{\mathrm{pre}}$ is still open.

As mentioned above, our general axioms feature as instances the laws of bisimulation and trace semantics known from the literature for nondeterministic, probabilistic, or stochastic processes. Moreover, axioms for pre-metaequivalences have never been investigated before; similarly, the instances of those for trace post-metaequivalence were not known for certain classes of processes. To the best of our knowledge, this is the first work in which a concrete behavioral metamodel is employed in place of category theory - whose mathematics may be perceived as highly complex by researchers not familiar with it - to develop equational characterizations that are valid regardless of specific classes of processes.

This paper is organized as follows. In Sect. 2, we recall the ULTraS metamodel together with its bisimulation and trace metaequivalences revisited according to [89]. In Sect. 3, we present the ULTraS-inspired process calculus UPROC and show the full compositionality of the metaequivalences with respect to the selected operators thanks to the aforementioned revisitation. In Sect. 4, we incrementally develop equational characterizations over UPRoC for the considered behavioral metaequivalences and discuss their relationships and limits. Finally, in Sect. 5 we provide some concluding remarks.

## 2 Background

We recall from [7] the ULTRAS metamodel (Sect. 2.1), reachability-consistent semirings (Sect. 2.2), bisimulation metaequivalences (Sect 2.3), resolutions of nondeterminism (Sect. 2.4), reachability measures (Sect. 2.5), and a revisitation of trace metaequivalences based on [89] (Sect 2.6).

### 2.1 The ULTraS Metamodel

ULTraS is a discrete state-transition metamodel parameterized with respect to a set $D$, where $D$-values are interpreted as degrees of one-step reachability. These values are assumed to be ordered according to a reflexive and transitive relation $\sqsubseteq_{D}$, which is equipped with minimum $\perp_{D}$ expressing unreachability. Let us denote by $(S \rightarrow D)$ the set of functions from a set $S$ to $D$. When $S$ is a set
of states, every element $\Delta$ of $(S \rightarrow D)$ can be interpreted as a function that distributes reachability over all possible next states. We call support of $\Delta$ the set of states $\operatorname{supp}(\Delta)=\left\{s \in S \mid \Delta(s) \neq \perp_{D}\right\}$ that are reachable according to $\Delta$.

To represent transition targets, we use the set $(S \rightarrow D)_{\text {nefs }}$ of $D$-distributions $\Delta$ over $S$ with nonempty and $f$ inite support, i.e., satisfying $0<|\operatorname{supp}(\Delta)|<\omega$. The lower bound avoids distributions always returning $\perp_{D}$ and hence transitions leading to nowhere. The upper bound will enable a correct definition of reachability measures for trace metaequivalences in Sect. 2.5 .

Definition 1. Let $\left(D, \sqsubseteq_{D}, \perp_{D}\right)$ be a preordered set with minimum. A uniform labeled transition system on it, or $D$-ULTraS, is a triple $\mathcal{U}=(S, A, \longrightarrow)$ where $S \neq \emptyset$ is an at most countable set of states, $A \neq \emptyset$ is a countable set of transitionlabeling actions, and $\longrightarrow \subseteq S \times A \times(S \rightarrow D)_{\text {nefs }}$ is a transition relation.

Every transition $(s, a, \Delta)$ of $\mathcal{U}$ is written $s \xrightarrow{a} \Delta$, where $\Delta\left(s^{\prime}\right)$ is a $D$-value quantifying the degree of reachability of $s^{\prime}$ from $s$ via that $a$-transition, with $\Delta\left(s^{\prime}\right)=\perp_{D}$ meaning that $s^{\prime}$ is not reachable with that transition. In the directed graph description of $\mathcal{U}$ (see, e.g., the forthcoming Fig. (1), vertices represent states and action-labeled edges represent action-labeled transitions. Given a transition $s \xrightarrow{a} \Delta$, the corresponding $a$-labeled edge goes from the vertex representing state $s$ to a set of vertices linked by a dashed line, each of which represents a state $s^{\prime} \in \operatorname{supp}(\Delta)$ and is labeled with $\Delta\left(s^{\prime}\right)$.

Example 1. As shown in 1015$] 7$, we can use the set $\mathbb{B}=\{\perp, \top\}$ with $\perp \sqsubseteq_{\mathbb{B}}$ $T$ for capturing labeled transition systems [30] and timed automata [3], the set $\mathbb{R}_{[0,1]}$ with the usual $\leq$ for capturing action-labeled discrete-time Markov chains [46, Markov decision processes [22], probabilistic automata [40], probabilistic timed automata [32, and Markov automata [24], and the set $\mathbb{R}_{\geq 0}$ with the usual $\leq$ for capturing action-labeled continuous-time Markov chains [46] and continuous-time Markov decision processes 38.

### 2.2 Reachability-Consistent Semirings

To express the calculations needed by behavioral metaequivalences, we further assume that $D$ has a commutative semiring structure. This means that $D$ is equipped with two binary operations $\oplus$ and $\otimes$, with the latter distributing over the former, which satisfy the following properties:

- $\otimes$ is associative and commutative and admits neutral element $1_{D}$ and absorbing element $0_{D}$. This multiplicative operation enables the combination of $D$-values of consecutive single-step reachability along the same computation.
$-\oplus$ is associative and commutative and admits neutral element $0_{D}$. This additive operation is useful for aggregating $D$-values of different computations starting from the same state, as well as for shorthands like $\Delta\left(S^{\prime}\right)=$ $\bigoplus_{s^{\prime} \in S^{\prime}} \Delta\left(s^{\prime}\right)$ given $s \xrightarrow{a} \Delta$.

We also assume that these two binary operations are reachability consistent, in the sense that they satisfy the following additional properties in accordance with the intuition behind the concept of reachability:
$-0_{D}=\perp_{D}$, i.e., the zero of the semiring denotes unreachability.
$-d_{1} \otimes d_{2} \neq 0_{D}$ if $d_{1} \neq 0_{D} \neq d_{2}$, hence as expected two consecutive steps cannot result in unreachability.

- The sum via $\oplus$ of finitely many values $1_{D}$ is always different from $0_{D}$ (known as characteristic zero). It ensures that two nonzero values sum up to zero only if they are one the inverse of the other with respect to $\oplus$, thus avoiding inappropriate zero results when aggregating $D$-values of distinct computations departing from the same state.

Example 2. As shown in [7, we can use the reachability-consistent semirings $(\mathbb{B}, \vee, \wedge, \perp, \top)$ for nondeterministic models and $\left(\mathbb{R}_{\geq 0},+, \times, 0,1\right)$ for probabilistic and stochastic models, as well as for their respective behavioral equivalences. In contrast, characteristic zero rules out all semirings $\left(\mathbb{N}_{n},+_{n}, \times_{n}, 0,1\right)$ of the classes of natural numbers that are congruent modulo $n \in \mathbb{N}_{\geq 2}$.

### 2.3 Bisimulation Post-/Pre-Metaequivalences

For bisimulation semantics we have two different variants of metaequivalence in the ULTraS setting, $\sim_{B}^{\text {post }}$ and $\sim_{B}^{\text {pre }}$. They are both defined in the style of [33], which requires bisimulations to be equivalence relations, but deal with sets of equivalence classes, rather than only with individual classes, to avoid an undesirable decrease of the discriminating power of $\sim_{B}^{\text {pre }}$ in certain circumstances. The difference between the two variants lies in the position - underlined in the definition below - of the universal quantification over sets of equivalence classes.

In the first case, which is the approach of 42, the quantification occurs after selecting a transition from either considered state, hence for each class set the transition of the challenger state and the transition of the defender state must reach that set with the same degree (fully matching transitions). In the second case, inspired by [4745|13], the quantification occurs before selecting transitions, so that a transition of the challenger can be matched by different transitions of the defender with respect to different class sets (partially matching transitions). In the definition below, given an equivalence relation $\mathcal{B}$ over a state space $S$ together with a set of equivalence classes $\mathcal{G} \in 2^{S / \mathcal{B}}, \bigcup \mathcal{G} \subseteq S$ denotes the union of all the equivalence classes in $\mathcal{G}$.

Definition 2. Let $\left(D, \oplus, \otimes, 0_{D}, 1_{D}\right)$ be a reachability-consistent semiring, $\mathcal{U}=$ $(S, A, \longrightarrow)$ be a D-ULTRAS, and $s_{1}, s_{2} \in S$ :
$-s_{1} \sim_{\mathrm{B}}^{\text {post }} s_{2}$ iff there exists a post-bisimulation $\mathcal{B}$ over $S$ such that $\left(s_{1}, s_{2}\right) \in$ $\mathcal{B}$. An equivalence relation $\mathcal{B}$ over $S$ is a post-bisimulation iff, whenever $\left(s_{1}, s_{2}\right) \in \mathcal{B}$, then for all $a \in A$ it holds that for each $s_{1} \xrightarrow{a} \Delta_{1}$ there exists $s_{2} \xrightarrow{a} \Delta_{2}$ such that for all $\mathcal{G} \in 2^{S / \mathcal{B}}$ :

$$
\Delta_{1}(\bigcup \mathcal{G})=\Delta_{2}(\bigcup \mathcal{G})
$$



Fig. 1. Difference between bisimulation metaequivalences: $s_{1} \chi_{\mathrm{B}}^{\text {post }} s_{2}, s_{1} \sim_{\mathrm{B}}^{\text {pre }} s_{2}$
$-s_{1} \sim_{\mathrm{B}}^{\text {pre }} s_{2}$ iff there exists a pre-bisimulation $\mathcal{B}$ over $S$ such that $\left(s_{1}, s_{2}\right) \in$ $\mathcal{B}$. An equivalence relation $\mathcal{B}$ over $S$ is a pre-bisimulation iff, whenever $\left(s_{1}, s_{2}\right) \in \mathcal{B}$, then for all $a \in A$ and for all $\mathcal{G} \in 2^{S / \mathcal{B}}$ it holds that for each $s_{1} \xrightarrow{a} \Delta_{1}$ there exists $s_{2} \xrightarrow{a} \Delta_{2}$ such that:

$$
\Delta_{1}(\bigcup \mathcal{G})=\Delta_{2}(\bigcup \mathcal{G})
$$

The difference between the two bisimulation metaequivalences emerges in the presence of internal nondeterminism, i.e., identically labeled transitions departing from the same state. Consider the two $D$-ULTraS models in Fig. 1, which feature the same distinct $D$-values $d_{1}$ and $d_{2}$ as well as the same inequivalent continuations given by the $D$-ULTraS submodels rooted at $r_{1}, r_{2}, r_{3}$. Notice that both the $D$-values and the continuations are shuffled within each model, while only the $D$-values are shuffled across the two models too. It holds that $s_{1} \mathcal{\chi}_{\mathrm{B}}^{\text {post }} s_{2}$ because, e.g., the leftmost $a$-transition of $s_{1}$ is not matched by any of the three $a$-transitions of $s_{2}$. In contrast, we have that $s_{1} \sim_{B}^{\text {pre }} s_{2}$. For instance, the leftmost $a$-transition of $s_{1}$ is matched by the central (resp. rightmost) $a$ transition of $s_{2}$ with respect to the equivalence class of $r_{1}$ (resp. $r_{2}$ ), and by the leftmost $a$-transition of $s_{2}$ with respect to the union of the equivalence classes of $r_{1}$ and $r_{2}$ - see the dashed arrow-headed lines at the bottom of Fig. 1.

### 2.4 Resolutions of Nondeterminism

When several transitions depart from the same state, they describe a nondeterministic choice among different behaviors. While in the case of bisimulation semantics nondeterminism is solved stepwise, for trace semantics overall resolutions of nondeterminism have to be made explicit.

A resolution of a state $s$ of a $D$-ULTraS $\mathcal{U}$ is the result of a possible way of resolving nondeterministic choices starting from $s$, as if a deterministic scheduler were applied that, at the current state $s^{\prime}$, selects one of the outgoing transitions of $s^{\prime}$, or no transitions at all thus stopping the execution. The applicability of other classes of schedulers, like randomized [40] and interpolating 21] ones and


Fig. 2. Lack of bijectivity breaks structure preservation on the resolution side
combinations thereof [16], may depend on the specific $D$, hence we will not consider them here.

We formalize a resolution of $s$ as a $D$-ULTraS $\mathcal{Z}$ with a tree-like structure, whose branching points correspond to target distributions of transitions. It is obtained by unfolding from $s$ the graph structure of $\mathcal{U}$ and by selecting at each reached state $s^{\prime}$ at most one of its outgoing transitions, hence it is isomorphic to a submodel of the unfolding of the original model. Following [28], we make use of a correspondence function from the acyclic state space of $\mathcal{Z}$ to the original state space of $\mathcal{U}$. For each transition $z \xrightarrow{a} \mathcal{Z} \Delta$ in $\mathcal{Z}$, all the states in $\operatorname{supp}(\Delta)$ must preserve the reachability degrees of the corresponding states in the support of the target of the corresponding transition in $\mathcal{U}$.

Extending [14], this function must be bijective between $\operatorname{supp}(\Delta)$ and the support of the target distribution of the corresponding transition in $\mathcal{U}$. Requiring injectivity as in [7] ensures submodel isomorphism, whereas surjectivity additionally guarantees that $\Delta$ preserves the overall reachability of the target distribution of the corresponding transition in $\mathcal{U}$ (unlike number 1 in the probabilistic case, in general there is no predefined value for the total reachability of a target distribution). For instance, in Fig. 2 the association of the same value $d$ to $s_{1}^{\prime}$ and $s_{2}^{\prime}$ allows for a function that maps $z$ to $s, z_{1}^{\prime}$ and $z_{2}^{\prime}$ to $s_{1}^{\prime}$, and $z_{1}^{\prime \prime}$ and $z_{2}^{\prime \prime}$ to $s_{1}^{\prime \prime}$, which is not injective and would cause the central ULTRAS to be considered a legal resolution of the leftmost ULTRAS although the former is not isomorphic to any submodel of the latter. The situation is similar for the rightmost ULTraS under the function that maps $\bar{z}$ to $s, \bar{z}^{\prime}$ to $s_{1}^{\prime}$, and $\bar{z}^{\prime \prime}$ to $s_{1}^{\prime \prime}$, which is not surjective.

Definition 3. Let $\mathcal{U}=(S, A, \longrightarrow \mathcal{U})$ be a $D$-ULTraS and $s \in S$. An acyclic $D$-ULTraS $\mathcal{Z}=(Z, A, \longrightarrow \mathcal{Z})$ is a resolution of $s$, written $\mathcal{Z} \in \operatorname{Res}(s)$, iff there exists a correspondence function $\operatorname{corr}_{\mathcal{Z}}: Z \rightarrow S$ such that $s=\operatorname{corr}_{\mathcal{Z}}\left(z_{s}\right)$, for some $z_{s} \in Z$ acting as the initial state of $\mathcal{Z}$, and for all $z \in Z$ it holds that:

- If $z \xrightarrow{a}{ }_{\mathcal{Z}} \Delta$ then $\operatorname{corr}_{\mathcal{Z}}(z) \xrightarrow{a} \mathcal{U} \Gamma$, with $\operatorname{corr}_{\mathcal{Z}}$ being bijective between $\operatorname{supp}(\Delta)$ and $\operatorname{supp}(\Gamma)$ and $\Delta\left(z^{\prime}\right)=\Gamma\left(\operatorname{corr}_{\mathcal{Z}}\left(z^{\prime}\right)\right)$ for all $z^{\prime} \in \operatorname{supp}(\Delta)$.
- At most one transition departs from $z$.


### 2.5 Reachability Measures

The definition of trace metaequivalences requires the measurement of multistep reachability, i.e., the degree of reachability of a given set of states from a given
state when executing a sequence of transitions labeled with a certain sequence of actions. We therefore provide a notion of measure schema for a D-ULTraS $\mathcal{U}$ as a set of homogeneously defined measure functions, one for each resolution $\mathcal{Z}$ of $\mathcal{U}$. In the following, we denote by $A^{*}$ the set of finite traces over an action set $A$, by $\varepsilon$ the empty trace, and by $|\alpha|$ the length of a trace $\alpha \in A^{*}$.

Definition 4. Let $\left(D, \oplus, \otimes, 0_{D}, 1_{D}\right)$ be a reachability-consistent semiring and $\mathcal{U}=(S, A, \longrightarrow \mathcal{U})$ be a $D$-ULTraS. A $D$-measure schema $\mathcal{M}$ for $\mathcal{U}$ is a set of measure functions of the form $\mathcal{M}_{\mathcal{Z}}: Z \times A^{*} \times 2^{Z} \rightarrow D$, one for each $\mathcal{Z}=$ $\left(Z, A, \longrightarrow_{\mathcal{Z}}\right) \in \operatorname{Res}(s)$ and $s \in S$, which are inductively defined on the length of their second argument as follows:
$\mathcal{M}_{\mathcal{Z}}\left(z, \alpha, Z^{\prime}\right)= \begin{cases}\bigoplus_{z^{\prime} \in \operatorname{supp}(\Delta)} \Delta\left(z^{\prime}\right) \otimes \mathcal{M}_{\mathcal{Z}}\left(z^{\prime}, \alpha^{\prime}, Z^{\prime}\right) & \text { if } \alpha=\text { a } \alpha^{\prime} \text { and } z \xrightarrow{a}{ }_{\mathcal{Z}} \Delta \\ 1_{D} & \text { if } \alpha=\varepsilon \text { and } z \in Z^{\prime} \\ 0_{D} & \text { otherwise }\end{cases}$
In the first clause, the value of $\mathcal{M}_{\mathcal{Z}}\left(z, \alpha, Z^{\prime}\right)$ is built as a sum of products of $D$-values - a formal power series in the semiring terminology - with the summation being well defined because $\operatorname{supp}(\Delta)$ is finite as established in Def. 1 . For simplicity, we will often indicate with $\mathcal{M}$ both the measure schema and any of its measure functions $\mathcal{M}_{\mathcal{Z}}$, using $\mathcal{M}_{\text {nd }}$ when the reachability-consistent semiring is $(\mathbb{B}, \vee, \wedge, \perp, \top)$ and $\mathcal{M}_{\mathrm{pb}}$ when it is $\left(\mathbb{R}_{\geq 0},+, \times, 0,1\right)$ as in [7].

### 2.6 Coherency-Based Trace Post-/Pre-Metaequivalences

Also for trace semantics we have two distinct metaequivalence variants in the ULTRAS framework, $\sim_{T}^{\text {post }}$ and $\sim_{T}^{\text {pre }}$, with the difference being the position of the universal quantification over traces. In the first case, which is the approach of [41], the quantification occurs after selecting resolutions, hence for each trace the resolution of the challenger and the resolution of the defender must execute that trace with the same degree (fully matching resolutions). In the second case, inspired by [11], the quantification occurs before selecting resolutions, so that a resolution of the challenger can be matched by different resolutions of the defender with respect to different traces (partially matching resolutions).

Trace metaequivalences tend to be overdiscriminating because of the freedom of schedulers of making different decisions in states enabling the same actions. To avoid this, we limit the excessive power of schedulers by restricting them to yield coherent resolutions. Intuitively, this means that, if several states in the support of the target distribution of a transition are equivalent, then the decisions made by the scheduler in those states have to be coherent with each other, so that the states to which they correspond in any resolution are equivalent too.

Coherent resolutions, introduced in [8] for nondeterministic and probabilistic processes, are extended to ULTraS in the following. They rely on coherent trace distributions, which are suitable families of sets of traces weighted with their execution degrees in a given resolution, built through the operations below.

Definition 5. Let $A \neq \emptyset$ be a countable set and $\left(D, \oplus, \otimes, 0_{D}, 1_{D}\right)$ a reachabilityconsistent semiring. For $a \in A, d \in D, T D \subseteq 2^{A^{*} \times D}$, and $T \subseteq A^{*} \times D$ we define:

$$
a \cdot T D=\{a . T \mid T \in T D\} \quad a . T=\left\{\left(a \alpha, d^{\prime}\right) \mid\left(\alpha, \overline{d^{\prime}}\right) \in T\right\}
$$

$d \otimes T D=\{d \otimes T \mid T \in T D\} \quad d \otimes T=\left\{\left(\alpha, d \otimes d^{\prime}\right) \mid\left(\alpha, d^{\prime}\right) \in T\right\}$
$\operatorname{tr}(T D)=\{\operatorname{tr}(T) \mid T \in T D\} \quad \operatorname{tr}(T)=\left\{\alpha \in A^{*} \mid \exists d^{\prime} \in D .\left(\alpha, d^{\prime}\right) \in T\right\}$
while for $T D_{1}, T D_{2} \subseteq 2^{A^{*} \times D}$ we define:
$T D_{1} \oplus T D_{2}= \begin{cases}\left\{T_{1} \oplus T_{2} \mid T_{1} \in T D_{1} \wedge T_{2} \in T D_{2} \wedge \operatorname{tr}\left(T_{1}\right)=\operatorname{tr}\left(T_{2}\right)\right\} \\ \left\{T_{1} \oplus T_{2} \mid T_{1} \in T D_{1} \wedge T_{2} \in T D_{2}\right\} & \text { if } \operatorname{tr}\left(T D_{1}\right)=\operatorname{tr}\left(T D_{2}\right)\end{cases}$
where for $T_{1}, T_{2} \subseteq A^{*} \times D$ we define:

$$
\begin{aligned}
T_{1} \oplus T_{2}= & \left\{\left(\alpha, d_{1} \oplus d_{2}\right) \mid\left(\alpha, d_{1}\right) \in T_{1} \wedge\left(\alpha, d_{2}\right) \in T_{2}\right\} \cup \\
& \left\{(\alpha, d) \in T_{1} \cup T_{2} \mid \alpha \notin \operatorname{tr}\left(T_{1}\right) \cap \operatorname{tr}\left(T_{2}\right)\right\}
\end{aligned}
$$

Weighted trace set addition $T_{1} \oplus T_{2}$ is commutative and associative, with degrees of identical traces in the two summands being always added up for coherency purposes. In constrast, trace distribution addition is only commutative. Essentially, the two summands in $T D_{1} \oplus T D_{2}$ represent two families of sets of weighted traces executable in the resolutions of two states in the support of a target distribution. Every weighted trace set $T_{1} \in T D_{1}$ is summed with every weighted trace set $T_{2} \in T D_{2}$ - to characterize an overall resolution - unless $T D_{1}$ and $T D_{2}$ have the same family of trace sets, in which case summation is restricted to weighted trace sets featuring the same traces for the sake of coherency. Due to the lack of associativity, in the definition below all trace distributions $\Delta\left(s^{\prime}\right) \cdot T D_{n-1}^{\mathrm{c}}\left(s^{\prime}\right)$ exhibiting the same family $\Theta$ of trace sets have to be summed up first, which is ensured by the presence of a double summation.

Definition 6. Let $\left(D, \oplus, \otimes, 0_{D}, 1_{D}\right)$ be a reachability-consistent semiring and $(S, A, \longrightarrow)$ be a D-ULTraS. The coherent trace distribution of $s \in S$ is the subset of $2^{A^{*} \times\left(D \backslash\left\{0_{D}\right\}\right)}$ defined as follows:

$$
T D^{\mathrm{c}}(s)=\bigcup_{n \in \mathbb{N}} T D_{n}^{\mathrm{c}}(s)
$$

with the coherent trace distribution of $s$ whose traces have length at most $n$ being defined as:
where $\operatorname{tr}(\Delta, n-1)=\left\{\operatorname{tr}\left(T D_{n-1}^{\mathrm{c}}\left(s^{\prime}\right)\right) \mid s^{\prime} \in \operatorname{supp}(\Delta)\right\}$ and the operator $\left(\varepsilon, 1_{D}\right) \dagger_{-}$ is such that $\left(\varepsilon, 1_{D}\right) \dagger T D=\left\{\left\{\left(\varepsilon, 1_{D}\right)\right\} \cup T \mid T \in T D\right\}$.

As shown by several examples in [8], the coherency constraints should involve all $T D_{n}^{\mathrm{c}}()_{-}$distributions separately - rather than $\left.T D^{\mathrm{c}}()_{-}\right)$- and should not consider the degrees contained in those trace distributions, i.e., they should rely on $\operatorname{tr}\left(T D_{n}^{\mathrm{c}}(-)\right)$ sets. In 99 it was further shown that the coherency constraints should be based on a monotonic construction in which any $T D_{n}^{c}(-)$ incrementally builds on $T D_{n-1}^{\mathrm{c}}(-)$, in the sense that every weighted trace set in the former should
include as a subset a weighted trace set in the latter. This is achieved through a variant of coherent trace distribution, called fully coherent trace distribution.

Definition 7. Let $\left(D, \oplus, \otimes, 0_{D}, 1_{D}\right)$ be a reachability-consistent semiring and $(S, A, \longrightarrow)$ be a D-ULTraS. The fully coherent trace distribution of $s \in S$ is the subset of $2^{A^{*} \times\left(D \backslash\left\{0_{D}\right\}\right)}$ defined as follows:

$$
T D^{\mathrm{fc}}(s)=\bigcup_{n \in \mathbb{N}} T D_{n}^{\mathrm{fc}}(s)
$$

with the fully coherent trace distribution of $s$ whose traces have length at most $n$ being the subset of $T D_{n}^{\mathrm{c}}(s)$ defined as:

$$
T D_{n}^{\mathrm{fc}}(s)= \begin{cases}\left\{T \in T D_{n}^{\mathrm{c}}(s) \mid \exists T^{\prime} \in T D_{n-1}^{\mathrm{fc}}(s) \cdot T^{\prime} \subseteq T\right\} \\ \left\{\left\{\left(\varepsilon, 1_{D}\right)\right\}\right\} & \text { if } n>0 \text { and } s \text { has outgoing transitions } \\ & \text { otherwise }\end{cases}
$$

We now adapt to ULTraS the two coherency constraints of [89]. The former preserves the equality of trace set families of any length $n$ between original states and the resolutions states to which they correspond. The latter requires a complete presence in each resolution of traces of length $n$ if any, including possible shorter maximal traces, which is looser than requiring resolution maximality.

Definition 8. Let $(S, A, \longrightarrow \mathcal{U})$ be a $D$-ULTraS, $s \in S$, and $\mathcal{Z}=(Z, A, \longrightarrow \mathcal{Z})$ $\in \operatorname{Res}(s)$ with correspondence function $\operatorname{corr}_{\mathcal{Z}}: Z \rightarrow S$. We say that $\mathcal{Z}$ is a coherent resolution of $s$, written $\mathcal{Z} \in \operatorname{Res}^{c}(s)$, iff for all $z \in Z$, whenever $z \xrightarrow{a}{ }_{\mathcal{Z}} \Delta$, then for all $n \in \mathbb{N}$ :

1. $\operatorname{tr}\left(\operatorname{TD}_{n}^{\mathrm{fc}}\left(\operatorname{corr}_{\mathcal{Z}}\left(z^{\prime}\right)\right)\right)=\operatorname{tr}\left(\operatorname{TD}_{n}^{\mathrm{fc}}\left(\operatorname{corr}_{\mathcal{Z}}\left(z^{\prime \prime}\right)\right)\right) \Longrightarrow \operatorname{tr}\left(T D_{n}^{\mathrm{fc}}\left(z^{\prime}\right)\right)=\operatorname{tr}\left(T D_{n}^{\mathrm{fc}}\left(z^{\prime \prime}\right)\right)$ for all $z^{\prime}, z^{\prime \prime} \in \operatorname{supp}(\Delta)$.
2. For all $z^{\prime} \in \operatorname{supp}(\Delta)$, the only $T \in T D_{n}^{\mathrm{fc}}\left(z^{\prime}\right)$ admits $\bar{T} \in T D_{n}^{\mathrm{fc}}\left(\operatorname{corr}_{\mathcal{Z}}\left(z^{\prime}\right)\right)$ such that $\operatorname{tr}(T)=\operatorname{tr}(\bar{T})$.

We can now define the two trace metaequivalences by making use of coherent resolutions of nondeterminism arising from deterministic schedulers. As in the case of bisimilarity, the difference between the two emerges in the presence of internal nondeterminism and is illustrated in Fig. 3. In the definition below, $z_{s_{i}}$ denotes both the initial state of $\mathcal{Z}_{i}$ and the state to which $s_{i}$ corresponds.

Definition 9. Let $\left(D, \oplus, \otimes, 0_{D}, 1_{D}\right)$ be a reachability-consistent semiring, $\mathcal{U}=$ $(S, A, \longrightarrow \mathcal{U})$ be a $D$-ULTraS, $\mathcal{M}$ be a $D$-measure schema for $\mathcal{U}$, and $s_{1}, s_{2} \in S$ :
$-s_{1} \sim_{\mathrm{T}, \mathcal{M}}^{\text {post }} s_{2}$ iff it holds that for each $\mathcal{Z}_{1}=\left(Z_{1}, A, \longrightarrow \mathcal{Z}_{1}\right) \in \operatorname{Res}^{\mathrm{c}}\left(s_{1}\right)$ there exists $\mathcal{Z}_{2}=\left(Z_{2}, A, \longrightarrow \mathcal{Z}_{2}\right) \in \operatorname{Res}^{\mathrm{c}}\left(s_{2}\right)$ such that for all $\alpha \in A^{*}$ :

$$
\mathcal{M}\left(z_{s_{1}}, \alpha, Z_{1}\right)=\mathcal{M}\left(z_{s_{2}}, \overline{\alpha, Z_{2}}\right)
$$

and also the condition obtained by exchanging $\mathcal{Z}_{1}$ with $\mathcal{Z}_{2}$ is satisfied.
$-s_{1} \sim_{\mathrm{T}, \mathcal{M}}^{\mathrm{pre}} s_{2}$ iff for all $\alpha \in A^{*}$ it holds that for each $\mathcal{Z}_{1}=\left(Z_{1}, A, \longrightarrow \mathcal{Z}_{1}\right) \in$ $\operatorname{Res}^{\mathrm{c}}\left(s_{1}\right)$ there exists $\mathcal{Z}_{2}=\left(Z_{2}, A, \longrightarrow \mathcal{Z}_{2}\right) \in \operatorname{Res}^{\mathrm{c}}\left(s_{2}\right)$ such that:
$\mathcal{M}\left(z_{s_{1}}, \alpha, Z_{1}\right)=\mathcal{M}\left(z_{s_{2}}, \alpha, Z_{2}\right)$
and also the condition obtained by exchanging $\mathcal{Z}_{1}$ with $\mathcal{Z}_{2}$ is satisfied.


Fig. 3. Difference between trace metaequivalences: $s_{i} \chi_{\mathrm{T}, \mathcal{M}}^{\text {post }} s_{j}, s_{i} \sim_{\mathrm{T}, \mathcal{M}}^{\mathrm{pre}} s_{j}$

The reader is referred to [7] to see that well-known specific equivalences are captured by the four metaequivalences introduced so far when instantiated with the semirings $(\mathbb{B}, \vee, \wedge, \perp, \top)$ and $\left(\mathbb{R}_{\geq 0},+, \times, 0,1\right)$ along with their measure functions $\mathcal{M}_{\mathrm{nd}}$ and $\mathcal{M}_{\mathrm{pb}}$. We finally revisit the comparison of the discriminating power of the four metaequivalences because the adoption of coherency rectifies a flaw in the proof of Prop. 3.5(3) in [7]. As shown in Fig. 4] where $s_{1} \sim_{\mathrm{B}}^{\text {post }} s_{2}$, the inclusion of $\sim_{\mathrm{B}}^{\text {post }}$ in $\sim_{\mathrm{T}, \mathcal{M}}^{\text {post }}$ would be prevented by incoherent resolutions. The resolution of $s_{2}$ starting with $z_{2}$, which cannot be matched by any resolution of $s_{1}$ with respect to trace $a b$, is not coherent because $\operatorname{tr}\left(T D_{1}^{\mathrm{fc}}\left(s_{2}^{\prime}\right)\right)=\{\{\varepsilon, b\},\{\varepsilon, c\}\}=$ $\operatorname{tr}\left(T D_{1}^{\mathrm{fc}}\left(s_{2}^{\prime \prime}\right)\right)$ whereas $\operatorname{tr}\left(T D_{1}^{\mathrm{fc}}\left(z_{2}^{\prime}\right)\right)=\{\{\varepsilon, b\}\} \neq\{\{\varepsilon, c\}\}=\operatorname{tr}\left(T D_{1}^{\mathrm{fc}}\left(z_{2}^{\prime \prime}\right)\right)$.

Proposition 1. Let $\left(D, \oplus, \otimes, 0_{D}, 1_{D}\right)$ be a reachability-consistent semiring, $\mathcal{U}=$ $(S, A, \longrightarrow \mathcal{U})$ be a D-ULTraS, and $\mathcal{M}$ be a $D$-measure schema for $\mathcal{U}$. Then:

1. $\sim_{\mathrm{B}}^{\mathrm{p}} \mathrm{p}_{\mathrm{p} \text {. }}^{\text {post }} \subseteq \sim_{\mathrm{B}}^{\mathrm{p} e}$, with $\sim_{\mathrm{B}}^{\text {post }}=\sim_{\mathrm{B}}^{\text {pre }}$ if $\mathcal{U}$ has no internal nondeterminism.
2. $\sim_{\mathrm{T}, \mathcal{M}}^{\mathrm{post}} \subseteq \sim_{\mathrm{T}, \mathcal{M}}^{\mathrm{pre}}$.
3. $\sim_{B}^{\text {post }} \subseteq \sim_{T, \mathcal{M}}^{\text {post }}$.
4. $\sim_{\mathrm{B}}^{\mathrm{pre}}$ is incomparable with $\sim_{\mathrm{T}, \mathcal{M}}^{\text {post }}$ and $\sim_{\mathrm{T}, \mathcal{M}}^{\text {pre }}$.

## 3 A Process Algebraic View of ULTraS

We introduce a very simple process calculus inspired by the ULTRAS metamodel, which we call UProC - uniform process calculus. In order to focus on the essence of the axiomatization for the various ULTRAS behavioral metaequivalences, we only admit dynamic process operators such as action prefix and choice.


Fig. 4. Validity of the inclusion of $\sim_{B}^{\text {post }}$ in $\sim_{T, \mathcal{M}}^{\text {post }}$ thanks to coherent resolutions

Given a preordered set $D$ equipped with minimum that yields a reachabilityconsistent semiring $\left(D, \oplus, \otimes, 0_{D}, 1_{D}\right)$, together with a countable set $A$ of actions, the syntax for UProC features two levels, one for the set $\mathbb{P}$ of process terms and one for the set $\mathbb{D}$ of reachability distribution terms:

$$
P::=\underline{0}|a \cdot \mathcal{D}| P+P \quad \mathcal{D}::=d \triangleright P \mid \mathcal{D} \phi \mathcal{D}
$$

where $a \in A, d \in D \backslash\left\{0_{D}\right\}$, and unary operators take precedence over binary ones. We let $\operatorname{init}(\underline{0})=\emptyset, \operatorname{init}(a . \mathcal{D})=\{a\}$, and $\operatorname{init}\left(P_{1}+P_{2}\right)=\operatorname{init}\left(P_{1}\right) \cup \operatorname{init}\left(P_{2}\right)$. We denote by $d \otimes \mathcal{D}$ the distribution term obtained from $\mathcal{D}$ by $\otimes$-multiplying each of its initial $D$-values by $d$.

The operational semantic rules below generate a $D$-ULTRAS $(\mathbb{P}, A, \longrightarrow)$ :

$$
\begin{gathered}
\frac{\mathcal{D} \longmapsto \Delta}{a . \mathcal{D} \xrightarrow{a} \Delta} \quad \frac{P_{1} \xrightarrow{a} \Delta}{P_{1}+P_{2} \xrightarrow{a} \Delta} \quad \frac{P_{2} \xrightarrow{a} \Delta}{P_{1}+P_{2} \xrightarrow{a} \Delta} \\
d \triangleright P \longmapsto\{(P, d)\} \quad \frac{\mathcal{D}_{1} \longmapsto \Delta_{1}}{\mathcal{D}_{1} \oplus \mathcal{D}_{2} \longmapsto \Delta_{1} \mapsto \Delta_{2}}
\end{gathered}
$$

The primary transition relation $\longrightarrow$ is defined as the smallest subset of $\mathbb{P} \times$ $A \times(\mathbb{P} \rightarrow D)_{\text {nefs }}$ satisfying the rules in the upper part. The secondary transition relation $\longmapsto$ is the smallest subset of $\mathbb{D} \times(\mathbb{P} \rightarrow D)_{\text {nefs }}$ satisfying the rules in the lower part, with $\{(P, d)\}$ being a shorthand for the reachability distribution identically equal to $0_{D}$ except in $P$ where its value is $d$; furthermore, $\oplus$ is lifted to reachability distributions by letting $\left(\Delta_{1} \oplus \Delta_{2}\right)(P)=\Delta_{1}(P) \oplus \Delta_{2}(P)$. Whenever $\mathcal{D} \longmapsto \Delta$, we let $\operatorname{supp}(\mathcal{D})=\operatorname{supp}(\Delta)$ and $\bigoplus \mathcal{D}=\bigoplus_{P \in \operatorname{supp}(\Delta)} \Delta(P)$.

To proceed with the axiomatization of the four behavioral metaequivalences, we need to show that they are congruences with respect to all the operators of UProC. Due to the two-level format of the syntax, as a preliminary step we have to lift the metaequivalences from processes to reachability distributions over processes. Extending [33], this can be done by considering $\mathcal{D}_{1}, \mathcal{D}_{2} \in \mathbb{D}$ related by an equivalence relation $\sim$ over $\mathbb{P}$ when they assign the same reachability degree to the same equivalence class, i.e., $\Delta_{1}(C)=\Delta_{2}(C)$ for all $C \in \mathbb{P} / \sim$ with $\mathcal{D}_{1} \longmapsto \Delta_{1}$ and $\mathcal{D}_{2} \longmapsto \Delta_{2}$.

Compositionality with respect to the two reachability distribution operators $\triangleright$ and $\phi$ can be established by abstracting from the specific behavioral metaequivalence. As for the two process operators, we have instead different proofs for bisimulation and trace semantics. These are minor reworkings of those in [7], except for the case of action prefix under trace semantics, for which we achieve full compositionality thanks to the use of coherent resolutions.

Theorem 1. Let $\sim_{\mathcal{M}} \in\left\{\sim_{\mathrm{B}}^{\text {post }}, \sim_{\mathrm{B}}^{\text {pre }}, \sim_{\mathrm{T}, \mathcal{M}}^{\text {post }}, \sim_{\mathrm{T}, \mathcal{M}}^{\text {pre }}\right\}$ for a measure schema $\mathcal{M}$ over the D-ULTraS semantics of UPRoC. Let $P_{1}, P_{2} \in \mathbb{P}$ and $\mathcal{D}_{1}, \mathcal{D}_{2} \in \mathbb{D}$. Then for all $d \in D \backslash\left\{0_{D}\right\}, \mathcal{D} \in \mathbb{D}, a \in A, P \in \mathbb{P}$ :

1. If $P_{1} \sim_{\mathcal{M}} P_{2}$, then $d \triangleright P_{1} \sim_{\mathcal{M}} d \triangleright P_{2}$.
2. If $\mathcal{D}_{1} \sim_{\mathcal{M}} \mathcal{D}_{2}$, then $\mathcal{D}_{1} \notin \mathcal{D} \sim_{\mathcal{M}} \mathcal{D}_{2} \phi \mathcal{D}$ and $\mathcal{D} \phi \mathcal{D}_{1} \sim_{\mathcal{M}} \mathcal{D} \phi \mathcal{D}_{2}$.
3. If $\mathcal{D}_{1} \sim_{\mathcal{M}} \mathcal{D}_{2}$, then $a . \mathcal{D}_{1} \sim_{\mathcal{M}} a . \mathcal{D}_{2}$.
4. If $P_{1} \sim_{\mathcal{M}} P_{2}$, then $P_{1}+P \sim_{\mathcal{M}} P_{2}+P$ and $P+P_{1} \sim_{\mathcal{M}} P+P_{2}$.


Fig. 5. Compositionality of trace semantics w.r.t. action prefix thanks to coherency

If in Def. 9 ordinary resolutions had been used instead of coherent ones, then similar to Thm. 4.2 of $[7$ in property 3 above we should have added "provided that all the processes in $\operatorname{supp}\left(\mathcal{D}_{i}\right), i \in\{1,2\}$, are pairwise $\sim_{\mathcal{M}}$-inequivalent" when $\sim_{\mathcal{M}}$ is a trace metaequivalence. In other words, compositionality of trace semantics with respect to action prefix would be partial without the restriction to coherent resolutions. The need for this trace-inequivalence constraint would emerge in our general setting because the continuation after an action is not a single process, but a reachability distribution over processes.

This can be illustrated through the following UProC terms $P_{1}$ and $P_{2}$ :

$$
\begin{aligned}
& P_{1}=a .\left(d_{1} \triangleright Q_{1} \notin d_{2} \triangleright Q_{2}\right) \quad P_{2}=a .\left(d_{1} \triangleright Q_{2} \oplus d_{2} \triangleright Q_{2}\right) \\
& Q_{1}=a^{\prime} \cdot b \cdot \underline{0}+a^{\prime} . c \cdot \underline{0} \quad Q_{2}=a^{\prime} .(b \cdot \underline{0}+c \cdot \underline{0})
\end{aligned}
$$

where a sequence of action prefixes like $a^{\prime} . b \cdot \underline{0}$ is a shorthand for $a^{\prime} .(\hat{d} \triangleright b \cdot(\hat{d} \triangleright \underline{0}))$, with the same value $\hat{d} \in D \backslash\left\{0_{D}\right\}$ being used here in all such sequences for simplicity. Their underlying $D$-ULTraS models are shown in the leftmost part of Fig. 5 . It is easy to see that $Q_{1}$ and $Q_{2}$ are trace equivalent, hence the two distributions describing the $a$-continuations of $P_{1}$ and $P_{2}$ are trace equivalent too. However, if we consider the resolution of $P_{1}$ starting with $z_{1}$ in the rightmost part of Fig. 5 . in which trace $\alpha=a a^{\prime} b$ is executable with degree $d_{1} \otimes \hat{d} \otimes \hat{d}$, we have that no resolution of $P_{2}$ is capable of matching it, as the executability degree of $\alpha$ would be $\left(d_{1} \oplus d_{2}\right) \otimes \hat{d} \otimes \hat{d}$ or $0_{D}$, unless $D=\mathbb{B}$ in which case $d_{1}=d_{2}=\top$ and $d_{1} \oplus d_{2}=\mathrm{T} \vee \top=\mathrm{T}$. As can be noted, the considered resolution of $P_{1}$ is not coherent because $\operatorname{tr}\left(T D_{2}^{\mathrm{fc}}\left(Q_{1}\right)\right)=\left\{\left\{\varepsilon, a^{\prime}, a^{\prime} b\right\},\left\{\varepsilon, a^{\prime}, a^{\prime} c\right\}\right\}=\operatorname{tr}\left(T D_{2}^{\mathrm{fc}}\left(Q_{2}\right)\right)$ but $\operatorname{tr}\left(T D_{2}^{\mathrm{fc}}\left(z_{1}^{\prime}\right)\right)=\left\{\left\{\varepsilon, a^{\prime}, a^{\prime} b\right\}\right\} \neq\left\{\left\{\varepsilon, a^{\prime}, a^{\prime} c\right\}\right\}=\operatorname{tr}\left(T D_{2}^{\mathrm{fc}}\left(z_{1}^{\prime \prime}\right)\right)$.

## 4 Axiomatizations of Behavioral Metaequivalences

In this section, we incrementally provide axioms in the UProC language for the four behavioral metaequivalences defined over the ULTraS metamodel. Since these axioms do not depend on any specific reachability-consistent semiring $\left(D, \oplus, \otimes, 0_{D}, 1_{D}\right)$, nor on any specific $D$-measure schema $\mathcal{M}$, from now on we omit $\mathcal{M}$ from trace metaequivalence symbols. Within examples, we will sometimes use subterms of the form $a . \underline{0}$ as abbreviation of $a .(\hat{d} \triangleright \underline{0})$, where the same value $\hat{d} \in D \backslash\left\{0_{D}\right\}$ is employed in all those subterms.

We start with the core axioms and the basic normal form used for all the metaequivalences (Sect. 4.1), then we single out additional axioms for $\sim_{B}^{\text {post }}$ (Sect. 4.2) and $\sim_{\mathrm{B}}^{\text {pre }}$ (Sect. 4.3) on the one hand, as well as different additional axioms for $\sim_{T}^{\text {post }}$ (Sect. 4.4 and $\sim_{T}^{\text {pre }}$ on the other hand. Before presenting soundness and completeness results, each set of axioms will be either compared with those known in the literature for specific classes of processes, or mentioned to yield the first equational characterization in a certain setting. Because of the absence of a completeness result, the axioms for $\sim_{T}^{\text {pre }}$ require further investigation and are not shown in the paper due to lack of space.

### 4.1 Core Axioms: Associativity, Commutativity, Neutral Element

Thanks to the format of the semantic rules in Sect. 3 and the associativity and commutativity of $\oplus$, for each metaequivalence the two UProC operators + and $\phi$ turn out to be associative and commutative - hence we can use their generalized versions respectively denoted by $\sum$ and $\sum-$ with $\underline{0}$ being the neutral element for operator + . Our starting point is thus a deduction system $\mathcal{A}$ that, in addition to reflexivity, symmetry, transitivity, and substitutivity, is based on the following core axioms:

$$
\begin{array}{|cc|}
\hline\left(\mathcal{A}_{1}\right) & \left(P_{1}+P_{2}\right)+P_{3}=P_{1}+\left(P_{2}+P_{3}\right) \\
\left(\mathcal{A}_{2}\right) & P_{1}+P_{2}=P_{2}+P_{1} \\
\left(\mathcal{A}_{3}\right) & P+\underline{0}=P \\
\left(\mathcal{A}_{4}\right) & \left(\mathcal{D}_{1} \notin \mathcal{D}_{2}\right) \notin \mathcal{D}_{3}=\mathcal{D}_{1} \notin\left(\mathcal{D}_{2} \oplus \mathcal{D}_{3}\right) \\
\left(\mathcal{A}_{5}\right) & \mathcal{D}_{1} \oplus \mathcal{D}_{2}=\mathcal{D}_{2} \oplus \mathcal{D}_{1} \\
\hline
\end{array}
$$

Axioms $\mathcal{A}_{1}$ to $\mathcal{A}_{3}$ are typical of nondeterministic process calculi 36, while axioms $\mathcal{A}_{4}$ and $\mathcal{A}_{5}$ encode those typical of probabilistic process calculi [29]4]. The latter calculi usually employ a probabilistic choice operator ${ }_{p}+$, so that associativity is represented as $\left(P^{\prime}{ }_{p}+P^{\prime \prime}\right)_{q}+P^{\prime \prime \prime}=P^{\prime}{ }_{p \cdot q}+\left(P^{\prime \prime}{ }_{(1-p) \cdot q /(1-p \cdot q)}+P^{\prime \prime \prime}\right)$ and commutativity is represented as $P^{\prime}{ }_{p}+P^{\prime \prime}=P^{\prime \prime}{ }_{1-p}+P^{\prime}$, with $p, q \in \mathbb{R}_{0,1[ }$. In $\mathcal{A}_{4}$ and $\mathcal{A}_{5}$, probabilities decorating operators like ${ }_{p}+$ are instead expressed by degrees within distributions, which avoids calculations when moving between the two distribution terms of either axiom.

To prove the completeness of the equational characterizations for the various metaequivalences, we introduce as usual a normal form to which each term is shown to be reducible, then we work with normal forms only. Extending [36, we say that $P \in \mathbb{P}$ is in sum normal form (snf) iff it is equal to $\underline{0}$ or $\sum_{i \in I} a_{i} \cdot\left(\sum_{j \in J_{i}} d_{i, j} \triangleright P_{i, j}\right)$ where $I$ and $J_{i}$ are finite nonempty index sets and every $P_{i, j}$ is in snf. The axiom system $\mathcal{A}$ is sufficient for snf reducibility.

Lemma 1. Let $P \in \mathbb{P}$. Then there exists $Q \in \mathbb{P}$ in snf such that $\mathcal{A} \vdash P=Q$.

### 4.2 Equational Characterization of $\sim_{B}^{\text {post }}$ : Idempotency

The additional laws for $\sim_{\mathrm{B}}^{\text {post }}$ are given by the following idempotency-related axioms, where we emphasize in boldface the occurrences of identical subterms:

$$
\begin{array}{|cc|}
\hline\left(\mathcal{A}_{\mathrm{B}, 1}^{\text {post }}\right) & \boldsymbol{P}+\boldsymbol{P}=\boldsymbol{P} \\
\left(\mathcal{A}_{\mathrm{B}, 2}^{\text {post }}\right) & d_{1} \triangleright \boldsymbol{P} \phi d_{2} \triangleright \boldsymbol{P}=\left(d_{1} \oplus d_{2}\right) \triangleright \boldsymbol{P}
\end{array}
$$

Axiom $\mathcal{A}_{\mathrm{B}, 1}^{\text {post }}$ expresses idempotency of choice and is typical of bisimilarity over nondeterministic process calculi [36]. Axiom $\mathcal{A}_{\mathrm{B}, 2}^{\text {post }}$ expresses a summationbased variant of idempotency that involves operator $\triangleright$ too; it encodes the axioms typical of bisimilarity over probabilistic process calculi [29], i.e., $P_{p}+P=P$, and over stochastic process calculi [27|26, i.e., $\lambda_{1} \cdot P+\lambda_{2} \cdot P=\left(\lambda_{1}+\lambda_{2}\right) . P$ with $\lambda_{1}, \lambda_{2} \in \mathbb{R}_{>0}$ being rates of exponential distributions. The two axioms are in agreement with those developed in the coalgebraic framework of 43 for various classes of probabilistic processes possibly including nondeterminism.

It is immediate to establish the soundness with respect to $\sim_{B}^{\text {post }}$ of the deduction system $\mathcal{A}_{\mathrm{B}}^{\text {post }}$ obtained from $\mathcal{A}$ by adding the two idempotency-related axioms $\mathcal{A}_{\mathrm{B}, 1}^{\text {post }}$ and $\mathcal{A}_{\mathrm{B}, 2}^{\text {post }}$.

Theorem 2. Let $P_{1}, P_{2} \in \mathbb{P}$. If $\mathcal{A}_{\mathrm{B}}^{\text {post }} \vdash P_{1}=P_{2}$, then $P_{1} \sim_{\mathrm{B}}^{\text {post }} P_{2}$.
As far as the completeness of $\mathcal{A}_{\mathrm{B}}^{\text {post }}$ with respect to $\sim_{\mathrm{B}}^{\text {post }}$ is concerned, we exploit Lemma 1, i.e., reducibility to snf.

Theorem 3. Let $P_{1}, P_{2} \in \mathbb{P}$. If $P_{1} \sim_{\mathrm{B}}^{\text {post }} P_{2}$, then $\mathcal{A}_{\mathrm{B}}^{\text {post }} \vdash P_{1}=P_{2}$.
Corollary 1. Let $P_{1}, P_{2} \in \mathbb{P}$. Then $P_{1} \sim_{\mathrm{B}}^{\text {post }} P_{2}$ iff $\mathcal{A}_{\mathrm{B}}^{\text {post }} \vdash P_{1}=P_{2}$.

### 4.3 Equational Characterization of $\sim_{B}^{\text {pre }}$ : B-Shuffling

When $P_{1} \sim_{\mathrm{B}}^{\text {post }} P_{2}$, every $a$-transition of either term is matched by an $a$-transition of the other with respect to all sets of equivalence classes, so that the target distributions $\Delta_{1}$ and $\Delta_{2}$ of the two $a$-transitions satisfy $\Delta_{1} \sim_{B}^{\text {post }} \Delta_{2}$. If instead $P_{1} \sim_{\mathrm{B}}^{\text {pre }} P_{2}$, every $a$-transition of either term is matched by an $a$-transition of the other with respect to a specific set of equivalence classes, hence $\Delta_{1} \sim_{\mathrm{B}}^{\text {pre }} \Delta_{2}$ is not necessarily true.

This is witnessed by the example shown in Fig. 1, which yields the balanced equality $a .\left(d_{1} \triangleright P_{1} \not d_{2} \triangleright P_{2}\right)+a .\left(d_{2} \triangleright P_{1} \not d_{1} \triangleright P_{3}\right)+a .\left(d_{1} \triangleright P_{2} \not d_{2} \triangleright P_{3}\right)=$ $a .\left(d_{2} \triangleright P_{1} \not d_{1} \triangleright P_{2}\right)+a .\left(d_{1} \triangleright P_{1} \not d_{2} \triangleright P_{3}\right)+a .\left(d_{2} \triangleright P_{2} \not d_{1} \triangleright P_{3}\right)$ where $d_{1}, d_{2}$ and $P_{1}, P_{2}, P_{3}$ are shuffled within either term, while only $d_{1}$ and $d_{2}$ are shuffled across the two terms too. An example of unbalanced equality - with unbalanced meaning that the number of $\phi$-summands is not the same within all $a$-summands - is given by $a \cdot\left(d_{1} \triangleright P_{1}\right)+a \cdot\left(d_{2} \triangleright P_{1} \triangleright d_{1} \triangleright P_{2}\right)+a \cdot\left(d_{2} \triangleright P_{2}\right)=a \cdot\left(d_{2} \triangleright P_{1}\right)+$ $a .\left(d_{1} \triangleright P_{1} \not d_{2} \triangleright P_{2}\right)+a .\left(d_{1} \triangleright P_{2}\right)$.

$$
\begin{aligned}
& \left(\mathcal{A}_{\mathrm{B}, 1}^{\mathrm{pre}}\right) \quad \sum_{i \in I_{1}} \boldsymbol{a} \cdot\left(\sum_{j \in J_{1, i}} d_{1, i, j} \triangleright P_{1, i, j}\right)=\sum_{i \in I_{2}} \boldsymbol{a} \cdot\left(\sum_{j \in J_{2, i}} d_{2, i, j} \triangleright P_{2, i, j}\right) \\
& \text { subject to: } \\
& \text { for all } i_{1} \in I_{1} \text { and } J_{1} \subseteq J_{1, i_{1}} \text { s.t. }\left(j \in J_{1} \wedge P_{1, i_{1}, k}=P_{1, i_{1}, j}\right) \Longrightarrow k \in J_{1} \\
& \text { there exist } i_{2} \in I_{2} \text { and } J_{2} \subseteq J_{2, i_{2}} \text { s.t. }\left(j \in J_{2} \wedge P_{2, i_{2}, k}=P_{2, i_{2}, j}\right) \Longrightarrow k \in J_{2} \\
& \text { such that the following three constraints are met: } \\
& \text { 1. } \forall j_{1} \in J_{1} .\left(\exists j_{2} \in J_{2} . P_{1, i_{1}, j_{1}}=P_{2, i_{2}, j_{2}} \vee \nexists j_{2} \in J_{2, i_{2}} . P_{1, i_{1}, j_{1}}=P_{2, i_{2}, j_{2}}\right) \\
& \text { 2. }\left\{P_{1, i_{1}, j} \mid j \in J_{1}\right\} \supseteq\left\{P_{2, i_{2}, j} \mid j \in J_{2}\right\} \\
& \text { 3. } \bigoplus_{j \in J_{1}} d_{1, i_{1}, j}=\bigoplus_{j \in J_{2}} d_{2, i_{2}, j} \\
& \text { and also the condition obtained by exchanging } i_{1}, J_{1} \text { with } i_{2}, J_{2} \text { is satisfied }
\end{aligned}
$$

Table 1. Axiom characterizing $\sim_{B}^{\text {pre }}$

In the identifications made possible by $\sim_{B}^{\text {pre }}$, no regularity can be assumed in general about the number of $a$-summands (internal nondeterminism) and the number of $\phi$-summands inside every $a$-summand. The two identifications exemplified above turn out to be among the simplest instances of the $B$-shuffling axiom in Table 1 characterizing the identification power of $\sim_{\mathrm{B}}^{\text {pre }}$, where $I_{1}, J_{1, i}$, $J_{1}, I_{2}, J_{2, i}, J_{2}$ are finite nonempty index sets and we emphasize in boldface the occurrences of identical actions.

All + -summands on both sides of the axiom start with the same action $a$. For each $a$-summand on the lefthand side indexed by $i_{1}$ and on the righthand side indexed by $i_{2}$, in the three constraints we use the maximal subsets $J_{1}$ of $J_{1, i_{1}}$ and $J_{2}$ of $J_{2, i_{2}}$ whose elements index all the occurrences of certain process terms, so as to consider every set of equivalence classes of process terms reached after performing $a$. More precisely:

1. The first constraint guarantees that, for each equivalence class $C$ such that (i) $C$ is reached via the $a$-summand on the lefthand side indexed by $i_{1}$ and
(ii) the $a$-derivative terms in $C$ are all indexed by elements of $J_{1}$, it holds that either $C$ is reached also via the $a$-summand on the righthand side indexed by $i_{2}$ and the $a$-derivative terms in $C$ are all indexed by elements of $J_{2}$, or $C$ is not reachable at all as no $P_{2, i_{2}, j_{2}}$ belongs to it.
2. The second constraint ensures that the elements of $J_{2}$ do not index process terms of further equivalence classes with respect to those singled out by $J_{1}$. It cannot be expressed as $\left\{P_{1, i_{1}, j} \mid j \in J_{1}\right\}=\left\{P_{2, i_{2}, j} \mid j \in J_{2}\right\}$ otherwise $a .\left(d \triangleright P_{1} \notin d \triangleright P_{2}\right)=a \cdot\left(d \triangleright P_{1}\right)+a .\left(d \triangleright P_{2}\right)$ subject to $d=d \oplus d$ would not be derivable because, for $J_{1}$ indexing both $P_{1}$ and $P_{2}$, we would have $J_{2}$ indexing at most one of those two $a$-derivative terms.
3. The maximality of $J_{1}$ and $J_{2}$ with respect to the process terms indexed by their elements, together with the first two constraints, causes the third constraint to state that, for an arbitrary set of equivalence classes identified by $J_{1}$, this set is reached via $a$ with the same overall $D$-value from both the $a$-summand on the lefthand side indexed by $i_{1}$ and the $a$-summand on the righthand side indexed by $i_{2}$.

The B-shuffling axiom $\mathcal{A}_{\mathrm{B}, 1}^{\text {pre }}$ subsumes the following laws that we have already encountered:
$-\mathcal{A}_{\mathrm{B}, 1}^{\text {post }}$, because in $P+P$ each subterm $a \cdot \mathcal{D}+a \cdot \mathcal{D}$ composed of two identical summands placed next to each other can be trivially equated to subterm $a \cdot \mathcal{D}$ of $P$ via $\mathcal{A}_{\mathrm{B}, 1}^{\mathrm{pre}}$.
$-\mathcal{A}_{\mathrm{B}, 2}^{\text {post }}$, because $a \cdot\left(d_{1} \triangleright P \not d_{2} \triangleright P\right)$ can be trivially equated to $a \cdot\left(\left(d_{1} \oplus d_{2}\right) \triangleright P\right)$ via $\mathcal{A}_{\mathrm{B}, 1}^{\mathrm{pre}}$.
$-a \cdot \mathcal{D}_{1}+a \cdot \mathcal{D}_{2}=a \cdot\left(\mathcal{D}_{1} \notin \mathcal{D}_{2}\right)$ under the same conditions as $\mathcal{A}_{\mathrm{B}, 1}^{\text {pre }}$.
We also point out that $\mathcal{A}_{\mathrm{B}, 1}^{\text {pre }}$ yields the first axiomatization for the bisimilarities over nondeterministic and probabilistic processes studied in [1345], which have the interesting property of being characterized by the probabilistic modal and temporal logics of 33|25].

We now show that $\mathcal{A}_{\mathrm{B}}^{\text {pre }}$, the deduction system obtained from $\mathcal{A}$ by adding the B-shuffling axiom $\mathcal{A}_{\mathrm{B}, 1}^{\text {pre }}$, is sound and complete with respect to $\sim_{\mathrm{B}}^{\text {pre }}$ by exploiting again Lemma 1 .

Theorem 4. Let $P_{1}, P_{2} \in \mathbb{P}$. If $\mathcal{A}_{\mathrm{B}}^{\text {pre }} \vdash P_{1}=P_{2}$, then $P_{1} \sim_{\mathrm{B}}^{\text {pre }} P_{2}$.
Theorem 5. Let $P_{1}, P_{2} \in \mathbb{P}$. If $P_{1} \sim_{\mathrm{B}}^{\text {pre }} P_{2}$, then $\mathcal{A}_{\mathrm{B}}^{\text {pre }} \vdash P_{1}=P_{2}$.
Corollary 2. Let $P_{1}, P_{2} \in \mathbb{P}$. Then $P_{1} \sim_{\mathrm{B}}^{\text {pre }} P_{2}$ iff $\mathcal{A}_{\mathrm{B}}^{\text {pre }} \vdash P_{1}=P_{2}$.

### 4.4 Equational Characterization of $\sim_{T}^{\text {post }}$ : Choice Deferral

The additional identification power of $\sim_{T}^{\text {post }}$ with respect to $\sim_{\mathrm{B}}^{\text {post }}$ is given by the choice-deferring axioms in Table 2, where $\mathcal{D}$ may have an empty support (abuse of notation), $J$ is a finite nonempty index set, and we emphasize in boldface the occurrences of noteworthy subterms, actions, and operators.

Axiom $\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}$ expresses the deferral of a nondeterministic choice. Its simplest instance $a \cdot\left(d \triangleright P^{\prime}\right)+a \cdot\left(d \triangleright P^{\prime \prime}\right)=a \cdot\left(d \triangleright\left(P^{\prime}+P^{\prime \prime}\right)\right)$ is reminiscent of the axiom typical of trace equivalence over nondeterministic process calculi [18]39] and is in agreement with axioms in the coalgebraic setting of [17. The axiom would not be valid if several distinct terms were considered in either $\phi$-choice, as for instance $a .\left(d_{1} \triangleright P_{1}^{\prime} \phi d_{2} \triangleright P_{2}^{\prime}\right)+a .\left(d_{1} \triangleright P_{1}^{\prime \prime} \phi d_{2} \triangleright P_{2}^{\prime \prime}\right) \chi_{\mathrm{T}}^{\text {post }} a .\left(d_{1} \triangleright\left(P_{1}^{\prime}+P_{1}^{\prime \prime}\right) \notin d_{2} \triangleright\left(P_{2}^{\prime}+P_{2}^{\prime \prime}\right)\right)$ because on the righthand side a resolution of $P_{1}^{\prime}$ and a resolution of $P_{2}^{\prime \prime}$ could be jointly taken into account whereas this is not possible on the lefthand side.

The condition to which $\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}$ is subject is necessary because, whenever $P^{\prime}+P^{\prime \prime}$ has the same initial actions as a term $P$ in the support of $\mathcal{D}$, then all resolutions of the righthand side term of the axiom have to satisfy the first coherency constraint of Def. 8 with respect to $P^{\prime}+P^{\prime \prime}$ and $P$, whereas this is not the case for the resolutions of the lefthand side term of the axiom, thus hampering resolution matching. This can be seen, for $\mathcal{D}$ given by $d^{\prime} \triangleright(b \cdot \underline{0}+c \cdot \underline{0})$, by considering the two process terms $a \cdot(\mathcal{D} \phi d \triangleright(b \cdot \underline{0}))+a \cdot(\mathcal{D} \phi d \triangleright(c \cdot \underline{0}))$ and $a \cdot(\mathcal{D} \phi d \triangleright(b \cdot \underline{0}+c \cdot \underline{0}))$, because after performing $a$ every resolution of the latter

$$
\begin{array}{|ll}
\left(\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}\right) & a \cdot\left(\mathcal{D} \notin d \triangleright \boldsymbol{P}^{\prime}\right)+a \cdot\left(\mathcal{D} \phi d \triangleright \boldsymbol{P}^{\prime \prime}\right)=a \cdot\left(\mathcal{D} \phi d \triangleright\left(\boldsymbol{P}^{\prime}+\boldsymbol{P}^{\prime \prime}\right)\right) \\
& \text { if } \operatorname{init}\left(P^{\prime}+P^{\prime \prime}\right) \neq \operatorname{init}(P) \text { for all } P \in \operatorname{supp}(\mathcal{D}), \text { unless } \operatorname{init}\left(P^{\prime}\right)=\operatorname{init}\left(P^{\prime \prime}\right) \\
\left(\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}\right) & a \cdot\left(\mathcal{D} \notin d_{1} \triangleright\left(\sum_{j \in J} \boldsymbol{b}_{j} \cdot \mathcal{D}_{1, j}\right) \not d_{2} \triangleright\left(\sum_{j \in J} \boldsymbol{b}_{j} \cdot \mathcal{D}_{2, j}\right)\right) \\
= & a \cdot\left(\mathcal{D} \phi\left(d_{1} \oplus d_{2}\right) \triangleright\left(\sum_{j \in J} \boldsymbol{b}_{j} \cdot\left(\mathcal{D}_{1, j}^{\prime} \notin \mathcal{D}_{2, j}^{\prime}\right)\right)\right) \\
& \text { if for } i=1,2 \text { there exists } d_{i}^{\prime} \in D \text { such that }\left(d_{1} \oplus d_{2}\right) \otimes d_{i}^{\prime}=d_{i}, \\
& \text { where } \mathcal{D}_{i, j}^{\prime}=d_{i}^{\prime} \otimes \mathcal{D}_{i, j}
\end{array}
$$

Table 2. Axioms characterizing $\sim_{T}^{\text {post }}$
can execute either $b$ or $c$ with degree $d^{\prime} \oplus d$ due to coherency, while the former has resolutions in which both $b$ and $c$ are executable after $a$, which therefore cannot be matched. The condition is not needed only if $P^{\prime}$ and $P^{\prime \prime}$ share the same initial actions.

Axiom $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ expresses instead the deferral of a distribution choice. The degrees $d_{1}$ and $d_{2}$ in it may be different and are summed up anyhow, instead of being equal and preserved like in $\mathcal{A}_{\mathrm{T}, 1}^{\mathrm{post}}$. This is analogous to what happens with the two idempotency-related axioms, as $\mathcal{A}_{\mathrm{B}, 1}^{\text {post }}$ preserves degrees while $\mathcal{A}_{\mathrm{B}, 2}^{\text {post }}$ sums them up. The $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ instance $a \cdot\left(d_{1} \triangleright\left(b \cdot\left(1_{D} \triangleright P_{1}\right)\right) \notin d_{2} \triangleright\left(b \cdot\left(1_{D} \triangleright P_{2}\right)\right)\right)=$ $a \cdot\left(1_{D} \triangleright b \cdot\left(d_{1} \triangleright P_{1} \not d_{2} \triangleright P_{2}\right)\right)$, for $d_{1} \oplus d_{2}=1_{D}$, is reminiscent of identifications typical of trace equivalence over fully probabilistic processes 29 .

We observe that there is no connection with the axiomatization of trace semantics for nondeterministic and probabilistic processes in [37], because the equivalence considered there is the simulation equivalence that turns out to be the coarsest (with respect to parallel composition) congruence [35] contained in the trace equivalence of 41]. There is some relationship with the axiomatization of trace semantics developed for fully probabilistic processes in the coalgebraic framework of 44, even though only complete traces are considered there. Since we are not aware of any other axiomatization related to probabilistic trace semantics, ours seems to be the first one that can be applied to the probabilistic trace equivalences of [29/41].

The embedding in the action prefix context $a .\left(\mathcal{D} \phi_{-}\right)$of both distribution terms on the two sides of $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ is due to the fact that the two distribution terms themselves are not necessarily identified by $\sim_{\mathrm{T}}^{\text {post }}$. For instance, the probabilistic terms $P_{1}^{\prime}, P_{1}^{\prime \prime}$, and $P_{2}$ respectively given by $b .\left(0.5 \triangleright\left(c_{1} \cdot \underline{0}+c . \underline{0}\right) \phi 0.5 \triangleright\left(c_{2} \cdot \underline{0}\right)\right)$, $b .\left(0.5 \triangleright\left(c_{1} \cdot \underline{0}\right) \notin 0.5 \triangleright\left(c . \underline{0}+c_{2} \cdot \underline{0}\right)\right)$, and $b .\left(0.25 \triangleright\left(c_{1} . \underline{0}+c . \underline{0}\right) \phi 0.25 \triangleright\left(c_{2} \cdot \underline{0}\right) \phi\right.$ $\left.0.25 \triangleright\left(c_{1} \cdot \underline{0}\right) \notin 0.25 \triangleright\left(c . \underline{0}+c_{2} \cdot \underline{0}\right)\right)$ are pairwise $\sim_{\mathrm{T}}^{\text {post }}$-inequivalent, hence so are the distribution terms $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ respectively given by $0.5 \triangleright P_{1}^{\prime} \oplus 0.5 \triangleright P_{1}^{\prime \prime}$ and $1 \triangleright P_{2}$, but $a . \mathcal{D}_{1} \sim_{\mathrm{T}}^{\text {post }} a . \mathcal{D}_{2}$ as correctly captured by $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$.

We now show that $\mathcal{A}_{\mathrm{T}}^{\text {post }}$, the deduction system obtained from $\mathcal{A}_{\mathrm{B}}^{\text {post }}$ by adding the two choice-deferring axioms $\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}$ and $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$, is sound with respect to $\sim_{\mathrm{T}}^{\text {post }}$.

Theorem 6. Let $P_{1}, P_{2} \in \mathbb{P}$. If $\mathcal{A}_{\mathrm{T}}^{\text {post }} \vdash P_{1}=P_{2}$, then $P_{1} \sim_{\mathrm{T}}^{\text {post }} P_{2}$.
As far as completeness is concerned, we extend to UPRoC the technique used in [548] for nondeterministic processes. It reduces the problem of establishing the completeness of an axiomatization on arbitrary terms with respect to some behavioral equivalence $\sim$ (which in our case is $\sim_{T}^{\text {post }}$ ) to the problem of establishing the completeness of the same axiomatization on terms in a $\sim$-specific normal form with respect to bisimilarity (which in our case is $\sim_{B}^{\text {post }}$ ).

We use each of the choice-deferring axioms $\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}$ and $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ as a graph rewriting rule (applied to the ULTraS model underlying the considered UProC term) that transforms its lefthand side into its righthand side. Given $P \in \mathbb{P}$, we then say that it is in $\sim_{\mathrm{T}}^{\text {post }}-s n f$ iff it is equal to $\underline{0}$ or $\sum_{i \in I} a_{i} \cdot\left(\sum_{j \in J_{i}} d_{i, j} \triangleright P_{i, j}\right)$ where $I$ and $J_{i}$ are finite nonempty index sets, $\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}$ is not applicable to any pair of +-summands starting with the same action, $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ is not applicable to any pair of $\phi$-summands sharing the same initial actions, and every $P_{i, j}$ is in $\sim_{\mathrm{T}}^{\text {post }}{ }^{\text {-snf. }}$.

Lemma 2. Let $P \in \mathbb{P}$. Then there exists $Q \in \mathbb{P}$ in $\sim_{\mathrm{T}}^{\text {post }}$-snf such that $\mathcal{A}_{\mathrm{T}}^{\text {post }} \vdash$ $P=Q$.

The completeness of $\mathcal{A}_{\mathrm{T}}^{\text {post }}$ holds only for reachability-consistent semirings whose support $D$ always admits the existence of values $d_{i}^{\prime}$ that make $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ applicable in the presence of any distribution term complying with the one on the lefthand side of the axiom. This is the case with $\mathbb{B}$ and $\mathbb{R}_{>0}$, but not with $\mathbb{N}$, because for instance $a .\left(5 \triangleright\left(b .\left(2 \triangleright P_{1}\right)\right) \notin 2 \triangleright\left(b .\left(3 \triangleright P_{2}\right)\right) \notin 6 \triangleright\left(b .\left(5 \triangleright P_{3}\right)\right)\right)$ would be equated to $a \cdot\left(13 \triangleright\left(b \cdot\left(\frac{10}{13} \triangleright P_{1} \not \frac{6}{13} \triangleright P_{2} \not \frac{30}{13} \triangleright P_{3}\right)\right)\right)$ where $\frac{10}{13}, \frac{6}{13}, \frac{30}{13} \notin \mathbb{N}$. To achieve completeness over $\mathbb{N}$, we should add to $\mathcal{A}_{\mathrm{T}}^{\text {post }}$ some further axiom equating for instance $a .\left(5 \triangleright\left(b .\left(2 \triangleright P_{1}\right)\right) \notin 2 \triangleright\left(b \cdot\left(3 \triangleright P_{2}\right)\right) \notin 6 \triangleright\left(b \cdot\left(5 \triangleright P_{3}\right)\right)\right)$ to $a .\left(1 \triangleright\left(b .\left(2 \triangleright P_{1}\right)\right) \nrightarrow 8 \triangleright\left(b .\left(3 \triangleright P_{2}\right)\right) \nmid 4 \triangleright\left(b .\left(5 \triangleright P_{3}\right)\right)\right)$ for suitable $P_{1}, P_{2}, P_{3}$ such as $\underline{0}$, because in both terms the degree of executability of trace $a$ is 13 and the one of trace $a b$ is 46 . In the following, for the sake of brevity we denote with $D_{\mathrm{T}, 2}^{\text {post }}$ the predicate asserting that a reachability-consistent semiring is considered whose support $D$ always enables the applicability of $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$.

Lemma 3. Let $P_{1}, P_{2} \in \mathbb{P}$ in $\sim_{\mathrm{T}}^{\text {post }}$-snf. Then $P_{1} \sim_{\mathrm{T}}^{\text {post }} P_{2}$ iff $P_{1} \sim_{\mathrm{B}}^{\text {post }} P_{2}$ under condition $D_{\mathrm{T}, 2}^{\text {post }}$.

Theorem 7. Let $P_{1}, P_{2} \in \mathbb{P}$. If $P_{1} \sim_{\mathrm{T}}^{\text {post }} P_{2}$, then $\mathcal{A}_{\mathrm{T}}^{\text {post }} \vdash P_{1}=P_{2}$ under condition $D_{\mathrm{T}, 2}^{\text {post }}$.

Corollary 3. Let $P_{1}, P_{2} \in \mathbb{P}$. Then $P_{1} \sim_{\mathrm{T}}^{\text {post }} P_{2}$ iff $\mathcal{A}_{\mathrm{T}}^{\text {post }} \vdash P_{1}=P_{2}$ under condition $D_{\mathrm{T}, 2}^{\text {post }}$.

## 5 Conclusions

We have incrementally developed general axiomatizations of bisimulation and trace semantics by working with the corresponding post-/pre-metaequivalences
on UProC terms. We have also revised according to [8] the notion of resolution of nondeterminism - originally introduced in [7] for the ULTRAS metamodel to ensure the inclusion of $\sim_{\mathrm{B}}^{\text {post }}$ in $\sim_{\mathrm{T}}^{\text {post }}$ as well as the full compositionality of action prefix for both trace metaequivalences $\sim_{\mathrm{T}}^{\text {post }}$ and $\sim_{\mathrm{T}}^{\text {pre }}$.

We plan to expand our axiomatizations to exhibit also the general laws for static process operators, such as the expansion law for parallel composition, and recursion. It would then be interesting to search for general axiomatizations of other semantics in the branching-time - linear-time spectrum, including weak ones. However, we believe that the most challenging open problems are (i) the investigation of the completeness of the axiomatization of $\sim_{\mathrm{T}}^{\text {pre }}$ and (ii) the extension of the axiomatization of $\sim_{T}^{\text {post }}$ for achieving completeness over reachabilityconsistent semirings like $\mathbb{N}$ for which axiom $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ is not always applicable.

We finally observe that our general approach has allowed us to discover the first axiomatization of a behavioral equivalence in several situations. This is important because, when moving from nondeterministic processes to processes including also probabilistic and timing aspects, there are several different ways of defining the same semantics - of which the post-/pre-approaches are two notable options - and the spectrum consequently becomes much more variegated, as shown for instance in [12]. In this respect, the ULTraS metamodel has thus proven to be a useful tool.

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## A Equational Characterization of $\sim_{T}^{\text {pre }}$ : T-Shuffling

When $P_{1} \sim_{\mathrm{T}}^{\text {post }} P_{2}$, every resolution of either term is matched by a resolution of the other with respect to all traces. However, unlike $\sim_{B}^{\text {post }}$, the target distributions $\Delta_{1}$ and $\Delta_{2}$ of the initial transitions of two matching resolutions do not necessarily satisfy $\Delta_{1} \sim_{T}^{\text {post }} \Delta_{2}$, as can be seen from the discussion in Sect. 4.4 about the embedding in an action prefix context of the distribution terms occurring in $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$. If instead $P_{1} \sim_{\mathrm{T}}^{\text {pre }} P_{2}$, every resolution of either term is matched by a resolution of the other with respect to a specific trace $\alpha$.

One may therefore expect that the axioms characterizing $\sim_{\mathrm{T}}^{\text {pre }}$ are capable of splitting a distribution $\mathcal{D}$ into an alternative composition among its embedded process terms, thus leading to $a \cdot\left(\mathcal{D}_{1} \oplus \mathcal{D}_{2}\right)=a \cdot \mathcal{D}_{1}+a \cdot \mathcal{D}_{2}$ that we have already encountered towards the end of Sect. 4.3. Like for $\sim_{B}^{\text {pre }}$, such an identification is not valid in general as we need to impose $\bigoplus\left(\mathcal{D}_{1} \phi \mathcal{D}_{2}\right)=\bigoplus \mathcal{D}_{1}=\bigoplus \mathcal{D}_{2}$ to ensure that trace $a$ has the same executability degree in the various resolutions of the two process terms. We can say that both $\sim_{B}^{\text {pre }}$ and $\sim_{T}^{\text {pre }}$ establish a connection between the two UProC operators + and $\phi$, expressed by the conditional transformability of one into the other, which holds for instance when $D=\mathbb{B}$.

The identification power of $\sim_{T}^{\text {pre }}$ is actually more general than the aforementioned conditional transformability, as well as different from the one of $\sim_{B}^{\text {pre }}$ due to Prop. 1(4). It is given by a shuffling capability distinct from the one of $\sim_{B}^{\text {pre }}$ because, while $\sim_{\mathrm{B}}^{\text {pre }}$ focuses on sets of equivalence classes that are reachable in one step, $\sim_{\mathrm{T}}^{\text {pre }}$ has to take into account degrees of multistep reachability.

For example, the process terms $a \cdot\left(d \triangleright\left(b_{1} \cdot \underline{0}\right) \notin d \triangleright\left(b_{2} \cdot \underline{0}\right)\right)+a \cdot\left(d \triangleright\left(b_{3} \cdot \underline{0}\right) \notin d \triangleright\right.$ $\left.\left(b_{4} \cdot \underline{0}\right)\right)$ and $a \cdot\left(d \triangleright\left(b_{1} \cdot \underline{0}\right) \phi d \triangleright\left(b_{3} \cdot \underline{0}\right)\right)+a \cdot\left(d \triangleright\left(b_{2} \cdot \underline{0}\right) \phi d \triangleright\left(b_{4} \cdot \underline{0}\right)\right)$ are considered equivalent to each other and to $a \cdot\left(d \triangleright\left(b_{1} \cdot \underline{0}+b_{2} \cdot \underline{0}\right) \phi d \triangleright\left(b_{3} \cdot \underline{0}+b_{4} \cdot \underline{0}\right)\right)$ by $\sim_{\mathrm{T}}^{\text {pre }}$, but told apart by $\sim_{\mathrm{B}}^{\text {pre }}$. In constrast, in Fig. 1, where $s_{1} \sim_{\mathrm{B}}^{\text {pre }} s_{2}$, we have $s_{1} \sim_{\mathrm{T}}^{\text {pre }} s_{2}$ if the sets of actions labeling the transitions departing from $r_{1}, r_{2}, r_{3}$ are disjoint from each other. However, if each of $r_{1}$ and $r_{2}$ has a $b$ transition towards a singleton distribution whose support contains a terminal state reached with degree $d_{b}^{\prime}$ from $r_{1}$ and $d_{b}^{\prime \prime}$ from $r_{2}$, then it may hold that $s_{1} \mathcal{\chi}_{\mathrm{T}}^{\text {pre }} s_{2}$. This is the case when the executability degrees $\left(d_{1} \otimes d_{b}^{\prime}\right) \oplus\left(d_{2} \otimes d_{b}^{\prime \prime}\right)$ and $\left(d_{2} \otimes d_{b}^{\prime}\right) \oplus\left(d_{1} \otimes d_{b}^{\prime \prime}\right)$ of trace $a b$ - which we assume not to be executable via $r_{3}$ - are different from each other as well as from $d_{1} \otimes d_{b}^{\prime}$ and $d_{2} \otimes d_{b}^{\prime \prime}$.

In the $T$-shuffling axiom for $\sim_{\mathrm{T}}^{\text {pre }}$ in Table A, we let $I_{1}, J_{1, i}, J_{1}, I_{2}, J_{2, i}, J_{2}$ be finite nonempty index sets and $\operatorname{der}(P, b)=\{\overline{\mathcal{D}} \in \mathbb{D} \mid P$ has a $b . \mathcal{D}$ summand $\}$ and we emphasize in boldface the occurrences of identical actions.

The first condition guarantees that the executability degree of trace $a$ in any resolution of the term on either side of the axiom is always matched by a resolution of the term on the other side of the axiom. The necessity of this condition can be understood from the case in which all $P_{1, i, j}$ and $P_{2, i, j}$ terms are $\underline{0}$, as the set of actions initially executable by those terms would be empty and hence the second condition would be trivially satisfied.

The second condition takes care of all the other traces starting with $a$. For each possible action $b$ following $a$ in a trace executable by either term on the two

$$
\begin{aligned}
& \left(\mathcal{A}_{\mathrm{T}, 1}^{\text {pre }}\right) \quad \sum_{i \in I_{1}} \boldsymbol{a} \cdot\left(\sum_{j \in J_{1, i}} d_{1, i, j} \triangleright P_{1, i, j}\right)=\sum_{i \in I_{2}} \boldsymbol{a} \cdot\left(\sum_{j \in J_{2, i}} d_{2, i, j} \triangleright P_{2, i, j}\right) \\
& \text { subject to: } \\
& \text { for all } i_{1} \in I_{1} \text { there exists } i_{2} \in I_{2} \text { such that } \bigoplus_{j \in J_{1, i_{1}}} d_{1, i_{1}, j}=\bigoplus_{j \in J_{2, i_{2}}} d_{2, i_{2}, j} \\
& \text { and also the condition obtained by exchanging } i_{1} \text { with } i_{2} \text { is satisfied } \\
& \text { and: } \\
& \text { for each } b \in \bigcup_{i \in I_{1}} \bigcup_{j \in J_{1, i}} \operatorname{init}\left(P_{1, i, j}\right) \text { it holds that: } \\
& \text { for all } i_{1} \in I_{1}, J_{1} \subseteq\left\{j \in J_{1, i_{1}} \mid b \in \operatorname{init}\left(P_{1, i_{1}, j}\right)\right\},\left(\mathcal{D}_{1, i_{1}, j ; b}\right)_{j \in J_{1}} \in \mathrm{X}_{j \in J_{1}} \operatorname{der}\left(P_{1, i_{1}, j}, b\right) \\
& \text { satisfying }\left(j \in J_{1} \wedge\left(\operatorname{init}\left(P_{1, i_{1}, k}\right)=\operatorname{init}\left(P_{1, i_{1}, j}\right) \vee \operatorname{init}\left(P_{1, i_{1}, k}\right)=\{b\}\right)\right) \Longrightarrow k \in J_{1} \\
& \text { there exist } i_{2} \in I_{2}, J_{2} \subseteq\left\{j \in J_{2, i_{2}} \mid b \in \operatorname{init}\left(P_{2, i_{2}, j}\right)\right\},\left(\mathcal{D}_{2, i_{2}, j ; b}\right)_{j \in J_{2}} \in \mathrm{X}_{j \in J_{2}} \operatorname{der}\left(P_{2, i_{2}, j}, b\right) \\
& \text { satisfying }\left(j \in J_{2} \wedge\left(\operatorname{init}\left(P_{2, i_{2}, k}\right)=\operatorname{init}\left(P_{2, i_{2}, j}\right) \vee \operatorname{init}\left(P_{2, i_{2}, k}\right)=\{b\}\right)\right) \Longrightarrow k \in J_{2} \\
& \text { such that the following constraint is met: } \\
& \sum_{j \in J_{1}} d_{1, i_{1}, j} \otimes \mathcal{D}_{1, i_{1}, j ; b}=\sum_{j \in J_{2}} d_{2, i_{2}, j} \otimes \mathcal{D}_{2, i_{2}, j ; b} \\
& \text { and also the condition obtained by exchanging } i_{1}, J_{1} \text { with } i_{2}, J_{2} \text { is satisfied } \\
& \text { and also the condition obtained by starting from } b \in \bigcup_{i \in I_{2}} \bigcup_{j \in J_{2, i}} i n i t\left(P_{2, i, j}\right) \text { is satisfied }
\end{aligned}
$$

Table 3. Axiom characterizing $\sim_{T}^{\text {pre }}$
sides of the axiom, the second condition checks whether that trace is executable with the same degree from the other term. To this purpose, we proceed as follows:

- In either term, we consider all possible subsets of $a$-derivative process terms enabling $b$ - respectively indexed by $J_{1}$ and $J_{2}$ - that fulfill the two coherency constraints of Def. 8 , For the first constraint, we require that if a term belongs to the subset, then so do all derivative terms with the same initial actions as that term. For the second constraint, we require that if a derivative term enables only $b$, then that term must belong to the subset.
- The corresponding tuples $\left(\mathcal{D}_{1, i_{1}, j ; b}\right)_{j \in J_{1}}$ and $\left(\mathcal{D}_{2, i_{2}, j ; b}\right)_{j \in J_{2}}$ of $b$-derivative distributions are taken from the Cartesian product of the $b$-derivative sets of the various $a$-derivative process terms, with any such set not being a singleton in the presence of internal nondeterminism, i.e., of several $b$-actions enabled by the corresponding $a$-derivative term.
- The constraint to meet by any pair of matching subsets of $a$-derivative process terms requires that after performing trace $a b$ the same distribution is reached with the same degree $\bigoplus_{j \in J_{1}}\left(d_{1, i_{1}, j} \otimes \bigoplus \mathcal{D}_{1, i_{1}, j ; b}\right)=\bigoplus_{j \in J_{2}}\left(d_{2, i_{2}, j} \otimes\right.$ $\bigoplus \mathcal{D}_{2, i_{2}, j ; b}$. This is jointly expressed by $\phi$-summing up the $b$-derivative distributions of the considered $a$-derivative process terms, with all $D$-values at the beginning of each $b$-derivative distribution being $\otimes$-multiplied by the $D$-value $\triangleright$-prefixing the corresponding $a$-derivative term.

The simplest instance of $\mathcal{A}_{\mathrm{T}, 1}^{\mathrm{pre}}$ is $a \cdot\left(d \triangleright P^{\prime}\right)+a \cdot\left(d \triangleright P^{\prime \prime}\right)=a \cdot\left(d \triangleright P^{\prime} \phi d \triangleright P^{\prime \prime}\right)$ subject to $d=d \oplus d$, thus leading to $a \cdot\left(d \triangleright\left(P^{\prime}+P^{\prime \prime}\right)\right)=a \cdot\left(d \triangleright P^{\prime} \phi d \triangleright P^{\prime \prime}\right)$ subject to $d=d \oplus d$ due to $\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}$ and transitivity, which amounts to the capability of splitting a term belonging to the support of a target distribution into its
summands. The T-shuffling axiom $\mathcal{A}_{\mathrm{T}, 1}^{\text {pre }}$ not only subsumes the two idempotencyrelated axioms $\mathcal{A}_{\mathrm{B}, 1}^{\text {post }}$ and $\mathcal{A}_{\mathrm{B}, 2}^{\text {post }}$ like the B-shuffling axiom $\mathcal{A}_{\mathrm{B}, 1}^{\text {pre }}$, but also the two choice-deferring axioms $\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}$ and $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ :

- For $\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}$, we observe that (i) trace $a$ can be executed with the same degree $d \oplus \bigoplus \mathcal{D}$ both in either summand on the lefthand side and on the righthand side and (ii) any longer trace passes through either $P^{\prime}$ or $P^{\prime \prime}$ with the same multistep degree on both sides.
- For $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$, we observe that (i) trace $a$ can be executed with the same degree $d_{1} \oplus d_{2} \oplus \bigoplus \mathcal{D}$ on both sides and (ii) any trace $a b_{j}$ reaches $d_{1} \otimes \mathcal{D}_{1, j} \oplus d_{2} \otimes \mathcal{D}_{2, j}$ on the lefthand side and $\left(d_{1} \oplus d_{2}\right) \otimes\left(d_{1}^{\prime} \otimes \mathcal{D}_{1, j} \oplus d_{2}^{\prime} \otimes \mathcal{D}_{2, j}\right)$ on the righthand side, which are equal to each other as $\left(d_{1} \oplus d_{2}\right) \otimes d_{i}^{\prime}=d_{i}$ for $i=1,2$.

It is worth noting that $\mathcal{A}_{\mathrm{T}, 1}^{\mathrm{pre}}$ paves the way to the first axiomatization for the trace equivalence over nondeterministic and probabilistic processes studied in [11], which has interesting congruence properties with respect to parallel composition.

We now prove that $\mathcal{A}_{\mathrm{T}}^{\text {pre }}$, the deduction system obtained from $\mathcal{A}$ by adding the T -shuffling axiom $\mathcal{A}_{\mathrm{T}, 1}^{\text {pre }}$, is sound with respect to $\sim_{\mathrm{T}}^{\text {pre }}$.

Theorem 8. Let $P_{1}, P_{2} \in \mathbb{P}$. If $\mathcal{A}_{\mathrm{T}}^{\text {pre }} \vdash P_{1}=P_{2}$, then $P_{1} \sim_{\mathrm{T}}^{\text {pre }} P_{2}$.
The combinatorial nature of $\mathcal{A}_{\mathrm{T}, 1}^{\mathrm{pre}}$ makes the graph rewriting technique used in [548] inapplicable to this axiom, hence the completeness of $\mathcal{A}_{\mathrm{T}}^{\text {pre }}$ is hard to establish. Note that the situation is quite different from the one of $\mathcal{A}_{\mathrm{B}}^{\text {pre }}$, where in the study of completeness the combinatorial nature of $\mathcal{A}_{\mathrm{B}, 1}^{\text {pre }}$ is compensated for by the coinductive nature of $\sim_{B}^{\text {pre }}$.

## B Proofs of Results

## Proof of Prop. 1.

Given a reachability-consistent semiring $\left(D, \oplus, \otimes, 0_{D}, 1_{D}\right)$, a $D$-ULTraS $\mathcal{U}=$ $(S, A, \longrightarrow \mathcal{U})$, and a $D$-measure schema $\mathcal{M}$ for $\mathcal{U}$, we proceed as follows:

1. See the proof of Prop. 3.5(1) of 7 .
2. See the proof of Prop. 3.5(2) of [7. Unlike bisimulation semantics, in the absence of internal nondeterminism we have that $\sim_{\mathrm{T}, \mathcal{M}}^{\text {post }}$ and $\sim_{\mathrm{T}, \mathcal{M}}^{\text {pre }}$ do not necessarily coincide. This can be seen by considering a $\mathbb{B}$-ULTraS starting with an $a$-transition that reaches a distribution with two states respectively having a $b$-transition and a $c$-transition, together with another $\mathbb{B}$-ULTraS starting with a choice between two $a$-transitions respectively followed by a $b$-transition and a $c$-transition. Their initial states are distinguished by $\sim_{\mathrm{T}, \mathcal{M}_{\mathrm{nd}}}^{\text {post }}$ but identified by $\sim_{\mathrm{T}, \mathcal{M}_{\mathrm{nd}}}^{\text {pre }}$ (see Fig. 1 of [7], where $s_{1} \mathcal{\chi}_{\mathrm{T}, \mathcal{M}_{\mathrm{nd}}}^{\text {post }} s_{2}$ while $s_{1} \sim_{\mathrm{T}, \mathcal{M}_{\text {nd }}}^{\text {pre }} s_{2}$ ).
3. We show that, from $\left(s_{1}, s_{2}\right) \in \mathcal{B}$ for some post-bisimulation $\mathcal{B}$, it follows that $(\star)$ for each $\mathcal{Z}_{1}=\left(Z_{1}, A, \longrightarrow \mathcal{Z}_{1}\right) \in \operatorname{Res}^{c}\left(s_{1}\right)-$ resp. $\mathcal{Z}_{2}=\left(Z_{2}, A, \longrightarrow \mathcal{Z}_{2}\right) \in$ Res $^{c}\left(s_{2}\right)$ - there exists $\mathcal{Z}_{2}=\left(Z_{2}, A, \longrightarrow \mathcal{Z}_{2}\right) \in$ Res $^{c}\left(s_{2}\right)-$ resp. $\mathcal{Z}_{1}=\left(Z_{1}, A\right.$, $\left.\longrightarrow \mathcal{Z}_{1}\right) \in \operatorname{Res}^{c}\left(s_{1}\right)$ - such that for all $\alpha \in A^{*}$ it holds that:

$$
\mathcal{M}\left(z_{s_{1}}, \alpha, Z_{1}\right)=\mathcal{M}\left(z_{s_{2}}, \alpha, Z_{2}\right)
$$

Starting from $s_{1}$, we focus on an arbitrary $\mathcal{Z}_{1}=\left(Z_{1}, A, \longrightarrow \mathcal{Z}_{1}\right) \in \operatorname{Res}^{c}\left(s_{1}\right)$, which we assume not to consist of a single state without transitions to avoid trivial cases. Let $z_{s_{1}} \xrightarrow{a} \mathcal{Z}_{1} \Delta_{1}$ be the initial transition of $\mathcal{Z}_{1}$, which we assume to derive from $s_{1} \xrightarrow{a} \mathcal{U} \Gamma_{1}$. Since $\left(s_{1}, s_{2}\right) \in \mathcal{B}$ and $\mathcal{B}$ is a postbisimulation, there must exist $\mathcal{Z}_{2}=\left(Z_{2}, A, \longrightarrow \mathcal{Z}_{2}\right) \in \operatorname{Res}^{c}\left(s_{2}\right)$ with initial transition $z_{s_{2}} \xrightarrow{a} \mathcal{Z}_{2} \Delta_{2}$, which we assume to derive from $s_{2} \xrightarrow{a} u \Gamma_{2}$, such that, in particular, for each $C \subseteq Z_{1} \cup Z_{2}$ whose image via $\operatorname{corr}_{\mathcal{Z}_{1}} \cup \operatorname{corr}_{\mathcal{Z}_{2}}$ is an equivalence class in $S / \mathcal{B}$, it holds that:

$$
\Delta_{1}(C)=\Gamma_{1}\left(\operatorname{corr}_{\mathcal{Z}_{1}}\left(C \cap Z_{1}\right)\right)=\Gamma_{2}\left(\operatorname{corr}_{\mathcal{Z}_{2}}\left(C \cap Z_{2}\right)\right)=\Delta_{2}(C)
$$

Among all the resolutions in $\operatorname{Res}^{c}\left(s_{2}\right)$ satisfying the equality above, we choose as $\mathcal{Z}_{2}$ one that can execute all the traces of $\mathcal{Z}_{1}$ (which must exist otherwise $s_{1}$ could execute a trace not executable by $s_{2}$ and hence $s_{1} \sim_{B}^{\text {post }} s_{2}$ would be contradicted) and only those traces (longer traces can be ruled out via pruning). Given an arbitrary $\alpha \in A^{*}$, we prove property $(\star)$ by proceeding by induction on $|\alpha| \in \mathbb{N}$ :

- If $|\alpha|=0$, i.e., $\alpha=\varepsilon$, then it trivially holds that:

$$
\mathcal{M}\left(z_{s_{1}}, \alpha, Z_{1}\right)=1_{D}=\mathcal{M}\left(z_{s_{2}}, \alpha, Z_{2}\right)
$$

- Let $|\alpha|=n+1$ for some $n \in \mathbb{N}$, with $\alpha=a^{\prime} \alpha^{\prime}$ and $\left|\alpha^{\prime}\right|=n$, and suppose that property ( $\star$ ) holds for each trace of length $n$ when starting from two post-bisimilar states. There are two cases:
- If $a^{\prime} \neq a$, since both $\mathcal{Z}_{1}$ and $\mathcal{Z}_{2}$ start with an $a$-transition, it trivially holds that:

$$
\mathcal{M}\left(z_{s_{1}}, \alpha, Z_{1}\right)=0_{D}=\mathcal{M}\left(z_{s_{2}}, \alpha, Z_{2}\right)
$$

- If $a^{\prime}=a$, we observe that an arbitrary $C \subseteq Z_{1} \cup Z_{2}$, whose image via $\operatorname{corr}_{\mathcal{Z}_{1}} \cup \operatorname{corr}_{\mathcal{Z}_{2}}$ is an equivalence class in $S / \mathcal{B}$, is either reachable via both $a$-transitions, or via neither; moreover, thanks to the coherency of $\mathcal{Z}_{1}$ and $\mathcal{Z}_{2}$, either $\alpha^{\prime}$ is executable in all the states of $C$, or in none of them (this does not necessarily hold in the case of a set of classes). Let $\mathcal{G}$ be the set of subsets of $Z_{1} \cup Z_{2}$, whose images via $\operatorname{corr}_{\mathcal{Z}_{1}} \cup \operatorname{corr}_{\mathcal{Z}_{2}}$ are equivalence classes in $S / \mathcal{B}$, that are reachable via both $a$-transitions (hence $\mathcal{G}$ is finite) and in which $\alpha^{\prime}$ is executable; note that the other subsets do not contribute to $\mathcal{M}\left(z_{s_{1}}, \alpha, Z_{1}\right)$ and $\mathcal{M}\left(z_{s_{2}}, \alpha, Z_{2}\right)$. For each $C \in \mathcal{G}$, given an arbitrary $z_{C, 1} \in C \cap \operatorname{supp}\left(\Delta_{1}\right)$ and an arbitrary $z_{C, 2} \in C \cap \operatorname{supp}\left(\Delta_{2}\right)$ whose corresponding states in $S$ are $s_{C, 1}$ and $s_{C, 2}$, since $s_{C, 1} \sim_{\mathrm{B}}^{\text {post }} s_{C, 2}$ and $\left|\alpha^{\prime}\right|=n$ by the induction hypothesis and the coherency of $\mathcal{Z}_{1}$ and $\mathcal{Z}_{2}$ we have that:

$$
\mathcal{M}\left(z_{C, 1}, \alpha^{\prime}, Z_{1}\right)=\mathcal{M}\left(z_{C, 2}, \alpha^{\prime}, Z_{2}\right)
$$

As a consequence, by the distributivity of $\otimes$ over $\oplus$ and the compositionality of equality with respect to both operations, we have that:

$$
\begin{aligned}
\mathcal{M}\left(z_{s_{1}}, \alpha, Z_{1}\right) & =\bigoplus_{C \in \mathcal{G}}\left(\Delta_{1}(C) \otimes \mathcal{M}\left(z_{C, 1}, \alpha^{\prime}, Z_{1}\right)\right) \\
& =\bigoplus_{C \in \mathcal{G}}\left(\Delta_{2}(C) \otimes \mathcal{M}\left(z_{C, 2}, \alpha^{\prime}, Z_{2}\right)\right) \\
& =\mathcal{M}\left(z_{s_{2}}, \alpha, Z_{2}\right)
\end{aligned}
$$

where finitely many $D$-values occur in both summations because $\mathcal{G}$ is finite.
4. If we consider a $\mathbb{B}$-ULTraS starting with a choice between two $a$-transitions respectively followed by a $b$-transition and a $c$-transition, together with another $\mathbb{B}$-ULTraS starting with an $a$-transition followed by a choice between a $b$-transition and a $c$-transition, then their initial states are distinguished by $\sim_{\mathrm{B}}^{\text {pre }}$ but identified by $\sim_{\mathrm{T}, \mathcal{M}_{\mathrm{nd}}}^{\text {post }}$ and $\sim_{\mathrm{T}, \mathcal{M}_{\mathrm{nd}}}^{\text {pre }}$ (see Fig. 1 of [7], where $s_{2} \chi_{\mathrm{B}}^{\text {pre }} s_{3}$ while $s_{2} \sim_{T, \mathcal{M}_{\mathrm{nd}}}^{\text {post }} s_{3}$ and $\left.s_{2} \sim_{\mathrm{T}, \mathcal{M}_{\mathrm{nd}}}^{\text {pre }} s_{3}\right)$.
On the other hand, in Fig. 11 it holds that $s_{1} \sim_{\mathrm{B}}^{\text {pre }} s_{2}$ whereas $s_{1} \mathcal{\chi}_{\mathrm{T}, \mathcal{M}}^{\text {post }} s_{2}$; moreover $s_{1} \mathcal{\chi}_{\mathrm{T}, \mathcal{M}}^{\text {pre }} s_{2}$ if $r_{1}$ (resp. $r_{2}$ ) has a $b$-transition that reaches with degree $d_{b}^{\prime}$ (resp. $\left.d_{b}^{\prime \prime}\right)$ a terminal state, whenever degrees $\left(d_{1} \otimes d_{b}^{\prime}\right) \oplus\left(d_{2} \otimes d_{b}^{\prime \prime}\right)$ and $\left(d_{2} \otimes d_{b}^{\prime}\right) \oplus\left(d_{1} \otimes d_{b}^{\prime \prime}\right)$ associated with trace $a b$ - which we assume not to be executable via $r_{3}$ - are different from each other as well as from $d_{1} \otimes d_{b}^{\prime}$ and $d_{2} \otimes d_{b}^{\prime \prime}$.

Proof of Thm. 1 .
Let $P_{1} \sim_{\mathcal{M}} P_{2}$ and $\mathcal{D}_{1} \sim_{\mathcal{M}} \mathcal{D}_{2}$ for $\sim_{\mathcal{M}} \in\left\{\sim_{\mathrm{B}}^{\text {post }}, \sim_{\mathrm{B}}^{\text {pre }}, \sim_{\mathrm{T}, \mathcal{M}}^{\text {post }}, \sim_{\mathrm{T}, \mathcal{M}}^{\text {pre }}\right\}$ :

1. From $d \triangleright P_{1} \longmapsto\left\{\left(P_{1}, d\right)\right\}=\Delta_{1}$ and $d \triangleright P_{2} \longmapsto\left\{\left(P_{2}, d\right)\right\}=\Delta_{2}$, it follows that $\Delta_{1}(C)=\Delta_{2}(C)=d$ for the only equivalence class containing $P_{1}$ and $P_{2}$, while $\Delta_{1}(C)=\Delta_{2}(C)=0_{D}$ for all the other classes $C \in \mathbb{P} / \sim_{\mathcal{M}}$. This means that $d \triangleright P_{1} \sim_{\mathcal{M}} d \triangleright P_{2}$.
2. From $\mathcal{D}_{1} \sim_{\mathcal{M}} \mathcal{D}_{2}$, it follows that $\Delta_{1}(C)=\Delta_{2}(C)$ for all $C \in \mathbb{P} / \sim_{\mathcal{M}}$, with $\mathcal{D}_{1} \longmapsto \Delta_{1}$ and $\mathcal{D}_{2} \longmapsto \Delta_{2}$. Assuming $\mathcal{D} \longmapsto \Delta$, we have that $\mathcal{D}_{1} \oplus \mathcal{D} \longmapsto$ $\Delta_{1} \oplus \Delta$ and $\mathcal{D}_{2} \oplus \mathcal{D} \longmapsto \Delta_{2} \oplus \Delta$ with $\left(\Delta_{1} \oplus \Delta\right)(C)=\Delta_{1}(C) \oplus \Delta(C)=$
$\Delta_{2}(C) \oplus \Delta(C)=\left(\Delta_{2} \oplus \Delta\right)(C)$ for all $C \in \mathbb{P} / \sim_{\mathcal{M}}$, i.e., $\mathcal{D}_{1} \notin \mathcal{D} \sim_{\mathcal{M}} \mathcal{D}_{2} \phi \mathcal{D}$. The proof of $\mathcal{D} \notin \mathcal{D}_{1} \sim_{\mathcal{M}} \mathcal{D} \phi \mathcal{D}_{2}$ is analogous.
3. From $\mathcal{D}_{1} \sim_{\mathcal{M}} \mathcal{D}_{2}$, it follows that $\Delta_{1}(C)=\Delta_{2}(C)$ for all $C \in \mathbb{P} / \sim_{\mathcal{M}}$, with $\mathcal{D}_{1} \longmapsto \Delta_{1}$ and $\mathcal{D}_{2} \longmapsto \Delta_{2}$, hence in particular it holds that for each $P_{1} \in \operatorname{supp}\left(\Delta_{1}\right)$ there must exist $P_{2} \in \operatorname{supp}\left(\Delta_{2}\right)$ such that $P_{1} \sim_{\mathcal{M}} P_{2}$, and vice versa.
If $\sim_{\mathcal{M}} \in\left\{\sim_{B}^{\text {post }}, \sim_{B}^{\text {pre }}\right\}$, we proceed as in the proof of Thm. 4.1 of [7], after observing that the starting point is a bisimulation witnessing all the aforementioned $\sim_{\mathcal{M}}$-identifications between processes in $\operatorname{supp}\left(\Delta_{1}\right)$ and processes in $\operatorname{supp}\left(\Delta_{2}\right)$.
If $\sim_{\mathcal{M}} \in\left\{\sim_{\mathrm{T}, \mathcal{M}}^{\text {post }}, \sim_{\mathrm{T}, \mathcal{M}}^{\text {pre }}\right\}$, the only interesting case is the one in which we consider a trace of the form $\alpha=a \alpha^{\prime} \in A^{*}$ and for $k \in\{1,2\}$ a resolution $\mathcal{Z}_{k}=\left(Z_{k}, A, \longrightarrow \mathcal{Z}_{k}\right) \in \operatorname{Res}^{c}\left(a . \mathcal{D}_{k}\right)$ that starts with an $a$-transition. This $a$ transition reaches with degree $d_{C}=\Delta_{k}(C)$ the set of processes in $\mathcal{Z}_{k}$ whose corresponding original processes via $\operatorname{corr}_{\mathcal{Z}_{k}}$ are in the same equivalence class $C \in \mathbb{P} / \sim_{\mathcal{M}}$. The reason is that, thanks to the coherency of $\mathcal{Z}_{k}$, two processes in the support of the target distribution of the considered $a$-transition of $\mathcal{Z}_{k}$ must possess the same traces if so do their corresponding processes in $\operatorname{supp}\left(\Delta_{k}\right)$, as is the case with the processes in $C$.
Given $P_{k, C} \in C \cap \operatorname{supp}\left(\Delta_{k}\right)$ for some $C \in \mathbb{P} / \sim_{\mathcal{M}}$ and $\mathcal{Z}_{k, C}=\left(Z_{k, C}, A\right.$, $\left.\longrightarrow \mathcal{Z}_{k, C}\right) \in \operatorname{Res}^{c}\left(P_{k, C}\right)$ part of $\mathcal{Z}_{k}$, for any other $P_{k, C}^{\prime} \in C \cap \operatorname{supp}\left(\Delta_{k}\right)$ we observe that $\mathcal{Z}_{k, C}^{\prime}=\left(Z_{k, C}^{\prime}, A, \longrightarrow \mathcal{Z}_{k, C}^{\prime}\right) \in \operatorname{Res}^{c}\left(P_{k, C}^{\prime}\right)$ part of $\mathcal{Z}_{k}$ must match $\mathcal{Z}_{k, C}$ with respect to all traces as $P_{k, C} \sim_{\mathcal{M}} P_{k, C}^{\prime}$ implies $z_{P_{k, C}} \sim_{\mathcal{M}} z_{P_{k, C}^{\prime}}$ due to the coherency of $\mathcal{Z}_{k}$ and the absence of nondeterminism in $\mathcal{Z}_{k, C}$ and $\mathcal{Z}_{k, C}^{\prime}$.
Starting from $a \cdot \mathcal{D}_{1}$, there are two cases:

- When $\sim_{\mathcal{M}}=\sim_{T, \mathcal{M}}^{\text {post }}$, we have that:

$$
\begin{aligned}
\mathcal{M}\left(z_{a, \mathcal{D}_{1}}, \alpha, Z_{1}\right) & =\underset{\operatorname{C\cap supp}\left(\Delta_{1}\right) \neq \emptyset}{\bigoplus}\left(d_{C} \otimes \mathcal{M}\left(z_{P_{1, C}}, \alpha^{\prime}, Z_{1, C}\right)\right) \\
& =\underset{C \cap s u p p\left(\Delta_{2}\right) \neq \emptyset}{ }\left(d_{C} \otimes \mathcal{M}\left(z_{P_{2, C}}, \alpha^{\prime}, Z_{2, C}\right)\right) \\
& =\mathcal{M}\left(z_{a} \cdot \mathcal{D}_{2}, \alpha, Z_{2}\right)
\end{aligned}
$$

where the existence of $\mathcal{Z}_{2, C}=\left(Z_{2, C}, A, \longrightarrow \mathcal{Z}_{2, C}\right) \in \operatorname{Res}^{c}\left(P_{2, C}\right)$ matching $\mathcal{Z}_{1, C}$ with respect to all traces is a consequence of the existence mentioned at the beginning of the proof - of $P_{2, C} \in \operatorname{supp}\left(\Delta_{2}\right)$ such that
 with an $a$-transition and continues as $\mathcal{Z}_{2, C}$ for $P_{2, C} \in C \cap \operatorname{supp}\left(\Delta_{2}\right)$, matches $\mathcal{Z}_{1}$ with respect to all traces.

- When $\sim_{\mathcal{M}}=\sim_{\mathrm{T}, \mathcal{M}}^{\text {pre }}$, we have that:

$$
\begin{aligned}
\mathcal{M}\left(z_{a, \mathcal{D}_{1}}, \alpha, Z_{1}\right) & =\underset{\operatorname{C\cap supp(\Delta _{1})\neq \emptyset }}{\bigoplus}\left(d_{C} \otimes \mathcal{M}\left(z_{P_{1, C}}, \alpha^{\prime}, Z_{1, C}\right)\right) \\
& =\underset{\substack{ \\
\oplus \\
\text { spp } p\left(\Delta_{2}\right) \neq \emptyset}}{ }\left(d_{C} \otimes \mathcal{M}\left(z_{P_{2, C},}, \alpha^{\prime}, Z_{2, C, \alpha^{\prime}}\right)\right) \\
& =\mathcal{M}\left(z_{a \cdot \mathcal{D}_{2}}, \alpha, Z_{2, \alpha}\right)
\end{aligned}
$$

where the existence of $\mathcal{Z}_{2, C, \alpha^{\prime}}=\left(Z_{2, C, \alpha^{\prime}}, A, \longrightarrow \mathcal{Z}_{2, C, \alpha^{\prime}}\right) \in \operatorname{Res}^{c}\left(P_{2, C}\right)$ matching $\mathcal{Z}_{1, C}$ with respect to $\alpha^{\prime}$ is a consequence of the existence -
mentioned at the beginning of the proof - of $P_{2, C} \in \operatorname{supp}\left(\Delta_{2}\right)$ such that $P_{1, C} \sim_{\mathrm{T}, \mathcal{M}}^{\text {pre }} P_{2, C}$. Therefore $\mathcal{Z}_{2, \alpha}=\left(Z_{2, \alpha}, A, \longrightarrow \mathcal{Z}_{2, \alpha}\right) \in \operatorname{Res}^{\mathrm{c}}\left(s_{2}\right)$, which starts with an $a$-transition and continues as $\mathcal{Z}_{2, C, \alpha^{\prime}}$ for $P_{2, C} \in$ $C \cap \operatorname{supp}\left(\Delta_{2}\right)$, matches $\mathcal{Z}_{1}$ with respect to $\alpha$.
4. If $\sim_{\mathcal{M}} \in\left\{\sim_{\mathrm{B}}^{\text {post }}, \sim_{\mathrm{B}}^{\text {pre }}\right\}$, we proceed as in the proof of Thm. 4.5 of [7].

If $\sim_{\mathcal{M}} \in\left\{\sim_{\mathrm{T}, \mathcal{M}}^{\text {post }}, \sim_{\mathrm{T}, \mathcal{M}}^{\text {pre }}\right\}$, we proceed as in the proof of Thm. 4.6 of [7].
Proof of Lemma 1,
We proceed by induction on the syntactical structure of $P$ :

- If $P$ is $\underline{0}$, then the result follows by taking $Q$ equal to $\underline{0}$ and using reflexivity.
- If $P$ is $a . \mathcal{D}$, i.e., $a .\left(\sum_{j \in J} d_{j} \triangleright P_{j}\right)$ with $J$ being a finite nonempty index set due to the syntax rule for $\mathcal{D}$, then by the induction hypothesis for all $j \in J$ there exists $Q_{j}$ in snf such that $\mathcal{A} \vdash P_{j}=Q_{j}$. The result follows by substitutivity with respect to $\triangleright, \notin$, and action prefix, after observing that $a \cdot \sum_{j \in J} d_{j} \triangleright Q_{j}$ is in snf.
- If $P$ is $P_{1}+P_{2}$, then by the induction hypothesis there exist $Q_{1}$ and $Q_{2}$ in snf such that $\mathcal{A} \vdash P_{1}=Q_{1}$ and $\mathcal{A} \vdash P_{2}=Q_{2}$. The result follows by substitutivity with respect to choice, as $Q_{1}+Q_{2}$ is in snf after removing via $\mathcal{A}_{3}$ a possible $\underline{0}$-summand between $Q_{1}$ and $Q_{2}$.


## Proof of Thm. 2.

A straightforward consequence of the definition of $\mathcal{A}_{\mathrm{B}}^{\text {post }}$, the fact that $\sim_{B}^{\text {post }}$ is an equivalence relation, Thm. 11 and the fact that the lefthand side term of each core axiom as well as of $\mathcal{A}_{\mathrm{B}, 1}^{\text {post }}$ and $\mathcal{A}_{\mathrm{B}, 2}^{\text {post }}$ is $\sim_{\mathrm{B}}^{\text {post }}$-equivalent to the term on the righthand side of the same axiom.

## Proof of Thm. 3.

Without loss of generality, we suppose that $P_{1}$ and $P_{2}$ are both in snf. Should this not be the case, thanks to Lemma 1 we could find $Q_{1}$ and $Q_{2}$ in snf such that $\mathcal{A}_{\mathrm{B}}^{\text {post }} \vdash P_{1}=Q_{1}$ and $\mathcal{A}_{\mathrm{B}}^{\text {post }} \vdash P_{2}=Q_{2}$. Due to Thm. 2 we would then derive $P_{1} \sim_{\mathrm{B}}^{\text {post }} Q_{1}$, hence $Q_{1} \sim_{\mathrm{B}}^{\text {post }} P_{1}$ as $\sim_{\mathrm{B}}^{\text {post }}$ is symmetric, and $P_{2} \sim_{\mathrm{B}}^{\text {post }} Q_{2}$. Since $P_{1} \sim_{\mathrm{B}}^{\text {post }} P_{2}$, we would also derive $Q_{1} \sim_{\mathrm{B}}^{\text {post }} Q_{2}$ as $\sim_{\mathrm{B}}^{\text {post }}$ is transitive. As a consequence, proving $Q_{1} \sim_{\mathrm{B}}^{\text {post }} Q_{2} \Longrightarrow \mathcal{A}_{\mathrm{B}}^{\text {post }} \vdash Q_{1}=Q_{2}$ would finally entail $\mathcal{A}_{\mathrm{B}}^{\text {post }} \vdash P_{1}=P_{2}$ by symmetry (applied to $\mathcal{A}_{\mathrm{B}}^{\text {post }} \vdash P_{2}=Q_{2}$ ) and transitivity. We thus proceed by induction on the syntactical structure of $P_{1}$ in snf:

- If $P_{1}$ is $\underline{0}$, then from $P_{1} \sim_{\mathrm{B}}^{\text {post }} P_{2}$ and $P_{2}$ in snf we derive that $P_{2}$ can only be $\underline{0}$, from which the result follows by reflexivity.
- If $P_{1}$ is $\sum_{i \in I_{1}} a_{1, i} \cdot\left(\sum_{j \in J_{1, i}} d_{1, i, j} \triangleright P_{1, i, j}\right)$, then from $P_{1} \sim_{\mathrm{B}}^{\text {post }} P_{2}$ and $P_{2}$ in snf we derive that $P_{2}$ can only be $\sum_{i \in I_{2}} a_{2, i} .\left(\sum_{j \in J_{2, i}} d_{2, i, j} \triangleright P_{2, i, j}\right)$.
Since $P_{1} \sim_{\mathrm{B}}^{\text {post }} P_{2}$, for each $i_{1} \in I_{1}$ there exists $i_{2} \in I_{2}$ such that $a_{2, i_{2}}=a_{1, i_{1}}$ and $\bigoplus_{P_{1, i_{1}, j} \in \cup \mathcal{G}} d_{1, i_{1}, j}=\bigoplus_{P_{2, i_{2}, j} \in \cup \mathcal{G}} d_{2, i_{2}, j}$ for each set $\mathcal{G}$ of equivalence classes, and vice versa, implying that both summands respectively indexed by $i_{1}$ and $i_{2}$ have process terms in the same classes. Since the various $P_{1, i_{1}, j}$ and $P_{2, i_{2}, j}$ related by $\sim_{\mathrm{B}}^{\text {post }}$ are in snf and hence can be equated by
$\mathcal{A}_{\mathrm{B}}^{\text {post }}$ thanks to the induction hypothesis, $a_{1, i_{1}} \cdot\left(\sum_{j \in J_{1, i_{1}}} d_{1, i_{1}, j} \triangleright P_{1, i_{1}, j}\right)$ and $a_{2, i_{2}} \cdot\left(\sum_{j \in J_{2, i_{2}}} d_{2, i_{2}, j} \triangleright P_{2, i_{2}, j}\right)$ can be equated by $\mathcal{A}_{\mathrm{B}}^{\text {post }}$ due to substitutivity with respect to $\triangleright, \notin$, and action prefix as well as possible applications of $\mathcal{A}_{\mathrm{B}, 2}^{\text {post }}$ (preceded by the necessary applications of $\mathcal{A}_{4}$ and $\mathcal{A}_{5}$ to make identically targeted -summands next to each other). Should there exist $k_{2} \in I_{2} \backslash\left\{i_{2}\right\}$ such that $a_{2, k_{2}}=a_{1, i_{1}}$ and $\bigoplus_{P_{1, i_{1}, j} \in \cup \mathcal{G}} d_{1, i_{1}, j}=\bigoplus_{P_{2, k_{2}, j} \in \cup \mathcal{G}} d_{2, k_{2}, j}$ for each set $\mathcal{G}$ of equivalence classes, due to the same reasons $a_{2, i_{2}} \cdot\left(\sum_{j \in J_{2, i_{2}}} d_{2, i_{2}, j} \triangleright P_{2, i_{2}, j}\right)$ and $a_{2, k_{2}} \cdot\left(\sum_{j \in J_{2, k_{2}}} d_{2, k_{2}, j} \triangleright P_{2, k_{2}, j}\right)$ could be equated by $\mathcal{A}_{\mathrm{B}}^{\text {post }}$. The result finally follows from substitutivity with respect to + as well as possible applications of $\mathcal{A}_{\mathrm{B}, 1}^{\text {post }}$ (preceded by the necessary applications of $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ to make identical + -summands next to each other).

Proof of Thm. 4.
A straightforward consequence of the definition of $\mathcal{A}_{\mathrm{B}}^{\text {pre }}$, the fact that $\sim_{B}^{\text {pre }}$ is an equivalence relation, Thm. 1] and the fact that the lefthand side term of each core axiom as well as of $\mathcal{A}_{\mathrm{B}, 1}^{\mathrm{pre}}$ is $\sim_{\mathrm{B}}^{\text {pre }}$-equivalent to the term on the righthand side of the same axiom (under the conditions of the axiom itself in the case of $\left.\mathcal{A}_{\mathrm{B}, 1}^{\mathrm{pre}}\right)$.

## Proof of Thm. 5.

The proof is similar to the one of Thm. 3. but when $P_{1}$ is $\sum_{i \in I_{1}} a_{1, i} \cdot\left(\sum_{j \in J_{1, i}} d_{1, i, j}\right.$ $\left.\triangleright P_{1, i, j}\right)$ and $P_{2}$ is $\sum_{i \in I_{2}} a_{2, i} .\left(\sum_{j \in J_{2, i}} d_{2, i, j} \triangleright P_{2, i, j}\right)$, with both terms in snf, it changes as follows.
Since $P_{1} \sim_{\mathrm{B}}^{\text {pre }} P_{2}$, for each action $a$ and for each set $\mathcal{G}$ of equivalence classes it holds that for each $i_{1} \in I_{1}$ there exists $i_{2} \in I_{2}$ such that $a_{1, i_{1}}=a_{2, i_{2}}=a$ and $\bigoplus_{P_{1, i_{1}, j} \in \cup \mathcal{G}} d_{1, i_{1}, j}=\bigoplus_{P_{2, i_{2}, j} \in \cup \mathcal{G}} d_{2, i_{2}, j}$, and vice versa, implying that both summands respectively indexed by $i_{1}$ and $i_{2}$ have process terms in $\bigcup \mathcal{G}$ or neither has and that, more generally, $\left\{a_{1, i} \mid i \in I_{1}\right\}=\left\{a_{2, i} \mid i \in I_{2}\right\}$. Since the various $P_{1, i_{1}, j}$ and $P_{2, i_{2}, j}$ related by $\sim_{\mathrm{B}}^{\text {pre }}$ are in snf and hence can be equated by $\mathcal{A}_{\mathrm{B}}^{\text {pre }}$ thanks to the induction hypothesis, all those derivative terms belonging to the same equivalence class can be equated via $\mathcal{A}_{\mathrm{B}}^{\text {pre }}$ to the same process term, with the latter being used to replace all the former within $P_{1}$ and $P_{2}$ to obtain $P_{1}^{\prime}$ and $P_{2}^{\prime}$ in snf such that $\mathcal{A}_{\mathrm{B}}^{\text {pre }} \vdash P_{1}^{\prime}=P_{1}$ and $\mathcal{A}_{\mathrm{B}}^{\text {pre }} \vdash P_{2}^{\prime}=P_{2}$ due to substitutivity with respect to $\triangleright, \notin$, action prefix, and + .
From Thm. 4 we obtain $P_{1}^{\prime} \sim_{\mathrm{B}}^{\text {pre }} P_{1}$ and $P_{2}^{\prime} \sim_{\mathrm{B}}^{\text {pre }} P_{2}$, hence $P_{1}^{\prime} \sim_{\mathrm{B}}^{\text {pre }} P_{2}^{\prime}$ because $P_{1} \sim_{\mathrm{B}}^{\text {pre }} P_{2}$ and $\sim_{\mathrm{B}}^{\text {pre }}$ is symmetric and transitive. From $P_{1}^{\prime} \sim_{\mathrm{B}}^{\text {pre }} P_{2}^{\prime}$ and $\sim_{\mathrm{B}}^{\text {pre }}$-equivalent derivatives of $P_{1}^{\prime}$ and $P_{2}^{\prime}$ being identical, we then derive $\mathcal{A}_{\mathrm{B}}^{\text {pre }} \vdash P_{1}^{\prime}=P_{2}^{\prime}$ via possible applications of $\mathcal{A}_{\mathrm{B}, 1}^{\text {pre }}$ to the choices among all summands in $P_{1}^{\prime}$ and $P_{2}^{\prime}$ starting with the same action (preceded by the necessary applications of $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ to make those summands next to each other), followed by substitutivity with respect to + . The result finally stems from symmetry (applied to $\mathcal{A}_{\mathrm{B}}^{\text {pre }} \vdash P_{1}^{\prime}=P_{1}$ ) and transitivity.
Proof of Thm. 6.
A straightforward consequence of the definition of $\mathcal{A}_{\mathrm{T}}^{\text {post }}$, the fact that $\sim_{\mathrm{T}}^{\text {post }}$ is an equivalence relation, Thm. 1, and Thm. 2 together with Prop. 11(3) and
the fact that the lefthand side term of each of the axioms $\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}$ and $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ is $\sim_{\mathrm{T}}^{\text {post }}$-equivalent to the term on the righthand side of the same axiom under the conditions of the axiom itself.
As far as $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ is concerned, we observe that the deferral of the distribution choice preserves the trace distributions when moving from the lefthand side term to the righthand side one, hence coherency fits smoothly with the deferral both in the case in which within some $\mathcal{D}_{1, j_{1}}$ or $\mathcal{D}_{2, j_{2}}$ there are $\sim_{\mathrm{T}}^{\text {post }}$-equivalent process terms, and in the case in which some $\mathcal{D}_{1, j_{1}}$ is $\sim_{\mathrm{T}}^{\text {post }}$-equivalent to some $\mathcal{D}_{2, j_{2}}$. We point out that coherency would not fit if looser coherency constraints, based on weighted trace sets rather than trace distributions as in Def. 8, would have been adopted. Similar to $T D^{c}(s)$ in Def. 6, one may define $T^{c}(s)$ by considering all weighted traces executable from $s$ at once - i.e., without keeping track of the resolutions in which they are feasible - and use it for coherency purposes, but then axiom $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ would not be sound as can be seen from the example in Fig. 9 of [8].

## Proof of Lemma 2.

Since the ULTraS $\mathcal{U}$ underlying $P$ is a directed acyclic finite graph, the application to $\mathcal{U}$ of the graph rewriting system based on $\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}$ and $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ is terminating, i.e., any reduction sequence leads in finitely many steps to a directed acyclic finite graph $\mathcal{U}^{\prime}$ that cannot be further transformed. Moreover, due to the soundness of $\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}$ and $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ with respect to $\sim_{\mathrm{T}}^{\text {post }}$ established by Thm. 6, any graph transformation step preserves $\sim_{\mathrm{T}}^{\text {post }}$. Thanks to $\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}$ and $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ being part of $\mathcal{A}_{\mathrm{T}}^{\text {post }}$ and substitutivity, each such step is derivable in $\mathcal{A}_{\mathrm{T}}^{\text {post }}$, to which Lemma 1 applies too as $\mathcal{A}_{\mathrm{T}}^{\text {post }}$ includes $\mathcal{A}$, thereby yielding a $\sim_{\mathrm{T}}^{\text {post }}$-snf.

## Proof of Lemma 3.

Given $P_{1}, P_{2} \in \mathbb{P}$ in $\sim_{\mathrm{T}}^{\text {post }}$-snf, we only need to prove $P_{1} \sim_{\mathrm{T}}^{\text {post }} P_{2} \Longrightarrow P_{1} \sim_{\mathrm{B}}^{\text {post }} P_{2}$ as the reverse implication is Prop. 1(3). Denoting with $\mathbb{P}^{\prime}$ the set of UProC terms in $\sim_{\mathrm{T}}^{\text {post }}$-snf, the result will follow by showing that the equivalence relation below is a post-bisimulation:

$$
\mathcal{B}=\left\{\left(Q_{1}, Q_{2}\right) \in \mathbb{P}^{\prime} \times \mathbb{P}^{\prime} \mid Q_{1} \sim_{\mathrm{T}}^{\text {post }} Q_{2}\right\}
$$

where with respect to Def. 2 we will consider only individual equivalence classes because unlike pre-bisimilarity, for which addressing also sets of several equivalence classes matters in terms of discriminating power, for post-bisimilarity individual classes are sufficient.
Observing that $(\underline{0}, \underline{0}) \in \mathcal{B}$, we examine $\left(Q_{1}, Q_{2}\right) \in \mathcal{B}$ such that $Q_{1}$ and $Q_{2}$ are both of the form $\sum_{i \in I} a_{i} \cdot\left(\sum_{j \in J_{i}} d_{i, j} \triangleright Q_{i, j}\right)$ with $I$ and $J_{i}$ finite and nonempty. Since $Q_{1} \sim_{\mathrm{T}}^{\text {post }} Q_{2}$, there cannot be any action enabled by only one of the two considered terms. Thus, in the post-bisimulation game, we can focus on an arbitrary action $a$ enabled by both $Q_{1}$ and $Q_{2}$. There are three cases related to the number of $a$-transitions departing from $Q_{1}$ and $Q_{2}$ :

1. If $Q_{1} \xrightarrow{a} \Delta_{1}$ and $Q_{2} \xrightarrow{a} \Delta_{2}$ are the only $a$-transitions of $Q_{1}$ and $Q_{2}$, then they must match each other in the post-bisimulation game under $\mathcal{B}$, meaning that $\Delta_{1} \sim_{\mathrm{T}}^{\text {post }} \Delta_{2}$. This can be shown by examining the reachability via
those two $a$-transitions of an arbitrary equivalence class $C \in \mathbb{P}^{\prime} / \mathcal{B}$ satisfying $C \cap \operatorname{supp}\left(\Delta_{1}\right) \neq \emptyset$ and $C \cap \operatorname{supp}\left(\Delta_{2}\right) \neq \emptyset$.
Note that there cannot be any class intersecting only one of those two supports. Indeed, due to $D_{\mathrm{T}, 2}^{\text {post }}$ and $\sim_{\mathrm{T}}^{\text {post }}$-snf, in either support any two terms $Q^{\prime}$ and $Q^{\prime \prime}$ must enable two different sets of actions - otherwise $\mathcal{A}_{\mathrm{T}, 2}^{\text {post }}$ would be applicable - and hence $Q^{\prime} \chi_{\mathrm{T}}^{\text {post }} Q^{\prime \prime}$. This means that either support intersects as many classes in $\mathbb{P}^{\prime} / \mathcal{B}$ as there are process terms in the support itself so, by virtue of the uniqueness of the two $a$-transitions of $Q_{1}$ and $Q_{2}$ and $Q_{1} \sim_{\mathrm{T}}^{\text {post }} Q_{2}$, we have that every class intersecting either support must then intersect also the other, with each intersection being a singleton.
To exploit the aforementioned information in the post-bisimulation game under $\mathcal{B}$, for $k \in\{1,2\}$ we build a coherent resolution $\mathcal{Z}_{k}$ starting with an $a$-transition, to which the only $a$-transition of $Q_{k}$ corresponds, that leads to a distribution $\Gamma_{k}$ whose support contains terms from which only traces of length 1 can be executed. Since the various terms in $\operatorname{supp}\left(\Delta_{k}\right)$ enable different sets of actions, the first coherency constraint of Def. 8 is trivially satisfied by $\mathcal{Z}_{k}$, while the second coherency constraint requires that every term in $\operatorname{supp}\left(\Gamma_{k}\right)$ has an outgoing transition unless its corresponding original term is $\underline{0}$.
Since we are not guaranteed that different actions can be selected from different terms in $\operatorname{supp}\left(\Delta_{k}\right)$ - think, e.g., of $\operatorname{supp}\left(\Delta_{k}\right)$ containing a term enabling only $a$, a term enabling only $b$, and a term enabling both $a$ and $b$ only - we consider each term of $\operatorname{supp}\left(\Delta_{k}\right)$ in the context of ${ }_{-}+b_{C} \cdot\left(d_{b_{C}} \triangleright \underline{0}\right)$, where the fresh action $b_{C}$ and its corresponding degree $d_{b_{C}}$ depend on the equivalence class $C$ to which the original term belongs. Thanks to Thm. 1, the term derived from the one in $C \cap \operatorname{supp}\left(\Delta_{1}\right)$ and the term derived from the one in $C \cap \operatorname{supp}\left(\Delta_{2}\right)$ are $\sim_{\mathrm{T}}^{\text {post }}$-equivalent. As a consequence, the terms $Q_{1}^{\prime}$ and $Q_{2}^{\prime}$ respectively derived from $Q_{1}$ and $Q_{2}$ are $\sim_{\mathrm{T}}^{\text {post }}$-equivalent too, as well as in $\sim_{\mathrm{T}}^{\text {post }}{ }_{\text {-snf. }}$.
Let $\Delta_{1}^{\prime}$ and $\Delta_{2}^{\prime}$ be the reachability distributions respectively derived from $\Delta_{1}$ and $\Delta_{2}$, so that $Q_{1}^{\prime} \xrightarrow{a} \Delta_{1}^{\prime}$ and $Q_{2}^{\prime} \xrightarrow{a} \Delta_{2}^{\prime}$ with $\left|\operatorname{supp}\left(\Delta_{k}^{\prime}\right)\right|=\left|\operatorname{supp}\left(\Delta_{k}\right)\right|$ and $\Delta_{k}^{\prime}\left(P^{\prime}\right)=\Delta_{k}(P)$ for $P^{\prime}$ being $P+b_{C} \cdot\left(d_{b_{C}} \triangleright \underline{0}\right)$. From $Q_{1}^{\prime} \sim_{\mathrm{T}}^{\text {post }} Q_{2}^{\prime}$ and the uniqueness of the two $a$-transitions of $Q_{1}^{\prime}$ and $Q_{2}^{\prime}$, it follows that $\mathcal{Z}_{1} \in \operatorname{Res}^{\mathrm{c}}\left(Q_{1}^{\prime}\right)$ must be matched by $\mathcal{Z}_{2} \in \operatorname{Res}^{\mathrm{c}}\left(Q_{2}^{\prime}\right)$ with respect to all traces, and vice versa, where $z_{Q_{1}^{\prime}} \xrightarrow{a} \Gamma_{1}$ and $z_{Q_{2}^{\prime}} \xrightarrow{a} \Gamma_{2}$ with every term in $\operatorname{supp}\left(\Gamma_{k}\right)$ enabling only the corresponding action $b_{C}$. In particular, for every equivalence class $C^{\prime}$ derived from $C$ intersecting both $\operatorname{supp}\left(\Delta_{1}\right)$ and $\operatorname{supp}\left(\Delta_{2}\right)$, it holds that:

$$
\mathcal{M}\left(z_{Q_{1}^{\prime}}, a b_{C}, Z_{1}\right)=\mathcal{M}\left(z_{Q_{2}^{\prime}}, a b_{C}, Z_{2}\right)
$$

where for $k \in\{1,2\}$ :

$$
\mathcal{M}\left(z_{Q_{k}^{\prime}}, a b_{C}, Z_{k}\right)=\Delta_{k}^{\prime}\left(C^{\prime}\right) \otimes d_{b_{C}}
$$

and hence:

$$
\Delta_{1}(C)=\Delta_{1}^{\prime}\left(C^{\prime}\right)=\Delta_{2}^{\prime}\left(C^{\prime}\right)=\Delta_{2}(C)
$$

2. Suppose that $Q_{1}$ has a single $a$-transition $Q_{1} \xrightarrow{a} \Delta_{1}$ while $Q_{2}$ has two $a$ transitions $Q_{2} \xrightarrow{a} \Delta_{2}^{\prime}$ and $Q_{2} \xrightarrow{a} \Delta_{2}^{\prime \prime}$, where $\Delta_{2}^{\prime} \neq \Delta_{2}^{\prime \prime}$, i.e., $\operatorname{supp}\left(\Delta_{2}^{\prime}\right) \neq$
$\operatorname{supp}\left(\Delta_{2}^{\prime \prime}\right)$ or $\Delta_{2}^{\prime}(Q) \neq \Delta_{2}^{\prime \prime}(Q)$ for some $Q \in \operatorname{supp}\left(\Delta_{2}^{\prime}\right) \cup \operatorname{supp}\left(\Delta_{2}^{\prime \prime}\right)$, by virtue of the operational semantics rules of UProC. Since $Q_{2}$ is in $\sim_{T}^{\text {post }}$-snf, it also holds that $\Delta_{2}^{\prime} \chi_{\mathrm{T}}^{\text {post }} \Delta_{2}^{\prime \prime}$, i.e., $\Delta_{2}^{\prime}(C) \neq \Delta_{2}^{\prime \prime}(C)$ for some $C \in \mathbb{P}^{\prime} / \mathcal{B}$, otherwise $\mathcal{A}_{\mathrm{T}, 1}^{\mathrm{post}}$ would be applicable to $Q_{2}$ yielding a single $a$-derivation for $Q_{2}$ itself. The validity of $\Delta_{2}^{\prime} \neq \Delta_{2}^{\prime \prime}$ and $\Delta_{2}^{\prime} \chi_{\mathrm{T}}^{\text {post }} \Delta_{2}^{\prime \prime}$ avoids any overlapping with case 1 , as it excludes the possibility for both $a$-transitions of $Q_{2}$ of being matched by the only $a$-transition of $Q_{1}$ in the post-bisimulation game under $\mathcal{B}$.
We show that the existence of two such $a$-transitions from $Q_{2}$ contradicts $Q_{1} \sim_{\mathrm{T}}^{\text {post }} Q_{2}$ or $Q_{2}$ being in $\sim_{\mathrm{T}}^{\text {post }}$-snf with respect to the applicability of $\mathcal{A}_{\mathrm{T}, 1}^{\mathrm{post}}$. There are two subcases related to the number of maximal resolutions of $Q_{1}$ starting with an $a$-transition:

- If $Q_{1}$ has a single maximal resolution starting with an $a$-transition, after performing the initial $a$-transition there are no nondeterministic choices in the ULTraS underlying $Q_{1}$. As a consequence, from $\Delta_{2}^{\prime} \chi_{\mathrm{T}}^{\text {post }} \Delta_{2}^{\prime \prime}$ it follows that at most one of the two $a$-transitions of $Q_{2}$ can be matched by the only $a$-transition of $Q_{1}$ in the post-bisimulation game under $\mathcal{B}$, hence $Q_{1} \sim_{\mathrm{T}}^{\text {post }} Q_{2}$ is contradicted.
- Suppose that $Q_{1}$ has several maximal resolutions each starting with an $a$-transition, meaning that after performing the initial $a$-transition at least one nondeterministic choice is present in the ULTraS underlying $Q_{1}$. We have two further subcases:
- If only one of the two $a$-transitions of $Q_{2}$ is matched by the only $a$ transition of $Q_{1}$ in the post-bisimulation game under $\mathcal{B}$, then $Q_{1} \sim_{\mathrm{T}}^{\text {post }}$ $Q_{2}$ is contradicted.
- Suppose that neither $a$-transition of $Q_{2}$ is matched by the only $a$ transition of $Q_{1}$. To allow for several maximal resolutions each starting with an $a$-transition to which the only $a$-transition of $Q_{1}$ corresponds, $\operatorname{supp}\left(\Delta_{1}\right)$ has to contain for $n \in \mathbb{N}_{\geq 1}$ a term of the form $P_{1}+a_{2} \cdot\left(\mathcal{D}_{2} \phi d_{2} \triangleright\left(\ldots \triangleright\left(P_{n-1}+a_{n} \cdot\left(\mathcal{D}_{n} \phi \bar{d}_{n} \triangleright\left(\boldsymbol{Q}^{\prime}+\boldsymbol{Q}^{\prime \prime}\right)\right)\right) \ldots\right)\right)$ where $\operatorname{init}\left(Q^{\prime}\right) \neq \operatorname{init}\left(Q^{\prime \prime}\right)$. From $Q_{1} \sim_{\mathrm{T}}^{\text {post }} Q_{2}$ and $\Delta_{2}^{\prime} \mathcal{\chi}_{\mathrm{T}}^{\text {post }} \Delta_{2}^{\prime \prime}$, it follows that $\operatorname{supp}\left(\Delta_{2}^{\prime}\right)$ has to contain $P_{1}+a_{2} \cdot\left(\mathcal{D}_{2} \notin d_{2} \triangleright(\ldots \triangleright\right.$ $\left.\left.\left(P_{n-1}+a_{n} \cdot\left(\mathcal{D}_{n} \phi d_{n} \triangleright \boldsymbol{Q}^{\prime}\right)\right) \ldots\right)\right)$ and $\operatorname{supp}\left(\Delta_{2}^{\prime \prime}\right)$ has to contain $P_{1}+$ $a_{2} \cdot\left(\mathcal{D}_{2} \phi d_{2} \triangleright\left(\ldots \triangleright\left(P_{n-1}+a_{n} \cdot\left(\mathcal{D}_{n} \phi d_{n} \triangleright Q^{\prime \prime}\right)\right) \ldots\right)\right)$, with $\operatorname{init}\left(Q^{\prime}+\right.$ $\left.Q^{\prime \prime}\right) \neq \operatorname{init}(Q)$ for all $Q \in \operatorname{supp}\left(\mathcal{D}_{n}\right)$, but then $\mathcal{A}_{\mathrm{T}, 1}^{\text {post }}$ could have been applied $n$ times within $Q_{2}$ thus contradicting $Q_{2}$ being in $\sim_{\mathrm{T}}^{\text {post }}$-snf.

3. Suppose that both $Q_{1}$ and $Q_{2}$ have several $a$-transitions each. Similar to case 2 , the supports of the target reachability distributions of the $a$-transitions departing from the same term are pairwise different and $\sim_{\mathrm{T}}^{\text {post }}$-inequivalent. There are three subcases related to $a$-transition matching:

- If each $a$-transition of $Q_{1}$ is matched by an $a$-transition of $Q_{2}$ in the post-bisimulation game under $\mathcal{B}$, and vice versa, then we are done.
- If only $Q_{1}$ (resp. $Q_{2}$ ) has $a$-transitions not matched by any $a$-transition of $Q_{2}\left(\right.$ resp. $\left.Q_{1}\right)$, or both $Q_{1}$ and $Q_{2}$ have a single $a$-transition not matched by any $a$-transition of the other, then $Q_{1} \sim_{\mathrm{T}}^{\text {post }} Q_{2}$ is contradicted.
- If both $Q_{1}$ and $Q_{2}$ have several $a$-transitions not matched by any $a$ transition of the other, then we proceed on these $a$-transitions as in case 2 , with the difference that both $Q_{1}$ being in $\sim_{\mathrm{T}}^{\text {post }}$-snf and $Q_{2}$ being in $\sim_{\mathrm{T}}^{\text {post }}$-snf may be contradicted.

Proof of Thm. 7,
Without loss of generality, thanks to Lemma 2 we can assume that $P_{1}$ and $P_{2}$ are in $\sim_{\mathrm{T}}^{\text {post }}$-snf. From $P_{1} \sim_{\mathrm{T}}^{\text {post }} P_{2}$ and Lemma 3, it then follows that $P_{1} \sim_{\mathrm{B}}^{\text {post }} P_{2}$. From Thm. 3, we obtain $\mathcal{A}_{\mathrm{B}}^{\text {post }} \vdash P_{1}=P_{2}$, hence $\mathcal{A}_{\mathrm{T}}^{\text {post }} \vdash P_{1}=P_{2}$ due to Prop. 1(3).

Proof of Thm. 8 .
A straightforward consequence of the definition of $\mathcal{A}_{\mathrm{T}}^{\text {pre }}$, the fact that $\sim_{\mathrm{T}}^{\text {pre }}$ is an equivalence relation, Thm. 1, and the fact that the lefthand side term of each core axiom as well as of $\mathcal{A}_{\mathrm{T}, 1}^{\mathrm{pre}}$ is $\sim_{\mathrm{T}}^{\mathrm{pre}}$-equivalent to the term on the righthand side of the same axiom (under the conditions of the axiom itself in the case of $\mathcal{A}_{\mathrm{T}, 1}^{\mathrm{pre}}$ ).

