The Geometry of Lattice Cryptography

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August 29-30, 2011 (FOSAD '11 – Bertinoro, Italy)

Cryptography, Complexity and Lattices

Cryptography: exploiting hard computational problems to build computer systems that are hard to break.

Good news

There are plenty of hard computational problems in computer science.

Bad news

Finding cryptographically useful hard problems seems hard.

Cryptography requires problems that

- are very hard to solve: solution should take enormous time
- are hard to solve on average, even with small probability
- have extra features, e.g., trapdoors, regularity, etc.

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Point Lattices and Cryptography

Lattice problems

- appear to be very hard (solution takes exponential time),
- have been widely studied by mathematicians since 19th century (Lagrange, Gauss, Dirichlet, . . .),
- provably yield hard on average problems, from worst-case complexity assumptions.

Lattice related constructions and cryptographic functions

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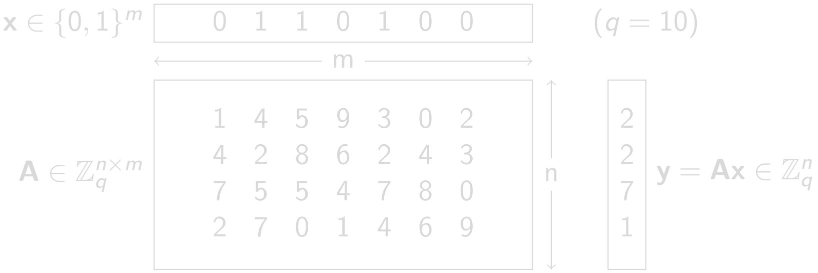
Ajtai's function

Definition (Ajtai's function)

$f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} \bmod q \qquad$ where $\mathbf{A} \in \mathbb{Z}_q^{n\times m}$ and $\mathbf{x} \in \{0,1\}^m$



$\mathbf{x} \in \{0,1\}^m$ $\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$ $(q = 10)$

$\xleftrightarrow{\quad m \quad}$

$$\mathbf{A} \in \mathbb{Z}_q^{n\times m} \quad \begin{bmatrix} 1 & 4 & 5 & 9 & 3 & 0 & 2 \\ 4 & 2 & 8 & 6 & 2 & 4 & 3 \\ 7 & 5 & 5 & 4 & 7 & 8 & 0 \\ 2 & 7 & 0 & 1 & 4 & 6 & 9 \end{bmatrix} \; n \quad \begin{bmatrix} 2 \\ 2 \\ 7 \\ 1 \end{bmatrix} \; \mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{Z}_q^n$$

Security (One-wayness)

Given $\mathbf{A}$ and $\mathbf{y}$, it is hard to find $\mathbf{x} \in \{0,1\}^m$ s.t. $f_{\mathbf{A}}(\mathbf{x}) = \mathbf{y}$.

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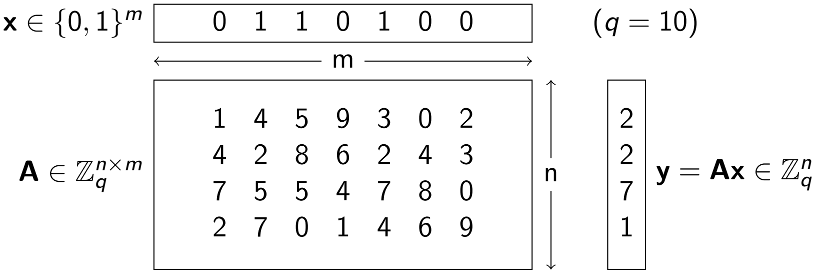

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1. Point Lattices
   - Computational Problems
   - The dual lattice
2. Lattice Cryptography
   - Average Case Hardness
   - Random Lattices
   - Cryptographic functions

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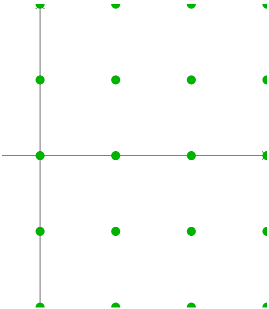
Point Lattices

- The simplest example of lattice is $\mathbb{Z}^n = \{(x_1, \ldots, x_n) \colon x_i \in \mathbb{Z}\}$
- Other lattices are obtained by applying a linear transformation

$$\mathbf{B} \colon \mathbf{x} = (x_1, \ldots, x_n) \mapsto \mathbf{Bx} = x_1 \cdot \mathbf{b}_1 + \cdots + x_n \cdot \mathbf{b}_n$$



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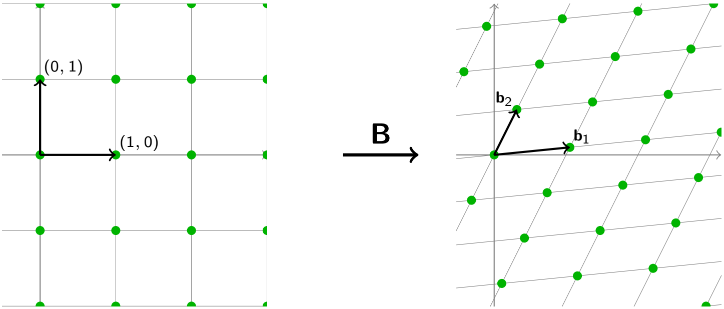


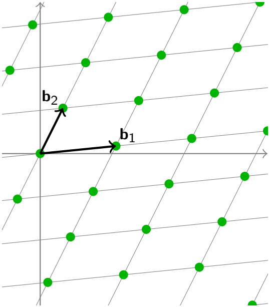
Lattices and Bases

A lattice is the set of all integer linear combinations of (linearly independent) basis vectors $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset \mathbb{R}^n$:

$$\mathcal{L} = \sum_{i=1}^{n} \mathbf{b}_i \cdot \mathbb{Z} = \{\mathbf{B}\mathbf{x} \colon \mathbf{x} \in \mathbb{Z}^n\}$$

The same lattice has many bases

$$\mathcal{L} = \sum_{i=1}^{n} \mathbf{c}_i \cdot \mathbb{Z}$$



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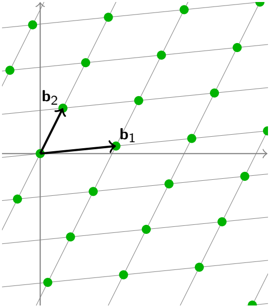

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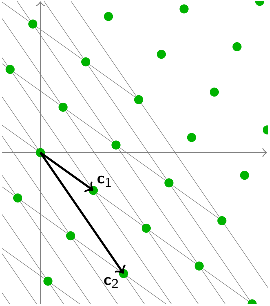

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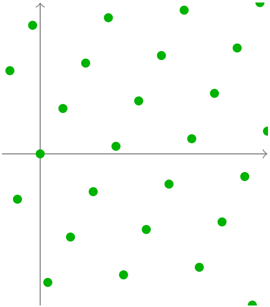


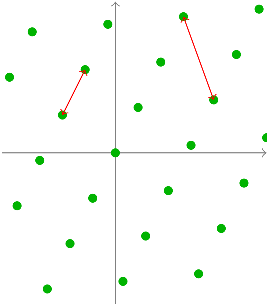
Definition (Lattice)

A discrete additive subgroup of $\mathbb{R}^n$

Minimum Distance and Successive Minima

- Minimum distance

$$\lambda_1 = \min_{\mathbf{x},\mathbf{y}\in\mathcal{L},\mathbf{x}\neq\mathbf{y}} \|\mathbf{x}-\mathbf{y}\| = \min_{\mathbf{x}\in\mathcal{L},\mathbf{x}\neq\mathbf{0}} \|\mathbf{x}\|$$



- Successive minima ($i = 1, \ldots, n$)

$$\lambda_i = \min\{r : \dim \operatorname{span}(\mathcal{B}(r) \cap \mathcal{L}) \geq i\}$$

- Examples
  - $\mathbb{Z}^n$: $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$
  - Always: $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$

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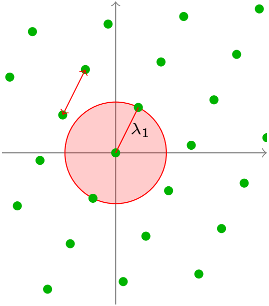

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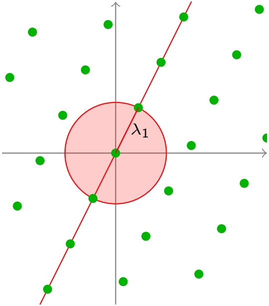

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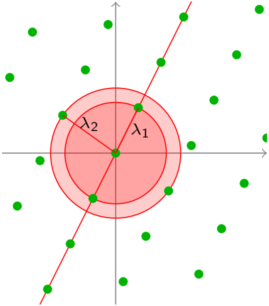

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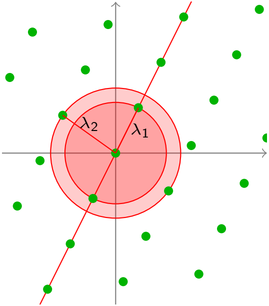

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Distance Function and Covering Radius

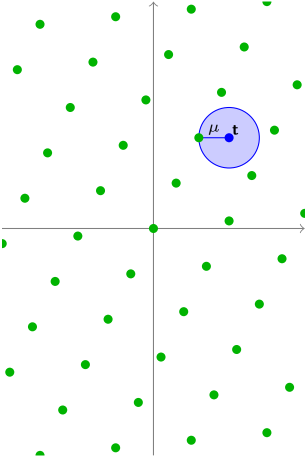
- Distance function

$$\mu(\mathbf{t}, \mathcal{L}) = \min_{\mathbf{x} \in \mathcal{L}} \|\mathbf{t} - \mathbf{x}\|$$

- Covering radius

$$\mu(\mathcal{L}) = \max_{\mathbf{t} \in span(\mathcal{L})} \mu(\mathbf{t}, \mathcal{L})$$

- Spheres or radius $\mu(\mathcal{L})$ centered around all lattice points cover the whole space



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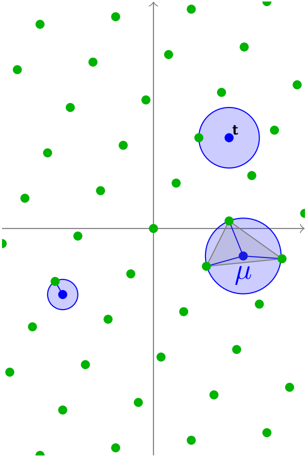

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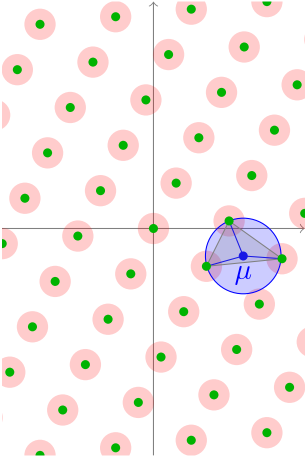

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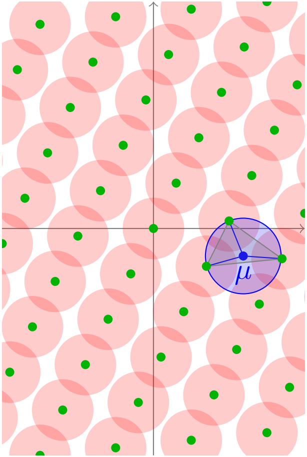

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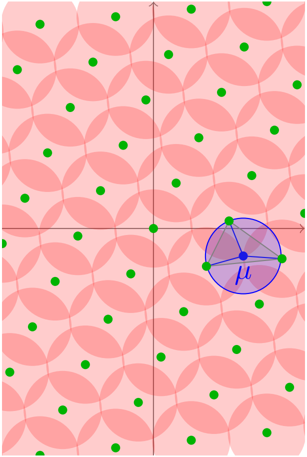


Bounding the covering radius

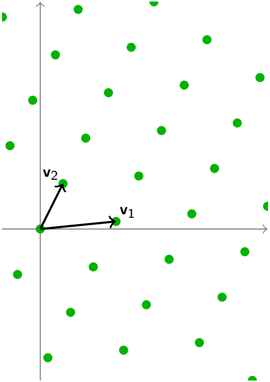
- Let $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ be linearly independent, $\|\mathbf{v}_i\| \leq \lambda_n$
- Tile $\mathbb{R}^n$ with copies of $\mathcal{P}(\mathbf{V}) = \mathbf{V}[0,1)^n$
- If $\mathbf{t} \in \mathbf{x} + \mathcal{P}(\mathbf{V})$, then

$$\|\mathbf{t} - \mathbf{x}\| \leq \sum \|\mathbf{v}_i\| \leq n\lambda_n.$$

- This proves $\mu(\mathcal{L}) \leq n\lambda_n(\mathcal{L})$, and can be further improved:

Theorem

For any lattice $\mathcal{L}$, $\mu(\mathcal{L}) \leq \frac{\sqrt{n}}{2}\lambda_n(\mathcal{L})$



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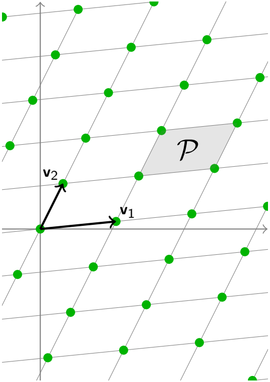

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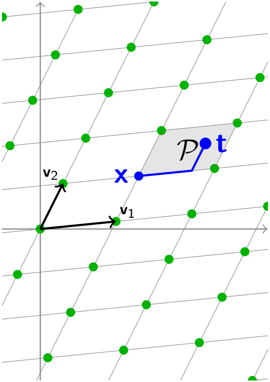

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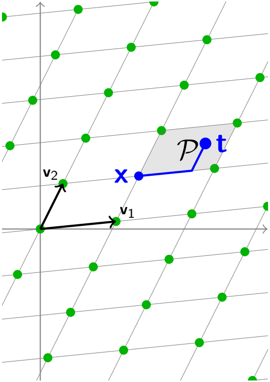

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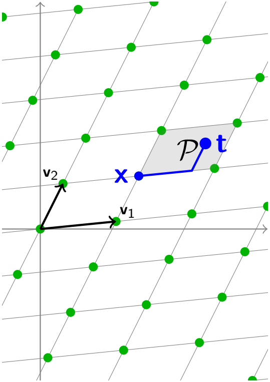
- Let $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ be linearly independent, $\|\mathbf{v}_i\| \leq \lambda_n$
- Tile $\mathbb{R}^n$ with copies of $\mathcal{P}(\mathbf{V}) = \mathbf{V}[0,1)^n$
- If $\mathbf{t} \in \mathbf{x} + \mathcal{P}(\mathbf{V})$, then

$$\|\mathbf{t} - \mathbf{x}\| \leq \sum \|\mathbf{v}_i\| \leq n\lambda_n.$$

- This proves $\mu(\mathcal{L}) \leq n\lambda_n(\mathcal{L})$, and can be further improved:



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Bounding the successive minima

- Let $\|\mathbf{b}_1\| = \lambda_1(\mathcal{L})$
- Let $\mathbf{t} = \frac{1}{2}\mathbf{b}_1$
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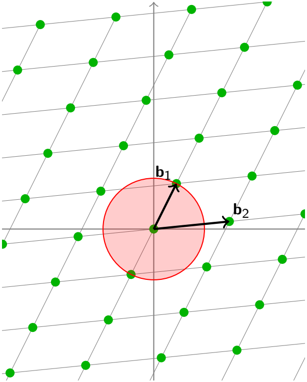

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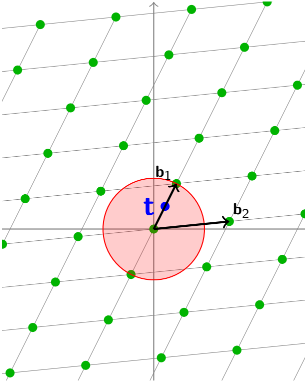

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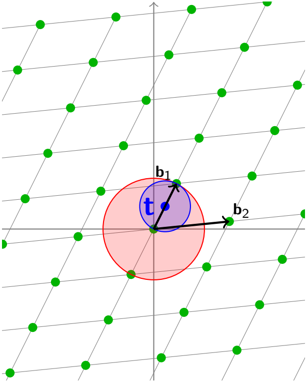

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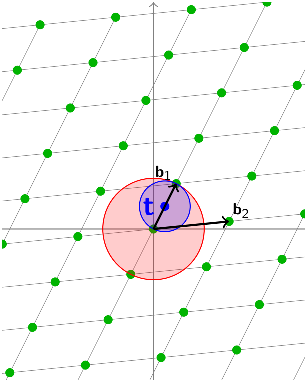

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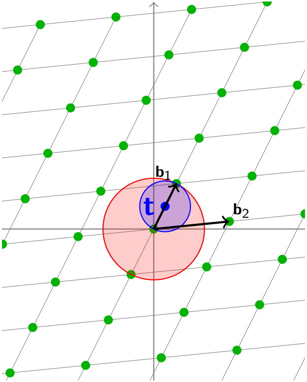

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For any lattice $\mathcal{L}$, $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq 2\mu \leq \sqrt{n}\lambda_n$

Remarks:

1. $\mu \approx \lambda_n$ (up to $\sqrt{n}$ factors)
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Give an explicit construction of a lattice satisfying $\mu \leq 2\lambda_1$

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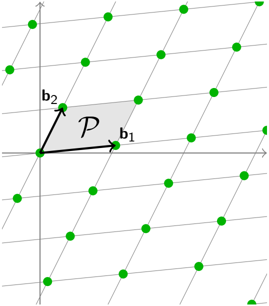
Determinant

Definition (Determinant)

$\det(\mathcal{L})$ = volume of the fundamental region $\mathcal{P} = \sum_i \mathbf{b}_i \cdot [0,1)$

- Different bases define different fundamental regions
- All fundamental regions have the same volume
- The determinant of a lattice can be efficiently computed from any basis.



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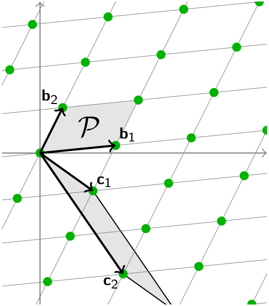

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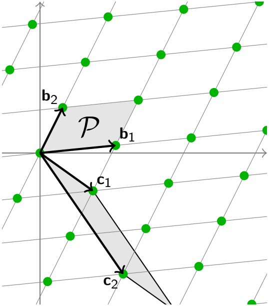

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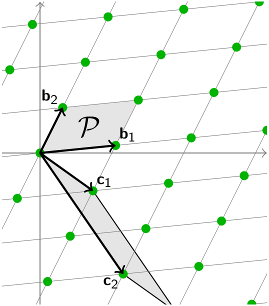


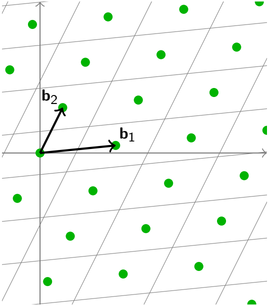
Density estimates

Definition (Centered Fundamental Parallelepiped)

$$\mathcal{P} = \sum_i \mathbf{b}_i \cdot [-1/2, 1/2)$$

- $\mathrm{vol}(\mathcal{P}(\mathbf{B})) = \det(\mathcal{L})$
- $\{\mathbf{x} + \mathcal{P}(\mathbf{B}) \mid \mathbf{x} \in \mathcal{L}\}$ partitions $\mathbb{R}^n$
- For all sufficiently large $S \subseteq \mathbb{R}^n$

$$|S \cap \mathcal{L}| \approx \mathrm{vol}(S)/\det(\mathcal{L})$$



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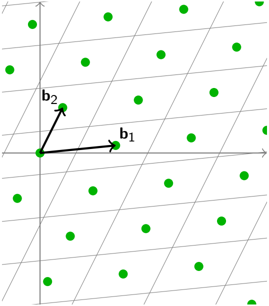

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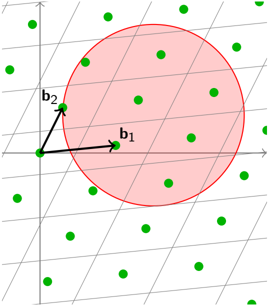
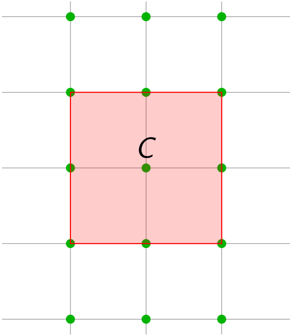


Minkowski's convex body theorem

Theorem (Convex Body)

Let $C \subset \mathbb{R}^n$ be a symmetric convex body. If vol$(C) > 2^n$, then C contains a nonzero integer vector

- $C = \mathbf{B}^{-1}[-r,r]^n$ has volume $\det(\mathbf{B})^{-1}(2r)^n = 2^n$
- C contains $\mathbf{x} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$
- $\mathbf{B}C = [-r,r]^n$ contains $\mathbf{Bx}$
- $\lambda_1(\mathcal{L}) \le \sqrt{n}r = \sqrt{n}\det(\mathcal{L})^{1/n}$



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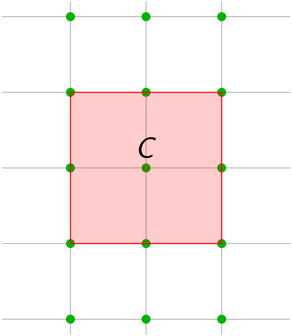

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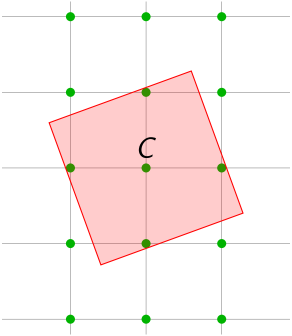

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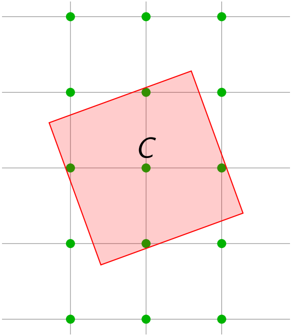

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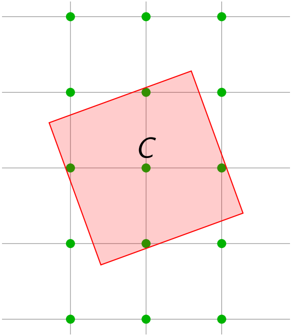

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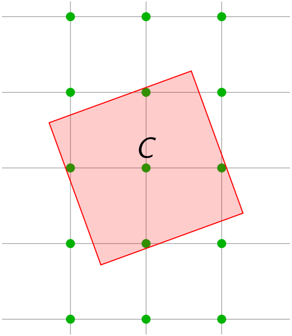

Minkowski's second theorem

Theorem (Minkowski)

$$\lambda_1(\mathcal{L}) \leq \left(\prod_i \lambda_i(\mathcal{L})\right)^{1/n} \leq \sqrt{n}\,\det(\mathcal{L})^{1/n}$$

- For $\mathbb{Z}^n$, $\lambda_1 = (\prod_i \lambda_i)^{1/n} = 1$ is smaller than Minkowski's bound by $\sqrt{n}$
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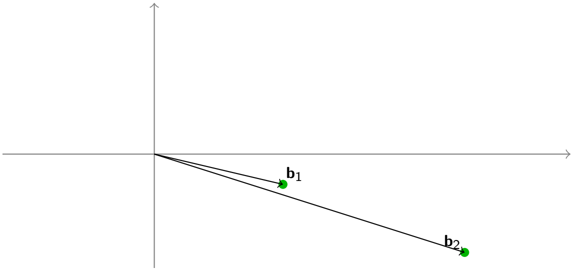
Outline

1. Point Lattices
   - Computational Problems
   - The dual lattice
2. Lattice Cryptography
   - Average Case Hardness
   - Random Lattices
   - Cryptographic functions

Shortest Vector Problem

Definition (Shortest Vector Problem, SVP)

Given a lattice $\mathcal{L}(\mathbf{B})$, find a (nonzero) lattice vector $\mathbf{Bx}$ (with $\mathbf{x} \in \mathbb{Z}^k$) of length (at most) $\|\mathbf{Bx}\| \leq \lambda_1$



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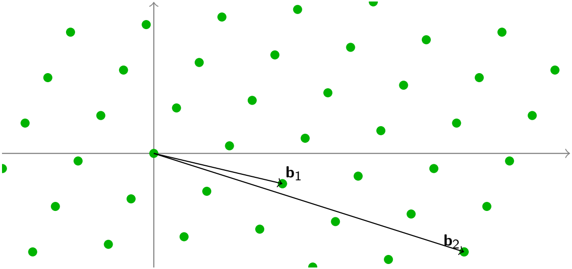

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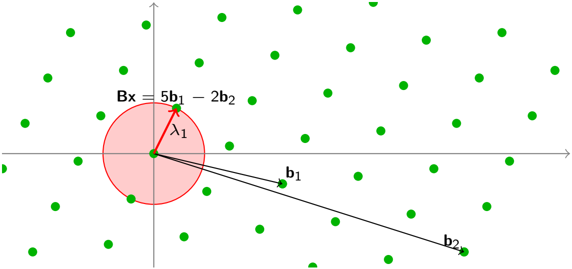

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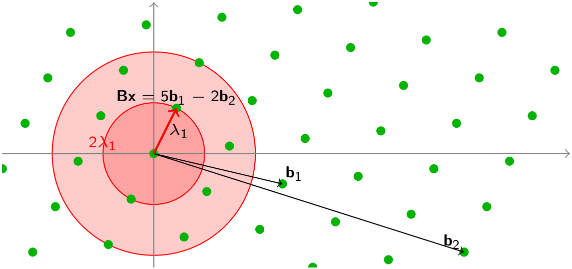
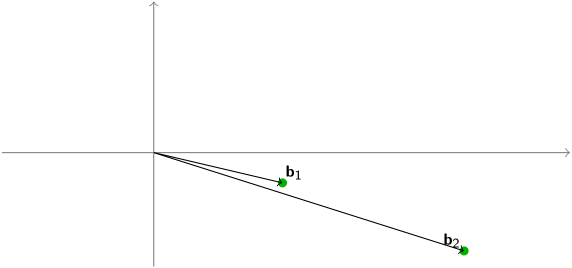


Shortest Independent Vectors Problem

Definition (Shortest Independent Vectors Problem, SIVP)

Given a lattice $\mathcal{L}(\mathbf{B})$, find n linearly independent lattice vectors $\mathbf{Bx}_1, \ldots, \mathbf{Bx}_n$ of length (at most) $\max_i \|\mathbf{Bx}_i\| \leq \lambda_n$



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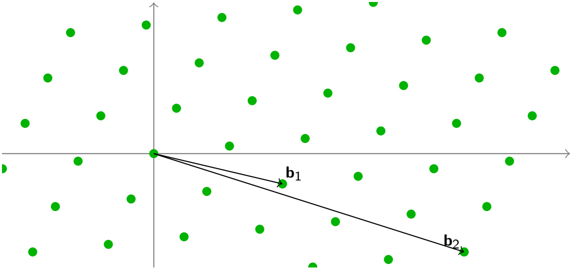

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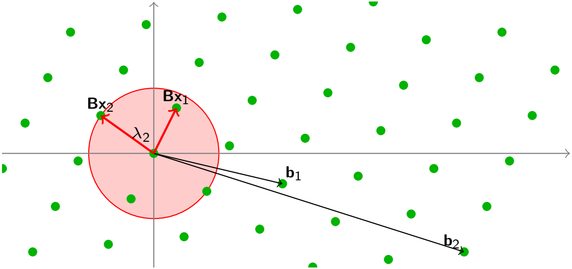

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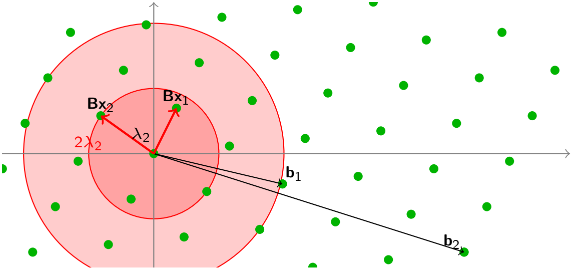
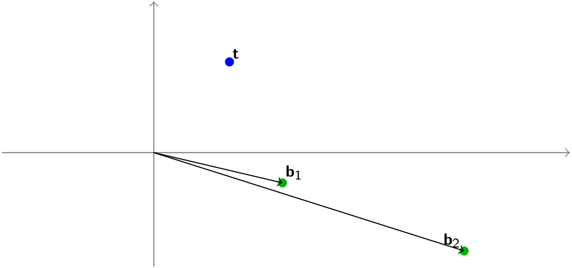


Closest Vector Problem

Definition (Closest Vector Problem, CVP)

Given a lattice $\mathcal{L}(\mathbf{B})$ and a target point $\mathbf{t}$, find a lattice vector $\mathbf{Bx}$ within distance $\|\mathbf{Bx} - \mathbf{t}\| \leq \mu$ from the target



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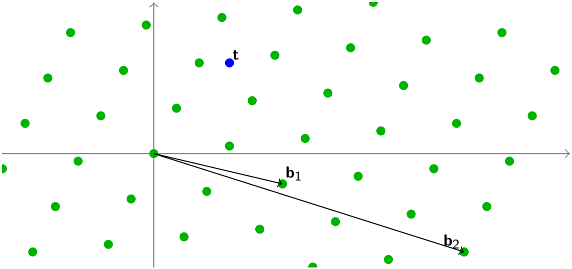

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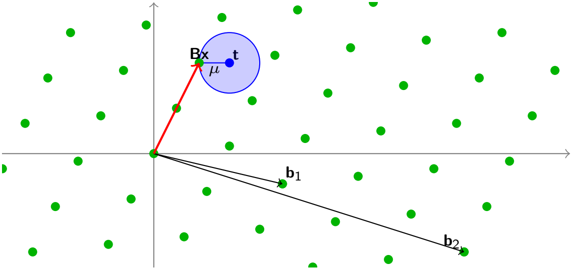

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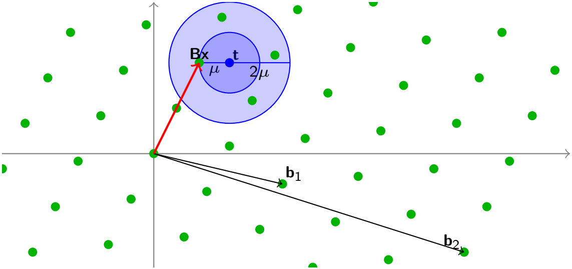
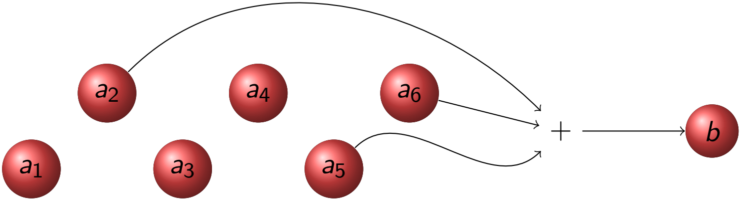


NP-hardness of CVP

Definition (Subset Sum)

Given $a_1, \ldots, a_n, b \in \mathbb{Z}$ find $S \subseteq \{1, \ldots, n\}$ s.t. $\sum_{i \in S} a_i = b$



Theorem

$\|\mathbf{Bx} - \mathbf{t}\| \leq \sqrt{n}$ *if and only if* $\mathbf{x} \in \{0,1\}^n$ *and* $\sum_{x_i=1} a_i = b$.

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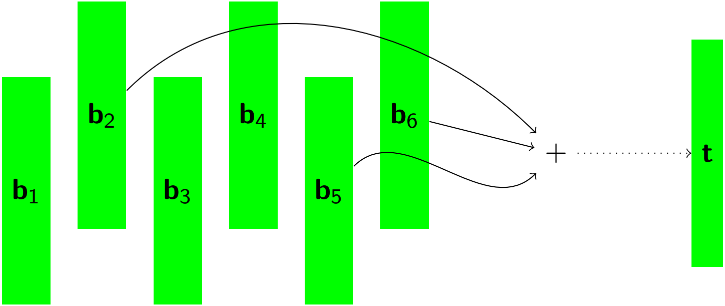

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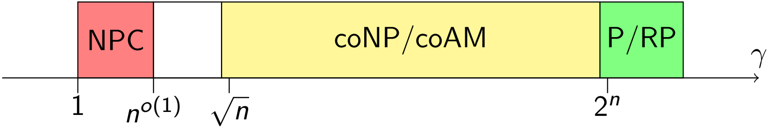
Complexity of CVP, SVP, SIVP



- Best algorithm for exact solution takes time 2^n [MV10]
- (Almost) NP-hard for factors up to $\gamma = n^{1/loglogn}$. [Ajtai96,. . . ,HR07]
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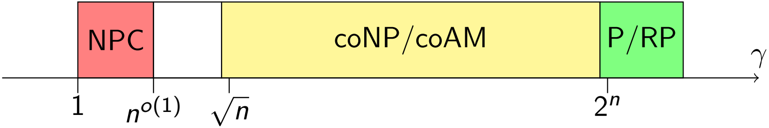

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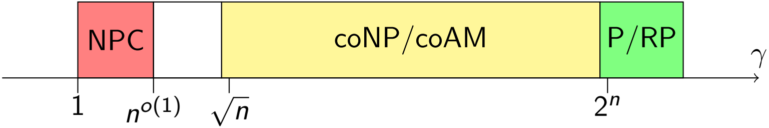

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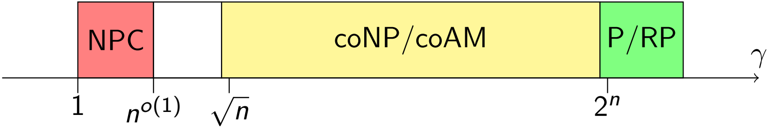

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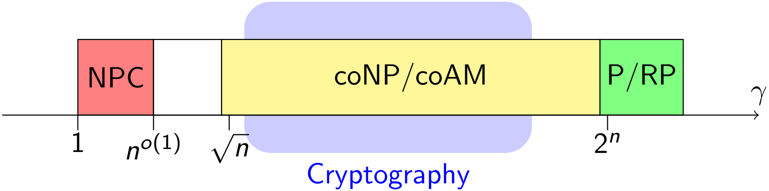

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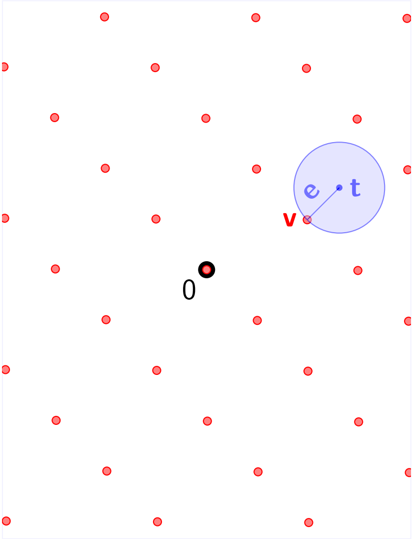
CVP and lattice cosets



- Lattice Λ, target $\mathbf{t}$
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Definition (Coset CVP)

Given a lattice coset $\mathbf{t} + \mathcal{L}$, find the (approximately) shortest element of $\mathbf{t} + \mathcal{L}$.

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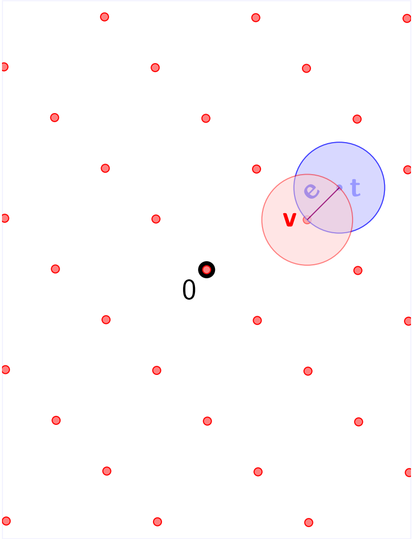

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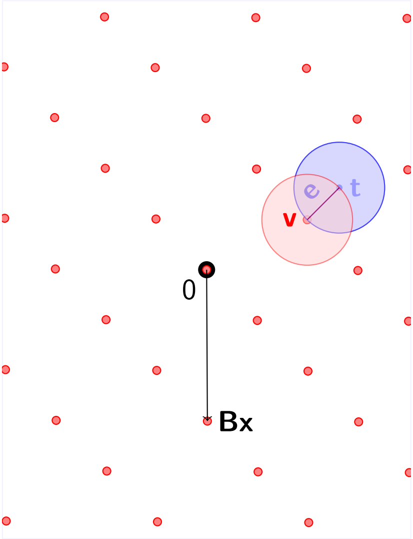

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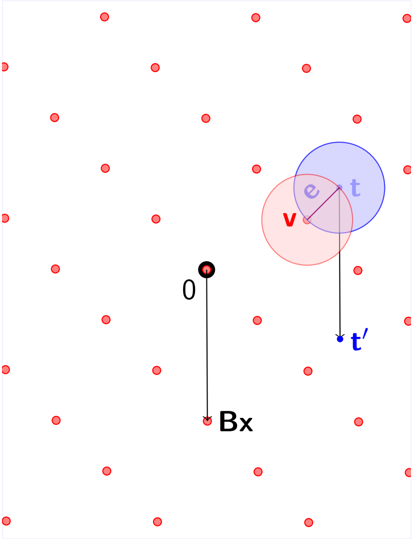

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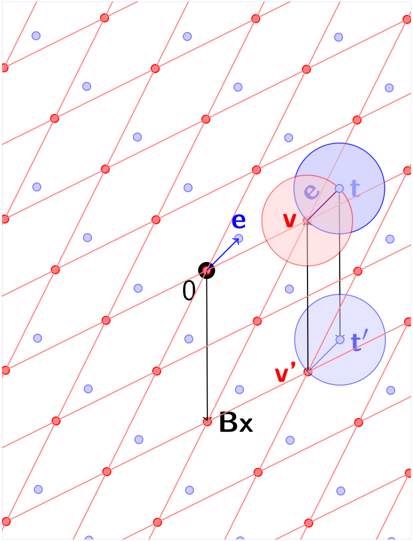

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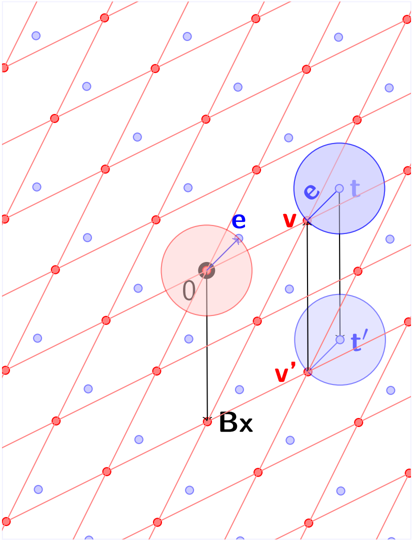

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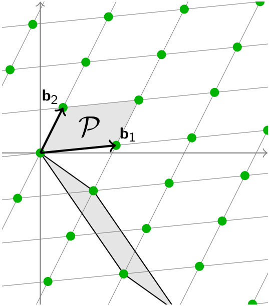
Working modulo a lattice

Definition (Fundamental Region)

$D \subset \mathbb{R}^n$ is a fundamental region for $\mathcal{L}$ if $\{D + \mathbf{x} \mid \mathbf{x} \in \mathcal{L}\}$ is a partition of $\mathbb{R}^n$.

- $(\mathcal{L}, +)$ is a subgroup of $(\mathbb{R}^n, +)$
- One can form the quotien group $\mathbb{R}^n/\mathcal{L}$
- Elements of $\mathbb{R}^n/\mathcal{L}$ are cosets $\mathbf{t} + \mathcal{L}$
- Any fundamental region D gives a set of standard representatives
- $\mathcal{P} = \sum_i \mathbf{b}_i \cdot [0, 1) \equiv \mathbb{R}^n/\mathcal{L}$



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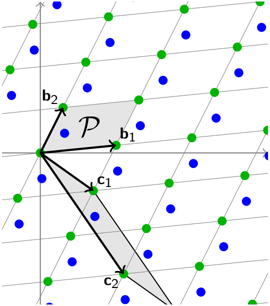

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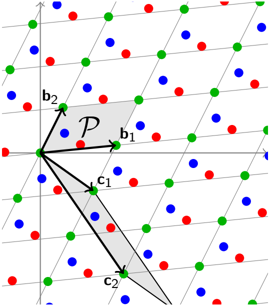

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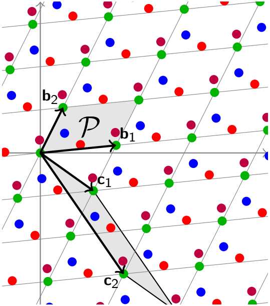


Interlude: CVP One-way Function?

Candidate OWF

Key: a hard lattice $\mathcal{L}$

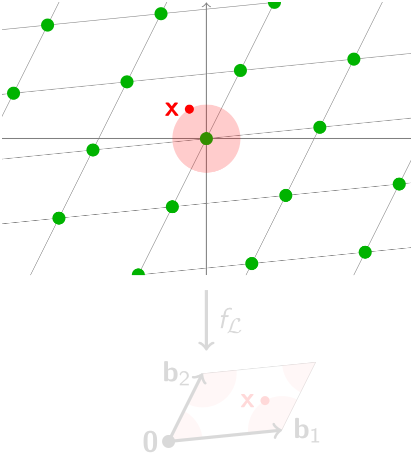
Input: $\mathbf{x}$, $\|\mathbf{x}\| \leq \beta$

Output: $f_{\mathcal{L}}(\mathbf{x}) = \mathbf{x} \bmod \mathcal{L}$

- $\beta < \lambda_1/2$: $f_{\mathcal{L}}$ is injective
- $\beta > \lambda_1/2$: $f_{\mathcal{L}}$ is not injective
- $\beta \geq \mu$: $g_{\mathcal{L}}$ is surjective
- $\beta \gg \mu$: $g_{\mathcal{L}}(\mathbf{x})$ is almost uniform

Question

Is $f_{\mathcal{L}}$ hard to invert?



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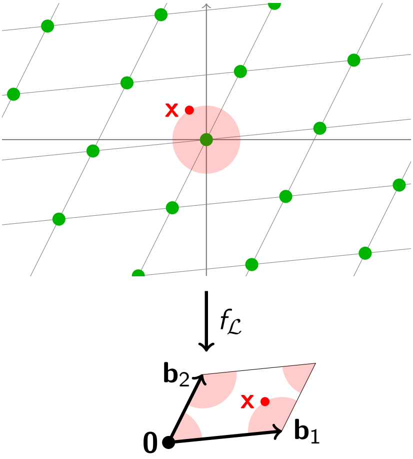

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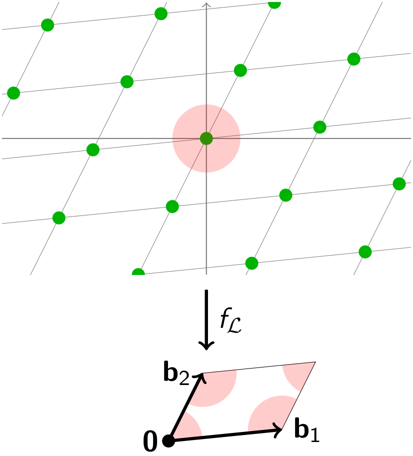

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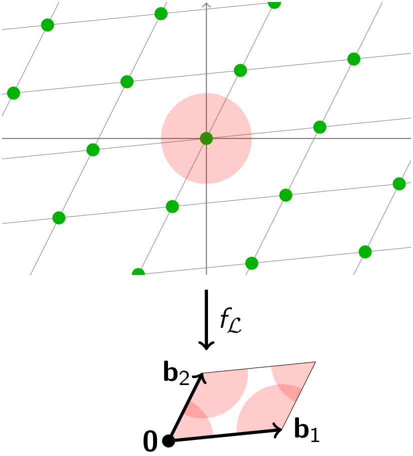

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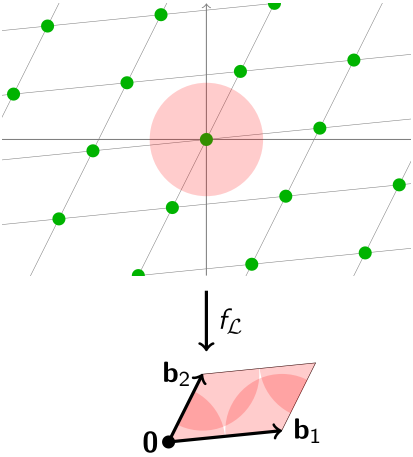

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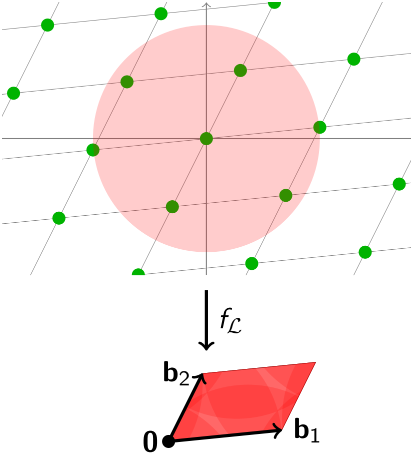

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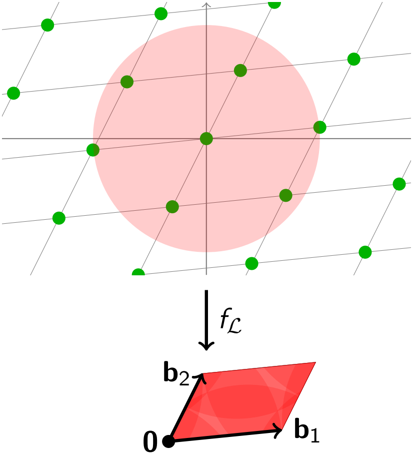
- $\beta > \lambda_1/2$: $f_{\mathcal{L}}$ is not injective
- $\beta \geq \mu$: $g_{\mathcal{L}}$ is surjective
- $\beta \gg \mu$: $g_{\mathcal{L}}(\mathbf{x})$ is almost uniform



Question

Is $f_{\mathcal{L}}$ hard to invert?

Outline

1. Point Lattices
   - Computational Problems
   - The dual lattice
2. Lattice Cryptography
   - Average Case Hardness
   - Random Lattices
   - Cryptographic functions

The Dual

- A vector space over $\mathbb{R}$ is a set of vectors V with
  - a vector addition operation $\mathbf{x} + \mathbf{y} \in V$
  - a scalar multiplication $a \cdot \mathbf{x} \in V$
- The dual of a vector space V is the set $V^* = Hom(V, \mathbb{R})$ of linear functions $\phi : V \to \mathbb{R}$, typically represented as vectors $\mathbf{x} \in V$, where $\phi_{\mathbf{x}}(\mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$
- The dual of a lattice Λ is defined similarly as the set of linear functions $\phi_{\mathbf{x}} : \Lambda \to \mathbb{Z}$ represented as vectors $\mathbf{x} \in span(\Lambda)$.

Definition (Dual lattice)

The dual of a lattice Λ is the set of all vectors $\mathbf{x} \in span(\Lambda)$ such that $\langle \mathbf{x}, \mathbf{v} \rangle \in \mathbb{Z}$ for all $\mathbf{v} \in \Lambda$

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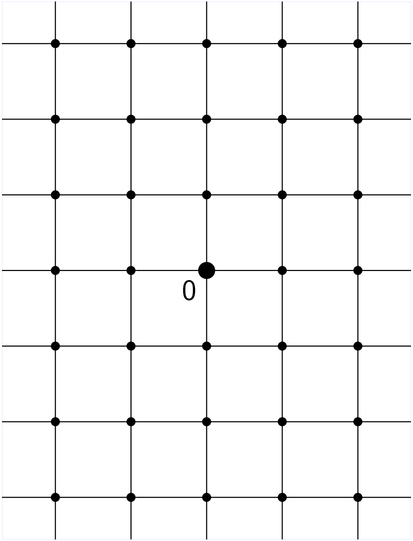
Dual lattice: Examples



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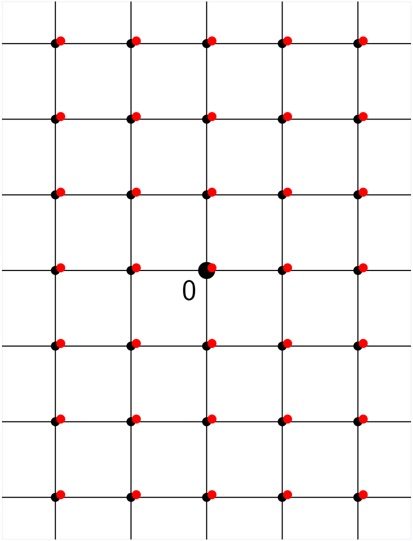

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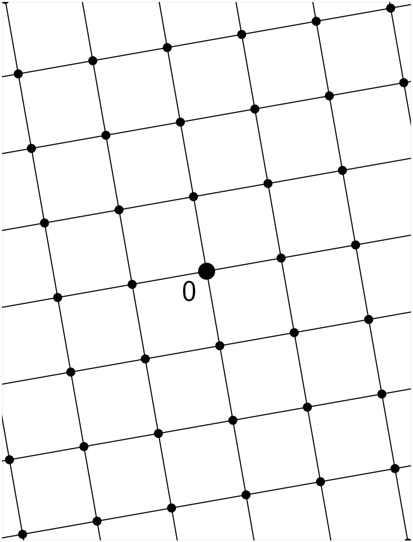

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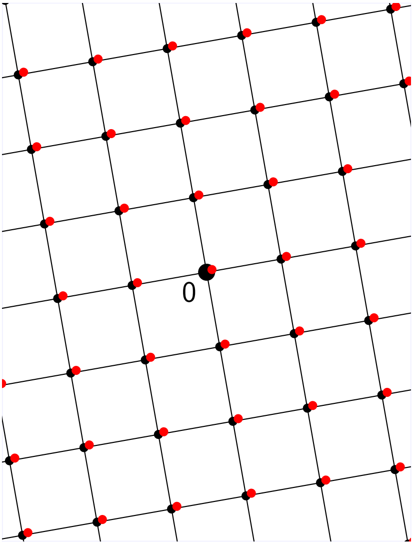

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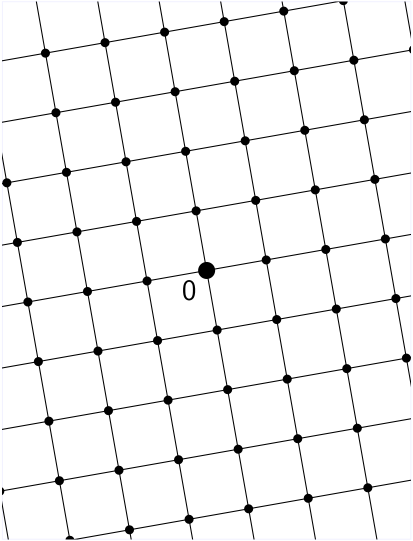

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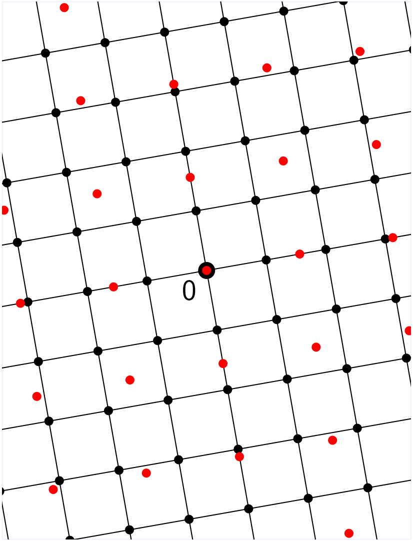

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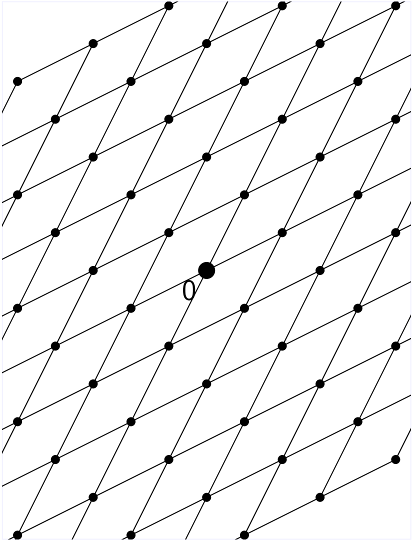

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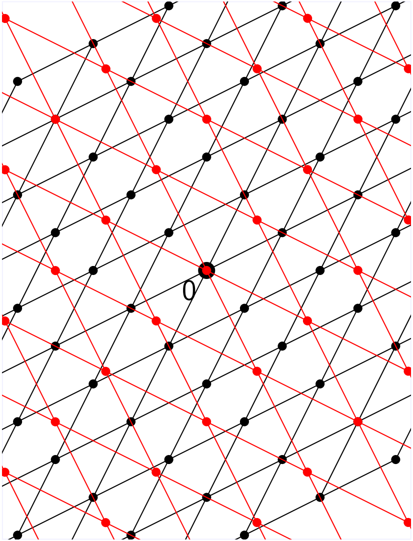

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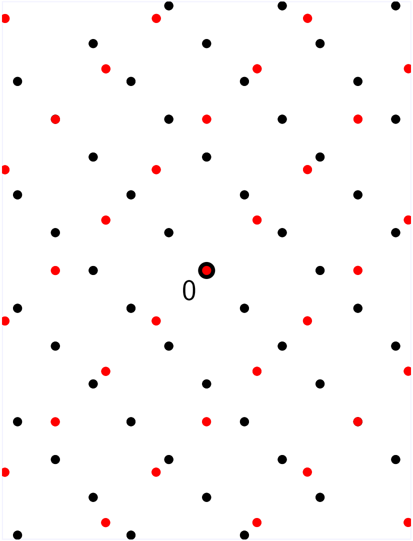

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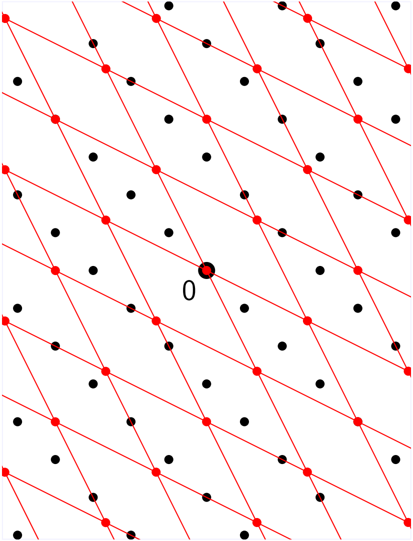

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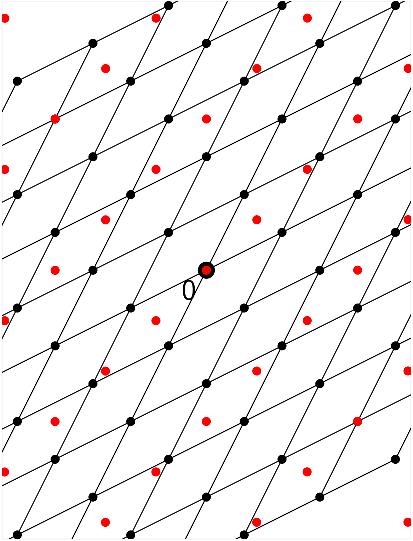

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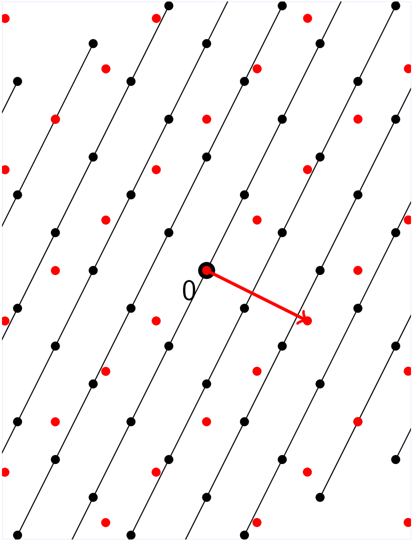

- Each dual vector $\mathbf{v} \in \mathcal{L}^*$, partitions the lattice $\mathcal{L}$ into layers orthogonal to $\mathbf{v}$

$$L_i = \{\mathbf{x} \in \mathcal{L} \mid \mathbf{x} \cdot \mathbf{v} = i\}$$

- Layers are at distance $1/\|\mathbf{v}\|$
- $\rho(\mathcal{L}) \geq \frac{1}{2\|\mathbf{v}\|}$
- If $\lambda_1(\mathcal{L}^*)$ is small, then $\rho(\mathcal{L})$ is large.

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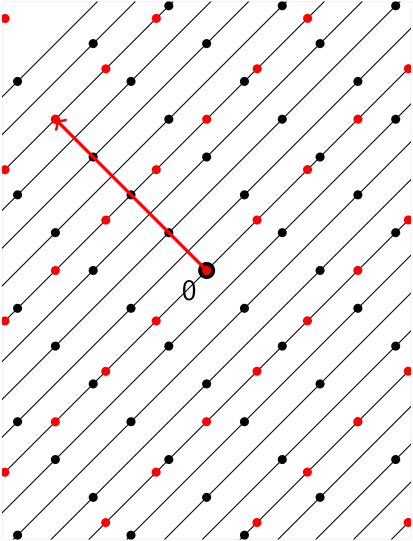

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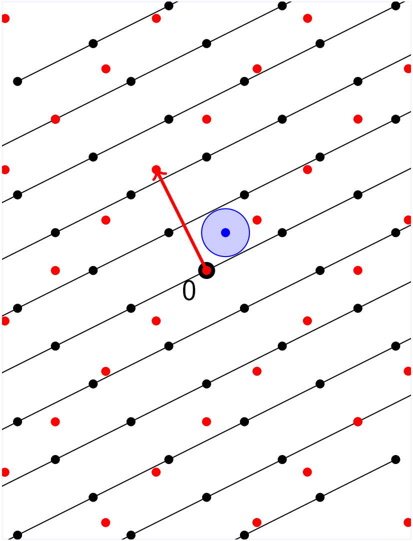

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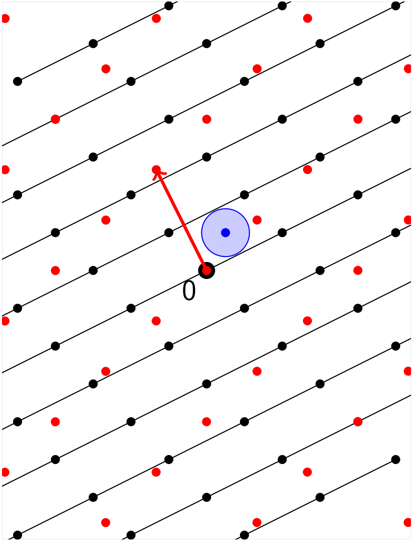

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Transference Theorems

Theorem (Banaszczyk)

For any lattice $\mathcal{L}$

$$1 \leq 2\lambda_1(\mathcal{L}) \cdot \rho(\mathcal{L}^*) \leq n.$$

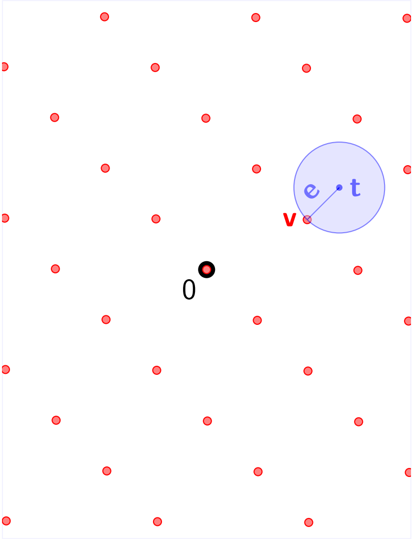
Theorem (Banaszczyk)

For every i,

$$1 \leq \lambda_i(\mathcal{L}) \cdot \lambda_{n-i+1}(\mathcal{L}^*) \leq n.$$

- Approximating $\lambda_1(\mathcal{L})$ within a factor n is in $NP \cap coNP$
- Same is true for $\lambda_i, \ldots, \lambda_n$ and ρ.

CVP and dual lattice



- Lattice Λ, target $\mathbf{t} = \mathbf{v} + \mathbf{e}$
- Dual lattice $\Lambda^* = \mathcal{L}(\mathbf{D})$.
- Syndrome of $\mathbf{t}$:

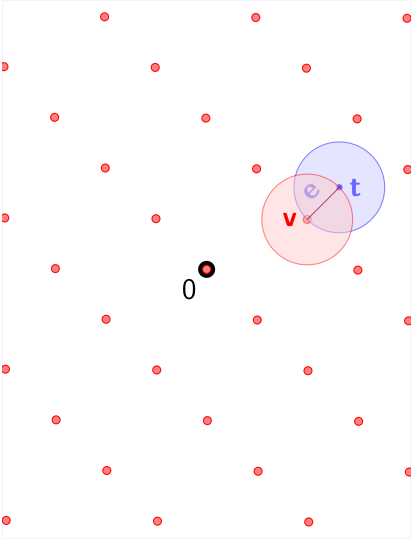
$$\begin{aligned} \mathbf{s} &= \langle \mathbf{D}, \mathbf{t} \rangle \bmod 1 \\ &= \langle \mathbf{D}, \mathbf{v} \rangle + \langle \mathbf{D}, \mathbf{e} \rangle \bmod 1 \\ &= \langle \mathbf{D}, \mathbf{e} \rangle \bmod 1. \end{aligned}$$

- All vectors in a coset $\mathbf{t} + \mathcal{L}$ have the same syndrome.

Definition (Syndrome CVP)

Find shortest $\mathbf{e}$ such that $\langle \mathbf{D}, \mathbf{e} \rangle = \mathbf{s} \bmod 1$

CVP and dual lattice



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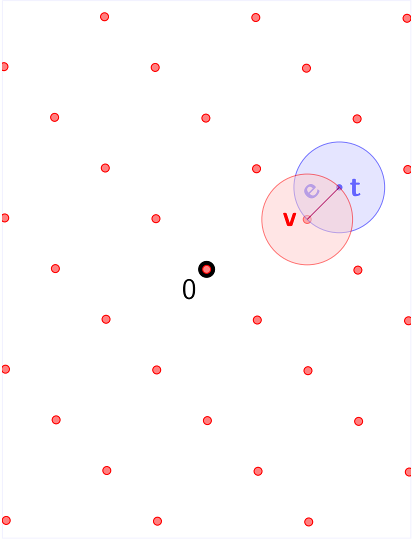

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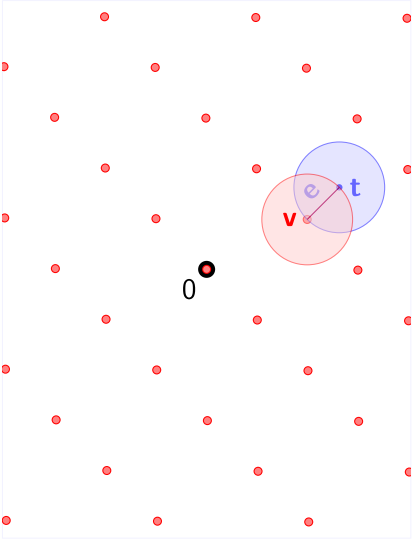

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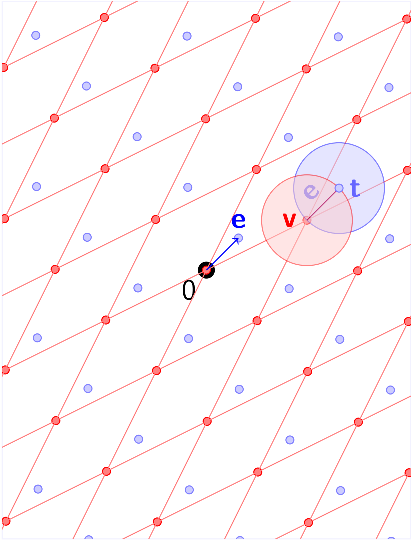

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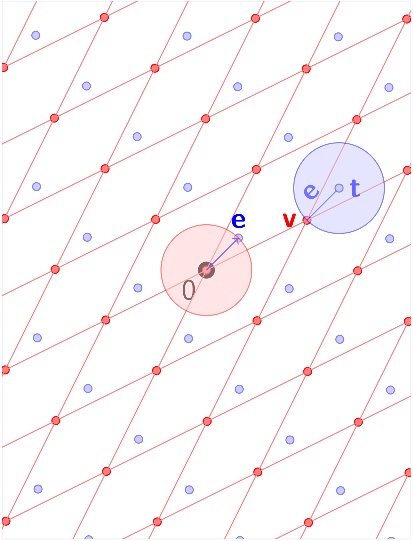

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Back to CVP One-way function

Candidate OWF

Key: a hard lattice $\mathcal{L}(\mathbf{D})^*$

Input: $\mathbf{x}$, $\|\mathbf{x}\| \leq \beta$

Output: $f_{\mathbf{D}}(\mathbf{x}) = \mathbf{D}\mathbf{x} \bmod 1$

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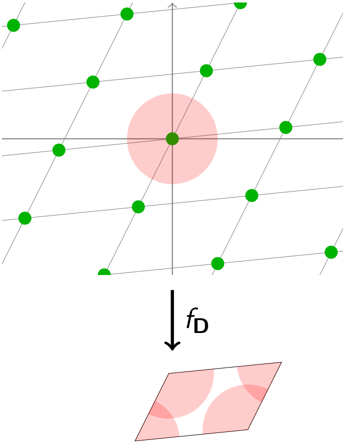

Special Versions of CVP

Definition (Decisional CVP)

Given $(\mathcal{L}, \mathbf{t}, d)$, with $\mu(\mathbf{t}, \mathcal{L}) \leq d$, find a lattice point within distance d from $\mathbf{t}$.

- If d is arbitrary, then one can find the closest lattice vector by binary search on d.
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ADD reduces to SIVP

ADD input: $\mathcal{L}$ and arbitrary $\mathbf{t}$

- Compute short vectors $\mathbf{V} = \mathrm{SIVP}(\mathcal{L})$
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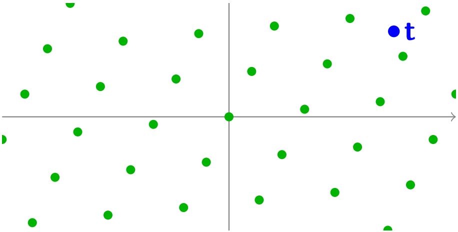

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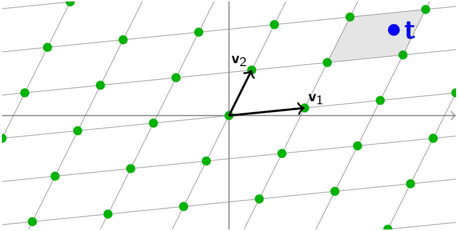

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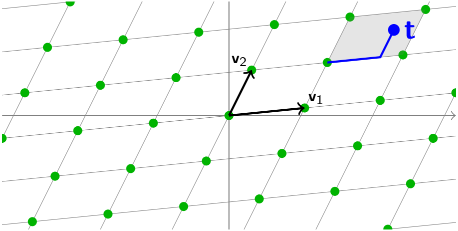

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- Output is correct as long as

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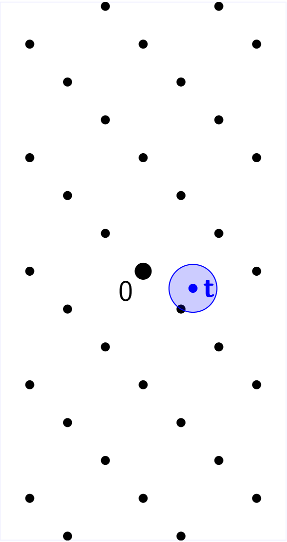

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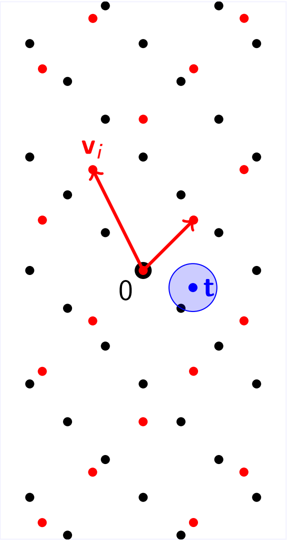

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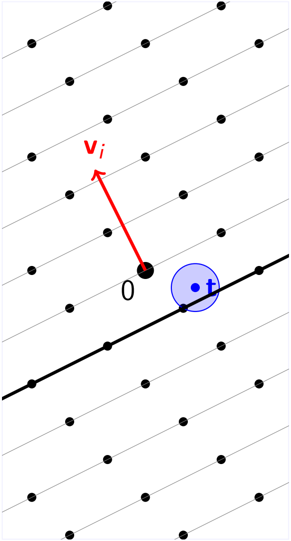

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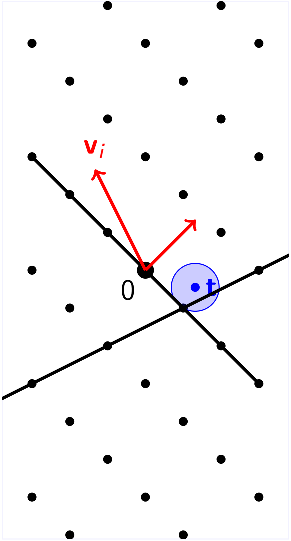

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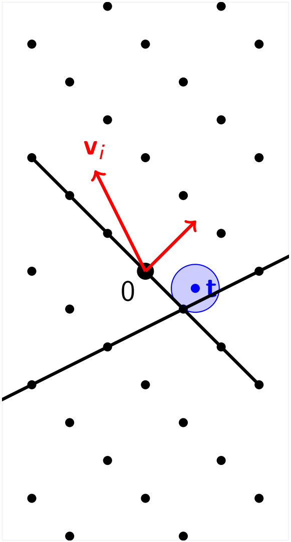

Special Versions of SVP and SIVP

- GapSVP: compute (or approximate) the value λ_1 without necessarily finding a short vector
- GapSIVP: compute (or approximate) the value λ_n without necessarily finding short linearly independent vectors
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Relations among lattice problems

- SIVP $\approx$ ADD [MG'01]
- SVP $\leq$ CVP [GMSS'99]
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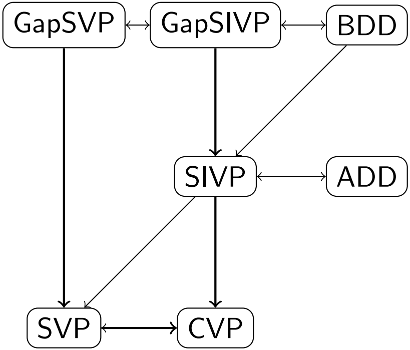

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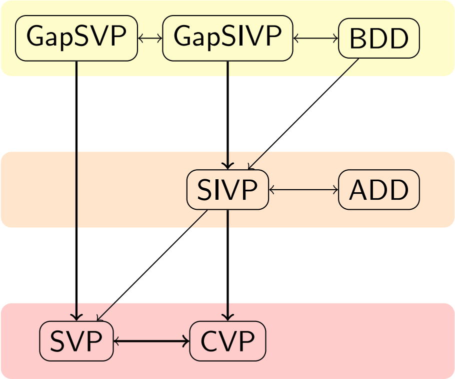


Outline

1. Point Lattices
   - Computational Problems
   - The dual lattice
2. Lattice Cryptography
   - Average Case Hardness
   - Random Lattices
   - Cryptographic functions

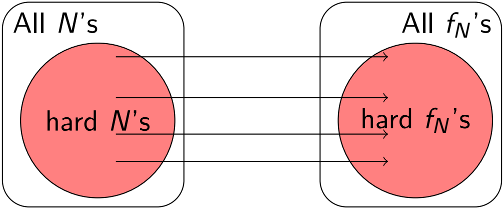
Provable security (from average case hardness)

Example 1: (Rabin) modular squaring

- $f_N(x) = x^2 \bmod N$, where $N = p \cdot q$
- Inverting f_N is at least as hard as factoring N

Theorem

f_N is cryptographically hard to invert, provided most $N = p \cdot q$ are hard to factor



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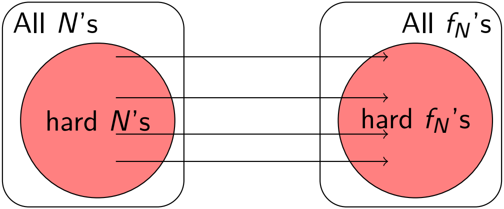

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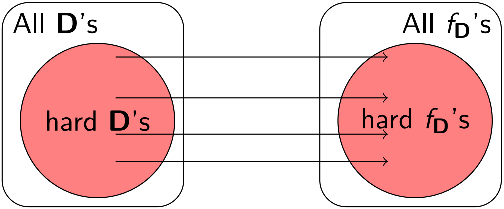

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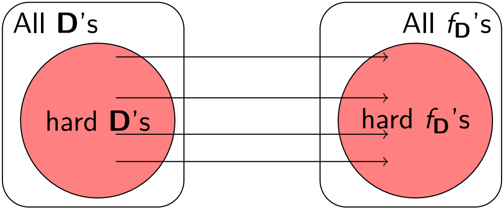

Average-case Complexity

Average-case complexity depends on input distribution

Example (Factoring problem)

Given a number N, output $a, b > 1$ such that $N = ab$

Factoring can be easy on average

if N is uniformly random, then $N = 2 \cdot \frac{N}{2}$ with probability 50%!

- Factoring $N = pq$ is believed to be hard when p, q are randomly chosen primes
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There is a probability distribution on **D** such that

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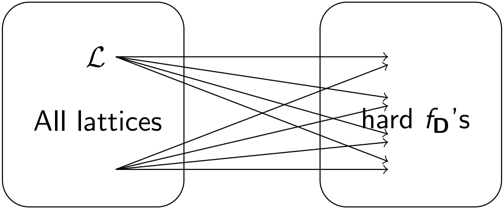
- **D** is also very easy to sample



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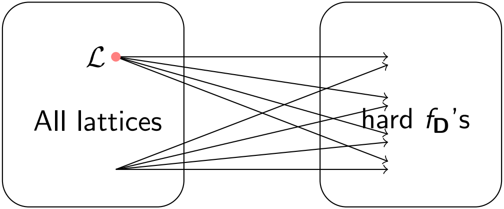

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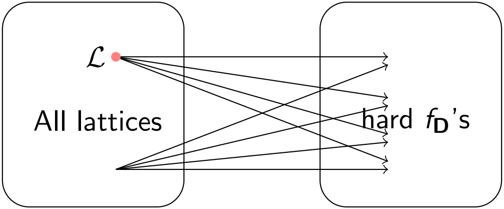

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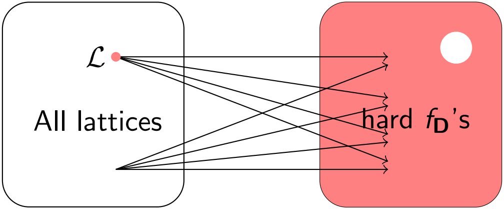

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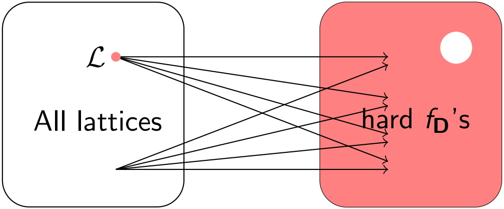

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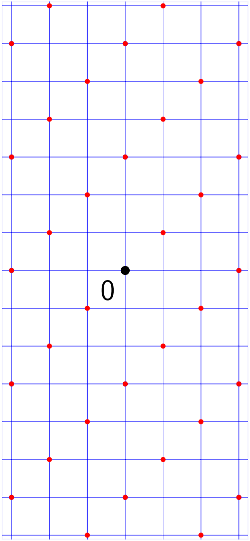
Random lattices in Cryptography



- Cryptography typically uses (random) lattices Λ such that
  - $\Lambda \subseteq \mathbb{Z}^d$ is an integer lattice
  - $q\mathbb{Z}^d \subseteq \Lambda$ is periodic modulo a small integer q.
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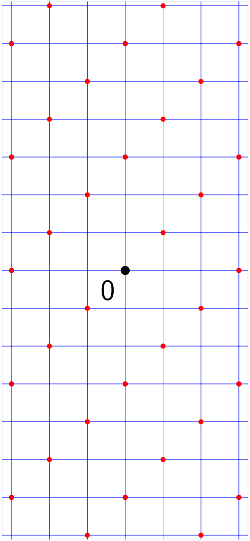

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- $\Lambda_q(\mathbf{A}) = \{\mathbf{x} \mid \mathbf{x} \bmod q \in \mathbf{A}^T \mathbb{Z}_q^n\} \subseteq \mathbb{Z}^d$
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For any lattice Λ the following conditions are equivalent:

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Duality of q-ary lattices

- The q-ary lattices associated to $\mathbf{A}$ are dual (up to scaling)

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Non-degenerate Matrices

Definition

$$\mathcal{M}_{k,n} = \{\mathbf{A} \in \mathbb{Z}_q^{k\times n} \mid \mathbf{A}\mathbb{Z}_q^n = \mathbb{Z}_q^k\}$$

- $\Pr\{\mathbf{A} \in \mathcal{M}_{k,n}\} \geq 1 - \frac{1}{q^{n-k}}$
- $\Lambda_q^\perp(\mathcal{M}_{k,n}) \equiv \Lambda_q(\mathcal{M}_{n-k,n})$ are the same distribution
- $\det(\Lambda_q^\perp(\mathcal{M}_{k,n})) = \det(\Lambda_q(\mathcal{M}_{n-k,n})) = q^k$
- Minkowki's bound $\lambda_1 \leq \sqrt{n}q^{k/n}$

Theorem

Almost every lattice in $\Lambda_q^\perp(\mathcal{M}_{k,n}) \equiv \Lambda_q(\mathcal{M}_{n-k,n})$ satisfies

$$\lambda_1, \ldots, \lambda_n, \rho = \Theta(\sqrt{n}q^{k,n})$$

Non-degenerate Matrices

Definition

$$\mathcal{M}_{k,n} = \{\mathbf{A} \in \mathbb{Z}_q^{k\times n} \mid \mathbf{A}\mathbb{Z}_q^n = \mathbb{Z}_q^k\}$$

- $\Pr\{\mathbf{A} \in \mathcal{M}_{k,n}\} \geq 1 - \frac{1}{q^{n-k}}$
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Are q-ary lattices hard?

Question

Are lattice problems on random q-ary lattices hard on average?

- GapSVP and GapSIVP are easy!
- Why? Just output Minkowki's bound $\sqrt{n}q^{k/n}$!
- What about BDD? (Remember BDD $\leq$ GapSVP.)
- BDD may still be hard! Reduction from BDD to GapSVP requires a wost-case GapSVP oracle.
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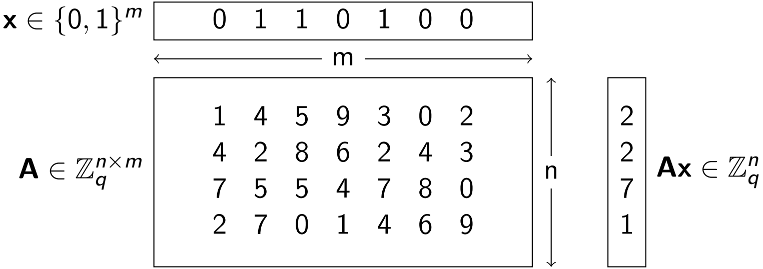
Ajtai's function

Definition (Ajtai's function)

Keyed function family

$$f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} \bmod q$$

where $\mathbf{A} \in \mathbb{Z}_q^{n\times m}$ and $\mathbf{x} \in \{0,1\}^m$.



$$\mathbf{x} \in \{0,1\}^m \quad \boxed{\begin{matrix}0 & 1 & 1 & 0 & 1 & 0 & 0\end{matrix}}$$

$$\xleftrightarrow{\quad m \quad}$$

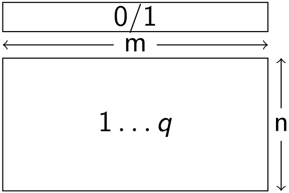
$$\mathbf{A} \in \mathbb{Z}_q^{n\times m} \quad \boxed{\begin{matrix}1 & 4 & 5 & 9 & 3 & 0 & 2\\ 4 & 2 & 8 & 6 & 2 & 4 & 3\\ 7 & 5 & 5 & 4 & 7 & 8 & 0\\ 2 & 7 & 0 & 1 & 4 & 6 & 9\end{matrix}} \quad \updownarrow n \quad \boxed{\begin{matrix}2\\ 2\\ 7\\ 1\end{matrix}} \quad \mathbf{Ax} \in \mathbb{Z}_q^n$$

Ajtai's function and q-ary lattices

- $f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} \bmod q$, where $\mathbf{x}$ is short
- The output of $f_{\mathbf{A}}(\mathbf{x})$ is the syndrome of $\mathbf{x}$
- Inverting $f_{\mathbf{A}}(\mathbf{x})$ is the same as CVP in its syndrome decoding formulation with lattice $\Lambda_q^{\perp}(\mathbf{A})$ and target $\mathbf{t} \in \mathbf{x} + \Lambda_q^{\perp}(\mathbf{A})$
- The q-ary lattice $\Lambda_q^{\perp}(\mathbf{A})$ is the kernel of $f_{\mathbf{A}}$
- Finding collisions $f_{\mathbf{A}}(\mathbf{x}) = f_{\mathbf{A}}(\mathbf{y})$ is equivalent to finding short vectors $\mathbf{x} - \mathbf{y} \in \Lambda_q^{\perp}(\mathbf{A})$

Parameters

- Parameters:
  - n: main security parameter
  - $q = n^2 = n^{O(1)}$ small modulus
  - $m = 2n \log_2 q = O(n \log n)$
  - e.g., $n = 256$, $q = 2^{16}$, $m = 8192$
- $f_{\mathbf{A}}$ is a compression function
  - It maps m bits to $n \log_2 q < m$ bits (e.g., $8192 \to 4096$)
  - There exist collisions $f_{\mathbf{A}}(\mathbf{x}) = f_{\mathbf{A}}(\mathbf{y})$


$0/1$

m

$1 \dots q$

n

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Is $f_{\mathbf{A}}$ collision resistant when $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ is chosen at random?

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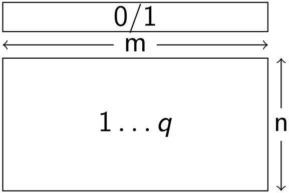

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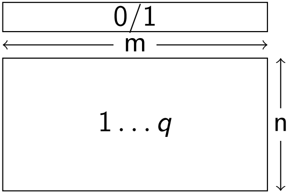

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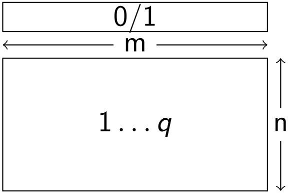

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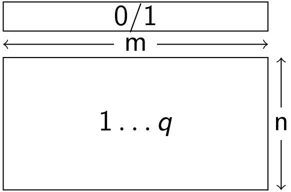
Efficiency issues

- $q = n^{O(1)}$, $m = 2n \log_2 q$
- Let's lower $n = 64$, $q = 2^8$, $m = 1024$
- $f_{\mathbf{A}}$ maps 1024 bits to 512.
- Key size: $nm \log q = O(n^2 \log^2 n) = 2^{19} = 64KB$
- Runtime: $nm = O(n^2 \log n) = 2^{16}$ arithmetic operations
- Still inefficient because of quadratic dependency in n



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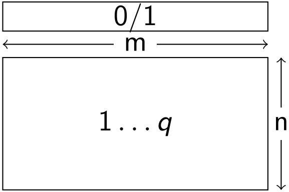

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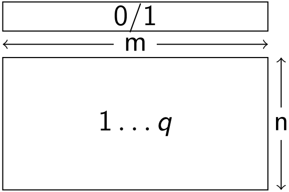

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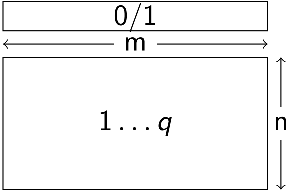

Efficient lattice based hashing

Idea

Use structured matrix

$$\mathbf{A} = [\mathbf{A}^{(1)} \mid \ldots \mid \mathbf{A}^{(m/n)}]$$

where $\mathbf{A}^{(i)} \in \mathbb{Z}_q^{n \times n}$ is circulant

$$\mathbf{A}^{(i)} = \begin{bmatrix} a_1^{(i)} & a_n^{(i)} & \cdots & a_2^{(i)} \\ a_2^{(i)} & a_1^{(i)} & \cdots & a_3^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ a_n^{(i)} & a_{n-1}^{(i)} & \cdots & a_1^{(i)} \end{bmatrix}$$

- Proposed by [M02], where it is proved that $f_{\mathbf{A}}$ is one-way under plausible complexity assumptions
- Similar idea first used by NTRU public key cryptosystem (1998), but with no proof of security
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Can you find a collision?

1	4	3	8	6	4	9	0	2	6	4	5	3	2	7	1
8	1	4	3	0	6	4	9	5	2	6	4	1	3	2	7
3	8	1	4	9	0	6	4	4	5	2	6	7	1	3	2
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3

Can you find a collision?

1	0	0	-1	-1	1	1	0	0	0	1	1	1	0	-1	0	
1	4	3	8	6	4	9	0	2	6	4	5	3	2	7	1	5
8	1	4	3	0	6	4	9	5	2	6	4	1	3	2	7	4
3	8	1	4	9	0	6	4	4	5	2	6	7	1	3	2	8
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3	6

Can you find a collision?

?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	
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8	1	4	3	0	6	4	9	5	2	6	4	1	3	2	7	0
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4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3	0

Can you find a collision?

$$
\left|\begin{array}{cccc|cccc|cccc|cccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 4 & 3 & 8 & 6 & 4 & 9 & 0 & 2 & 6 & 4 & 5 & 3 & 2 & 7 & 1 \\
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3 & 8 & 1 & 4 & 9 & 0 & 6 & 4 & 4 & 5 & 2 & 6 & 7 & 1 & 3 & 2 \\
4 & 3 & 8 & 1 & 4 & 9 & 0 & 6 & 6 & 4 & 5 & 2 & 2 & 7 & 1 & 3 \\
\hline
\end{array}\right|
\qquad
\left|\begin{array}{c}
\ \\
\hline
\ \\
\ \\
\ \\
\ \\
\hline
\end{array}\right|
$$

$$
+1\times\begin{array}{|c|}\hline 6\\ 6\\ 6\\ 6\\ \hline\end{array}
\qquad
-1\times\begin{array}{|c|}\hline 9\\ 9\\ 9\\ 9\\ \hline\end{array}
\qquad
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\qquad
\begin{array}{|c|}
\\
\hline
0 \\
0 \\
0 \\
0 \\
\hline
\end{array}
$$

$$
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\qquad
-1 \times \begin{array}{|c|} \hline 9 \\ 9 \\ 9 \\ 9 \\ \hline \end{array}
\qquad
+0 \times \begin{array}{|c|} \hline 7 \\ 7 \\ 7 \\ 7 \\ \hline \end{array}
\qquad
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$$

Remarks about proofs of security

- This function is essentially the compression function of hash function LASH, modeled after NTRU
- You can still "prove" security based on average case assumption: Breaking the above hash function is as hard as finding short vectors in a random lattice $\Lambda([\mathbf{A}^{(1)}|\ldots|\mathbf{A}^{(m/n)}])$
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Back to general lattices

- Finding short vectors in $\Lambda_q^\perp(\mathbf{A})$ when $\mathbf{A}$ is a random "block circulant" matrix is easy
- What about unstructured random $\mathbf{A} \in \mathbb{Z}_q^{k \times n}$?

Question

Is $f_\mathbf{A}$ collision resistant when $\mathbf{A} \in \mathbb{Z}_q^{k \times n}$ is random?

- Yes, provided SIVP/ADD/BDD are hard in the worst-case! [Ajtai96,...,MR04]
- We will give an oversimplified proof sketch, where $\mathbf{A} \in \mathbb{R}^{k \times n}$

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- What about unstructured random $\mathbf{A} \in \mathbb{Z}_q^{k \times n}$?

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- Yes, provided SIVP/ADD/BDD are hard in the worst-case! [Ajtai96,...,MR04]
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Consider an arbitrary lattice, and add noise to each lattice point until the entire space is covered. Increase the noise until the space is uniformly covered.

How much noise is needed? [MR]

$\|\mathbf{r}\| \leq (\log n) \cdot \sqrt{n} \cdot \lambda_n/2$

- Each point in $\mathbf{a} \in \mathbb{R}^n$ can be written $\mathbf{a} = \mathbf{v} + \mathbf{r}$ where $\mathbf{v} \in \mathcal{L}$ and $\|\mathbf{r}\| \approx \sqrt{n}\lambda_n$.
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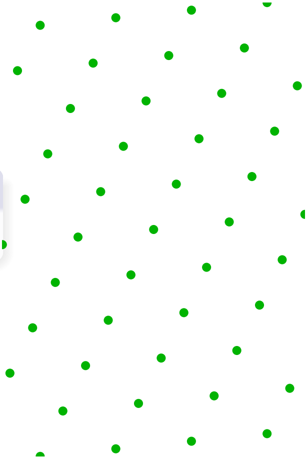

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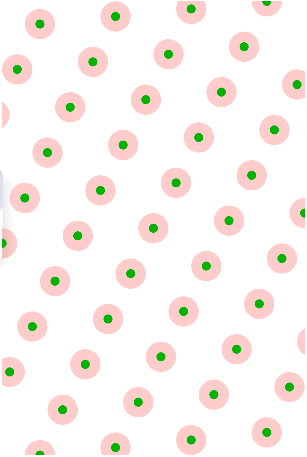

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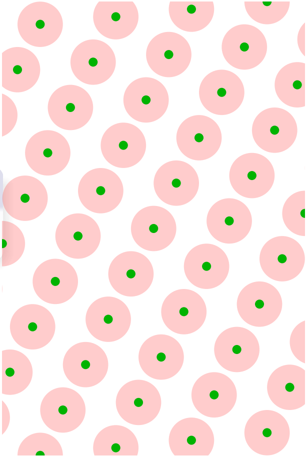

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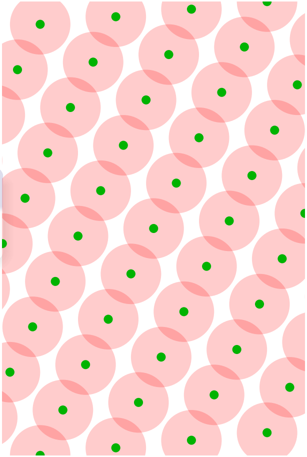

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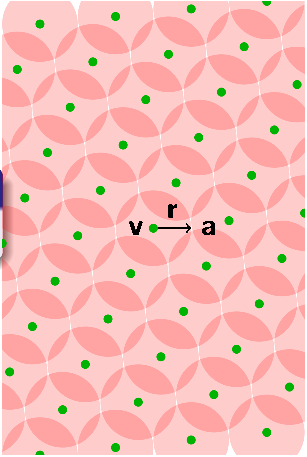

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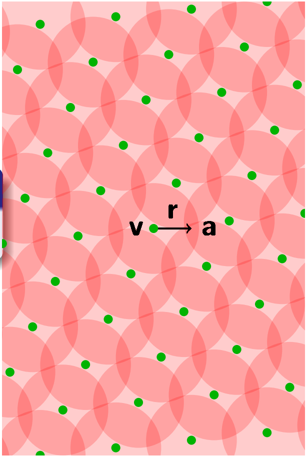

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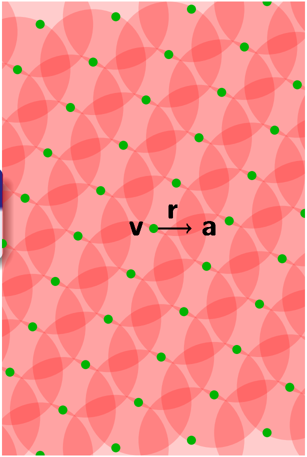

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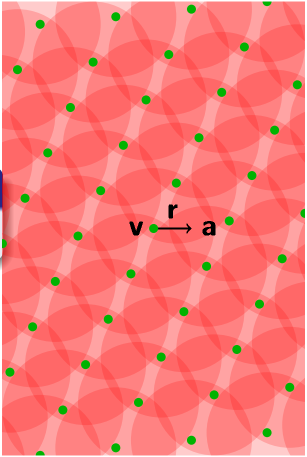

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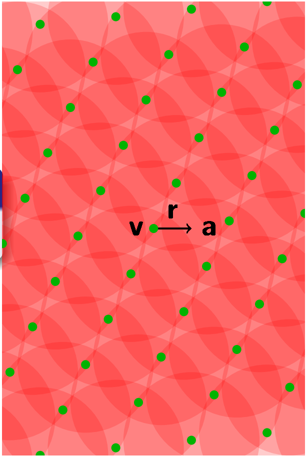

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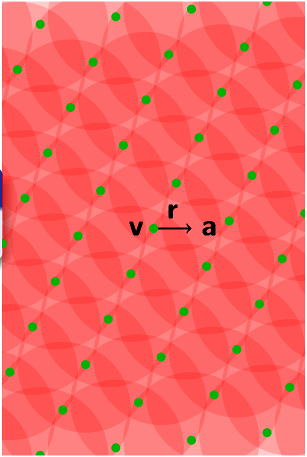

Security proof (sketch)

- Generate random points $\mathbf{a}_i = \mathbf{v}_i + \mathbf{r}_i$, where
    - $\mathbf{v}_i$ is a random lattice point
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- $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$ is distributed almost uniformly at random in $\mathbb{R}^{n\times m}$, so
    - if we can break Ajtai's function $f_{\mathbf{A}}$, then
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$$\sum(\mathbf{v}_i + \mathbf{r}_i)z_i = \sum \mathbf{a}_i z_i = \mathbf{0}$$

- Rearranging the terms yields a lattice vector

$$\sum \mathbf{v}_i z_i = -\sum \mathbf{r}_i z_i$$

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What about efficiency

?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?
1	-4	-3	-8	6	-4	-9	-0	2	-6	-4	-5	3	-2	-7	-1
8	1	-4	-3	0	6	-4	-9	5	2	-6	-4	1	3	-2	-7
3	8	1	-4	9	0	6	-4	4	5	2	-6	7	1	3	-2
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3

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Theorem (LM'07)

Provably collision resistant, assuming the worst case hardness of approximating SVP and SIVP over ideal lattices.

Efficiency of anti-cyclic hashing

- Key size: $(m/n) \cdot n \log q = m \cdot \log q = \tilde{O}(n)$ bits
- Anti-cyclic matrix-vector multiplication can be computed in quasi-linear time $\tilde{O}(n)$ using FFT
- The resulting hash function can also be computed in $\tilde{O}(n)$ time
- For approximate choice of parameters, this can be very practical (SWIFFT [LMPR])
- The hash function is linear: $\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y}$
- We will see that this can be a feature rather than a weakness

Outline

1. Point Lattices
   - Computational Problems
   - The dual lattice
2. Lattice Cryptography
   - Average Case Hardness
   - Random Lattices
   - Cryptographic functions

Hard Random Lattices

Theorem (Ajtai,MR04)

$f_{\mathbf{A}}$ is collision resistant, under the assumption that SIVP is hard to approximate in the worst-case withing a factor $\gamma \approx n$.

Equivalently, ...

Theorem

If ADD is hard to approximate in the worst case within $\gamma \approx n$, then ADD is hard on average for input distribution $\Lambda_q^\perp(\mathbb{Z}_q^{n \times m})$.

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- OTS: diginal signature scheme that allows to sign a single message (faster than a full fledged signature scheme)
- Global parameters: q-ary lattice $\mathbf{A}$
- Secret key: short error vectors $\mathbf{S}$
- Public key: syndromes $\mathbf{P} = \mathbf{AS}$ (Hash of secret key under homomorphic hash function)
- Message: short vector $\mathbf{m}$
- Signature: $\sigma = \mathbf{Sm}$
- Verify: Check if σ is short and $\mathbf{Pm} = \mathbf{A}\sigma$

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- Generate $\mathbf{A}$, $\mathbf{S}$, $\mathbf{P} = \mathbf{AS}$
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- Attacker produces a forgery $(\mathbf{m}', \sigma')$
- $(\mathbf{Sm}', \sigma')$ is a collision: $\mathbf{ASm}' = \mathbf{Pm}' = \mathbf{A}\sigma'$

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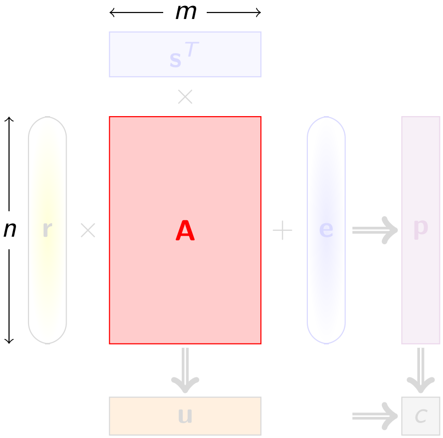
Regev (LWE) cryptosystem



- Parameters: $m, n, q \in \mathbb{Z}$, $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$
- Secret key: $\mathbf{s} \in \mathbb{Z}_q^n$, $\mathbf{e} \in \mathcal{E}^m$
- Public key: $\mathbf{p} = \mathbf{A}\mathbf{s} + \mathbf{e} \approx_c \mathbb{Z}_q^m$
- Encrypt$_{\mathbf{p}}(m;(\mathbf{r}))$:
$$\begin{aligned} \mathbf{u} &= \mathbf{r}^T \mathbf{A} \\ c &= \mathbf{r}^T \mathbf{p} + m - r_0 \end{aligned}$$
- Decrypt$_{\mathbf{s}}(\mathbf{u}, c) =$ $c - \mathbf{u} \cdot \mathbf{s} \approx m.$

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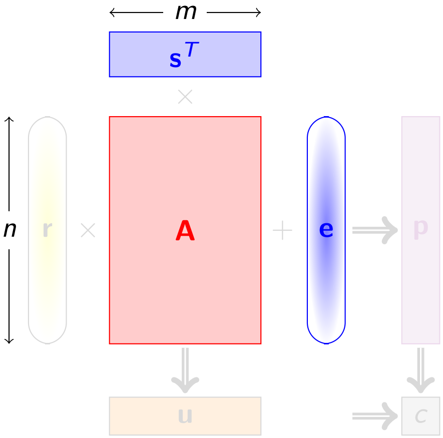

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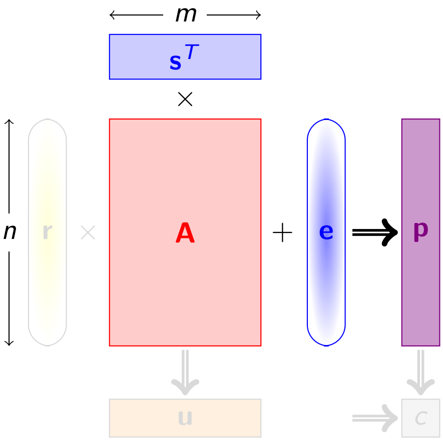

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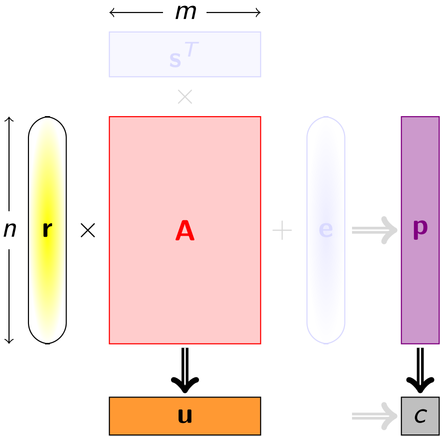

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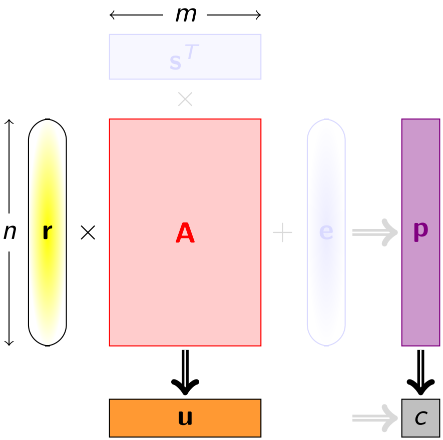

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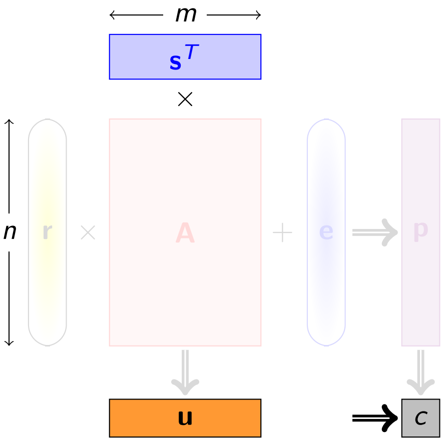

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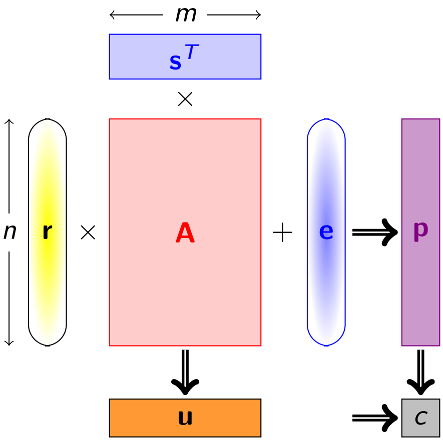
The geometry of LWE encryption



- Public key:
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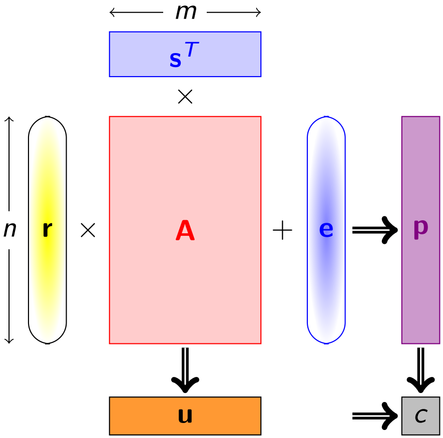

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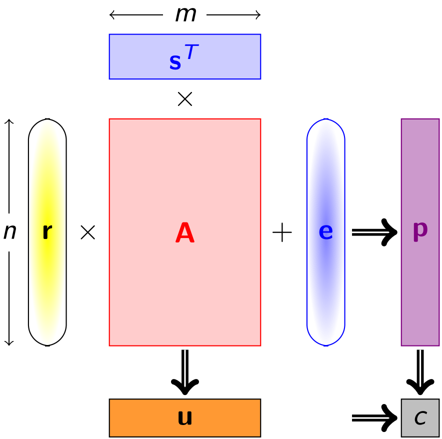

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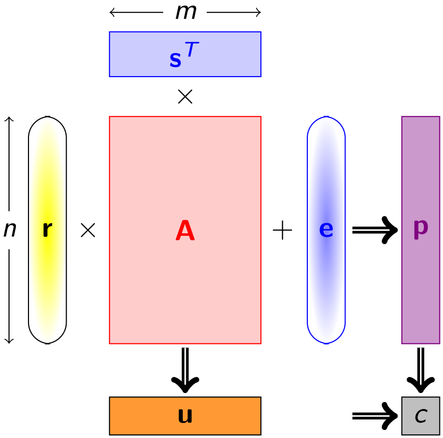

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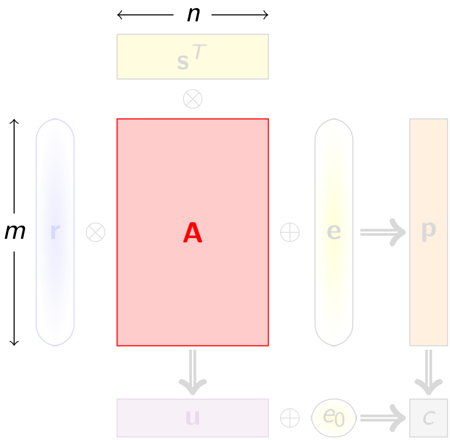
GPV (dual LWE) cryptosystem



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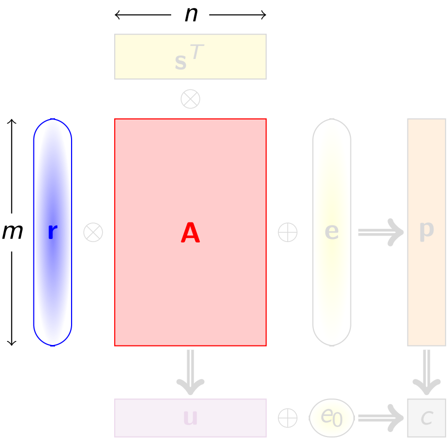

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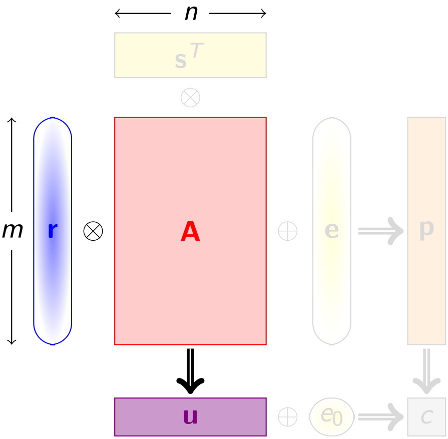

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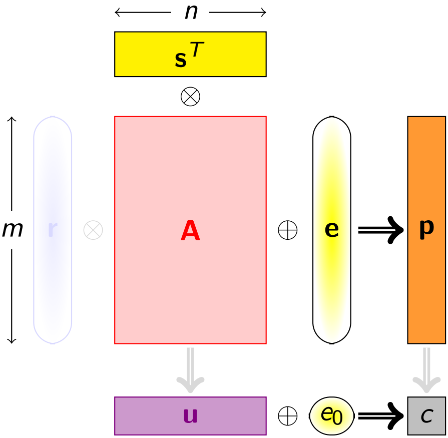

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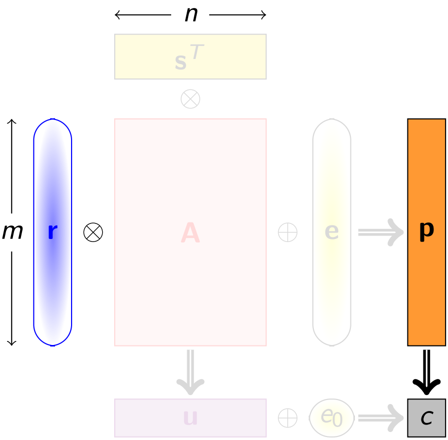

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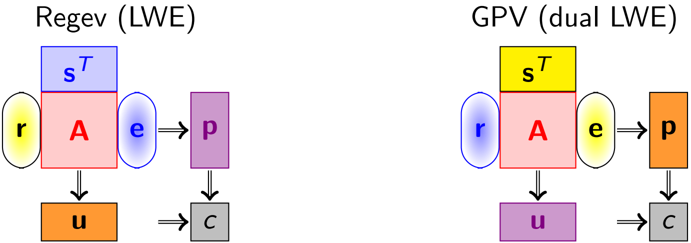
Comparing Regev and GPV encryption

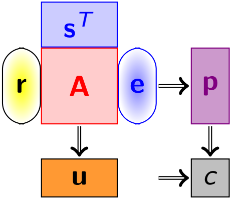


Regev and GPV cryptosystems use the same mathematical objects $\mathbf{A}, \mathbf{s}, \mathbf{r}, \mathbf{e}, \mathbf{p}, \mathbf{u}, c$, but operate on them in different roles:

Public key generation	$\Longleftrightarrow$	Encryption
Secret key	$\Longleftrightarrow$	Encryption randomness
Public key	$\Longleftrightarrow$	Ciphertext

Comparing Regev and GPV encryption

Regev (LWE)



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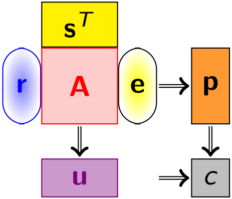

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Naive interpretation

- The schemes are syntactically similar: Regev and GPV cryptosystems operate on the same mathematical objects $\mathbf{A}, \mathbf{s}, \mathbf{r}, \mathbf{e}, \mathbf{p}, \mathbf{u}, c$.
- The scheme are semantically different:

Common parameters	$\mathbf{A}$	$\Longleftrightarrow$	$\mathbf{A}$	Common parameters
secret key	$\mathbf{s}, \mathbf{e}$	$\Longleftrightarrow$	$\mathbf{s}, \mathbf{e}$	encryption randomness
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The true answer: Lattices and Duality

- The schemes are syntactically different: The symbols $\mathbf{A}, \mathbf{s}, \mathbf{r}, \mathbf{e}, \mathbf{p}, \mathbf{u}, c$ in Regev and GPV cryptosystems represent different mathematical objects
- The two schemes are semantically equivalent:

Common parameters	$\mathbf{A}$	$\Longleftrightarrow$	$\mathbf{A}'$	Common parameters
secret key	$\mathbf{s}, \mathbf{e}$	$\Longleftrightarrow$	$\mathbf{r}'$	secret key
encryption randomness	$\mathbf{r}$	$\Longleftrightarrow$	$\mathbf{s}', \mathbf{e}'$	encryption randomness
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Trapdoor functions

Theorem (A99,AP09,MP11)

There is an algorithm to efficiently generate a random $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ *together with a short basis* $\mathbf{S} \in \mathbb{Z}^{m \times m}$ *of* $\Lambda_q^\perp(\mathbf{A})$.

Trapdoor function:

- Inverting $f_{\mathbf{A}}$ is a BDD problem
- BDD can be solved with a short dual basis
- $\mathbf{S}$ can be used as an inversion trapdoor

Injective trapdoor functions can be used for the construction of a wide range of other more complex cryptographic primitives.

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Conclusion

- Lattice cryptography allows to build a wide range of many other cryptographic primitives (Hierarchical identity based encryption, Fully homomorphic encryption, and much more)
- It has great potential for fast implementation due to simple operations and high parallelizability
- Most primitives can be described and explained in terms of a handful of basic geometric concepts
- Everything that can be done with number theoretic scheme can be done with lattice crypography as well
- Currently the only method known to build fully homomorphic encryption
- Not quite ready for use in practice, but moving fast in that direction
- Open problems: concrete efficiency, security evaluation, etc.