# Lessons to be learned from Stone Dualities for Markov Processes

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Based on joint work with

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Complex networks/systems are often modelled as stochastic processes

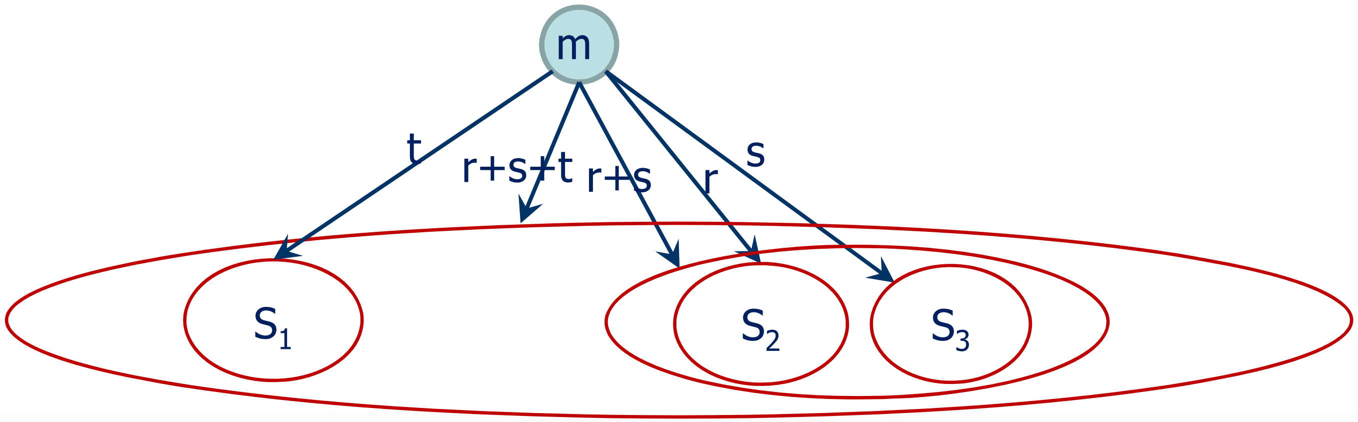
- to encapsulate a lack of knowledge or inherent non-determinism,
- to approximate the complex behaviour of real systems that cannot be modeled exactly since exact data are unknown

A **Markov process** is a measurable mapping

$$\theta : M \rightarrow \Delta(M,\Sigma)$$

where $(M,\Sigma)$ is a measurable (analytic) state space,

and $\Delta(M,\Sigma)$ is the space of measures on $(M,\Sigma)$.



## Background

Stochastic/probabilistic/subprobabilistic Bisimulation

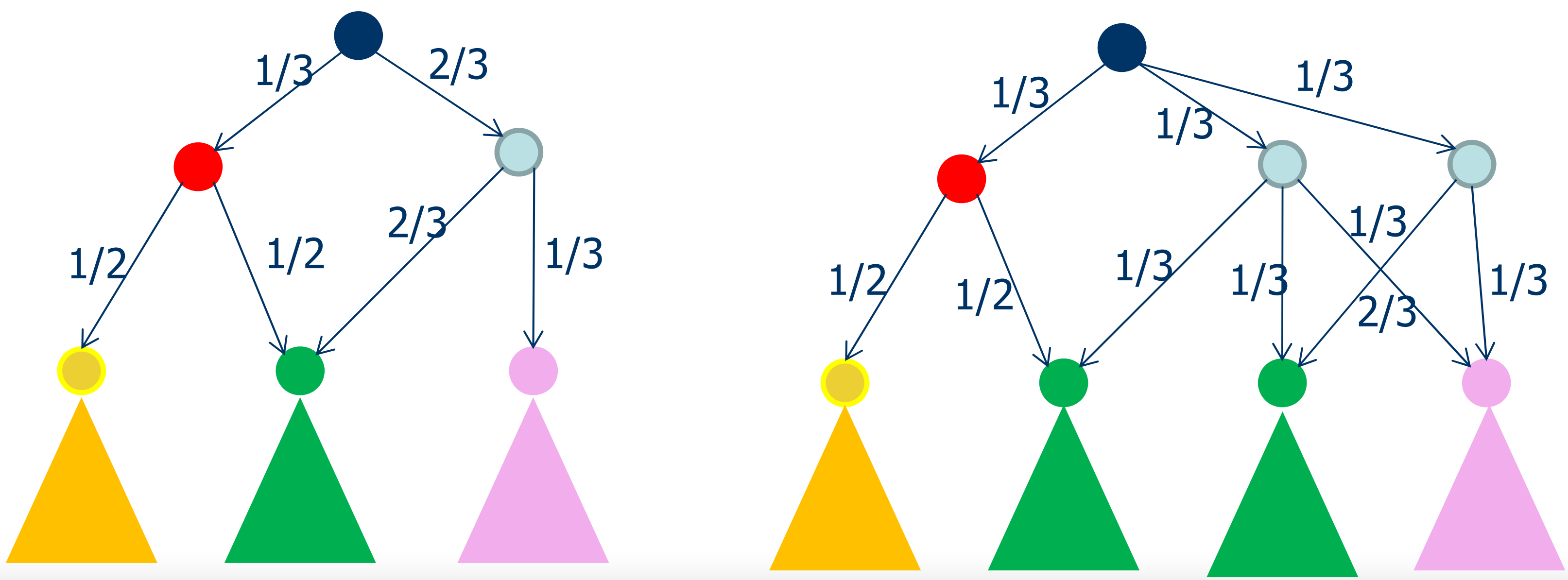
- equates systems with identical (probabilistic) behaviours

Given $\theta : M \rightarrow \Delta(M,\Sigma)$ , a **bisimulation** is a relation (equivalence)

$$R \subseteq M \times M$$

s.t. $mRn$ implies

- $\forall\ S \in \Sigma(R)\ ,\ \theta(m)(S)=\theta(n)(S)$



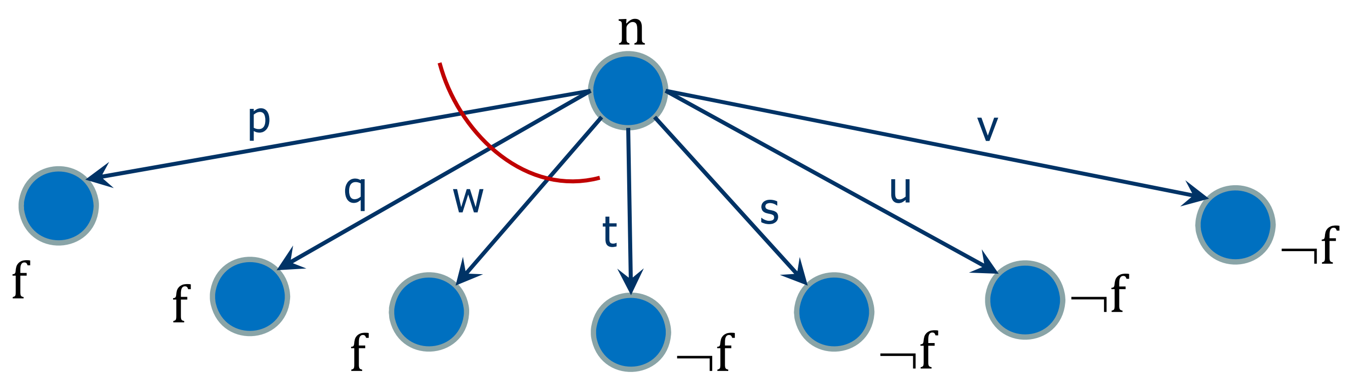
## Markovian Logics

**Syntax:** $f := \bot \mid f \to f \mid L_r f \mid M_r f \qquad r \in \mathbb{Q} \cap [0,1]$

**Semantics:** Given $\theta : M \to \Delta(M,\Sigma)$ , $m \in M$

$m \models L_r f$ iff $\theta(m)([f]) \geq r$ where $[f]=\{n \in M \mid n \models f\}$

$m \models M_r f$ iff $\theta(m)([f]) \leq r$



**MP, $\sim$** $\quad\models\quad$ **ML**

<u>Logical characterisation:</u>

Given $\theta : M \rightarrow \Delta(M,\Sigma)$,

$$m \sim n \quad \text{iff} \quad [\forall f \in \mathcal{L} ,\ m \models f \ \text{ iff } \ n \models f ].$$

[Desharnais, Edalat, Panangaden, LICS'98]

<u>Sound and Strongly-complete axiomatization:</u> For any $F \subseteq \mathcal{L}$ and $f \in \mathcal{L}$ ,

$$F \models f \ \text{ iff } \ F \vdash f$$

Model construction using maximal consistent sets of formulas.

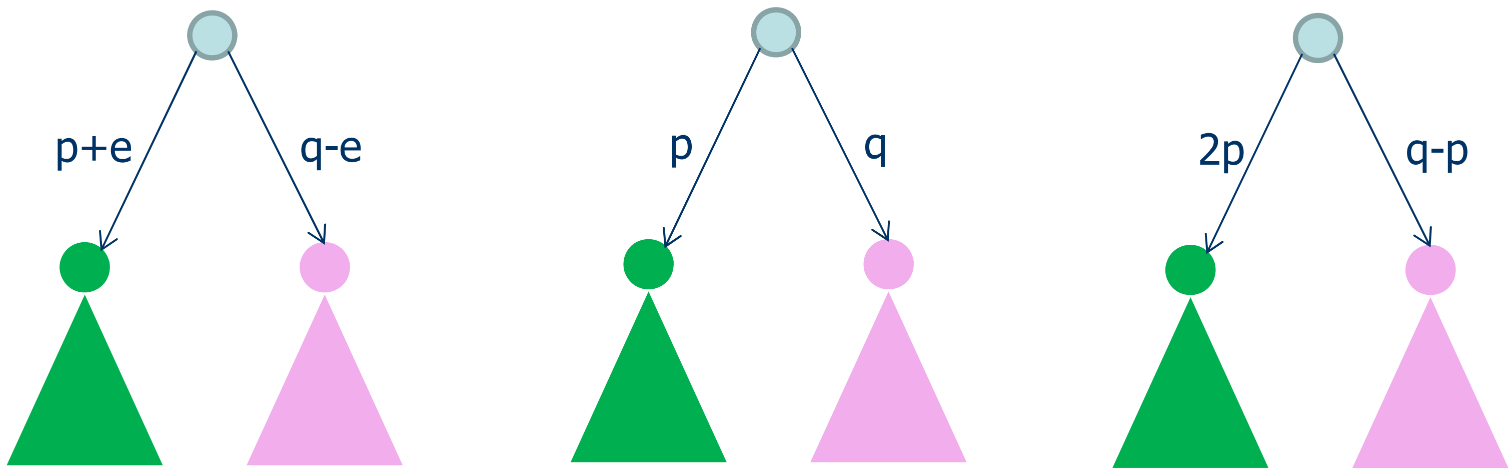
[Cardelli, Larsen, Mardare, CSL2011]

[Kozen, Mardare, Panangaden LICS'12, LICS'13]

## Practical perspective – the first argument

Bisimilarity is a too strict concept

- We would like to understand when two systems have <u>similar behaviours</u>



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bisimilarity => bisimilarity distance (pseudometric)

$$d: M \times M \rightarrow [0,1]$$

P1. $d(m,n)=d(n,m)$

P2. $d(m,n) \leq d(m,m')+d(m',n)$

P3. $d(m,n)=0$ iff $m \sim n$

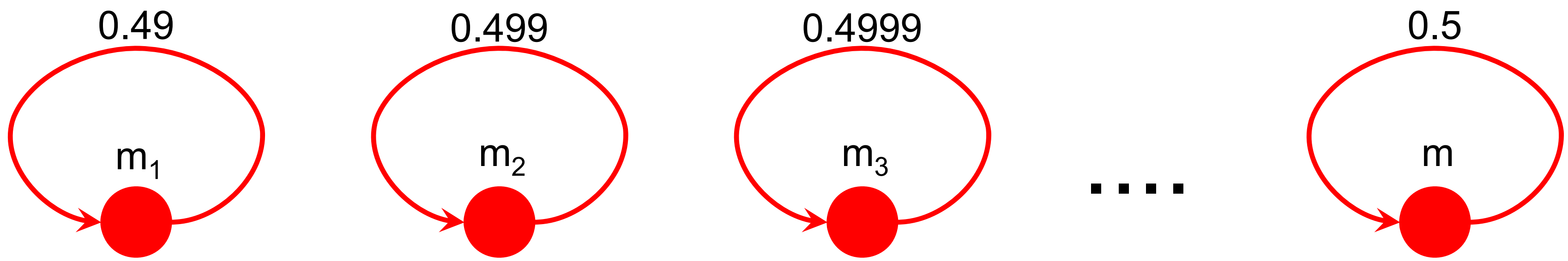
[van Bruegel, Worell, CONCUR'01.]
[van Bruegel, et.al, FOSSACS'03.]
[Desharnais, et.al., TCS 2004.]

## Practical perspective – the second argument

Often in science we

- approximate a real system
- check properties of better and better approximations
- extrapolate the results to the real system.

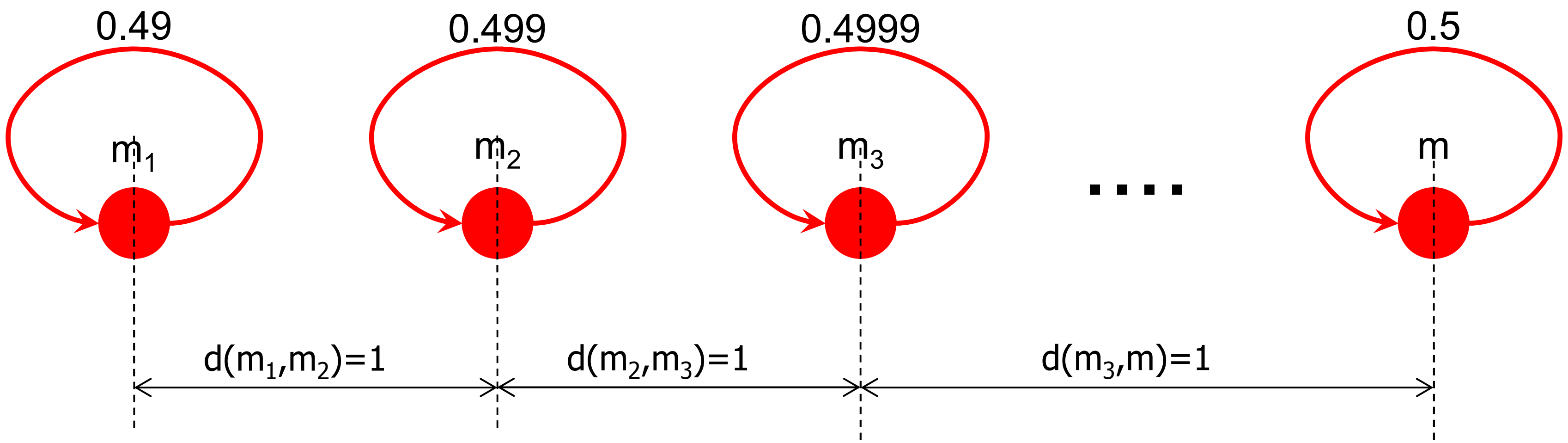
Assuming that we have a behavioural distance d, we implicitly assume some convergence properties




A proper bisimilarity distance must prove such a convergence in the open-ball topology!

An example of a *not so useful* bisimilarity distance

$$d(m,n) = \begin{cases} 0 & \text{if } m \sim n \\ 1 & \text{otherwise.} \end{cases}$$




The sequence is not Cauchy!

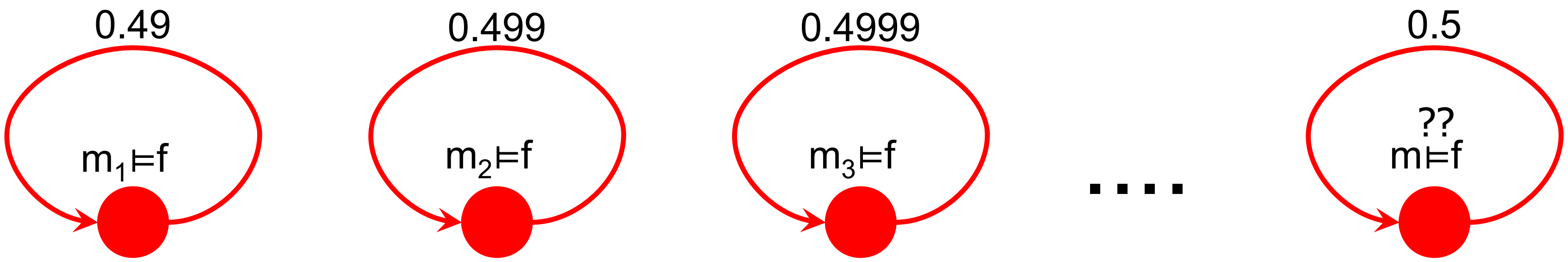
## Practical perspective – the third argument

Often in science we

- approximate a real system
- check properties of better and better approximations
- extrapolate the results to the real system.

Assuming that we have a behavioural distance d, we implicitly assume that:

Conjecture 1: If $\lim m_k = m$ and for each k, $m_k \vDash f$, then $m \vDash f$.

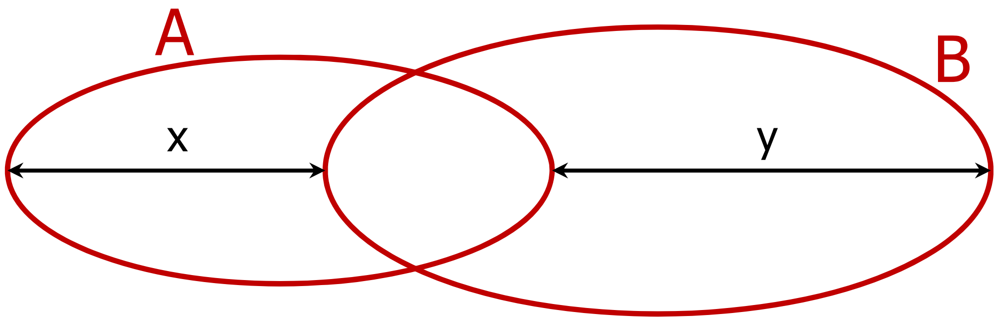


## Practical perspective – the fourth argument

Any pseudometric $d: M \times M \to [0,1]$

induces a Hausdorff pseudometric $d^H: 2^M \times 2^M \to [0,1]$

$$d^H(A,B) = \max\{x,y\}$$




$$x = \sup_{a \in A} \inf_{b \in B} d(a,b) \qquad y = \sup_{b \in B} \inf_{a \in A} d(a,b)$$

## Practical perspective – the fourth argument

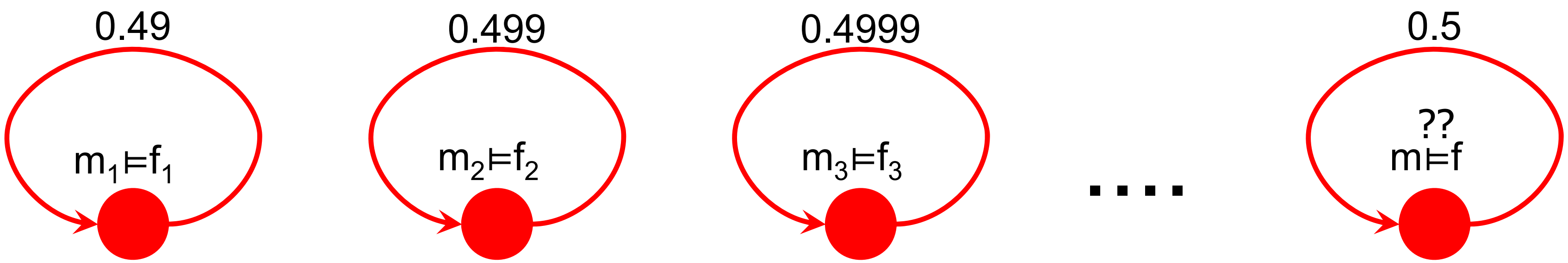
A logical property can be identified with the set of its models.
Hence, we get a pseudometric

$$d^H : \mathcal{L} \times \mathcal{L} \rightarrow [0,1]$$

and a topology over the space of formulas.

A possible convergence in $\mathcal{L}$:

$$L_{0.499..}T \xrightarrow{d^H} L_{0.5}T$$



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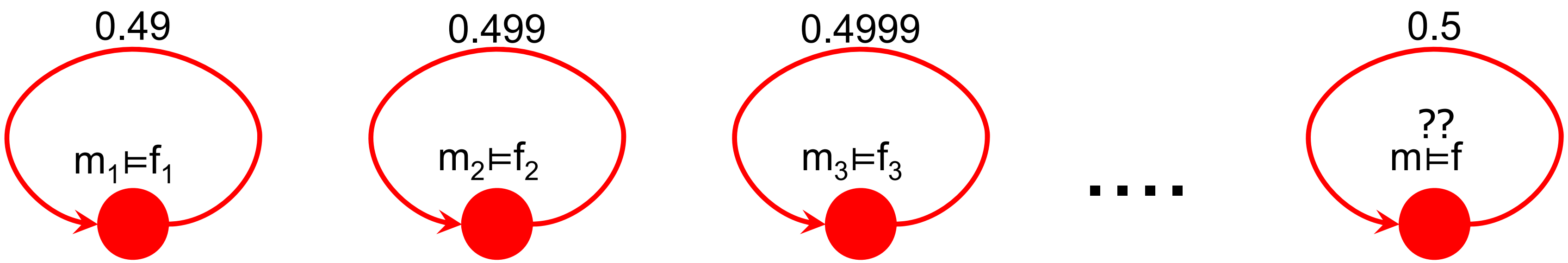
<u>Conjecture 1:</u> If $\lim m_k = m$ and for each k, $m_k \vDash f$, then $m \vDash f$.

<u>Conjecture 2:</u> If $\begin{cases} m_k \xrightarrow{d} m \\ f_k \xrightarrow{d^H} f \end{cases}$ and for each k, $m_k \vDash f_k$, then $m \vDash f$.



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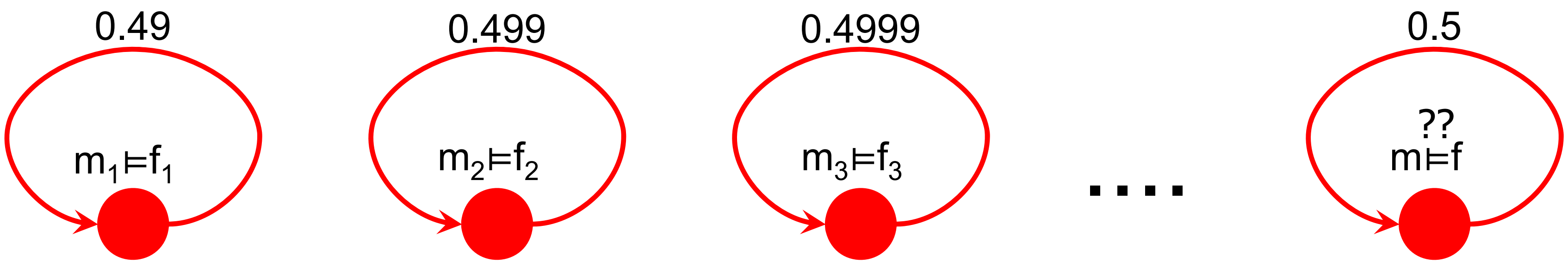
Conjecture 1: If $\lim m_k = m$ and for each k, $m_k \vDash f$, then $m \vDash f$. =>NO!

Conjecture 2: If $\left\{ \begin{array}{l} m_k \xrightarrow{d} m \\ f_k \xrightarrow{d^H} f \end{array} \right.$ and for each k, $m_k \vDash f_k$, then $m \vDash f$. =>NO!

[Larsen, Mardare, Panangaden MFCS2012]



## The topological space of logical formulas

The probabilistic case:

| | | |
|---|---|---|
| $\mathcal{L}$: | $f ::= \perp \mid f \to f \mid L_r f$ | $r \in \mathbb{Q} \cap [0,1]$ |
| $\mathcal{L}^+$ | $g := T \mid g \wedge g \mid g \vee g \mid Lrf \mid M_r f$ | $f \in \mathcal{L}$ |
| $\mathcal{L}^-$ | $\{\neg g \mid g \in \mathcal{L}^+\}$ | |

Proposition: Let $d$ be a bisimilarity distance on probabilistic MPs.

1. If $f \in \mathcal{L}^+$, then $[f]$ is a closed set in the open ball topology of d.
2. If $f \in \mathcal{L}^-$, then $[f]$ is an open set in the open ball topology of d.

[MFCS 2012]

## The topological space of logical formulas

The general (subprobabilistic, stochastic) case:

| | | |
|---|---|---|
| $\mathcal{L}$ | $f := \perp \mid f \to f \mid L_r f \mid M_r f$ | $r \in \mathbb{Q}_+$ |
| $\mathcal{L}_0$ | $f := \perp \mid f \to f \mid L_r f$ | $r \in \mathbb{Q}_+$ |
| $\mathcal{L}^+$ | $g := T \mid g \wedge g \mid g \vee g \mid L_r f \mid M_r f$ | $f \in \mathcal{L}$ |
| $\mathcal{L}_0^+$ | $g := T \mid g \wedge g \mid g \vee g \mid L_r f$ | $f \in \mathcal{L}$ |

$\mathcal{L}^- := \{\neg g \mid g \in \mathcal{L}^+\}$, $\mathcal{L}_0^- := \{\neg g \mid g \in \mathcal{L}_0^+\}$

Proposition: Let d be a dynamically-continuous bisimilarity distance on DMPs.

1. If $f \in \mathcal{L}_0^+$, then $[f]$ is a closed set in the open ball topology induced by d.
2. If $f \in \mathcal{L}_0^-$, then $[f]$ is an open set in the open ball topology induced by d.
3. If $f \in \mathcal{L}^+$, then $[f]$ is a $G_\delta$ set (countable intersection of open sets).
4. If $f \in \mathcal{L}^-$, then $[f]$ is a $F_\sigma$ set (countable union of closed sets).

[MFCS 2012]

## Mathematical perspective

Can we characterize the behavioural distances that behave correctly topologically?

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For $\theta : M \to \Delta(M,\Sigma)$ , a **bisimulation** is a relation $R \subseteq M \times M$ s.t. $mRn$ implies

- $\forall\, S \in \Sigma(R)$ , $\theta(m)(S) = \theta(n)(S)$

A first attempt

For $\theta : M \to \Delta(M,\Sigma)$ , a "good" bisimilarity distance is a pseudometric $d : M \times M \to [0,1]$ such that for any sequence $(m_k)_k$ with $m_k \xrightarrow{d} m$

- $\forall\, S \in \Sigma(\sim)$, $\theta(m_k)(S) \xrightarrow{\mathbb{R}} \theta(m)(S)$

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For $\theta : M \to \Delta(M,\Sigma)$ , a **bisimulation** is a relation $R \subseteq M \times M$ s.t. $mRn$ implies

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A first attempt - Wrong! It misses the "coinductive nature"!

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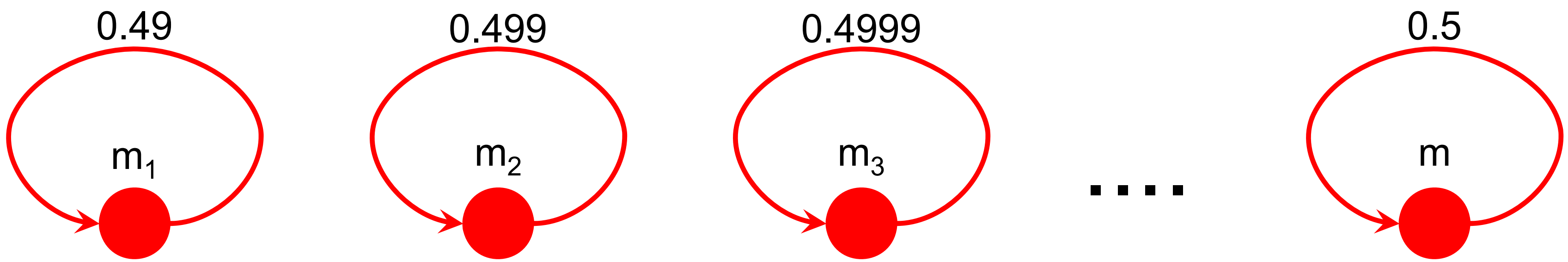


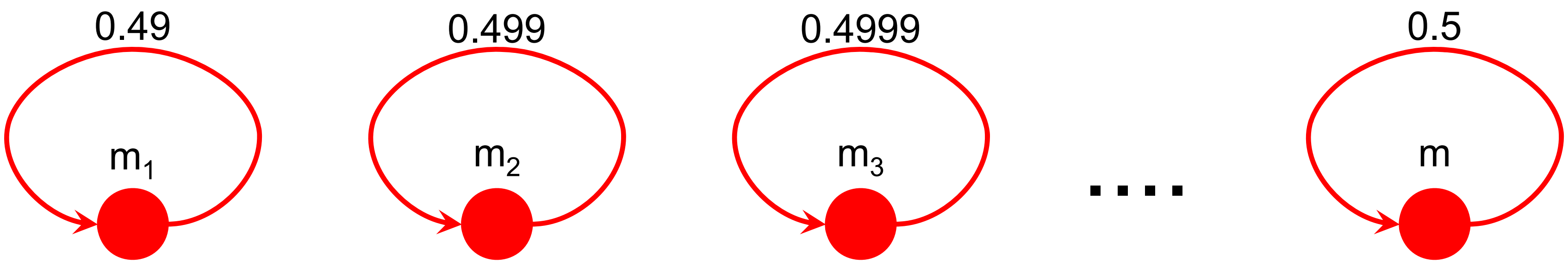

$\theta(m_k)(\{m\})=0$ for any $k$ and $\theta(m)(\{m\})=0.5$

## Mathematical perspective

A second attempt

For $\theta : M \to \Delta(M,\Sigma)$, a "good" bisimilarity distance is a pseudometric $d: M \times M \to [0,1]$ such that for any sequence $(m_k)_k$ with $m_k \xrightarrow{d} m$,

- $\forall S \in \Sigma(\sim)$, $\forall (S_k)_k \subseteq \Sigma(\sim)$ such that $S_k \xrightarrow{d^H} S$
  - $\theta(m_k)(S_k) \xrightarrow{\mathbb{R}} \theta(m)(S)$



## Mathematical perspective

A second attempt - Wrong quantifiers!

For $\theta : M \to \Delta(M,\Sigma)$ , a "good" bisimilarity distance is a pseudometric $d: M \times M \to [0,1]$ such that for any sequence $(m_k)_k$ with $m_k \xrightarrow{d} m$,

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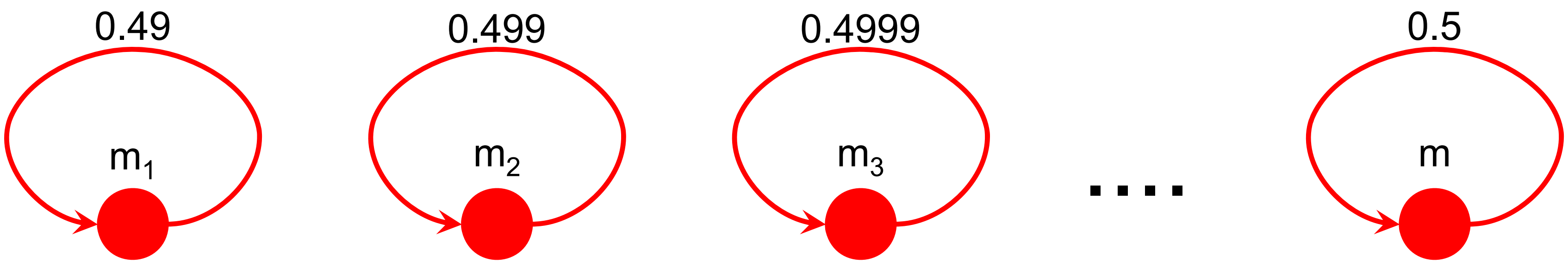


$S_1=S_2=\ldots=S=\{m\}$ $\theta(m_k)(S)=0$ for any k and $\theta(m)(S)=0.5$

## Mathematical perspective

### A third attempt

For $\theta : M \to \Delta(M,\Sigma)$ , a dynamically-continuous bisimilarity distance is a pseudometric $d{:}M \times M \to [0,1]$ such that for any sequence $(m_k)_k$ ,

$m_k \xrightarrow{d} m$ implies

- $\forall\, S \in \Sigma(\sim),\ \exists\ (S_k)_k \subseteq \Sigma(\sim)$ such that
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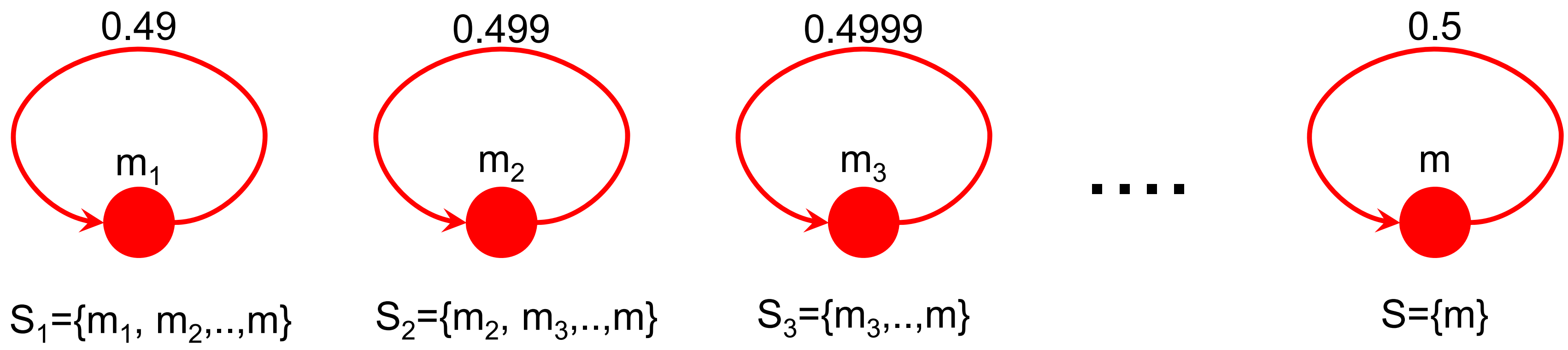
    - $\theta(m_k)(S_k) \xrightarrow{\mathbb{R}} \theta(m)(S)$



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What is the relation to bisimulation?

## Mathematical perspective

For $\theta : M \rightarrow \Delta(M,\Sigma)$ , a **bisimulation** is a relation $R \subseteq M \times M$
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- $\forall\, S \in \Sigma(R)$ , $\exists\, S' \in \Sigma(R)$ such that
  - $S = S'$
  - $\theta(m)(S) = \theta(n)(S')$

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- $\forall\, S \in \Sigma(R)$ , $\exists\, S' \in \Sigma(R)$ such that
    - $(S,S') \in =_{\Sigma}$
    - $(\theta(m)(S), \theta(n)(S')) \in =_{\mathbb{R}}$

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For $\theta : M \to \Delta(M,\Sigma)$ , $X \subseteq \Sigma \times \Sigma$ and $Y \subseteq \mathbb{R} \times \mathbb{R}$ a $(X,Y)$-**bisimulation** is a relation $R \subseteq M \times M$
s.t. $mRn$ implies

- $\forall\, S \in \Sigma(R)$ , $\exists\, S' \in \Sigma(R)$ such that
  - $(S,S') \in X$
  - $(\theta(m)(S), \theta(n)(S')) \in Y$

## Mathematical perspective

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  - $(\theta(m)(S), \theta(n)(S')) \in Y$

A “classic” bisimulation is nothing else but a $(=_{\Sigma},=_{\mathbb{R}})$-bisimuation.

## Mathematical perspective

For $\theta : M \rightarrow \Delta(M,\Sigma)$ , $X \subseteq \Sigma^{k+1}$ and $Y \subseteq \mathbb{R}^{k+1}$ a (X,Y)-**bisimulation** is a relation $R \subseteq M^{k+1}$
s.t. $(m, m_1, m_2, .., m_k) \in R$ implies

- $\forall\ S \in \Sigma(R)$ , $\exists\ S_1, S_2,..,S_k \in \Sigma(R)$ such that
  - $(S, S_1, S_2,..,S_k) \in X$
  - $(\theta(m)(S), \theta(m_1)(S_1), \theta(m_2)(S_2),\ldots, \theta(m_k)(S_k)) \in Y$

## Mathematical perspective

For $\theta : M \to \Delta(M,\Sigma)$ , $X \subseteq \Sigma^\omega$ and $Y \subseteq \mathbb{R}^\omega$ a (X,Y)-**bisimulation** is a relation $R \subseteq M^{k+1}$
s.t. $(m, m_1, m_2, .., m_{k,\ldots}) \in R$ implies

- $\forall\, S \in \Sigma(R)$ , $\exists\, S_1, S_2,..,S_k \ldots \in \Sigma(R)$ such that
  - $(S, S_1, S_2,..,S_k ,\ldots) \in X$
  - $(\theta(m)(S), \theta(m_1)(S_1), \theta(m_2)(S_2),\ldots, \theta(m_k)(S_k),\ldots) \in Y$

If we take

- $k=\omega$
- $(S, S_1, S_2,.. ) \in X$ iff $S_i \xrightarrow{d^H} S$
- $(r, r_1, r_2,..) \in Y$ iff $r_i \xrightarrow{\mathbb{R}} r$

then,

$$R=\{(m, m_1, m_2,..) \mid m_i \xrightarrow{d} m\}$$

is an (X,Y)-bisimulation iff d is dynamic-continuous.

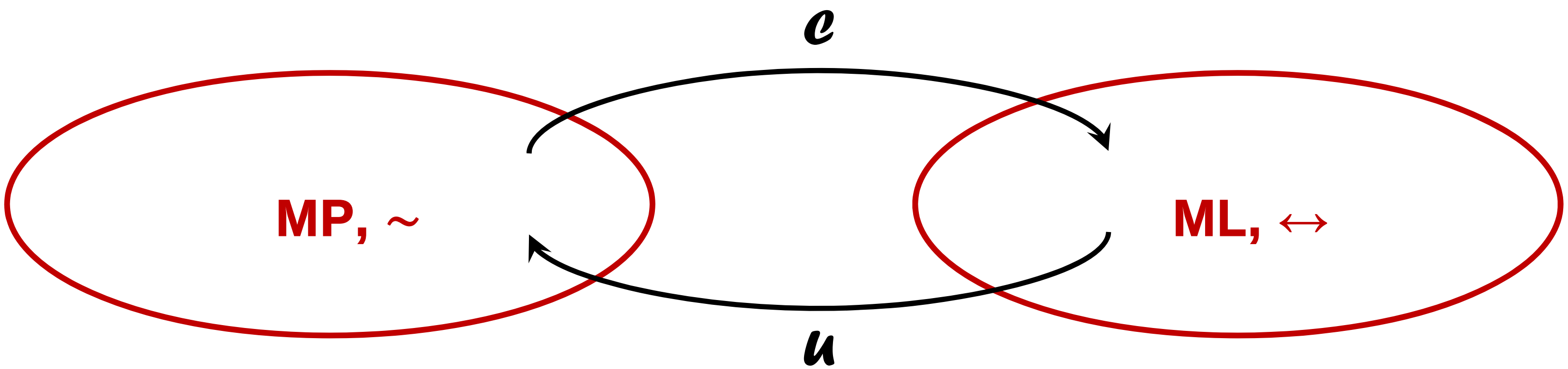
## Mathematical perspective

However, the concept of dynamic-continuity is not sufficient to solve our problem since the following distance

$$d(m,n) = \begin{cases} 0 & \text{if } m \sim n \\ 1 & \text{otherwise.} \end{cases}$$

is dynamic-continuous!

## The lesson of the "classic" Stone duality for MPs

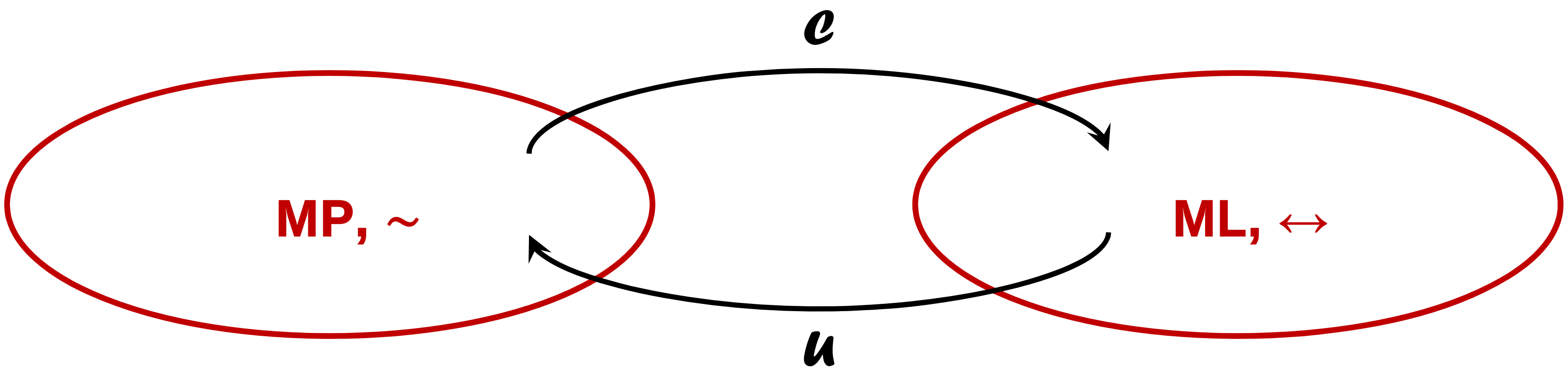



[Kozen, Larsen, Mardare, Panangaden LICS2013]

For $f \in \mathcal{L}$, let $[f]=\{u \in U \mid f \in u\}$.

- the set $\{[f] \mid f \in \mathcal{L}\}$ generates a "Stone" topology $\mathfrak{T}_{\mathcal{L}}$ on U
- we construct an MP on (U,B) where B is the Borel algebra of $\mathfrak{T}_{\mathcal{L}}$

## The lesson of the “classic” Stone duality for MPs




[Kozen, Larsen, Mardare, Panangaden LICS2013]

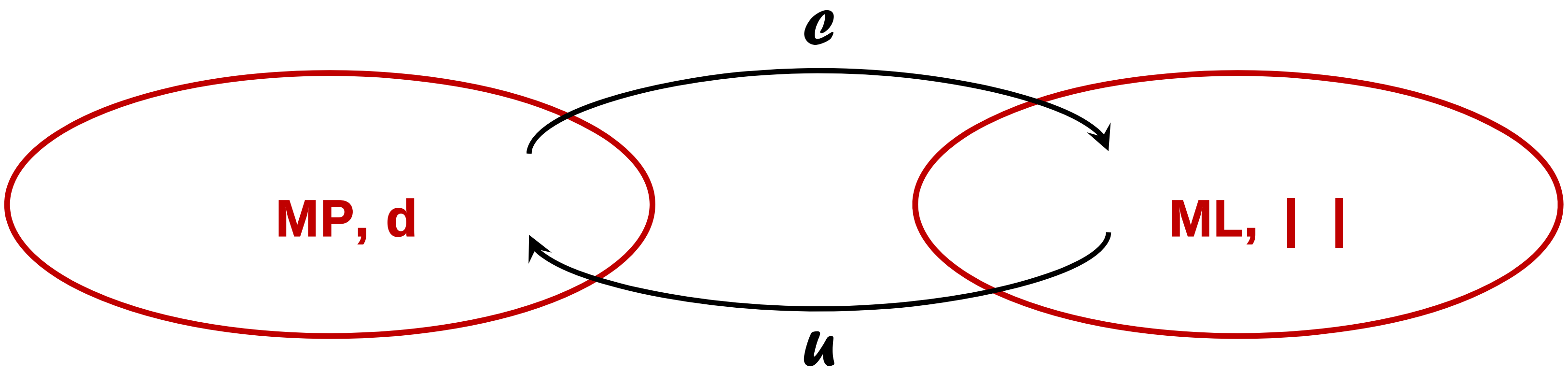
For $f \in \mathcal{L}$, let $[f]=\{u \in U \mid f \in u\}$.

- the set $\{[f] \mid f \in \mathcal{L}\}$ generates a “Stone” topology $\tau_{\mathcal{L}}$ on U
- we construct an MP on (U,B) where B is the Borel algebra of $\tau_{\mathcal{L}}$

There exists a complex relationship between $\tau_{\mathcal{L}}$ and $M^{\sim}$ :

$\sim$ is the separability relation induced by $\tau_{\mathcal{L}}$

## The lesson of the “classic” Stone duality for MPs




[Kozen, Mardare, Panangaden MFPS2014]

For $f \in \mathcal{L}$, let $|f| = \inf\{d(m,n) \mid m \vDash f,\ n \vDash f\}$.

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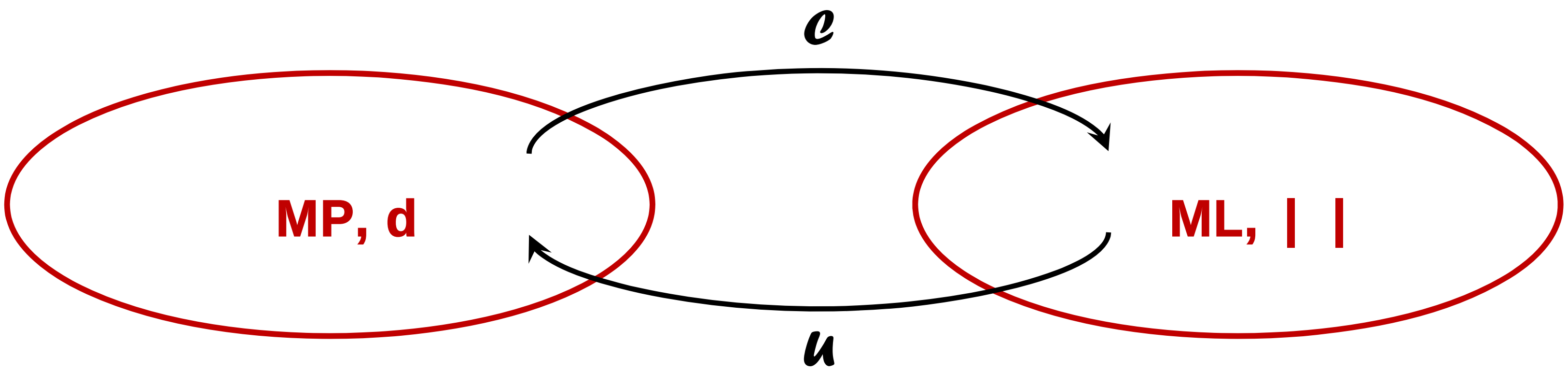


[Kozen, Mardare, Panangaden MFPS2014]

For $f \in \mathcal{L}$, let $|f| = \inf\{d(m,n) \mid m \models f, \; n \models f\}$.

There exist two topologies on MPs

$\mathcal{T}_{\mathcal{L}}$ and $\mathcal{T}_{d}$

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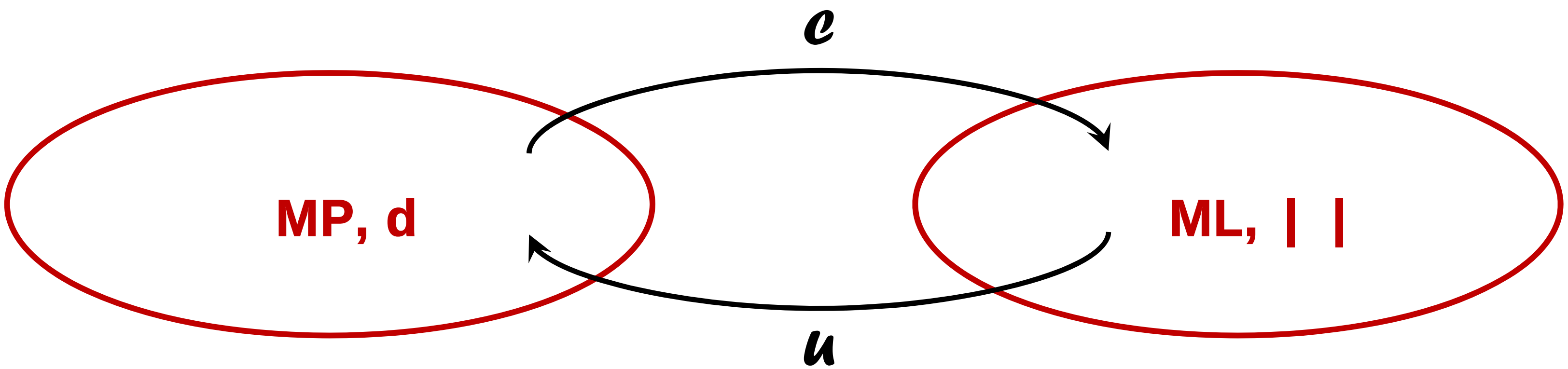


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$\mathfrak{T}_{\mathcal{L}}$ and $\mathfrak{T}_{d}$

What is the relationship between them?

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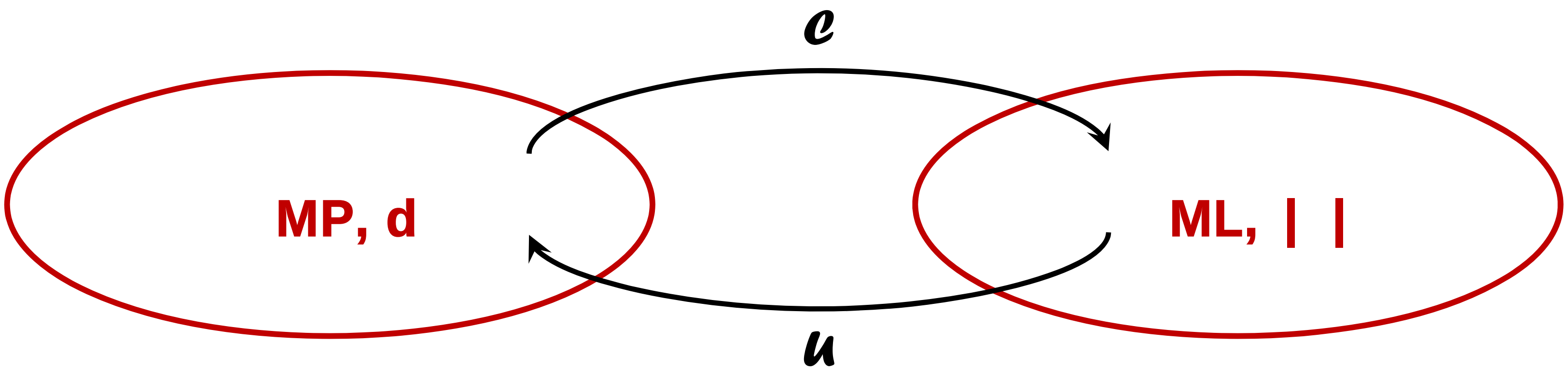



[Kozen, Mardare, Panangaden MFPS2014]

For $f \in \mathcal{L}$, let $|f| = \inf \{d(m,n) \mid m \models f,\ n \models f\}$.

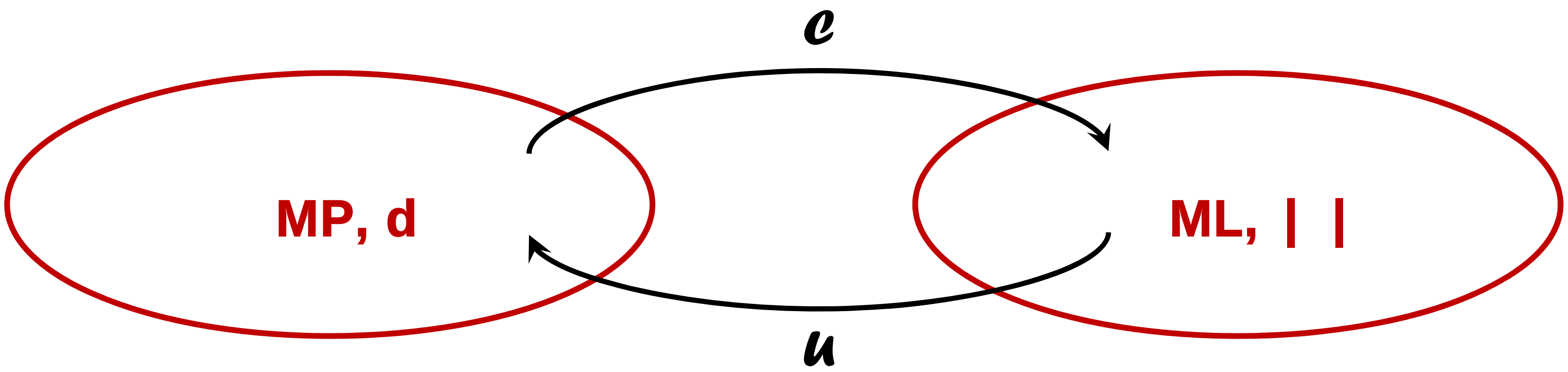
There exist two topologies on MPs

$\tau_{\mathcal{L}}$ and $\tau_d$

What is the relationship between them?

$\tau_{\mathcal{L}} \neq \tau_d$

## The lesson of the “classic” Stone duality for MPs

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$\tau_{\mathcal{L}}$ and $\tau_d$ induce the same separability relation which is $\sim$

## The lesson of the extended Stone duality for MPs

**Theorem:**

Given an SMP $(M,B,\theta)$ and a pseudometric $d : M \times M \to [0,1]$, the following statements are equivalent:

1. $\forall m, \inf_{c \in B,\, m \in c} \sup\{d(n,n') \mid n,n' \in c\} = 0$
2. $\forall m, m' \inf_{c \in B,\, m,m' \in c} \sup\{d(n,n') \mid n,n' \in c\} = d(m,m')$
3. The topology $\mho_B$ refines the topology $\mho_d$
4. The pseudometric $d$ is continuous in both arguments with respect to $\mho_B$.

[Kozen, Mardare, Panangaden, MFPS 2014]

The previous conditions enforce the concept of dynamic-continuity.

## Conclusions

- We provide a characterization of the behavioural distances that induce well-behaved topologies.

- The “classic” Stone duality for MPs do not only clarify the relation between MPs, Markovian logics and bisimilarity, but it also provides the right framework for allowing us to extend the bisimilarity-based semantics to a distance-based semantics.

- The relation between bisimilarity classes and the support topology of a (Stone-) MP can be generalized to understand the relation between the same topology and the open-ball topology induced by a behavioral distance.

- The metric duality underlines the importance of a concept of “diameter” for the elements of the Boolean algebra.