The Fellowship of the Semiring: Concerning Bisimulations for Quantitative Systems

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Motivation

I like *meta*models, like ULTraS.

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A good metamodel is useful insomuch as it provides

- unifying mathematical (categorical) theory of many models
- general results, logics and tools, which can be readily instantiated
- cross-fertilizing connections between models
- scenario for comparing models (cf. Gorla's talk about translations)
- deeper insights

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Problem (The Open Problem)

Can we define a good metamodel for concurrent systems with quantitative aspects?

Approaching the Open Problem

In the previous talk: **ULTraS**

- covers many kinds of quantitative models (non-determistic probabilistic, stochastic, timed . . .).
- provides a general definition of M-bisimilarity
- we got already general results about strong quantitative bisimulation [M. & Peressotti, QAPL'14]
  - general definition with coalgebraic characterization (coalgebraic bisimulation / kernel bisimulations)
  - GSOS rule format guaranteeing compositionality
  - general decidability algorithm

Sounds encouraging. . .

Can we get similar results about observational equivalences for quantitative systems? (weak, trace, branching, delay. . .)

Focusing the Open Problem: *weak bisimulation*

Other observational equivalences for quantitative systems (weak, trace, branching, delay. . .) are not as well understood as strong bisimulation.

- unobservable actions may have observable effects (e.g., execution times, probabilities, energy consumption)
- not a single definition, but many "ad hoc"
- sometimes, no agreement on what is the "right" definition
- no clear categorical characterization

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Focusing the Open Problem

How to give a general, good definition of *weak bisimulation*, for a wide range of labelled transition systems with quantitative aspects?

In this talk: *weak weighted bisimulation*

We give a general definition of *weak bisimulation* valid for a wide range of labelled transition systems, namely *LTS weighted over semirings*.

1. general: it encompasses many known systems
2. decidable: a uniform algorithm applicable to various semirings
3. with a categorical coalgebraic construction.

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Applications:

- obtaining weak bisimulations and decision algorithms for new kinds of systems
- generalize further to other classes of systems (beyond weighted LTS) and to other behavioural equivalences (beyond weak bisimilarity)

Weighted Transition Systems and Weak Bisimulations

Weighted Labelled Transition Systems

Let $\mathfrak{W} = (W, +, 0)$ be a commutative monoid.

Definition ([Klin, 2009])

A *($\mathfrak{W}$-weighted) labelled transition system* is a triple (X, A, ρ) where:

- X is a set of *states* (processes);
- A is a set of *labels* (actions);
- $\rho : X \times A \times X \to W$ is a *weight function*.

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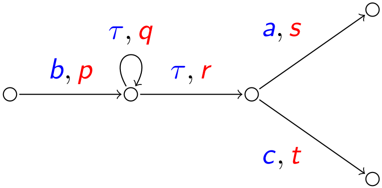
Transitions can be thought to be labelled with actions and weights drawn from $\mathfrak{W}$, with the unit 0 disabling transitions.



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Different $\mathfrak{W}$ yield different systems and bisimulation:

- usual non-deterministic LTS: $2 = (\{\mathtt{tt}, \mathtt{ff}\}, \vee, \mathtt{ff})$:
- stochastic LTS: $(\mathbb{R}_0^+, +, 0)$
- fully probabilistic LTS: $(\mathbb{R}_0^+, +, 0)$ such that $\forall x : \sum_{a,y} \rho(x \xrightarrow{a} y) \in \{0, 1\}$
- *etc.*

Weighted (strong) bisimulation

Definition ([Klin, 2009])

A *(strong)* $\mathfrak{W}$*-bisimulation* on (X, A, ρ) is an equivalence relation $R \subseteq X \times X$ such that $(x, x') \in R$ iff for each label $a \in A$ and each equivalence class C of R:

$$\sum_{y \in C} \rho(x \xrightarrow{a} y) = \sum_{y \in C} \rho(x' \xrightarrow{a} y).$$

Using different $\mathfrak{W}$ we can recover different systems and bisimulation:

- $(\{\mathtt{tt}, \mathtt{ff}\}, \vee, \mathtt{ff})$: strong non-deterministic bisimulation (Milner);
- $(\mathbb{R}_0^+, +, 0)$: strong stochastic bisimulation (Hillstone, Panangaden);
- $(\mathbb{R}_0^+, +, 0)$: strong probabilistic bisimulation (Larsen-Skou);
- *etc.*

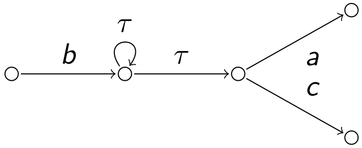
Weak bisimulation: the non-deterministic case via "double arrow" construction

Definition ([Milner, ages ago])

$R \subseteq X \times X$ is a *weak (non-deterministic) bisimulation on* $(X, A + \{\tau\}, \longrightarrow)$ iff for each $(x, x') \in R$, label $\alpha \in A + \{\tau\}$ and equivalence class $C \in X/R$:

$$\exists y \in C . x \stackrel{\alpha}{\Longrightarrow} y \iff \exists y' \in C . x' \stackrel{\alpha}{\Longrightarrow} y'$$

where $\Longrightarrow \subseteq X \times (A \uplus \{\tau\}) \times X$ is the τ-reflexive and τ-transitive closure of $\longrightarrow$.



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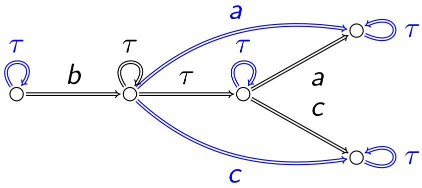

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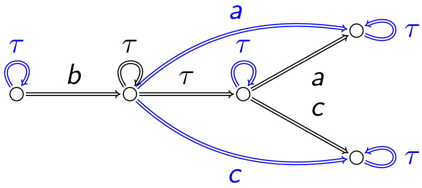
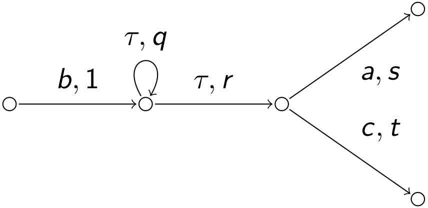


Generalizing the non-deterministic case?

What if we apply the same approach to a fully-probabilistic system ($\sum \rho \in 0, 1$)?



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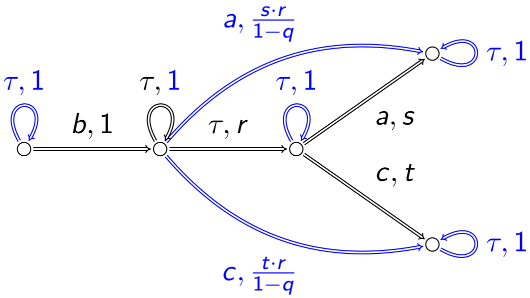

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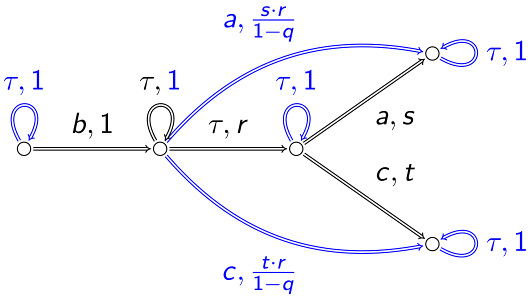


This is *not* probabilistic.
This is *not* a weak probabilistic bisimulation in the sense of Baier-Hermanns.

Weak bisimulation: the fully-probabilistic case

Definition ([Baier-Hermanns, 97])

$R \subseteq X \times X$ is a *weak (probabilistic) bisimulation* on $(X, A + \{\tau\}, \mathrm{P})$ iff for $(x, x') \in R$, $a \in A$ and equivalence class $C \in X/R$:

$$\mathrm{Prob}(x, \tau^* a \tau^*, C) = \mathrm{Prob}(x', \tau^* a \tau^*, C)$$
$$\mathrm{Prob}(x, \tau^*, C) = \mathrm{Prob}(x', \tau^*, C).$$

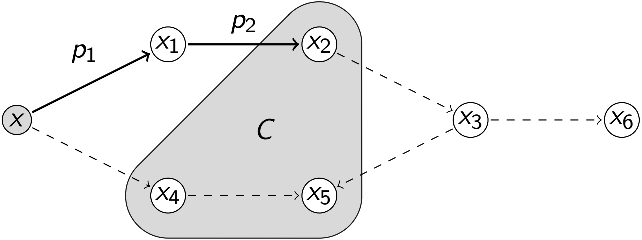
where Prob is the extension over finite execution paths of the unique probability measure induced by P.

Intuitively...

$\mathrm{Prob}(x, T, C)$ is the probability of **reaching** C from x generating some trace in T.

States of C cannot be considered separately because σ-additivity does not hold (i.e. $\mathrm{Prob}(x, T, C_1 \cup C_2) \neq \mathrm{Prob}(x, T, C_1) + \mathrm{Prob}(x, T, C_2)$)

τ-closure vs. reachability: probabilistic



Assuming p_i is the probability of an action, what is the probability to reach class C from x?

$$1 > (p_1 \cdot p_2)$$

(we ignored labels, but can be easily taken into account).

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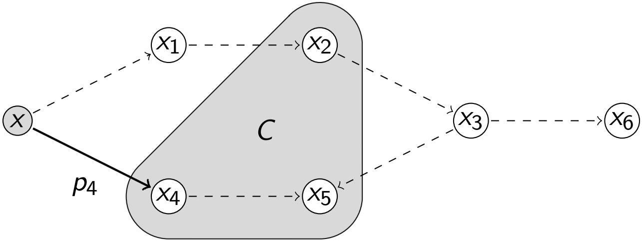

Assuming p_i is the probability of an action, what is the probability to reach class C from x?

$$1 = (p_1 \cdot p_2) + (p_4)$$

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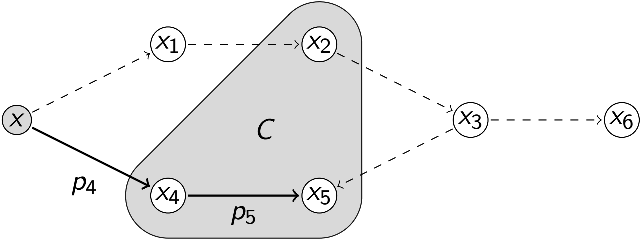

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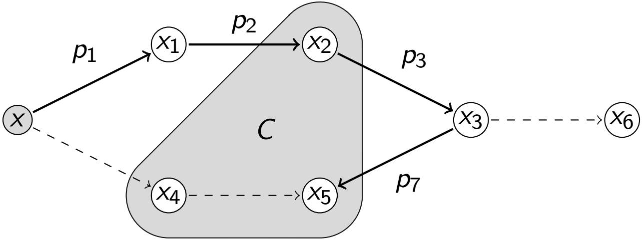

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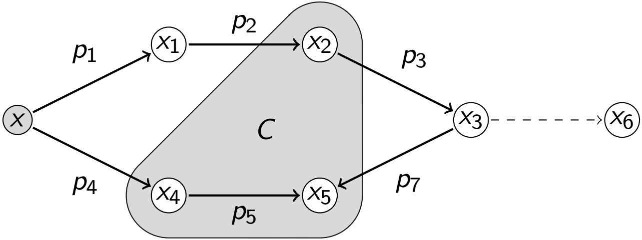

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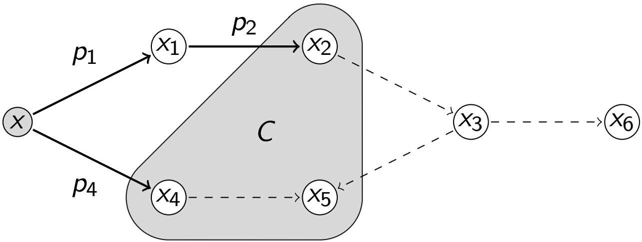

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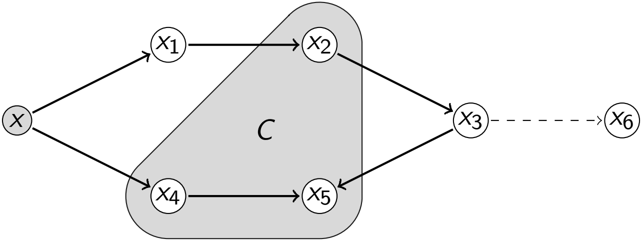
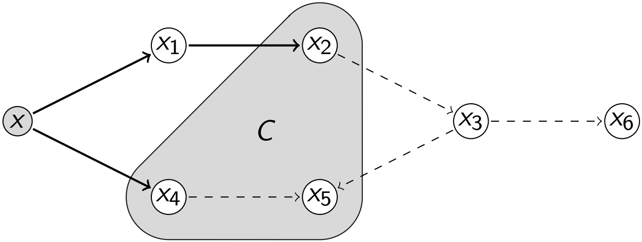


Assuming the non-deterministic case ($p_i = \mathtt{tt}$), can we reach C from x?

$$\mathtt{tt} = (\mathtt{tt} \wedge \mathtt{tt}) \vee (\mathtt{tt}) \vee (\mathtt{tt} \wedge \mathtt{tt}) \vee (\mathtt{tt} \wedge \mathtt{tt} \wedge \mathtt{tt} \wedge \mathtt{tt})$$

τ-closure vs. reachability: non-deterministic



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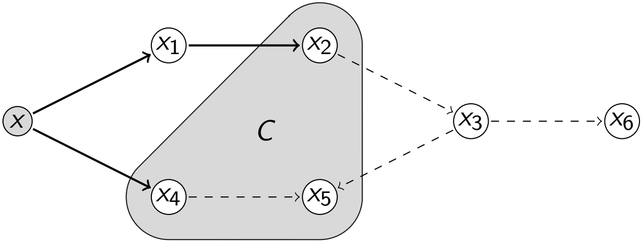

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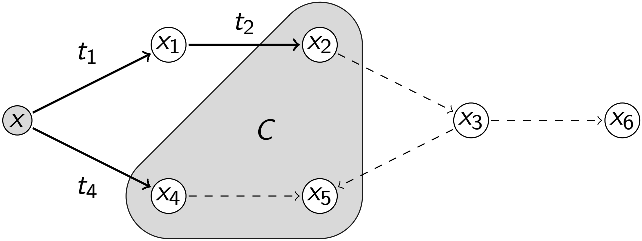
Here τ-closure and reachability coincide...

But this is very specific case (and there is a *very specific* reason.)

τ-closure vs. reachability: stochastic



Assuming t_i describes the time consumed by an action, how much time takes to go from x to C?

$$t = \min(t_1 + t_2, t_4)$$

Weighting execution paths

Previous examples used two operations on weights:

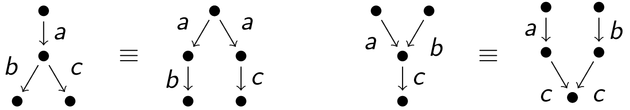
- $(W, +, 0)$ for **branching** (a commutative monoid)
- $(W, \cdot, 1)$ for **chaining** (a monoid)

Subject to some coherence conditions:

- 0 expresses **termination** (annihilates chaining)

$$0 \cdot a = 0 = a \cdot 0$$

- **independence** of execution paths



$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \qquad (a + b) \cdot c = (a \cdot c) + (b \cdot c)$$

Semirings of weights

Henceforth, let $\mathfrak{W} = (W, +, 0, \cdot, 1)$ be a **semiring** (*cf.* $\mathfrak{W}$-automata).

Definition (Path weight)

Given a weight function ρ, its extension to finite paths is:

$$\rho(x_0 \xrightarrow{a_1} x_1 \dots \xrightarrow{a_n} x_n) \triangleq \rho(x_0 \xrightarrow{a_1} x_1) \cdot \dots \cdot \rho(x_{n-1} \xrightarrow{a_n} x_n)$$

Weighting finite paths is enough for our aims since two (countably) infinite paths are observationally distinguished iff there is a finite path telling them apart *i.e.* by finite observation.

(Countably infinite paths require countable multiplication, or equivalently a sufficiently expressive notion of limits).

Categorically: WLTS are coalgebras

Define the Set monad of finitely supported $\mathfrak{W}$-valued functions s.t.:
For every set X:

$$\mathcal{F}_{\mathfrak{W}}(X) \triangleq \{\psi : X \to W \mid \psi \text{ is countably supported}\}$$

For every function $f : X \to Y$:

$$\mathcal{F}_{\mathfrak{W}}(f)(\varphi) \triangleq \lambda y{:}Y. \sum_{x \in f^{-1}(y)} \varphi(x)$$

$$\eta(x)(y) \triangleq \begin{cases} 1 & \text{if x = y} \\ 0 & \text{otherwise} \end{cases} \qquad \mu(\psi)(x) \triangleq \sum_{\varphi} \psi(\varphi) \cdot \varphi(x)$$

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- Strong weighted bisimulation is $\mathcal{F}_{\mathfrak{W}}(A \times \text{-})$-bisimulation.
- (ULTraS are $\mathcal{P}_f(\mathcal{F}_{\mathfrak{W}}(A \times \text{-}))$-Coalgebras.)

Categorically: the general setting

More generally we can consider TF_τ-coalgebras where:

- T is a monad yielding a CPPO-enriched Kleisli category
- F distributes over T
- $F_\tau \triangleq Id + F$ be the extension of F with silent action.

For WLTS, it is:

- $T = \mathcal{F}_{\mathfrak{W}} : Set \to Set$
- $F = A \times \_ : Set \to Set$
- $F_\tau X = X + A \times X = (\{\tau\} + A) \times X$

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Proposition ([M.&Peressotti 2013])

Given a coalgebra $\alpha : X \to TF_\tau X$ and an epic $f : X \to C$ (i.e. a partition of X), we can construct a **saturated** TF_τ **coalgebra** $\alpha : X \to TF_\tau X$ *representing the reachability of classes in C up-to τ-transitions.*

Weak bisimulation, categorically

Definition: A **weak bisimulation** between two TF_τ-coalgebras (X, α) and (Y, β), is a span of jointly monic arrows $X \xleftarrow{p} R \xrightarrow{q} Y$ such that there exists an epic cospan $X \xrightarrow{f} C \xleftarrow{g} Y$ such that (R, p, q) is the final span to make the following diagram commute:

$$\begin{tikzcd}
 & & R \arrow[dl, bend right, "p"'] \arrow[dr, bend left, "q"] & & \\
X \arrow[d, "\alpha"'] & X \arrow[r, "f"] \arrow[d, "\alpha^w"'] & C \arrow[d, "\gamma"'] & Y \arrow[l, "g"'] \arrow[d, "\beta^w"] & Y \arrow[d, "\beta"] \\
TF_\tau X & TF_\tau X \arrow[r, "TF_\tau f"'] & TF_\tau C & TF_\tau Y \arrow[l, "TF_\tau g"] & TF_\tau Y
\end{tikzcd}$$

where α^w, β^w are the *saturated TF_τ-coalgebras of α, β wrt f, g*.

Back to the concrete case: Weighting sets of paths

By instantiating the above construction in the WLTS case, saturation becomes weighting of (particular) sets of paths.

Definition (Finite paths to C)

For a state x, a set of traces T and a set of states C, the set of finite paths reaching C from x with trace in T is

$$\wr x, T, C \wr \triangleq \left\{ \pi \in \mathrm{FPaths}(x) \;\middle|\; \begin{array}{l} \mathrm{last}(\pi) \in C,\ \mathrm{trace}(\pi) \in T, \\ \forall \pi' \preceq \pi : \mathrm{trace}(\pi') \in T \Rightarrow \mathrm{last}(\pi') \notin C \end{array} \right\}$$

Weak $\mathfrak{W}$-bisimulation

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$R \subseteq X \times X$ is a *weak $\mathfrak{W}$-bisimulation* for $(X, A + \{\tau\}, \rho)$ iff for all $(x, x') \in R$, $a \in A$ and equivalence class $C \in X/R$, the following hold:

$$\rho(\wr x, \tau^*, C \int) = \rho(\wr x', \tau^*, C \int)$$
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$$\rho(\wr x, \tau^* a \tau^*, C \wr) = \rho(\wr x', \tau^* a \tau^*, C \wr).$$

Remark

- Weak $\mathfrak{W}$-bisimulation is just categorical weak bisimulation, concretely presented in the case of WLTS.
- Other bisimulations can be obtained by changing the set of paths (e.g., for delay bisimulation: $\wr x, \tau^*, C \wr$ and $\rho(\wr x, \tau^* a, C \wr)$)

Examples of weak $\mathfrak{W}$-bisimulation

- Non-deterministic systems and Milner's weak bisimulation: *Boolean semiring*: $(\{\mathtt{tt}, \mathtt{ff}\}, \vee, \mathtt{ff}, \wedge, \mathtt{tt})$
- Fully-probabilistic systems and Baier-Hermanns's weak bisimulation:
  - *Positive real semiring*: $(\overline{\mathbb{R}}_0^+, +, 0, \cdot, 1)$
  - *Probabilistic σ-semiring*: $([0,1], +, 0, \cdot, 1)$
- Stochastic systems (and a new weak bisimulation): *transition-time random variables semiring*: $\mathfrak{S} \triangleq (\mathbb{T}, \min, \mathcal{T}_{+\infty}, +, \mathcal{T}_0)$
- Troubleshooting: *Likelihood semiring*: $([0,1], \max, 0, \cdot, 1)$
- Optimization problems (especially scheduling):
  - *Tropical semiring*: $(\overline{\mathbb{R}}_0^+, \min, +\infty, +, 0)$
  - *Arctic semiring*: $(\overline{\mathbb{R}}, \max, -\infty, +, 0)$
  - *Bottleneck semiring*: $(\overline{\mathbb{R}}_0^+, \min, +\infty, \max, 0)$
- Formal languages: *Free language semiring*: $(\wp(\Sigma^*), \cup, \emptyset, \circ, \varepsilon)$
- And many more...

Deciding Weak Weighted Bisimulation

Computing weak $\mathfrak{W}$-bisimulation

We generalize Kanellakis-Smolka's algorithm for strong bisimulation of *finite* LTSs [Kanellakis-Smolka 1989].

Let $(X, A + \{\tau\}, \rho)$ be a finite $\mathfrak{W}$-LTS and let P be a partition of X.

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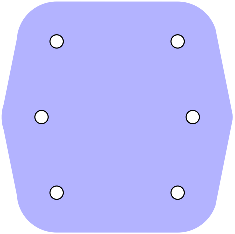

$$P_0 = \{X\}$$

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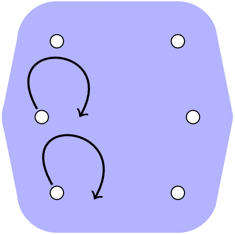

$$\rho(\wr x_0, \tau^* a \tau^*, X \int) = \rho(\wr x_1, \tau^* a \tau^*, X \int)$$

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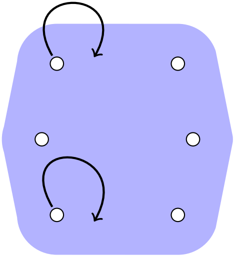

$$\rho(\wr x_0, \tau^* b \tau^*, X \wr) \neq \rho(\wr x_2, \tau^* b \tau^*, X \wr)$$

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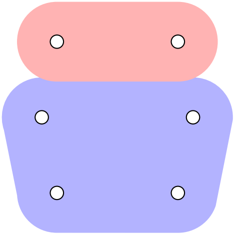

$$P_1 \triangleq \bigcup \left\{ B/\underset{b,X}{\approx} \mid B \in P_0 \right\} \qquad x \underset{b,X}{\approx} y \stackrel{\triangle}{\iff} \rho(x, \tau^* b \tau^*, X) = \rho(y, \tau^* b \tau^*, X$$

Computing weak $\mathfrak{W}$-bisimulation

We generalize Kanellakis-Smolka's algorithm for strong bisimulation of *finite* LTSs [Kanellakis-Smolka 1989].
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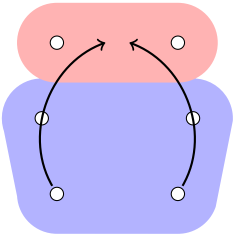

$$\rho(\wr x_0, \tau^*, C \wr) \neq \rho(\wr x_5, \tau^*, C \wr)$$

Computing weak $\mathfrak{W}$-bisimulation

We generalize Kanellakis-Smolka's algorithm for strong bisimulation of *finite* LTSs [Kanellakis-Smolka 1989].

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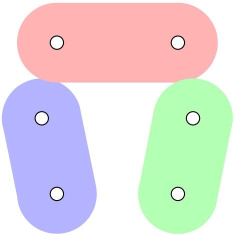


$$P_2 \triangleq \bigcup \left\{ B/\underset{\tau,C}{\approx} \mid B \in P_2 \right\} \quad x \underset{\tau,C}{\approx} y \overset{\triangle}{\iff} \rho(x, \tau^*, C) = \rho(y, \tau^*, C)$$

Computing the weight of redundancy-free sets

Question

Given x, a, C, how do we compute $\rho(\wr x, \tau^*, C \wr)$ and $\rho(\wr x, \tau^* a \tau^*, C \wr)$?

Computing the weight of redundancy-free sets

Question

Given x, a, C, how do we compute $\rho(\wr x, \tau^*, C \int)$ and $\rho(\wr x, \tau^* a \tau^*, C \int)$?

By solving a system of linear equations over $\mathfrak{W}$.
For each state x, let x_τ, x_a be two variable over $\mathfrak{W}$.
Equations:

$$x_\tau = \begin{cases} 1 & \text{if } x \in C \\ \sum_{y \in X} \rho(x, \tau, y) \cdot y_\tau & \text{otherwise} \end{cases}$$

$$x_a = \sum_{y \in X} \rho(x, a, y) \cdot y_\tau + \sum_{y \in X} \rho(x, \tau, y) \cdot y_a$$

Intuition: $\quad x_\tau = \rho(\wr x, \tau^*, C \int) \qquad x_a = \rho(\wr x, \tau^* a \tau^*, C \int)$

Solvability of the equation systems

The definitions of x_a's form a linear equation system $x = A \cdot x + b$, which defines an operator over W^n (A is $n \times n$).

$$F(y) = A \cdot y + b$$

The system has the same number of equations and unknowns, hence if there is a solution, it is unique (F has at most one fix-point).

Proposition

If $\mathfrak{W}$ is ω-continuous and admits a natural order (i.e. positively ordered), then F admits exactly one solution, which is its least fix point

$$c = F^*(0^n)$$

Complexity

The complexity is *almost* the same of Kanellakis-Smolka's original algorithm, but:

- No constant-time random-access data structures;
- No pre-computed transitions (and their weight).

Proposition (Time complexity)

The asymptotic upper bound for time complexity of the proposed algorithm is in

$$\mathcal{O}(nm(\mathcal{L}_{\mathfrak{W}}(n) + n^2))$$

where $n = |X|$ and $m = |A + \{\tau\}|$ and $\mathcal{L}_{\mathfrak{W}}(n)$ is the time complexity of solving a system of n linear equations with n variables over the $\mathfrak{W}$.

In presence of constant-time random-access data structures time complexity is in $\mathcal{O}(nm(\mathcal{L}_{\mathfrak{W}}(n) + n))$.

Conclusions: back to the Open Problem

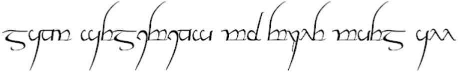
Done:

- framework for defining strong and weak bisimilarities (and beyond);
- coalgebraic characterization;
- general algorithm, parametric in the semiring.

format	example	Strong	Trace	Weak: τ-clos.	Weak: reach.
WLTS	CTMC, Fully prob.	✓[2]	✓[3]	✓[4]	✓[5]
ULTraS	MDP, Segala's	✓[6]	?[7]	?[8]	?
		Monoids	Semirings		

[2] [Klin, 2009]
[3] For ω-continuous semirings [Hasuo, 2007]
[4] For ω-continuous semirings [Brengos, 2014]
[5] [M. & Peressotti, 2013] (For fully probabilistic systems [Baier-Hermans 1997])
[6] [M. & Peressotti, 2014]
[7] For Segala systems [Varacca, Jacobs]
[8] For Segala systems [Segala 1994]

Thanks for your attention.



Thanks for your attention.

Many semirings to rule them all.

Appendix

The algorithm

$\mathcal{X} \leftarrow \{X\}$
$\mathcal{X}' \leftarrow \emptyset$
repeat
  changed $\leftarrow$ **false**
  $\mathcal{X}'' \leftarrow \mathcal{X}$
  for all $C \in \mathcal{X} \setminus \mathcal{X}'$ **do**
    for all $\alpha \in A + \{\tau\}$ **do**
      if $\langle \alpha, C \rangle$ is a split **then**
        $\mathcal{X} \leftarrow \bigcup \{B / \underset{\alpha,C}{\approx} \mid B \in \mathcal{X}\}$
        changed $\leftarrow$ **true**
      end if
    end for
  end for
  $\mathcal{X}' \leftarrow \mathcal{X}''$
until not *changed*
return $\mathcal{X}$

The algorithm II

Assumption: the carrier of the semiring has a total order.

$\mathcal{X} \leftarrow \{X\}$
$\mathcal{X}' \leftarrow \emptyset$
repeat
changed $\leftarrow$ **false**
for all $C \in \mathcal{X} \setminus \mathcal{X}'$ **do**
for all $\alpha \in A + \{\tau\}$ **do**
compute and sort $\rho(x, \hat{\alpha}, C)$ by block and weight
end for
if there is any split **then**
$\mathcal{X}' \leftarrow \mathcal{X}$
$\mathcal{X} \leftarrow \mathit{refine}(\mathcal{X}, C)$
changed $\leftarrow$ **true**
end if
end for
until not *changed*

Positively ordered semirings

A semiring $\mathfrak{W} = (W, +, 0, \cdot, 1)$ endowed with a partial order $(W, \leq)$ is *positivelly ordered* iff

- 0 is least element;
- $+$ and $\cdot$ respect $\leq$ *i.e.* for each a, b and c if $a \leq b$ then

$$a + c \leq b + c \qquad a \cdot c \leq b \cdot c \qquad c \cdot a \leq c \cdot b$$

Every PO semiring admits a "weakest" order $\trianglelefteq$:

$$a \trianglelefteq b \overset{\triangle}{\iff} \exists c : a + c = b.$$

This order is called *natural* and is the weakest in the sense that:

$$a \trianglelefteq b \implies a \leq b$$

for any $\leq$ rendering $\mathfrak{W}$ positively ordered.

PO semirings and Kleene fix point

Lemma

If $\mathfrak{W}$ admits countable sums then $(W, \leq, 0)$ is ω-CPO.

Lemma

F is Scott-continuous w.r.t. the pointwise extension of $\trianglelefteq$ to n-vectors.

Proposition

F has a least fix point and hence $x = A \cdot x + b$ has a unique solution.

Delay bisimulation

Definition

$R \subseteq X \times X$ is a *delay* $\mathfrak{W}$*-bisimulation* on R on X such that for all $(x, x') \in R$, $a \in A$ and $C \in X/R$:

$$\rho(x, \tau^* a, C) = \rho(x', \tau^* a, C)$$
$$\rho(x, \tau^*, C) = \rho(x', \tau^*, C).$$

The algorithm proposed can be used to compute delay bisimulations: just use the linear system:

$$\begin{aligned} x_\tau &= 1 && \text{for } x \in C \\ x_\tau &= \sum_{y \in X} \rho(x, \tau, y) \cdot y_\tau && \text{for } x \notin C \\ x_a &= \sum_{y \in X} \rho(x, \tau, y) \cdot y_a + \sum_{y \in X} \rho(x, a, y) \end{aligned}$$

whose solutions are precisely $\rho(x, \tau^* a, C)$.

A semiring for weak stochastic bisimulation

Stochastic bisimulation is $\mathbb{R}_0^+$-bisimulation [Klin-Sassone, FoSSaCS 2008].

$\mathbb{R}_0^+$ is used since exponentially distributed stochastic transitions can be expressed by rates (λ) and branching by arithmetic addition ($+$).

A semiring for weak stochastic bisimulation

Stochastic bisimulation is $\mathbb{R}_0^+$-bisimulation [Klin-Sassone, FoSSaCS 2008].

$\mathbb{R}_0^+$ is used since exponentially distributed stochastic transitions can be expressed by rates (λ) and branching by arithmetic addition ($+$).

Unfortunately...

there is no multiplication for $\mathbb{R}_0^+$ capturing chaining of stochastic transitions

A sequence of exponentially distributed stochastic transition is *hyperexponential*, not exponential. (Often this is *approximated* by an exponential distribution with the same average [Bernardo et al.]).

A semiring for weak stochastic bisimulation

The *stochastic semiring*:

$$\mathfrak{S} \triangleq (\mathbb{T}, \min, \mathcal{T}_{+\infty}, +, \mathcal{T}_0)$$

Carrier: $\mathfrak{S}$
The set of *transition-time random variables i.e.* random variables on $\overline{\mathbb{R}}_0^+$.

Branching: $(\mathfrak{S}, \min, \mathcal{T}_{+\infty})$
Random variables minimum express stochastic race (which is idempotent).
The unit is the constantly $+\infty$ random variable (which is self-independent).

Chaining: $(\mathfrak{S}, +, \mathcal{T}_0)$
Random variables sum express concatenation (which is commutative)
The unit is the constantly 0 random variable (which is self-independent).

Yet another tropical semiring!

A semiring for weak stochastic bisimulation

Idempotency of branching:

$$\begin{aligned}\mathbb{P}(\min(X, X) > t) &= \mathbb{P}(X > t \cap X > t) \\ &= \mathbb{P}(X > t) \cdot \mathbb{P}(X > t \mid X > t) \\ &= \mathbb{P}(X > t).\end{aligned}$$

By definition and idempotency of min and by definition and commutativity of $+$:

Termination: $\mathcal{T}_{+\infty} + X = \mathcal{T}_{+\infty}$

Distributivity: $X + \min(Y, Z) = \min(X + Y, X + Z)$

A semiring for weak stochastic bisimulation

Let $X, Y \in \mathbb{T}$ be continuous.

Branching: $\min(X, Y)$

$$f_{\min(X,Y)}(z) = f_X(z) + f_Y(z) - f_{X,Y}(z, z).$$

Assuming independence (not necessarily iid):

$$f_{\min(X,Y)}(z) = f_X(z) \cdot \int_z^{+\infty} f_Y(y)\mathrm{d}y + f_Y(z) \cdot \int_z^{+\infty} f_X(x)\mathrm{d}x.$$

Chaining: $X + Y$

$$f_{X+Y}(t) = \int_0^t f_{X,Y}(s, t-s)\mathrm{d}s$$

Assuming independence (not necessarily iid):

$$f_{X+Y}(t) = \int_0^t f_X(s) \cdot f_Y(t-s)\mathrm{d}s.$$

Weak stochastic bisimulation

Definition (Weak stochastic bisimulation)

Given a stochastic labelled transition system $(X, A + \{\tau\}, \theta)$, an equivalence relation $R \subseteq X \times X$ is a *weak stochastic bisimulation* for it iff for each pair of states $(x, x') \in R$, label $a \in A$ and equivalence class $C \in X/R$:

$$\theta(\wr x, \tau^* a \tau^*, C \int) = \theta(\wr x', \tau^* a \tau^*, C \int)$$
$$\theta(\wr x, \tau^*, C \int) = \theta(\wr x', \tau^*, C \int).$$

This is the **same definition** of non-deterministic and probabilistic systems, instantiated on a different semiring.

Coalgebraic saturation

In general we consider TF_τ-coalgebras where:

- T is a monad yielding a CPPO-enriched Kl (like $\mathcal{F}_{\mathfrak{W}}$ and semirings admitting a natural order)
- F distributes over T (like $A \times \_$).

Traces for a TF-coalgebra α can be obtained by means of the final map tr_α to the final $\overline{F}$-coalgebra in $Kl(T)$ [Hasuo, 2010].

Let $F_\tau \triangleq Id + F$ be the extension of F with silent action.
Delay-like $\tau^* a$ transitions described by a TF_τ-coalgebra α are single transitions of the *iterate* of α [Jacobs 2010; Silva, Westerbaan 2013]

$$\alpha^{\#} \triangleq \nabla_{FX} \circ \mathrm{tr}_\alpha$$

(Intuitively, consider α as a $Id + F$-coalgebra and drop the info about how many τ the trace has.)

Coalgebraic saturation

$\alpha^{\#}$ covers paths $\tau^* a$ (which form a *minimal* set "by definition").
What is missing is the (minimal) trailing τ^* part.

Every set of paths with trace $b^* a$ is minimal, because of its trace.

Idea

Make classes the observables, then use $(\_)^{\#}$ stopping as soon as the class is reached.

Then, $\wr x, \tau^*, C \int$ can be obtained as considering only τ-transitions where the only observable is C, the class to be reached.