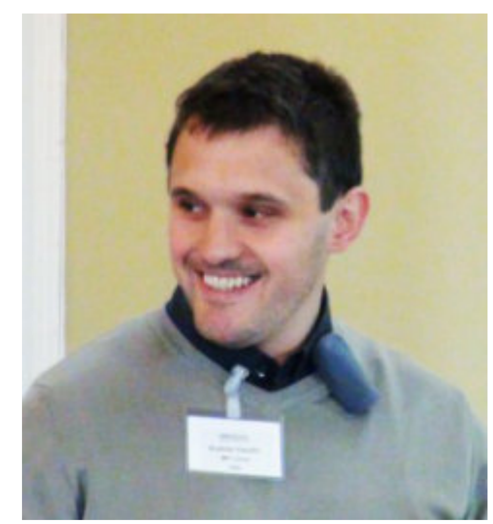
From Bisimulations to Dissimilarities for Linear Dynamical Systems

Giorgio Bacci, **Giovanni Bacci**, Kim G. Larsen,
Mirco Tribastone, Max Tschaikowski, and Andrea Vandin

OPCT 2023 - 27 June 2023

Equivalences vs. Pseudometrics

Specification $\sim$ Implementation

Specification $\overset{\varepsilon}{\longmapsto}$ Implementation

EQUIVALENCE RELATION:

Reflexive: $s \sim s$

Symmetric: $s \sim t \implies t \sim s$

Transitive: $s \sim u$ and $u \sim t \implies s \sim t$

- Reason about observational equivalence
- Often used to minimise the set of states of the system
- Not informative when the equivalence is not found

PSEUDOMETRIC:

Reflexive: $d(s, s) = 0$

Symmetric: $d(s, t) = d(t, s)$

Triangular inequality: $d(s, u) + d(u, t) \leq d(s, t)$

- Measure observational dissimilarities
- May be used to minimise the set of states beyond equivalence
- Provide information about the magnitude of dissimilarity

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DISSIMILARITY:

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Symmetric: $d(s, t) = d(t, s)$

~~Triangular inequality:~~ $\cancel{d(s, u) + d(u, t) \leq d(s, t)}$

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- Rea
  equi

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We consider Linear Dynamical Systems (LDS)

$$x(t+1) = Ax(t) + b$$

Some related work

- **Bisimilarity Pseudometrics for Markov Chains** [Desharnais et al.,CONCUR’99] [van Breugel & Worrell, ICALP’01]
- **Coagebraic Behavioural Metrics** [Baldan et al., LMCS’18]
- **Weighted Bisimulations for Linear WA** [Boreale, CONCUR’09]
- **Bisimulation Metrics for WA** [Balle, Gourdeau, Panangaden, ICALP’17]
- **Approximate Bisimulations for linear control systems** [Girard & Pappas, CDC’05—TAC’07]

…and many more

Backward Equivalence

adaptation from [Cardelli et al., LICS'16]

▶ **Definition 2** (Backward Equivalence). *Let $x(t+1) = Ax(t) + b$ be an LDS with n variables. An equivalence relation $R \subseteq [n] \times [n]$ is a* backward equivalence *if, for all $x \in \mathbb{R}^n$,*

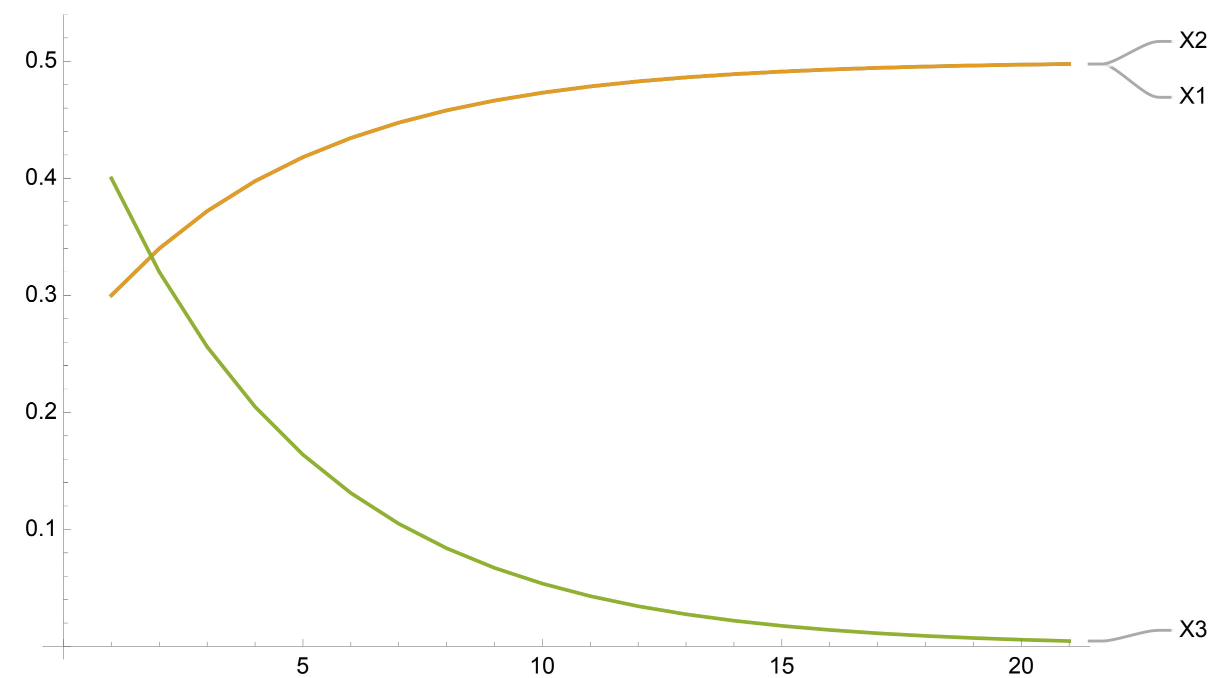
$$\bigwedge_{(i,j)\in R} (x_i = x_j) \implies \bigwedge_{(i,j)\in R} \Big(A_i x + b_i = A_j x + b_j\Big). \tag{1}$$

Example

Consider the LDS $x(t+1) = Ax(t)$ where $\tau = 0.1$ and

$$A = \begin{pmatrix} 1-\tau & \tau & \tau \\ \tau & 1-\tau & \tau \\ 0 & 0 & 1-2\tau \end{pmatrix}$$

$R = Id \cup \{(1,2),(2,1)\}$ is a BE



- Model reduction w.r.t. BE preserves the exact solutions!

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Relates variables with identical orbits when initialised equally

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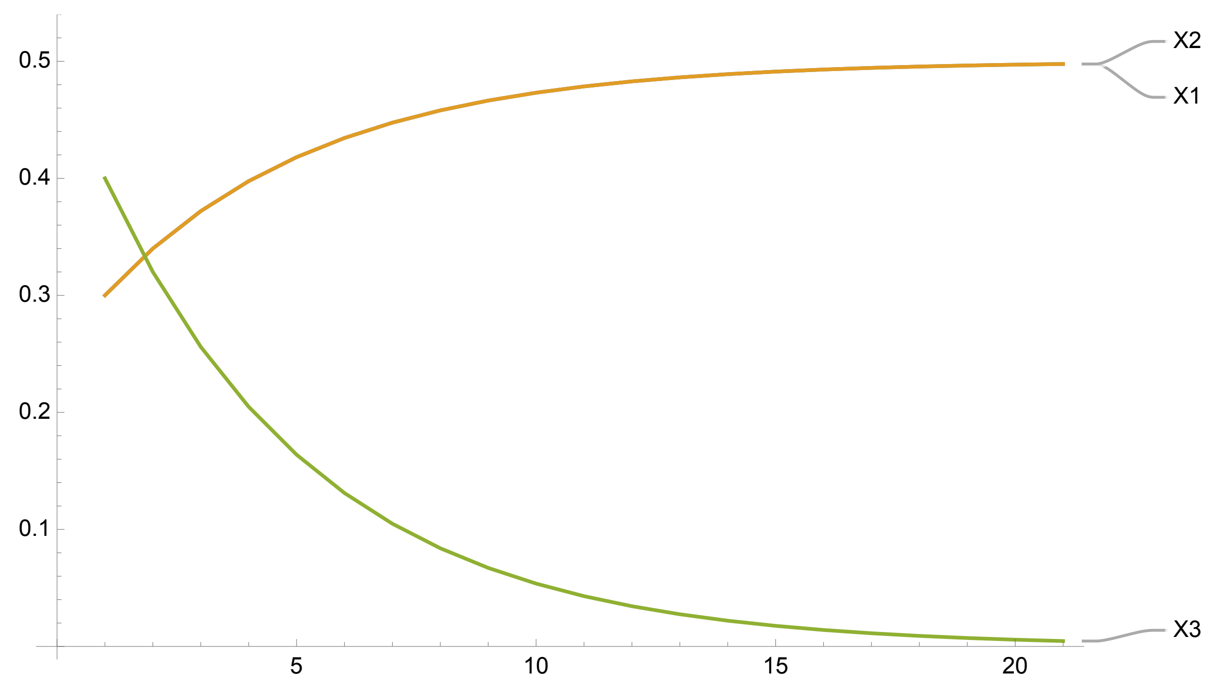

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Limitations of Backward Equivalence

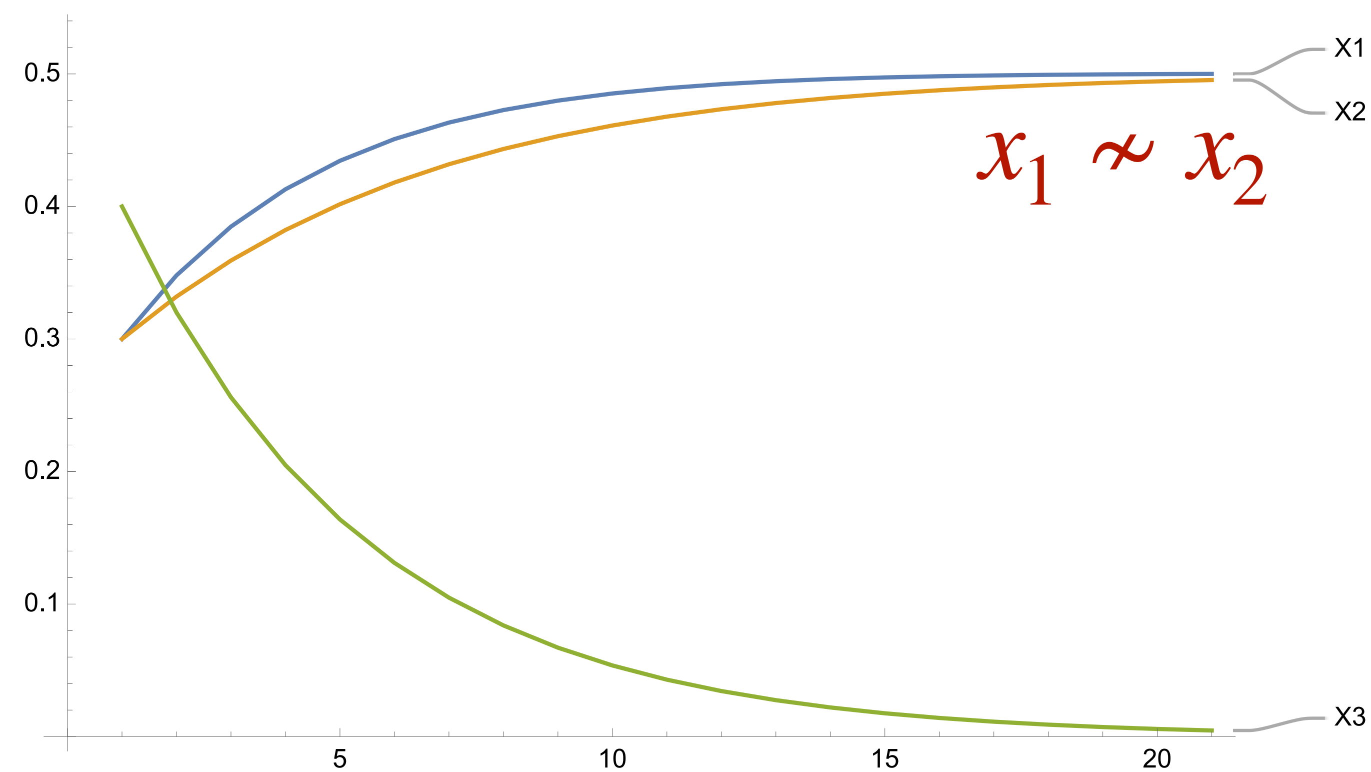
BE relies on strong assumptions

- **Initial conditions**: for x_i and x_j to be equivalent we need $x_i(0) = x_j(0)$
- **Small perturbations in the coefficients** break the equivalence

Perturbed Example

Consider the LDS $x(t+1) = Ax(t)$ where $\epsilon > 0$ and

$$A = \begin{pmatrix} 1-\tau & \tau & (1+\epsilon)\tau \\ \tau & 1-\tau & (1-\epsilon)\tau \\ 0 & 0 & 1-2\tau \end{pmatrix}$$



Backward Dissimilarity

▶ **Definition 5** (Backward dissimilarity). *Let $x(t+1) = Ax(t) + b$ be an LDS. A symmetric matrix $D \in \mathbb{R}^{n\times n}_{\geq 0}$ is a* backward dissimilarity *for a set $I \subseteq \mathbb{R}^n$ of initial conditions if, for all $x(0) \in I$ and $t \in \mathbb{N}$,*

$$\bigwedge_{1\leq i,j\leq n} \Big(|x_i(t) - x_j(t)| \leq D_{ij}\Big) \implies \bigwedge_{1\leq i,j\leq n} \Big(|x_i(t+1) - x_j(t+1)| \leq D_{ij}\Big).$$

Quantitative generalisation of backward equivalence: $D_{i,j} = 0 \iff i \sim j$

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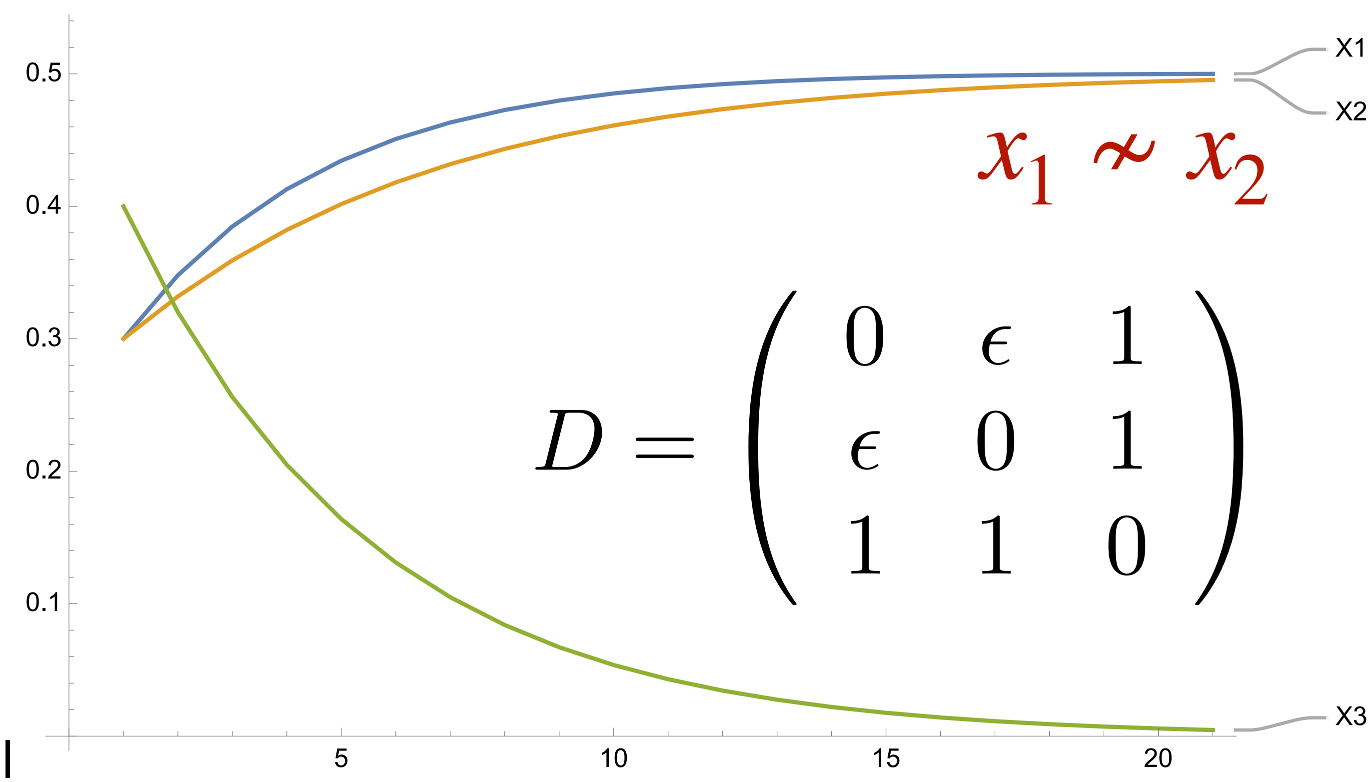
(*) for ϵ sufficiently small



$x_1 \nsim x_2$

$$D = \begin{pmatrix} 0 & \epsilon & 1 \\ \epsilon & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

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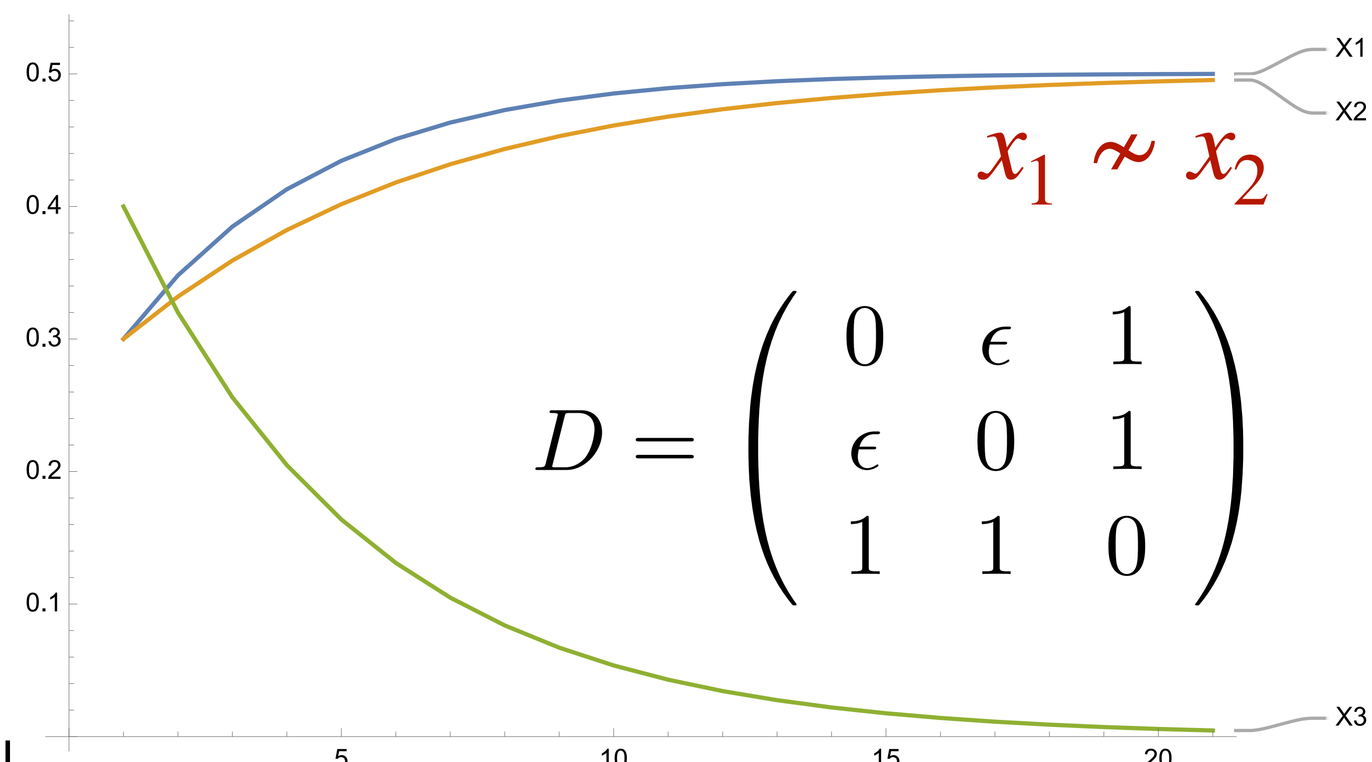


$x_1 \not\sim x_2$

$$D = \begin{pmatrix} 0 & \epsilon & 1 \\ \epsilon & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

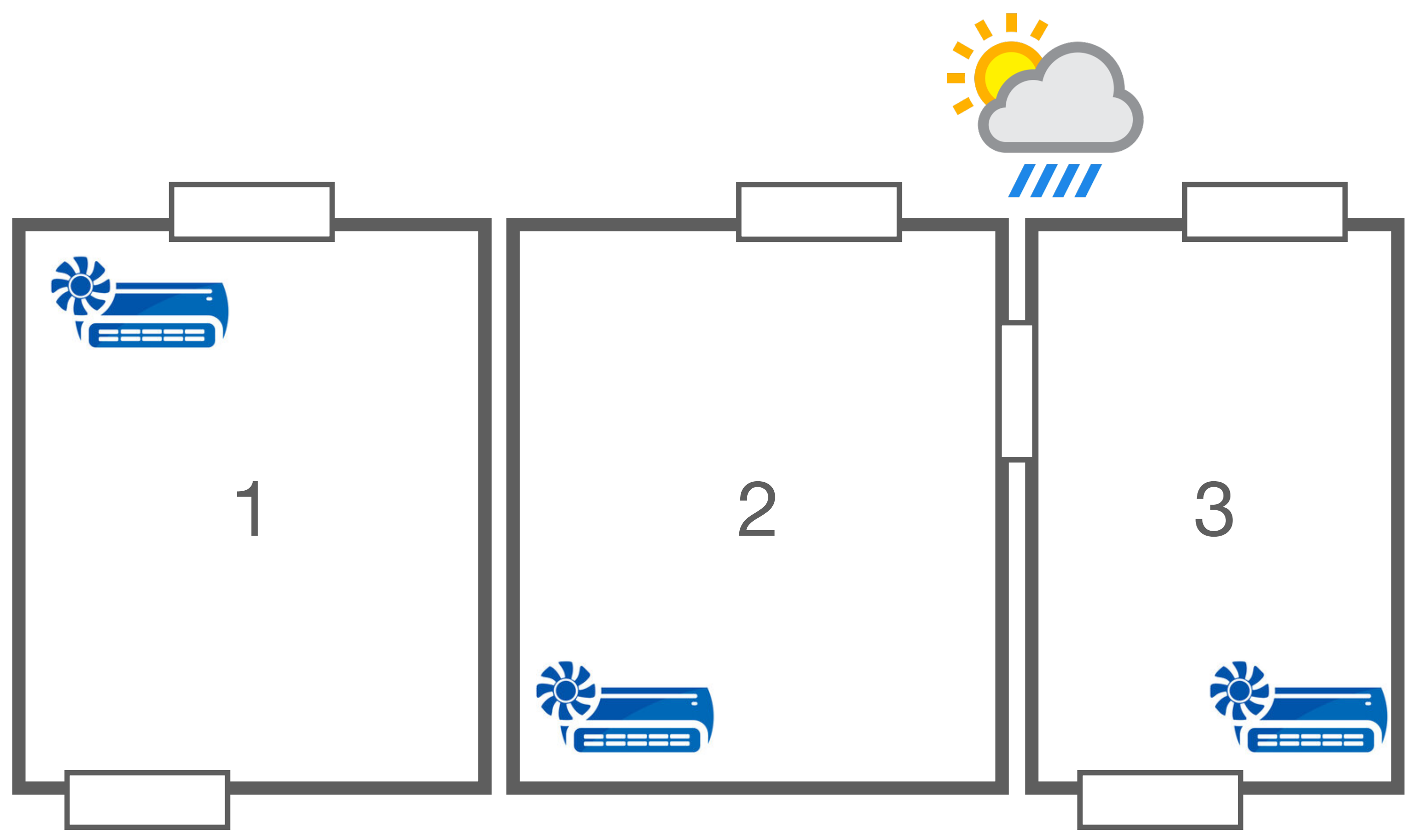
(*) for ϵ sufficiently small

Case Study: room heating

inspired from [Fehnker&Ivančić, HSCC'04]

The temperature $x_i(t)$ in room $i \in 1,2,3$ depends on:

- The temperature of the adjacent rooms
- Outside temperature
- Control of the air conditioning in each room $u_i(t)$



Linear dynamics

$$x(t+1) = f(x(t), u(t)) = Ax(t) + b + u(k)$$

$$A = \begin{pmatrix} 0.9910 & 0.0050 & 0 \\ 0.0050 & 0.9830 & 0.0055 \\ 0 & 0.0055 & 0.9915 \end{pmatrix} \quad b = \begin{pmatrix} 1.6 \\ 1.2 \\ 1.6 \end{pmatrix}$$

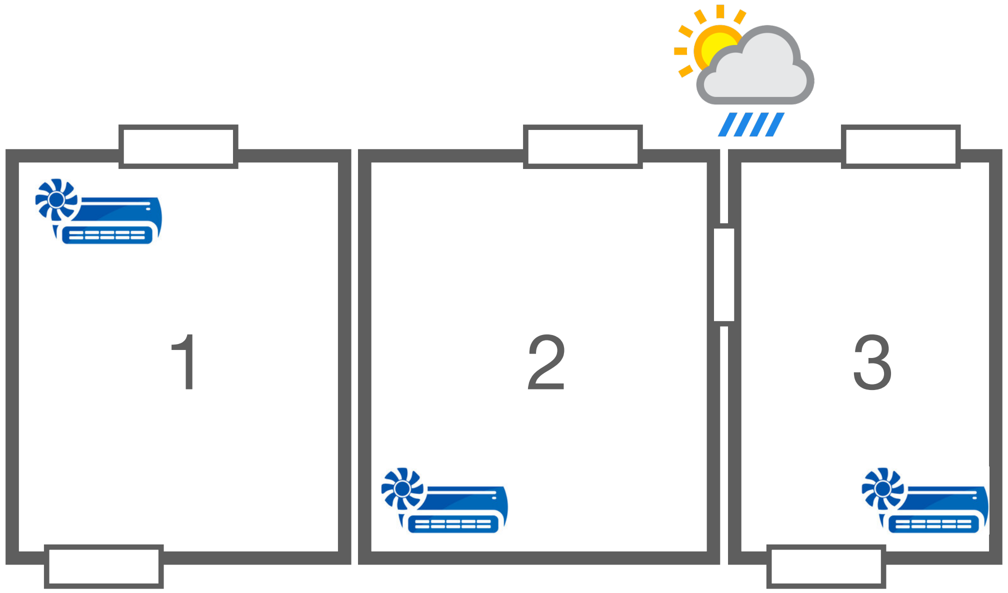
One obtains the **constant** control input $u^* = (I - A)x^* - b$ as the solution of $x^* = f(x^*, u^*)$ where $x^* = (20,20,20)$

Case Study: room heating

On-line Data Imputation

Assume one gives you the BD matrix D

	x_1	x_2	x_3
x_1	0	0.71	0.56
x_2	0.71	0	0.9
x_3	0.56	0.9	0



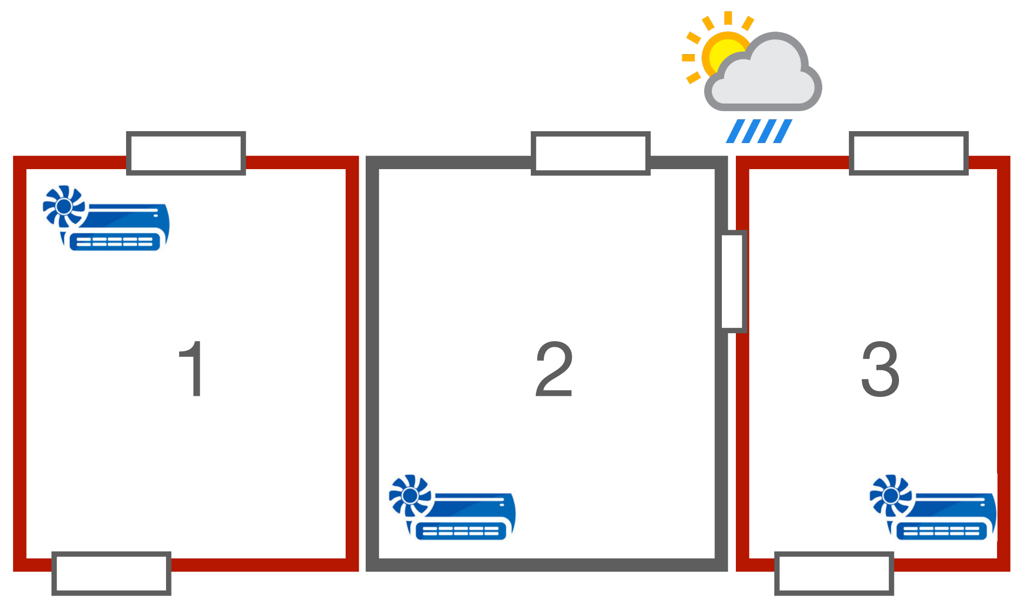
Assume that the thermometer in room 1 is malfunctioning. We can recover good estimates for the missing readings of $x_1(t)$ as

$$\max(x_2(t) - D_{12}, x_3(t) - D_{13}) \leq x_1(t) \leq \min(x_2(t) + D_{12}, x_3(t) + D_{13})$$

In particular we ensure that $x_1(t)$ is at most $0.56°C$ from $x_3(t)$

Case Study: room heating

Approximate Model Reduction



1. Perform **clustering** using D as underlying distance, obtaining the partitioning $\mathscr{H} = \{\{x_1, x_3\}, \{x_2\}\}$

2. Obtain from $\mathscr{H}$ the **reduced LDS** $y(t+1) = By(t) + c$

3. Get a BD D' for the "union" LDS $(x(t+1), y(t+1)) = (Ax(t) + b, By(t) + c)$

We approximately recover the original model as follows

$$\max_j y_j(t) - D'_{ij} \leq x_i(t) \leq \min_j y_j(t) + D'_{ij}$$

How do we compute backward dissimilarities?

Working Assumption

▶ **Definition 5** (Backward dissimilarity). *Let $x(t+1) = Ax(t) + b$ be an LDS. A symmetric matrix $D \in \mathbb{R}^{n\times n}_{\geq 0}$ is a* backward dissimilarity *for a set $I \subseteq \mathbb{R}^n$ of initial conditions if, for all $x(0) \in I$ and $t \in \mathbb{N}$,*

$$\bigwedge_{1\leq i,j\leq n} \Big(|x_i(t) - x_j(t)| \leq D_{ij}\Big) \implies \bigwedge_{1\leq i,j\leq n} \Big(|x_i(t+1) - x_j(t+1)| \leq D_{ij}\Big).$$

Working assumption: for the given set $I \subseteq \mathbb{R}^n$ of initial conditions, there exist $\lambda > 0$ such that, for any $x(0) \in I$, $\|x(t)\|_\infty \leq \lambda$ for all $t \geq 0$. (WA)

PROS

- Bounds the set of relevant dissimilarities
- $x(0) \in I \implies \{x_i(t)\}_{t\in\mathbb{N}} \subseteq [-\lambda, \lambda]$
- Simplifies our framework

CONS

- Estimation of λ may be tricky
  (I subset of (generalised) eigenspaces with eigenvalues γ s.t. $|\gamma| \leq 1$)
- Restriction on LDS we consider

(*) If the time horizon is bounded (i.e. $\{x_i(t)\}_{0\leq t\leq T}$), then WA can be dropped

Fixed point characterisation

For $x(t+1) = Ax(t) + b$ be an LDS, $I \subseteq \mathbb{R}^n$ and $\lambda > 0$ satisfying (WA)

$$\Delta_\lambda(D)_{ij} = \mathcal{T}_\lambda(D)(A_i, A_j) + |b_i - b_j|$$

Optimal solution of a transportation problem

Theorem (Fixed point characterisation of BD)

If $\Delta_\lambda(D) \sqsubseteq D$, then D is a backward dissimilarity for I

Theorem (Generalisation of BE)

Let δ be the least fixpoint of Δ_λ, and $\sim$ be the greatest BE, then

$$\delta_{ij} = 0 \iff i \sim j$$

Transportation Problem

▶ **Definition 9** (Transportation problem). *For two vectors $c, d \in \mathbb{R}^n$ and cost matrix $D \in \mathbb{R}^{n\times n}$ we define $\mathcal{T}_\lambda(D)(c,d)$ as the optimal value of the following linear program*

$$\mathcal{T}_\lambda(D)(c,d) = \min_{s,\bar{s},\omega}\left[\lambda \textstyle\sum_i (s_i + \bar{s}_i) + \sum_{i,j} D_{ij}\omega_{ij}\right]$$

$$\begin{aligned} \textit{subject to}\quad & \textstyle\sum_j \omega_{ij} + s_i = c_i^+ + d_i^- && i = 1\dots n \\ & \textstyle\sum_i \omega_{ij} + \bar{s}_j = c_j^- + d_j^+ && j = 1\dots n \\ & \omega_{ij} \geq 0,\ s_i \geq 0,\ \bar{s}_j \geq 0 && i,j = 1\dots n \end{aligned}$$

Example (Transport schedule)

$c(\bar{x}) = \frac{1}{10}x_1 - \frac{8}{10}x_3$

$d(\bar{x}) = \frac{2}{10}x_1 - \frac{9}{10}x_2$

$$D = \begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ & \begin{pmatrix} 0 & 2 & 5/3 \\ 2 & 0 & 1 \\ 5/3 & 1 & 0 \end{pmatrix} \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \end{array}$$

$\lambda = 2$

Transport schedule

	$\frac{2}{10}x_1$	$\frac{8}{10}x_3$	
$\frac{1}{10}x_1$	0: $\frac{1}{10}$	$5/3$: //	λ: //
$\frac{9}{10}x_2$	2: //	1: $\frac{8}{10}$	λ: $\frac{1}{10}$
	λ: $\frac{1}{10}$	λ: //	

Transport Policies

▶ **Definition 15.** *A* transport policy π *for* $x(t+1) = Ax(t) + b$ *is a map that assigns to each pair of indices* (i,j) *a transportation schedule* $\pi(i,j) \in \Gamma(A_i, A_j)$*. If* $\pi(i,j) \in \Gamma_V(A_i, A_j)$ *for all* (i,j)*,* π *is referred to as* vertex *transport policy.*

$$\Delta^{\pi}_{\lambda}(D)_{ij} = (\lambda \textstyle\sum_h s_h + \bar{s}_h + \sum_{h,k} D_{hk}\omega_{hk}) + |\, b_i - b_j \,|$$

where $\pi(i,j) = (s, \bar{s}, \omega)$

Proposition

Let π be a transport policy, then $\Delta^{\pi}_{\lambda}(D) \sqsubseteq D$ implies $\Delta_{\lambda}(D) \sqsubseteq D$

Theorem (Min Vertex Policy)

Let δ be the least fixpoint of Δ_{λ}, then

$$\delta = \min\{D \mid \pi \in \Pi_V(A, b) \text{ and } \Delta^{\pi}_{\lambda}(D) \sqsubseteq D\}$$

Simple policy iteration

SimplePolicyIteration(A, b, λ, R)

// Construct initial policy
let $D_{ij} = 0$ if $(i,j) \in R$ and $D_{ij} = 1$ if $(i,j) \notin R$.
for each $(i,j) \in [n] \times [n]$
    if $(i,j) \in R$
        $\pi[i,j] = k_\lambda(D)(A_i, A_j)$
    else $\pi[i,j] = (A_i^+ + A_j^-, A_i^- + A_j^+, \mathbf{0})$
// Iterative policy improvement
let D be the least fixpoint of Δ_λ^π
while $\exists(i,j).\, \Delta_\lambda(D)_{ij} < (D)_{ij}$
    $\pi[i,j] = k_\lambda(D)(A_i, A_j)$
    let D be the least fixpoint of Δ_λ^π
return D

Simple policy iteration

LDS satisfying (WA)

$\textsc{SimplePolicyIteration}(A, b, \lambda, R)$

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policy not optimal at (i, j)

Update policy at (i, j)

$$k_\lambda(D)(c, d) \in \operatorname*{argmin}_{(s, \bar{s}, \omega) \in \Gamma_V(c,d)} \lambda \sum_i (s_i + \bar{s}_i) + \sum_{i,j} D_{ij}\omega_{ij}$$

Lemma: $D^{(n+1)} \sqsubset D^{(n)}$

Fixpoint of Δ_λ

Simple policy iteration

BE for $x(t+1) = Ax(t) + b$

$\textsc{SimplePolicyIteration}(A, b, \lambda, R)$

```
// Construct initial policy
let D_ij = 0 if (i, j) ∈ R and D_ij = 1 if (i, j) ∉ R.
for each (i, j) ∈ [n] × [n]
    if (i, j) ∈ R
        π[i, j] = k_λ(D)(A_i, A_j)
    else π[i, j] = (A_i^+ + A_j^-, A_i^- + A_j^+, 0)
// Iterative policy improvement
let D be the least fixpoint of Δ_λ^π
while ∃(i, j). Δ_λ(D)_ij < (D)_ij
    π[i, j] = k_λ(D)(A_i, A_j)
    let D be the least fixpoint of Δ_λ^π
return D
```

$\pi(i,j) = (0,0,\omega)$ such that $w_{ij} = 0 \implies i \; R \; j$

$D^{(0)}_{ij} = 0 \iff i \; R \; j$

Open problems

- How to compute the **least backward dissimilarity**?
- Getting rid of the **working assumption**
    - We'll need to consider dissimilarities $D \in (\mathbb{R} \cup \{\infty\})^{n \times n}$
    - Can we generalise the fixpoint characterisation?
- Extend the dissimilarity framework to:
    - Continuous time models (e.g., ODEs, hybrid automata)
    - Non-linear dynamics (e.g., polynomials)