Reverse Bisimilarity vs. Forward Bisimilarity

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Reversible Computing

- Reversibility in mathematics: inverse function, inverse operation, ...
- More recent in informatics: seminal papers by Landauer (1961) and Bennett (1973) on IBM Journal of Research and Development.
- Landauer principle states that any *irreversible* manipulation of information such as:
  - erasure of bits
  - merging of computation paths

  must be accompanied by a corresponding *entropy increase*.
- Verified in 2012 and given a physical interpretation in 2018.

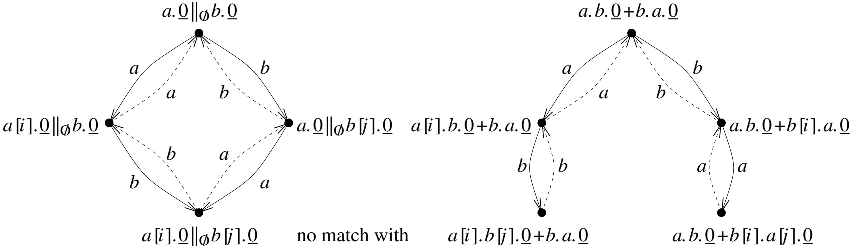
- Irreversible computations cause *heat dissipation* into circuits.
- Any reversible computation, where no information is lost, may be potentially carried out without releasing any heat.
- Low energy consumption could therefore be achieved by resorting to reversible computing.
- In addition, many applications of reversible computing:
    - Biochemical reaction modeling.
    - Parallel discrete-event simulation.
    - Robotics and control theory.
    - Fault tolerant computing systems.
    - Concurrent program debugging.

- Two directions of computation in a reversible system:
  - Forward: coincides with the normal way of computing.
  - Backward: the effects of the forward one are undone when needed.
- Different notions of reversibility in different settings:
  - Causal reversibility is the capability of going back to a past state in a way that is *consistent with the computational history* of the system (easy for sequential systems, hard for concurrent and distributed ones).
  - Time reversibility refers to the conditions under which the stochastic behavior remains the same when the *direction of time* is reversed (quantitative system models, efficient performance evaluation).

Reversibility in Process Algebra

- The *dynamic* approach of [DanosKrivine04] yielding RCCS uses stack-based memories attached to processes to record all the actions executed by those processes.
- A single transition relation is defined, while actions are divided into forward and backward resulting in forward and backward transitions.
- The *static* approach of [PhillipsUlidowski07] yielding CCSK is a method to reverse calculi by retaining within process syntax:
  - all executed actions, which are suitably decorated;
  - all dynamic operators, which are therefore treated as static.
- A forward transition relation and a backward transition relation are separately defined, labeled with communication keys as well.

- In [PU07] forward-reverse bisimilarity has been introduced too, which is truly concurrent as it does not satisfy the expansion law of parallel composition into a choice among all possible action sequencings:



- With back-and-forth bisimilarity [DeNicolaMontanariVaandrager90] the interleaving view can be restored as this bisimilarity is defined on computations instead of states to preserve both causality and history (one transition relation, viewed as bidirectional, outgoing/incoming).

- What are the properties of bisimilarity over reversible processes?
- *Neutrality* with respect to interleaving view vs. true concurrency, hence *only sequential processes* (no parallel composition).
- Minimal process calculus for reversible processes to *comparatively* investigate compositionality and equational characterizations of:
  - Forward-reverse bisimilarity.
  - Forward bisimilarity.
  - Reverse bisimilarity.
- Two different classes of processes:
  - Nondeterministic processes (causal reversibility).
  - Markovian processes (time reversibility, lumpability).
- We will discover several asymmetries among the three bisimilarities.

Nondeterministic Reversible Processes

- We usually describe only the future behavior of (sequential) processes:

$$P \ ::= \ \underline{0} \mid a \,.\, P \mid P + P$$

- Need for a syntax extended with information about the past behavior:

$$P \ ::= \ \underline{0} \mid a \,.\, P \mid a^{\dagger} .\, P \mid P + P$$

- Uniform action decoration instead of communication keys [PU07] due to the absence of parallel composition (keys are necessary to remember who synchronized with whom upon undoing actions).
- *Initial processes*: all the actions are unexecuted.
- They coincide with standard, future-only processes.
- *Final processes*: all the actions along a path have been executed.
- Several paths in the presence of $+$, only one is chosen.

- Countable set A of actions.
- Work with the set $\mathbb{P}$ of *reachable processes*:

$$
\begin{array}{rcl}
\textit{reachable}(\underline{0}) & & \\
\textit{reachable}(a\,.\,P) & \Longleftarrow & \textit{initial}(P) \\
\textit{reachable}(a^{\dagger}.\,P) & \Longleftarrow & \textit{reachable}(P) \\
\textit{reachable}(P_1 + P_2) & \Longleftarrow & (\textit{reachable}(P_1) \wedge \textit{initial}(P_2)) \vee \\
 & & (\textit{initial}(P_1) \wedge \textit{reachable}(P_2))
\end{array}
$$

- $\underline{0}$ is the only process that is both initial and final as well as reachable.
- Any initial or final process is reachable too.
- $\mathbb{P}$ also contains processes that are neither initial nor final: $a^{\dagger}.\,b\,.\,\underline{0}$.
- In $P_1 + P_2$ both subprocesses can be initial (at least one must be).
- Past actions can never follow future actions: $b\,.\,a^{\dagger}.\,\underline{0} \notin \mathbb{P}$.

- Must retain in the syntax all information needed to enable reversibility, so action prefix and choice are made static by the semantics [PU07].
- Semantics defined according to the structural operational approach: labeled transition system $(\mathbb{P}, A, \longrightarrow)$ where $\longrightarrow \subseteq \mathbb{P} \times A \times \mathbb{P}$.
- Single transition relation viewed as symmetric to meet loop property: *executed actions can be undone and undone actions can be redone* (necessary condition for any notion of reversibility).
- Forward/backward bisimilarity via outgoing/incoming transitions like in [DMV90].
- $P \xrightarrow{a} P'$ goes:
  - forward if viewed as an outgoing transition of P;
  - backward if viewed as an incoming transition of P'.

- Semantic rules for action prefix:

$$\frac{\textit{initial}(P)}{a\,.\,P \xrightarrow{a} a^{\dagger}\,.\,P} \qquad\qquad \frac{P \xrightarrow{b} P'}{a^{\dagger}\,.\,P \xrightarrow{b} a^{\dagger}\,.\,P'}$$

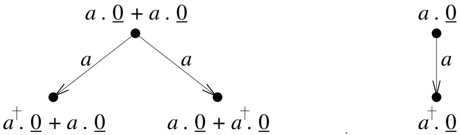
- The prefix related to the executed action is *not discarded*.
- It becomes a †-decorated part of the target process, necessary to offer again that action after coming back.
- Additional rule for performing unexecuted actions that are preceded by already executed actions (direct consequence of making prefix static).

- Semantic rules for alternative composition:

$$\frac{P_1 \xrightarrow{a} P_1' \quad \mathit{initial}(P_2)}{P_1 + P_2 \xrightarrow{a} P_1' + P_2} \qquad \frac{P_2 \xrightarrow{a} P_2' \quad \mathit{initial}(P_1)}{P_1 + P_2 \xrightarrow{a} P_1 + P_2'}$$

- The subprocess not involved in the executed action is *not discarded* but cannot proceed further (only the non-initial subprocess can).
- It becomes part of the target process, which is necessary for offering again the original choice after undoing all the executed actions.
- If both subprocesses are initial, both rules apply (nondet. choice).

- The labeled transition system underlying an initial process is a *tree*, whose branching points correspond to occurrences of $+$.
- Any non-final process has at least one outgoing transition.
- Any non-initial process has exactly one incoming transition due to decorations associated with executed actions.
- Consider the two initial processes $a\,.\,\underline{0} + a\,.\,\underline{0}$ and $a\,.\,\underline{0}$:



- Single a-transition on the left in a future-only process calculus.
- These two distinct processes should be considered equivalent.

Bisimilarities for Nondeterministic Reversible Processes

- Bisimulation game: *outgoing* transitions for forward direction and *incoming* transitions for backward direction [DMV90].
- A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a:
  - Forward bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
    - whenever $P_1 \xrightarrow{a} P'_1$, then $P_2 \xrightarrow{a} P'_2$ with $(P'_1, P'_2) \in \mathcal{B}$.
  - Reverse bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
    - whenever $P'_1 \xrightarrow{a} P_1$, then $P'_2 \xrightarrow{a} P_2$ with $(P'_1, P'_2) \in \mathcal{B}$.
  - Forward-reverse bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
    - whenever $P_1 \xrightarrow{a} P'_1$, then $P_2 \xrightarrow{a} P'_2$ with $(P'_1, P'_2) \in \mathcal{B}$.
    - whenever $P'_1 \xrightarrow{a} P_1$, then $P'_2 \xrightarrow{a} P_2$ with $(P'_1, P'_2) \in \mathcal{B}$.
- Largest such relations: $\sim_{\mathrm{FB}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$.

- $\sim_{\mathrm{FRB}} \subsetneq \sim_{\mathrm{FB}} \cap \sim_{\mathrm{RB}}$:
  - The inclusion is strict because the final processes $a^{\dagger} . \underline{0}$ and $a^{\dagger} . \underline{0} + c . \underline{0}$ are identified by $\sim_{\mathrm{FB}}$ and $\sim_{\mathrm{RB}}$, but distinguished by $\sim_{\mathrm{FRB}}$.
  - $\sim_{\mathrm{FB}}$ and $\sim_{\mathrm{RB}}$ are incomparable because $a^{\dagger} . \underline{0} \sim_{\mathrm{FB}} \underline{0}$ but $a^{\dagger} . \underline{0} \not\sim_{\mathrm{RB}} \underline{0}$ while $a . \underline{0} \sim_{\mathrm{RB}} \underline{0}$ but $a . \underline{0} \not\sim_{\mathrm{FB}} \underline{0}$.
- First asymmetry between $\sim_{\mathrm{FB}}$ and $\sim_{\mathrm{RB}}$:
  - $\sim_{\mathrm{FRB}} = \sim_{\mathrm{FB}}$ over initial processes, with $\sim_{\mathrm{RB}}$ strictly coarser.
  - $\sim_{\mathrm{FRB}} \neq \sim_{\mathrm{RB}}$ over final processes because, after going backward, discarded subprocesses come into play in the forward direction.
- $a . \underline{0} + a . \underline{0}$ and $a . \underline{0}$ are identified by all three bisimilarities as witnessed by any bisimulation containing the pairs $(a . \underline{0} + a . \underline{0}, a . \underline{0})$, $(a^{\dagger} . \underline{0} + a . \underline{0}, a^{\dagger} . \underline{0})$, $(a . \underline{0} + a^{\dagger} . \underline{0}, a^{\dagger} . \underline{0})$.

Compositionality Properties

- $\sim_{\rm FB}$ equates processes with different past: $a_1^{\dagger} \, . \, \underline{0} \sim_{\rm FB} a_2^{\dagger} \, . \, \underline{0} \sim_{\rm FB} \underline{0}$.
- $\sim_{\rm RB}$ equates processes with different future: $a_1 \, . \, \underline{0} \sim_{\rm RB} a_2 \, . \, \underline{0} \sim_{\rm RB} \underline{0}$.
- Second asymmetry between $\sim_{\rm FB}$ and $\sim_{\rm RB}$:
    - $a^{\dagger} . \, b \, . \, \underline{0} \sim_{\rm FB} b \, . \, \underline{0}$ but $a^{\dagger} . \, b \, . \, \underline{0} + c \, . \, \underline{0} \not\sim_{\rm FB} b \, . \, \underline{0} + c \, . \, \underline{0}$.
    - $a^{\dagger} . \, b \, . \, \underline{0} \not\sim_{\rm RB} b \, . \, \underline{0}$ hence no compositionality violation for $\sim_{\rm RB}$.
- $\sim_{\rm RB}$ and $\sim_{\rm FRB}$ never identify an initial process with a non-initial one, hence $\sim_{\rm FB}$ has to be made sensitive to the *presence of the past*.
- A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a past-sensitive forward bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ it holds that *initial*$(P_1) \iff$ *initial*(P_2) and for all $a \in A$, whenever $P_1 \xrightarrow{a} P_1'$, then $P_2 \xrightarrow{a} P_2'$ with $(P_1', P_2') \in \mathcal{B}$.
- $a_1^{\dagger} \, . \, \underline{0} \sim_{\rm FB,ps} a_2^{\dagger} \, . \, \underline{0}$, but $a^{\dagger} . \, \underline{0} \not\sim_{\rm FB,ps} \underline{0}$ and $a^{\dagger} . \, b \, . \, \underline{0} \not\sim_{\rm FB,ps} b \, . \, \underline{0}$.

- Let $P_1, P_2 \in \mathbb{P}$ be s.t. $P_1 \sim P_2$ and take arbitrary $a \in A$ and $P \in \mathbb{P}$.
- All the considered bisimilarities are congruences w.r.t. action prefix:
  - $a \,.\, P_1 \sim a \,.\, P_2$ provided that *initial*$(P_1) \wedge$ *initial*(P_2).
  - $a^\dagger \,.\, P_1 \sim a^\dagger \,.\, P_2$.
- $\sim_{\mathrm{FB,ps}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$ are congruences w.r.t. alternative composition:
  - $P_1 + P \sim P_2 + P$ and $P + P_1 \sim P + P_2$
    provided that *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.
- $\sim_{\mathrm{FB,ps}}$ is the coarsest congruence w.r.t. $+$ contained in $\sim_{\mathrm{FB}}$:
  - $P_1 \sim_{\mathrm{FB,ps}} P_2$ iff $P_1 + P \sim_{\mathrm{FB}} P_2 + P$
    for all $P \in \mathbb{P}$ s.t. *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.

Equational Characterizations

- Deduction system based on these axioms and inference rules on $\mathbb{P}$:
  - Reflexivity: $P = P$.
  - Symmetry: $\dfrac{P_1 = P_2}{P_2 = P_1}$.
  - Transitivity: $\dfrac{P_1 = P_2 \quad P_2 = P_3}{P_1 = P_3}$.
  - .-Substitutivity: $\dfrac{P_1 = P_2 \quad \mathit{initial}(P_1) \wedge \mathit{initial}(P_2)}{a\,.\,P_1 = a\,.\,P_2}$, $\dfrac{P_1 = P_2}{a^\dagger\,.\,P_1 = a^\dagger\,.\,P_2}$.
  - $+$-Substitutivity: $\dfrac{P_1 = P_2 \quad \mathit{initial}(P) \vee (\mathit{initial}(P_1) \wedge \mathit{initial}(P_2))}{P_1 + P = P_2 + P \quad P + P_1 = P + P_2}$.
- Correspond to $\sim_{\mathrm{FB,ps}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$ being equivalence relations as well as congruences w.r.t. action prefix and alternative composition.

$(\mathcal{A}_1)$		$(P_1 + P_2) + P_3$	$=$	$P_1 + (P_2 + P_3)$	
$(\mathcal{A}_2)$		$P_1 + P_2$	$=$	$P_2 + P_1$	
$(\mathcal{A}_3)$		$P + \underline{0}$	$=$	P	
$(\mathcal{A}_4)$	$[\sim_{\mathrm{FB,ps}}]$	$a^{\dagger} . P$	$=$	P	if $\neg \mathit{initial}(P)$
$(\mathcal{A}_5)$	$[\sim_{\mathrm{FB,ps}}]$	$a_1^{\dagger} . P$	$=$	$a_2^{\dagger} . P$	if $\mathit{initial}(P)$
$(\mathcal{A}_6)$	$[\sim_{\mathrm{FB,ps}}]$	$P + Q$	$=$	P	if $\neg \mathit{initial}(P)$, where $\mathit{initial}(Q)$
$(\mathcal{A}_7)$	$[\sim_{\mathrm{RB}}]$	$a . P$	$=$	P	where $\mathit{initial}(P)$
$(\mathcal{A}_8)$	$[\sim_{\mathrm{RB}}]$	$P + Q$	$=$	P	if $\mathit{initial}(Q)$
$(\mathcal{A}_9)$	$[\sim_{\mathrm{FB,ps}}]$	$P + P$	$=$	P	where $\mathit{initial}(P)$
$(\mathcal{A}_{10})$	$[\sim_{\mathrm{FRB}}]$	$P + Q$	$=$	P	if $\mathit{initial}(Q) \wedge \mathit{to\_initial}(P) = Q$

- $\mathcal{A}_8$ subsumes $\mathcal{A}_3$ (with $Q = \underline{0}$) and $\mathcal{A}_9$ (with $Q = P$).
- $\mathcal{A}_9$ and $\mathcal{A}_6$ apply in two different cases (P initial or not).
- $\mathcal{A}_{10}$ appeared for the first time in [LanesePhillips21].
- $\vdash_{4,5,6,9}^{1,2,3}$ / $\vdash_{7,8}^{1,2}$ / $\vdash_{10}^{1,2,3}$ sound and complete for $\sim_{\mathrm{FB,ps}}$ / $\sim_{\mathrm{RB}}$ / $\sim_{\mathrm{FRB}}$.
- Third asymmetry between $\sim_{\mathrm{FB}}$ and $\sim_{\mathrm{RB}}$: idempotency.

Markov Chains: Definition, Representation, Terminology

- A *random variable* X takes every value with a specific probability.
- A *stochastic process* describes the evolution over time of some random phenomenon through a sequence of random variables $X(t)$, one for each time instant t.
- A stochastic process $X(t)$ taking values from a discrete state space $\mathcal{S}$ for $t \in \mathbb{R}_{\geq 0}$ is a continuous-time Markov chain (CTMC) iff for all $n \in \mathbb{N}$, $t_0 < t_1 < \cdots < t_n < t_{n+1} \in \mathbb{R}_{\geq 0}$, $s_0, s_1, \ldots, s_n, s_{n+1} \in \mathcal{S}$:
$\Pr\{X(t_{n+1}) = s_{n+1} \mid X(t_0) = s_0, X(t_1) = s_1, \ldots, X(t_n) = s_n\} =$
$\Pr\{X(t_{n+1}) = s_{n+1} \mid X(t_n) = s_n\}$.
- The probability of moving from one state to another does not depend on the particular path that has been followed in the past to reach the current state, hence that path can be forgotten (*memorylessness*).

- A CTMC $X(t)$ is:
  - *Time homogeneous* iff $\Pr\{X(t + t') = s' \mid X(t) = s\}$ does not depend on the time instant t, so that $r_{s,s'} = \lim_{t' \to 0^+} \frac{\Pr\{X(t+t')=s' \mid X(t)=s\}}{t'}$.
  - *Irreducible* iff each of its states is reachable from every other state with probability greater than 0.
  - *Ergodic* iff it is irreducible and each of its states is positive recurrent, i.e., the CTMC will eventually return to it with probability 1 in an expected number of steps that is finite.
- Ergodicity coincides with irreducibility when the CTMC has finitely many states, as they form a finite strongly connected component.
- The sojourn time in state s is exponentially distributed with rate given by the sum r_s of the rates of the moves of s.
- The average sojourn time in s is $1/r_s$.
- The probability of moving from s to s' is $r_{s,s'}/r_s$.

- Every time-homogeneous and ergodic CTMC $X(t)$ is *stationary*, i.e., $(X(t_i + t'))_{1 \leq i \leq n}$ has the same joint distribution as $(X(t_i))_{1 \leq i \leq n}$ for all $n \in \mathbb{N}_{\geq 1}$ and $t_1 < \cdots < t_n, t' \in \mathbb{R}_{\geq 0}$.
- In this case $X(t)$ has a unique *steady-state probability distribution* $\boldsymbol{\pi} = (\pi(s))_{s \in \mathcal{S}}$ that fulfills $\pi(s) = \lim_{t \to \infty} \Pr\{X(t) = s \mid X(0) = s'\}$ for any $s' \in \mathcal{S}$ because it has reached equilibrium.
- Computed by solving the linear system of *global balance equations* $\boldsymbol{\pi} \cdot \mathbf{Q} = \mathbf{0}$ subject to $\sum_{s \in \mathcal{S}} \pi(s) = 1$ and $\pi(s) \in \mathbb{R}_{>0}$ for all $s \in \mathcal{S}$ (incoming probability flux equal to outgoing probability flux).
- The *infinitesimal generator matrix* $\mathbf{Q} = (q_{s,s'})_{s,s' \in \mathcal{S}}$ is such that $q_{s,s'} = r_{s,s'}$ for $s \neq s'$ while $q_{s,s} = - \sum_{s' \neq s} q_{s,s'}$.
- A CTMC can be represented through $\mathbf{Q}$ or as a state-transition graph in which every transition is labeled with the corresponding rate > 0.

Markov Chains: Time Reversibility and Lumpability

- A CTMC $X(t)$ is time reversible iff the behavior remains the same when the direction of time is reversed.
- $(X(t_i))_{1 \leq i \leq n}$ has the same joint distribution as $(X(t' - t_i))_{1 \leq i \leq n}$ for all $n \in \mathbb{N}_{\geq 1}$ and $t_1 < \cdots < t_n, t' \in \mathbb{R}_{\geq 0}$.
- $X(t)$ and its reversed version $X(-t)$ are stochastically identical: they are stationary and share the same steady-state distribution $\boldsymbol{\pi}$.
- In order for a stationary CTMC $X(t)$ to be time reversible, it is necessary and sufficient that one of the following holds [Kelly79]:
  - The *partial balance equations* $\pi(s) \cdot q_{s,s'} = \pi(s') \cdot q_{s',s}$ are satisfied for all distinct $s, s' \in \mathcal{S}$.
  - $q_{s_1,s_2} \cdot \ldots \cdot q_{s_{n-1},s_n} \cdot q_{s_n,s_1} = q_{s_1,s_n} \cdot q_{s_n,s_{n-1}} \cdot \ldots \cdot q_{s_2,s_1}$ for all $n \in \mathbb{N}_{\geq 2}$ and distinct $s_1, \ldots, s_n \in \mathcal{S}$.

- The sum of the partial balance equations for $s \in \mathcal{S}$ yields the global balance equation $\pi(s) \cdot |q_{s,s}| = \sum_{s' \neq s} \pi(s') \cdot q_{s',s}$.
- Time reversibility exploitable for efficient performance evaluation.
- The time-reversed version $X^{\mathrm{r}}(t)$ of a stationary CTMC $X(t)$ can be defined even when $X(t)$ is not reversible.
- Accomplished by using the steady-state distribution $\boldsymbol{\pi}$ of $X(t)$, with $X^{\mathrm{r}}(t)$ turning out to be a CTMC too and having the same $\boldsymbol{\pi}$.
- $q^{\mathrm{r}}_{s_j,s_i} = q_{s_i,s_j} \cdot \pi(s_i)/\pi(s_j)$ for all $s_i \neq s_j$.
- The time-reversed version of $X^{\mathrm{r}}(t)$ is $X(t)$.

- An *exact aggregation* of a CTMC partitions the state space in such a way that the probability of being in any of the aggregated states is equal to the sum of the probabilities of the original states it contains.
- The partition $\mathcal{P}$ induced by an equivalence relation $\mathcal{L}$ over $\mathcal{S}$ is an:
  - Ordinary lumping iff for all $(s_1, s_2) \in \mathcal{L}$ and $C \in \mathcal{P}$ with $s_1, s_2 \notin C$:

    $$\sum_{s' \in C} q_{s_1,s'} = \sum_{s' \in C} q_{s_2,s'}$$

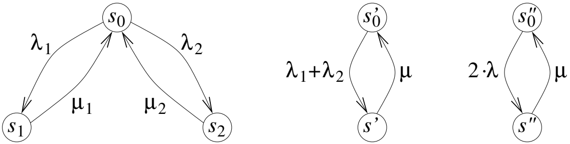
    in which case $q'_{C_1,C_2} = \sum_{s' \in C_2} q_{s,s'}$ for $s \in C_1$ [KemenySnell60].
  - Exact lumping iff for all $(s_1, s_2) \in \mathcal{L}$ and $C \in \mathcal{P}$:

    $$\sum_{s' \in C} q_{s',s_1} = \sum_{s' \in C} q_{s',s_2}$$

    in which case $q'_{C_1,C_2} = \frac{|C_2|}{|C_1|} \cdot \sum_{s' \in C_1} q_{s',s}$ for $s \in C_2$ [Schweitzer83].
  - Strict lumping iff it is both ordinary and exact [Buchholz94].

- Exact lumpability further guarantees that all the original states contained in the same aggregated state have the *same probability*, but does not necessarily imply ordinary lumpability.
- Rate equality check inside each class vs. no such a check.
- Incoming transitions vs. outgoing transitions.
- Relations between lumpability and time reversibility [MarinRossi17]:
  - An exact lumping of a CTMC corresponds to an ordinary lumping on the time-reversed CTMC.
  - An aggregation of a CTMC is a strict lumping iff it is a strict lumping for the time-reversed CTMC too.
  - An exact lumping of a CTMC is also an ordinary lumping whenever the CTMC is time reversible, the vice versa does not hold in general.

- Consider the following three time-reversible, ergodic CTMCs:

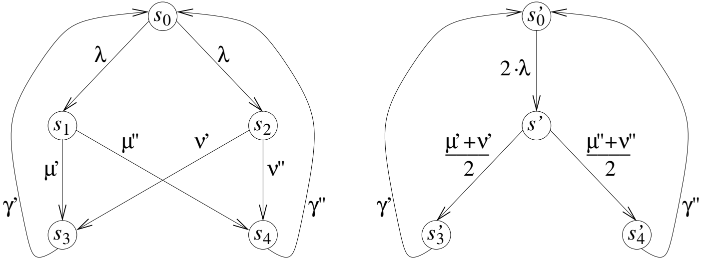


- Global balance equation solution for the first CTMC:

$\pi(s_0) = \frac{\mu_1 \cdot \mu_2}{\mu_1 \cdot \mu_2 + \lambda_1 \cdot \mu_2 + \lambda_2 \cdot \mu_1}$ $\quad \pi(s_1) = \frac{\lambda_1 \cdot \mu_2}{\mu_1 \cdot \mu_2 + \lambda_1 \cdot \mu_2 + \lambda_2 \cdot \mu_1}$ $\quad \pi(s_2) = \frac{\lambda_2 \cdot \mu_1}{\mu_1 \cdot \mu_2 + \lambda_1 \cdot \mu_2 + \lambda_2 \cdot \mu_1}$

- $\lambda_1 = \lambda_2$ but $\mu_1 \neq \mu_2$: no exact aggregation.
- $\mu_1 = \mu_2 \triangleq \mu$ but $\lambda_1 \neq \lambda_2$: the second CTMC is an ordinary lumping of the first one – $\pi(s') = \frac{\lambda_1 + \lambda_2}{\mu + \lambda_1 + \lambda_2} = \pi(s_1) + \pi(s_2)$, $\pi(s_1) \neq \pi(s_2)$ – but not an exact lumping.
- $\lambda_1 = \lambda_2 \triangleq \lambda$ and $\mu_1 = \mu_2 \triangleq \mu$: the third CTMC is a strict lumping of the first one – $\pi(s'') = \frac{2 \cdot \lambda}{\mu + 2 \cdot \lambda} = \pi(s_1) + \pi(s_2)$, $\pi(s_1) = \pi(s_2)$.

- Consider the following two non-time-reversible, ergodic CTMCs:



- The second CTMC is an exact lumping of the first one when $\mu' + \mu'' = \nu' + \nu''$ (i.e., $q_{s_1,s_1} + q_{s_2,s_1} = q_{s_1,s_2} + q_{s_2,s_2}$).
- It is not an ordinary lumping if $\mu' \neq \nu'$ and $\mu'' \neq \nu''$.
- Incomparability of ordinary lumpability and exact lumpability in the absence of time reversibility.

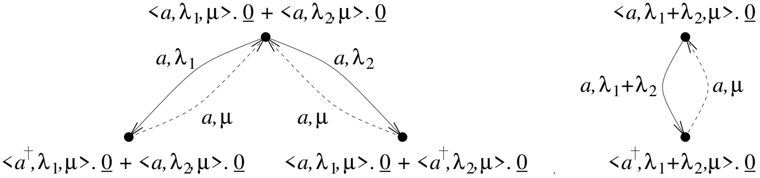
Markovian Reversible Processes

- Actions augmented with rates to describe their *quantitative aspects* such as time durations and execution probabilities (in case of choice).
- A transition relation in a single direction is no longer sufficient because every transition is now labeled with a rate too, where the rate may be different depending on whether the transition goes forward or backward (without necessarily affecting time reversibility).
- Syntax extended with forward and backward rates for actions:

$$P \ ::= \ \underline{0} \mid {<}a, \lambda, \mu{>} \,.\, P \mid {<}a^{\dagger}, \lambda, \mu{>} \,.\, P \mid P + P$$

- Predicates *initial*, *final*, *reachable* extended accordingly.
- $\mathbb{P}_{\mathrm{M}}$ denotes the set of reachable processes.

- For consistency with the CTMC theory it is not possible to use a transition relation with forward rates separated as in [PU07] from a transition relation with backward rates.
- $<a, \lambda_1, \mu> . \underline{0} + <a, \lambda_2, \mu> . \underline{0}$ and $<a, \lambda_1 + \lambda_2, \mu> . \underline{0}$ would be equated by a Markovian variant of $\sim_{\rm FRB}$ on two transition relations, but this would not be consistent with exact lumpability when $\lambda_1 \neq \lambda_2$:



- Keep using a single transition relation $\longrightarrow_{\rm M} \subseteq \mathbb{P}_{\rm M} \times (A \times \mathbb{R}_{>0}) \times \mathbb{P}_{\rm M}$ but embodying forward rate transitions and backward rate transitions:

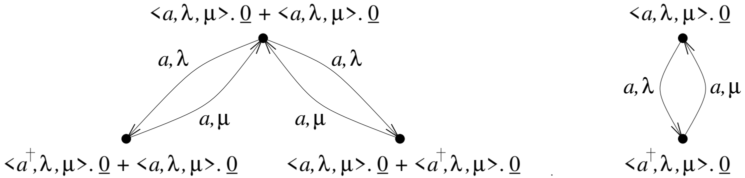
$$\frac{\mathit{initial}(P)}{<a, \lambda, \mu> . P \xrightarrow{a,\lambda}_{\rm M} <a^{\dagger}, \lambda, \mu> . P} \qquad \frac{\mathit{initial}(P)}{<a^{\dagger}, \lambda, \mu> . P \xrightarrow{a,\mu}_{\rm M} <a, \lambda, \mu> . P}$$

$$\frac{P \xrightarrow{b,\xi}_{\rm M} P'}{<a^{\dagger}, \lambda, \mu> . P \xrightarrow{b,\xi}_{\rm M} <a^{\dagger}, \lambda, \mu> . P'}$$

$$\frac{P_1 \xrightarrow{a,\xi}_{\rm M} P'_1 \quad \mathit{initial}(P_2)}{P_1 + P_2 \xrightarrow{a,\xi}_{\rm M} P'_1 + P_2} \qquad \frac{P_2 \xrightarrow{a,\xi}_{\rm M} P'_2 \quad \mathit{initial}(P_1)}{P_1 + P_2 \xrightarrow{a,\xi}_{\rm M} P_1 + P'_2}$$

- If both subprocesses of $+$ are initial, then the *race policy* applies: each action has an execution probability proportional to its rate.

- Any state corresponding to a process different from $\underline{0}$ can now have several incoming transitions too.
- The labeled transition system underlying an initial process turns out to be a *tree-like extension of a birth-death process*.
- The underlying CTMC turns out to be not only ergodic, but also time reversible due to its tree-like birth-death structure [Kelly79].
- Consider $<a, \lambda, \mu> . \underline{0} + <a, \lambda, \mu> . \underline{0}$ and $<a, \lambda, \mu> . \underline{0}$:



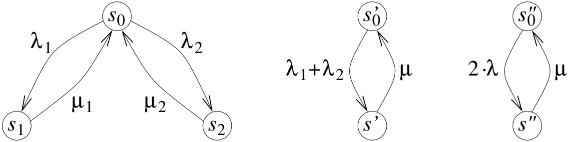
- Single a-transition on the left would have not reflected the total exit rate $2 \cdot \lambda$ of the source state, prevented by action decoration.

Bisimilarities for Markovian Reversible Processes

- In the forward case:
  - Probabilistic bisim. [LarsenSkou91] induces DTMC ordinary lumping.
  - Markovian bisim. [Hillston96] induces CTMC ordinary lumping.
- Markovian variants of $\sim_{\rm FB}$, $\sim_{\rm RB}$, $\sim_{\rm FRB}$ defined in such a way to induce CTMC ordinary lumping, exact lumping, strict lumping.
- The Markovian variant of $\sim_{\rm FB}$ will apply the rate equality check inside each class too, hence not all ordinary lumpings can be induced.
- While in Markov chain theory one is interested in state probabilities, in concurrency theory one experiments with processes by observing the labels of the transitions that are executed.
- A state with a self-looping λ-transition and a state with a self-looping μ-transition are ordinarily lumpable, although the more λ and μ are different, the easier it is for an observer to tell those two states apart.

- An equivalence relation $\mathcal{B}$ over $\mathbb{P}_{\mathrm{M}}$ is a:
  - Markovian forward bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$, $a \in A$, and $C \in \mathbb{P}_{\mathrm{M}}/\mathcal{B}$:
  $$\mathit{rate}_{\mathrm{out}}(P_1, a, C) \;=\; \mathit{rate}_{\mathrm{out}}(P_2, a, C)$$
  where $\mathit{rate}_{\mathrm{out}}(P, a, C) \;=\; \sum\{\!| \xi \in \mathbb{R}_{>0} \mid \exists P' \in C.\, P \xrightarrow{a,\xi}_{\mathrm{M}} P' |\!\}$.
  - Markovian reverse bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
  $$\mathit{rate}_{\mathrm{out}}(P_1, a, \mathbb{P}_{\mathrm{M}}) \;=\; \mathit{rate}_{\mathrm{out}}(P_2, a, \mathbb{P}_{\mathrm{M}})$$
  and for all $C \in \mathbb{P}_{\mathrm{M}}/\mathcal{B}$:
  $$\mathit{rate}_{\mathrm{in}}(P_1, a, C) \;=\; \mathit{rate}_{\mathrm{in}}(P_2, a, C)$$
  where $\mathit{rate}_{\mathrm{in}}(P, a, C) \;=\; \sum\{\!| \xi \in \mathbb{R}_{>0} \mid \exists P' \in C.\, P' \xrightarrow{a,\xi}_{\mathrm{M}} P |\!\}$.
  - Markovian forward-reverse bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$, $a \in A$, and $C \in \mathbb{P}_{\mathrm{M}}/\mathcal{B}$:
  $$\mathit{rate}_{\mathrm{out}}(P_1, a, C) \;=\; \mathit{rate}_{\mathrm{out}}(P_2, a, C)$$
  $$\mathit{rate}_{\mathrm{in}}(P_1, a, C) \;=\; \mathit{rate}_{\mathrm{in}}(P_2, a, C)$$
- Largest such relations: $\sim_{\mathrm{MFB}}$, $\sim_{\mathrm{MRB}}$, $\sim_{\mathrm{MFRB}}$.

- Consider again the following three time-reversible, ergodic CTMCs:



- It holds that:
  - $<a, \lambda_1, \mu> . \underline{0} + <a, \lambda_2, \mu> . \underline{0} \sim_{\mathrm{MFB}} <a, \lambda_1 + \lambda_2, \mu> . \underline{0}$
    $<a^\dagger, \lambda_1, \mu> . \underline{0} + <a, \lambda_2, \mu> . \underline{0} \sim_{\mathrm{MFB}} <a^\dagger, \lambda_1 + \lambda_2, \mu> . \underline{0}$
    $<a, \lambda_1, \mu> . \underline{0} + <a^\dagger, \lambda_2, \mu> . \underline{0} \sim_{\mathrm{MFB}} <a^\dagger, \lambda_1 + \lambda_2, \mu> . \underline{0}$
  - $<a^\dagger, \lambda, \mu> . \underline{0} + <a, \lambda, \mu> . \underline{0} \not\sim_{\mathrm{MRB}} <a^\dagger, 2 \cdot \lambda, \mu> . \underline{0}$
    $<a, \lambda, \mu> . \underline{0} + <a^\dagger, \lambda, \mu> . \underline{0} \not\sim_{\mathrm{MRB}} <a^\dagger, 2 \cdot \lambda, \mu> . \underline{0}$
    $<a, \lambda, \mu> . \underline{0} + <a, \lambda, \mu> . \underline{0} \not\sim_{\mathrm{MRB}} <a, 2 \cdot \lambda, \mu> . \underline{0}$
    $<a^\dagger, \lambda, \mu> . \underline{0} + <a, \lambda, \mu> . \underline{0} \sim_{\mathrm{MRB}} <a, \lambda, \mu> . \underline{0} + <a^\dagger, \lambda, \mu> . \underline{0}$
- Fourth asymmetry between forward and reverse bisimilarities:
  - Any aggregated state resulting from an ordinary lumping is $\sim_{\mathrm{MFB}}$-equivalent to each of the original states it contains.
  - In an exact lumping $\sim_{\mathrm{MRB}}$-equivalence certainly holds only among the original states contained in an aggregated state.

- $\sim_{\rm MFB}$ is sensitive to the presence of the past (unlike $\sim_{\rm FB}$):
  $<a^{\dagger}, \lambda, \mu> . \underline{0} \not\sim_{\rm MFB} \underline{0}$ due to the outgoing a-transition with rate μ.
- $\sim_{\rm MFB}$ cannot identify processes with a different past (unlike $\sim_{\rm FB,ps}$):
  $<a^{\dagger}, \lambda, \mu> . \underline{0} \not\sim_{\rm MFB} <b^{\dagger}, \delta, \gamma> . \underline{0}$ whenever $a \neq b$ or $\mu \neq \gamma$.
- $\sim_{\rm MRB}$ is sensitive to the presence of the future (unlike $\sim_{\rm RB}$):
  $<a, \lambda, \mu> . \underline{0} \not\sim_{\rm MRB} \underline{0}$ due to the incoming a-transition with rate μ.
- $\sim_{\rm MRB}$ cannot identify processes with a different future (unlike $\sim_{\rm RB}$):
  $<a, \lambda, \mu> . \underline{0} \not\sim_{\rm MRB} <b, \delta, \gamma> . \underline{0}$ whenever $a \neq b$ or $\mu \neq \gamma$.
- *Extension of the first asymmetry* between forward and reverse bisim.:
  - $\sim_{\rm MFRB} = \sim_{\rm MRB}$ over $\mathbb{P}_{\rm M}$, with $\sim_{\rm MFB}$ strictly coarser.
- A consequence of the time reversibility of the underlying CTMCs.

Compositionality Properties

- $\sim_{\rm MFB}/\sim_{\rm MRB}$ not totally sensitive to the past/future.
- Compositionality violation with respect to $+$ for $\sim_{\rm MFB}$ and $\sim_{\rm MRB}$:
  - $<a, \lambda, \lambda> . \underline{0} \ \sim_{\rm MFRB} \ <a^{\dagger}, \lambda, \lambda> . \underline{0}$.
  - $<a, \lambda, \lambda> . \underline{0} + <c, \kappa_1, \kappa_2> . \underline{0} \ \not\sim_{\rm MFRB} \ <a^{\dagger}, \lambda, \lambda> . \underline{0} + <c, \kappa_1, \kappa_2> . \underline{0}$.
- $\mathbb{P}'_{\rm M} = \mathbb{P}_{\rm M} \setminus \{<a, \lambda, \lambda> . \underline{0} \mid a \in A, \lambda \in \mathbb{R}_{>0}\}$.
- Let $P_1, P_2 \in \mathbb{P}_{\rm M}$ s.t. $P_1 \sim P_2$ and take arbitrary $a \in A$ and $P \in \mathbb{P}_{\rm M}$.
- All the considered bisimilarities are congruences w.r.t. action prefix:
  - $<a, \lambda, \mu> . P_1 \sim <a, \lambda, \mu> . P_2$ provided that *initial*$(P_1) \wedge$ *initial*(P_2).
  - $<a^{\dagger}, \lambda, \mu> . P_1 \sim <a^{\dagger}, \lambda, \mu> . P_2$.
- They also are congruences w.r.t. alternative composition over $\mathbb{P}'_{\rm M}$:
  - $P_1 + P \sim P_2 + P$ and $P + P_1 \sim P + P_2$ with $P_1, P_2, P \in \mathbb{P}'_{\rm M}$ provided that *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.

Equational Characterizations

$(\mathcal{A}_{\mathrm{M},1})$		$(P_1 + P_2) + P_3$	$=$	$P_1 + (P_2 + P_3)$
$(\mathcal{A}_{\mathrm{M},2})$		$P_1 + P_2$	$=$	$P_2 + P_1$
$(\mathcal{A}_{\mathrm{M},3})$		$P + \underline{0}$	$=$	P
$(\mathcal{A}_{\mathrm{M},4})$	$[\sim_{\mathrm{MFB}}]$	$<a, \lambda_1, \mu> . P + <a, \lambda_2, \mu> . P$	$=$	$<a, \lambda_1 + \lambda_2, \mu> . P$ where $initial(P)$
$(\mathcal{A}_{\mathrm{M},5})$	$[\sim_{\mathrm{MFB}}]$	$<a^{\dagger}, \lambda_1, \mu> . P + <a, \lambda_2, \mu> . Q$	$=$	$<a^{\dagger}, \lambda_1 + \lambda_2, \mu> . P$ if $to\_initial(P) = Q$, where $initial(Q)$

- Markovian variants of $\mathcal{A}_4$ to $\mathcal{A}_6$ are not valid for $\sim_{\mathrm{MFB}}$ because it is sensitive to the presence of the past, cannot identify processes with a different past, and views all the transitions as outgoing.
- Markovian variants of $\mathcal{A}_7$ and $\mathcal{A}_8$ are not valid for $\sim_{\mathrm{MRB}}$ because it is sensitive to the presence of the future, cannot identify processes with a different future, and views all the transitions as incoming.
- $\vdash_{4,5}^{1,2,3}$ / $\vdash^{1,2,3}$ sound and complete for $\sim_{\mathrm{MFB}}$ / $\sim_{\mathrm{MRB}}$.

Concluding Remarks and Future Work

- Asymmetries discovered among the considered bisimilarities:
  - In the nondeterministic case $\sim_{\mathrm{FRB}} = \sim_{\mathrm{FB}}$ over initial processes only. In the Markovian case $\sim_{\mathrm{MFRB}} = \sim_{\mathrm{MRB}}$ over all reachable processes.
  - Insensitivity to the presence of the past breaks $\sim_{\mathrm{FB}}$ compositionality. Partial sensitivity breaks the compositionality of $\sim_{\mathrm{MFB}}$ and $\sim_{\mathrm{MRB}}$.
  - Forward bisimilarity needs explicit idempotency axioms. Reverse bisimilarity does not, especially in the nondeterministic case.
  - Any aggregated state resulting from an ordinary lumping is $\sim_{\mathrm{MFB}}$-equivalent to each of the original states it contains. In an exact lumping $\sim_{\mathrm{MRB}}$-equivalence certainly holds only among the original states contained in an aggregated state.
- Modal/temporal logic characterizations?
- Properties of weak variants of these bisimilarities?
- What changes when admitting irreversible actions?