Quantitative Weakest Pre vs. Strongest Post

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Programming Principles, Logic
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Open Problems in Concurrency Theory

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Roadmap

1. Revisiting Weakest Preconditions
2. Quantitative Weakest Pre
3. Quantitative Strongest Post
4. On Liberal vs. Non-liberal predicate/quantity transformers
   - Even Strongest Liberal Postconditions
     (the ones Dijkstra & Scholten forgot (or didn't care) about)
5. Thoughts on Hoare logic and Incorrectness Logic

Weakest Preconditions

— revisited —

Weakest Preconditions – A Coarse Weather Forecast

Objective: Forecast tomorrow's weather in Bertinoro

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- **Predicates ($F, G, \ldots$):** *Mappings* φ: States $\to \{\text{true}, \text{false}\}$ describing sets of states

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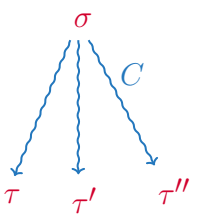

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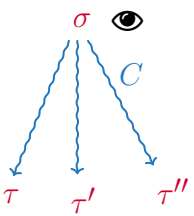

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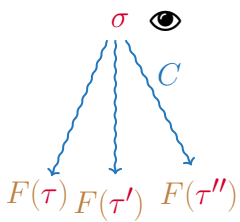

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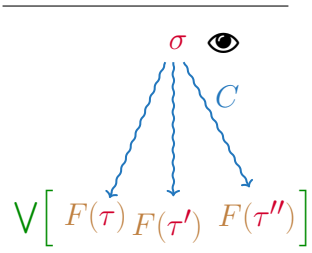

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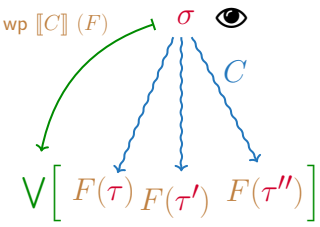

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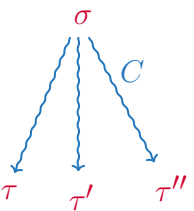

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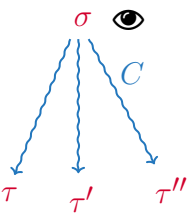

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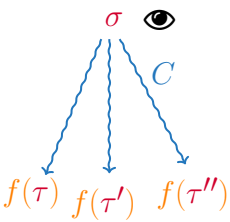

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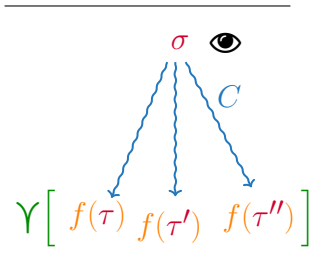

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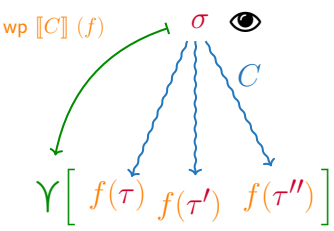

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— our main contribution —

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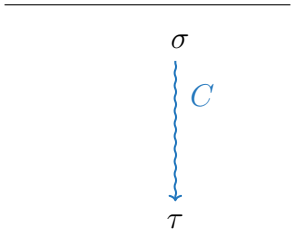

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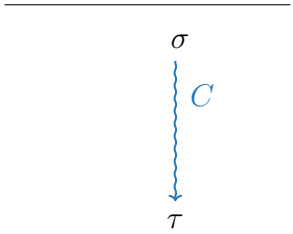

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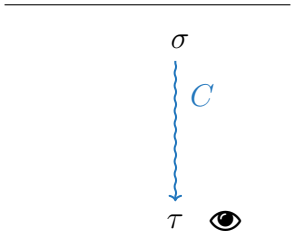

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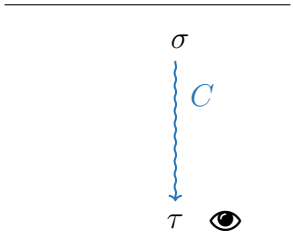

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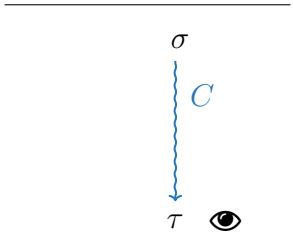

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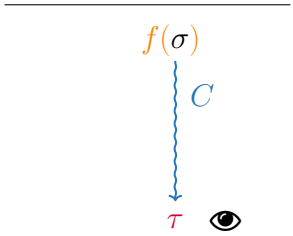

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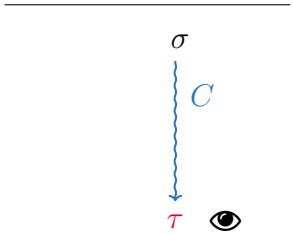

Quantitative Strongest Post – A Fine-grained Weather *Back*cast [OOPSLA'22]

Question: What *was yesterday's* temperature in Bertinoro?

More Formally:

- **Given:** Program $[\![C]\!]$: States $\to$ States
  describing how the atmosphere evolves within 24h
- **Given:** Quantity f: States $\to \mathbb{R} \cup \{-\infty, +\infty\}$
  mapping atmosphere state
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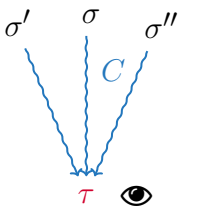

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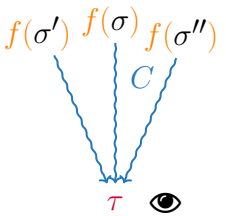

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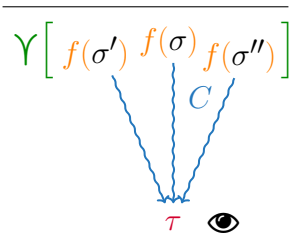

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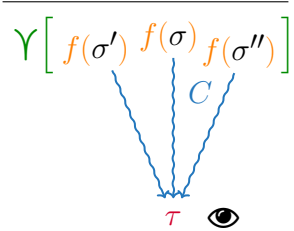
$\mathsf{sp}\,[\![C]\!]\,(f) : \mathsf{Quant} \to \mathsf{Quant}$



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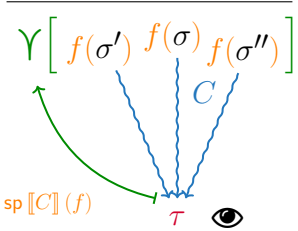

Quantitative Strongest Post Example

$$x := x + 1$$

Quantitative Strongest Post Example

$/\!/\!/ \ x$

$x := x + 1$

Quantitative Strongest Post Example

$$
\begin{array}{l}
/\!/\!/\ \boldsymbol{x} \\
x := x + 1 \\
/\!/\!/\ \partial v\colon\ [x = (x+1)\,[x/v]] \between x\,[x/v]
\end{array}
$$

Quantitative Strongest Post Example

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\begin{array}{l}
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/\!/\!/\ \partial v\colon\ [x = (x+1)\,[x/v]] \between x\,[x/v] \\
/\!/\!/\ \partial v\colon\ [x = v + 1] \between v
\end{array}
$$

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/\!/\!/\ \boldsymbol{x - 1}
\end{array}
$$

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/\!/\!/\ \boldsymbol{x - 1} \\
x := x - y
\end{array}
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/\!/\!/\ \boldsymbol{x + y - 1}
\end{array}
$$

Quantitative Strongest Post and the Flow of Quantitative Information

$$
\begin{array}{l}
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x := x + 1\,; \\
/\!/\!/\ \partial v\colon\ [x = (x+1)\,[x/v]] \between x\,[x/v] \\
/\!/\!/\ \partial v\colon\ [x = v + 1] \between v \\
/\!/\!/\ \boldsymbol{x - 1} \\
x := x - y \\
/\!/\!/\ \boldsymbol{x + y - 1}
\end{array}
$$

Quantitative Strongest Post and the Flow of Quantitative Information

	Concrete State	**Flowing Quantity**
$/\!/\!/\ \boldsymbol{x}$		
$x := x + 1\,;$		
$/\!/\!/\ \partial v\colon [x = (x+1)[x/v]] \curlywedge x[x/v]$		
$/\!/\!/\ \partial v\colon [x = v + 1] \curlywedge v$		
$/\!/\!/\ \boldsymbol{x - 1}$		
$x := x - y$		
$/\!/\!/\ \boldsymbol{x + y - 1}$		

Quantitative Strongest Post and the Flow of Quantitative Information

Concrete State	Flowing Quantity
$x\colon 16 \quad y\colon 3$	

$$
\begin{array}{l}
/\!/\!/\ \boldsymbol{x} \\
x := x + 1\,; \\
/\!/\!/\ \partial v\colon\ [x = (x+1)\,[x/v]] \curlywedge x\,[x/v] \\
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/\!/\!/\ \boldsymbol{x + y - 1}
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$$

Quantitative Strongest Post and the Flow of Quantitative Information

	Concrete State	Flowing Quantity
$/\!/\!/\ \boldsymbol{x}$	$x\colon 16 \quad y\colon 3$	$\mathbf{16}$
$x := x + 1\,;$		
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$/\!/\!/\ \partial v\colon\ [x = v + 1] \between v$		
$/\!/\!/\ \boldsymbol{x - 1}$		
$x := x - y$		
$/\!/\!/\ \boldsymbol{x + y - 1}$		

Quantitative Strongest Post and the Flow of Quantitative Information

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$x := x + 1\,;$		
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$/\!/\!/\ \partial v\colon\ [x = v + 1] \lightning v$		
$/\!/\!/\ \boldsymbol{x - 1}$	$x\colon 17 \quad y\colon 3$	
$x := x - y$		
$/\!/\!/\ \boldsymbol{x + y - 1}$		

Quantitative Strongest Post and the Flow of Quantitative Information

	Concrete State	Flowing Quantity
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$/\!/\!/\ \partial v\colon [x = v + 1] \lightning v$		
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$/\!/\!/\ \boldsymbol{x - 1}$	$x\colon 17 \quad y\colon 3$	**16**
$x := x - y$		
$/\!/\!/\ \boldsymbol{x + y - 1}$	$x\colon 14 \quad y\colon 3$	

Quantitative Strongest Post and the Flow of Quantitative Information

	Concrete State	Flowing Quantity
$/\!/\!/\ \boldsymbol{x}$	$x\colon 16 \quad y\colon 3$	**16**
$x := x + 1\,;$		
$/\!/\!/\ \partial v\colon\ [x = (x+1)\,[x/v]] \between x\,[x/v]$		
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$/\!/\!/\ \boldsymbol{x - 1}$	$x\colon 17 \quad y\colon 3$	**16**
$x := x - y$		
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Liberal vs. Non-liberal

Quantitative Weakest Pre

[Dijkstra'75, Kozen'83, McIver&Morgan'05, OOPSLA'22]

- **Given:** Program C and (post)quantity f: States $\to \mathbb{R} \cup \{-\infty, +\infty\}$
- **Question:** What is the anticipated value of f after executing C?

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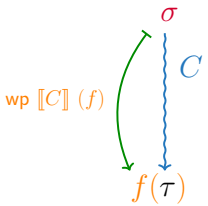

- **Weakest pre:** Mapping from initial state σ to anticipated value of f evaluated in final state reached after execution of C on σ.

Quantitative Weakest Pre

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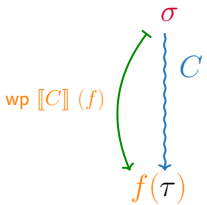

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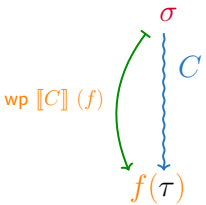

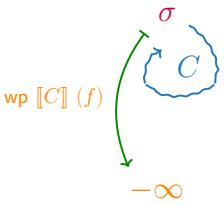

- **Weakest pre:** Mapping from initial state σ to anticipated value of f evaluated in final state reached after execution of C on σ.
  - Assign $-\infty$ to wp $[\![C]\!]$ (f) (σ) if C does not terminate on σ.

Quantitative Weakest Liberal Pre [Dijkstra'75, Kozen'83, McIver&Morgan'05, OOPSLA'22]

- **Given:** Program C and (post)quantity f: States $\to \mathbb{R} \cup \{-\infty, +\infty\}$
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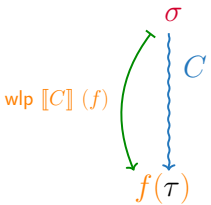

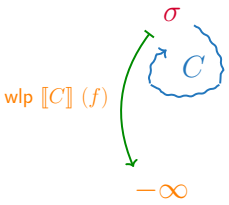

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  - Assign $-\infty$ to wlp $[\![C]\!]\ (f)\,(\sigma)$ if C does not terminate on σ.

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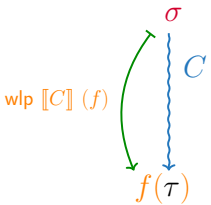

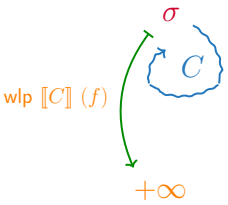

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Quantitative Strongest Post

[Dijkstra'75, ~~Kozen'83, McIver&Morgan'05,~~ OOPSLA'22]

- **Given:** Program C and (pre)quantity $f : \text{States} \to \mathbb{R} \cup \{-\infty, +\infty\}$
- **Question:** What is the retrocipated value of f before executing C?

Quantitative Strongest Post

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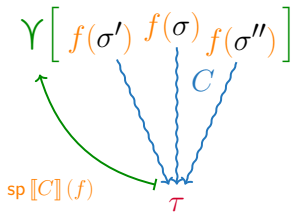

- **Strongest post:** Mapping from final state τ to maximal retrocipated value of f evaluated in initial state(s) that can reach τ by executing C.

Quantitative Strongest Post

[Dijkstra'75, ~~Kozen'83, McIver&Morgan'05,~~ OOPSLA'22]

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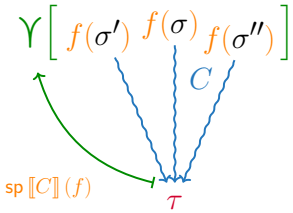

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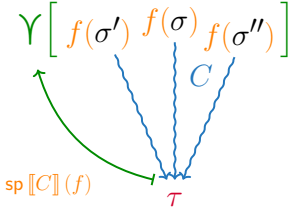

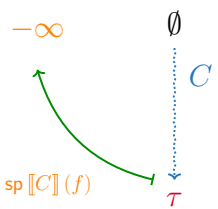

- **Strongest post:** Mapping from final state τ to maximal retrocipated value of f evaluated in initial state(s) that can reach τ by executing C.
  - Assign $-\infty$ to $\text{sp}\,[\![C]\!]\,(f)\,(\tau)$ if τ is unreachable by executing C.

Quantitative Strongest Liberal Post [~~Dijkstra'75, Kozen'83, McIver&Morgan'05,~~ OOPSLA'22]

- **Given:** Program C and (pre)quantity f: States $\to \mathbb{R} \cup \{-\infty, +\infty\}$
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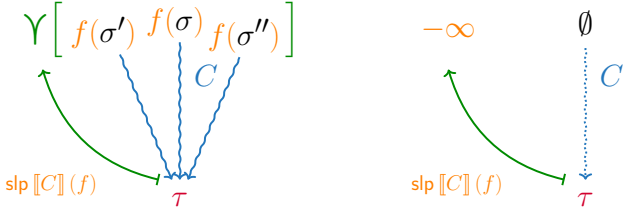

- **Strongest liberal post:** Mapping from final state τ to min. retrocip. value of f evaluated in initial state(s) that can reach τ by executing C.
  - Assign $-\infty$ to slp $[\![C]\!]\,(f)\,(\tau)$ if τ is unreachable by executing C.

Quantitative Strongest Liberal Post [~~Dijkstra'75, Kozen'83, McIver&Morgan'05,~~ OOPSLA'22]

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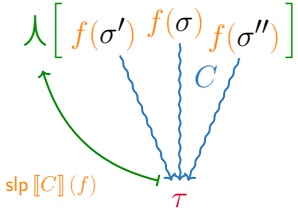

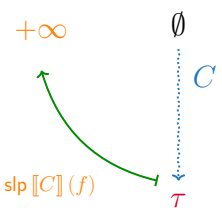

- **Strongest liberal post:** Mapping from final state τ to min. retrocip. value of f evaluated in initial state(s) that can reach τ by executing C.
  - Assign $+\infty$ to slp $[\![C]\!](f)(\tau)$ if τ is unreachable by executing C.

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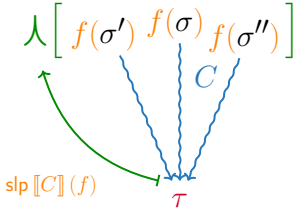

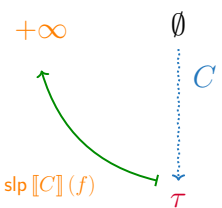

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Quantitative Strongest (Liberal) Post Examples

Reversible Assignment:

sp:

$$x := x + 1$$

slp:

$$x := x + 1$$

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$/\!/\!/\ \boldsymbol{x}$

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Quantitative Strongest (Liberal) Post Examples

Reversible Assignment:

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$x := x + 1$

$/\!/\!/\ \exists v\colon [x = v + 1] \cdot v$

slp:

$/\!/\!/\ x$

$x := x + 1$

Quantitative Strongest (Liberal) Post Examples

Reversible Assignment:

sp:

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$x := x + 1$
$/\!/\!/\ \wr v\colon [x = v + 1] \curlywedge v$
$/\!/\!/\ x - 1$

slp:

$/\!/\!/\ x$
$x := x + 1$

Quantitative Strongest (Liberal) Post Examples

Reversible Assignment:

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$$
\begin{array}{l}
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x := x + 1 \\
/\!/\!/\ \exists v\colon [x = v + 1] \curlywedge v \\
/\!/\!/\ x - 1
\end{array}
$$

slp:

$$
\begin{array}{l}
/\!/\!/\ x \\
x := x + 1 \\
/\!/\!/\ \forall v\colon [x \neq v + 1] \curlyvee v
\end{array}
$$

Quantitative Strongest (Liberal) Post Examples

Reversible Assignment:

sp:

$$
\begin{array}{l}
/\!/\!/\ x \\
x := x + 1 \\
/\!/\!/\ \mathfrak{Q} v \colon [x = v + 1] \curlywedge v \\
/\!/\!/\ x - 1
\end{array}
$$

slp:

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\begin{array}{l}
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x := x + 1 \\
/\!/\!/\ \mathfrak{L} v \colon [x \neq v + 1] \curlyvee v \\
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\end{array}
$$

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/\!/\!/\ \exists v\colon [x = v + 1] \curlywedge v \\
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Irreversible Assignment:

sp:

$$x := 2$$

slp:

$$x := 2$$

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/\!/\!/\ x - 1
\end{array}
$$

slp:

$$
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$/\!/\!/\ x - 1$

slp:

$/\!/\!/\ x$
$x := x + 1$
$/\!/\!/\ \forall v\colon [x \neq v + 1] \curlyvee v$
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Irreversible Assignment:

sp:

$/\!/\!/\ x$
$x := 2$
$/\!/\!/\ \exists v\colon [x = 2] \curlywedge v$
$/\!/\!/\ [x = 2]$

slp:

$/\!/\!/\ x$
$x := 2$

Quantitative Strongest (Liberal) Post Examples

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$$
\begin{array}{l}
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x := x + 1 \\
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/\!/\!/\ [x = 2]
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$$

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$$
\begin{array}{l}
/\!/\!/\ x \\
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/\!/\!/\ \exists v\colon [x = 2] \curlywedge v \\
/\!/\!/\ [x = 2]
\end{array}
$$

slp:

$$
\begin{array}{l}
/\!/\!/\ x \\
x := 2 \\
/\!/\!/\ \forall v\colon [x \neq 2] \curlyvee v \\
/\!/\!/\ [x \neq 2]
\end{array}
$$

Quantitative Strongest Post Loop Example

$$hi := hi + 5$$

$$\texttt{while}\,(\,lo < hi\,)\,\{$$
$$\qquad lo := lo + 1 \quad \}$$

Quantitative Strongest Post Loop Example

$$
\begin{array}{l}
/\!/\!/\ \boldsymbol{hi} \\
hi \texttt{ := } hi + 5 \\
\\
\texttt{while}\,(\, lo < hi \,)\,\{ \\
\qquad lo \texttt{ := } lo + 1 \quad \}
\end{array}
$$

Quantitative Strongest Post Loop Example

$$
\begin{array}{l}
/\!/\!/\ \boldsymbol{hi} \\
hi := hi + 5 \\
/\!/\!/\ \boldsymbol{hi - 5} \\
\texttt{while}\,(\, lo < hi \,)\,\{ \\
\qquad lo := lo + 1 \quad \}
\end{array}
$$

Quantitative Strongest Post Loop Example

$$
\begin{array}{l}
/\!/\!/\ \boldsymbol{hi} \\
hi := hi + 5 \\
/\!/\!/\ \boldsymbol{hi - 5} \\
\texttt{while}\,(\, lo < hi \,)\,\{ \\
\qquad lo := lo + 1 \quad \} \\
/\!/\!/\ \boldsymbol{[lo \geq hi] \curlywedge (hi - 5)}
\end{array}
$$

Weakest Pre vs. Strongest Post

	Does what?	liberal vs. non-liberal	is about

Weakest Pre vs. Strongest Post

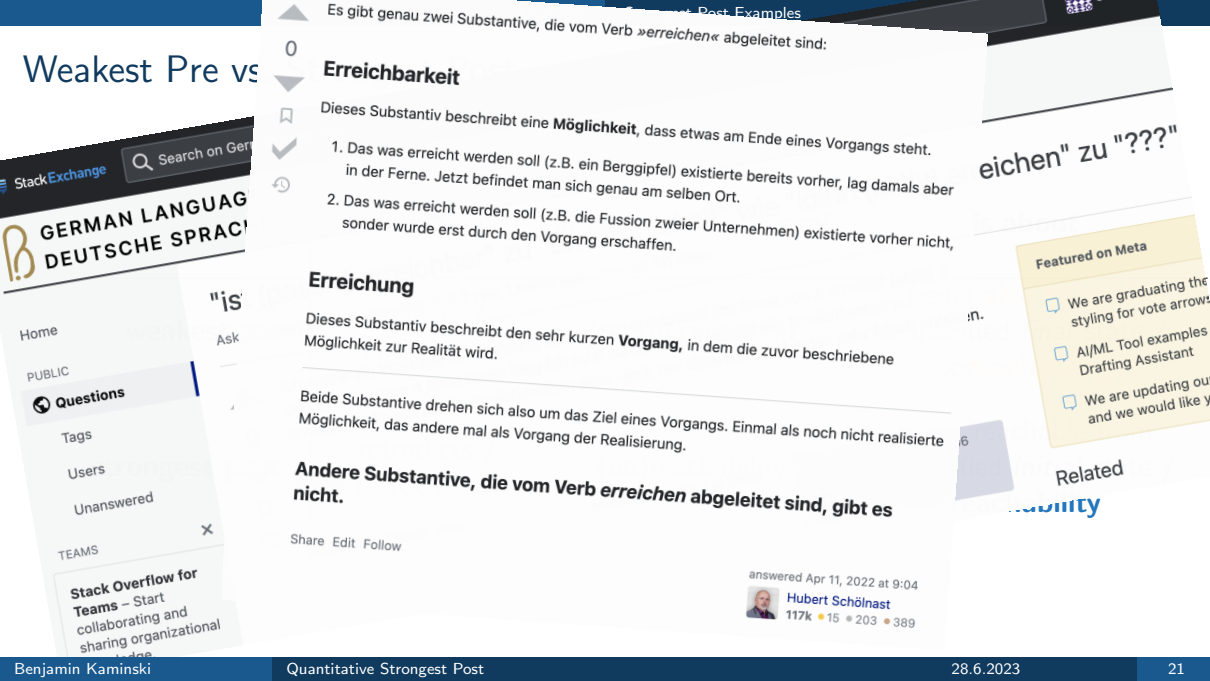
	Does what?	**liberal vs. non-liberal**	**is about**
weakest pre	predicts / anticipates	(non)termination	being able to reach a specified *final* state / **coreachability**

Weakest Pre vs. Strongest Post

	Does what?	**liberal vs. non-liberal**	**is about**
weakest pre	predicts / anticipates	(non)termination	being able to reach a specified *final* state / **coreachability**
strongest post	retrodicts / retrocipates	(un)reachability	being reachable from a specified *initial* state / **reachability**

Weakest Pre vs. Strongest Post

Weakest Pre vs



Galois Connections between Weakest Pre and Strongest Post

Dijkstra's classical predicate transformers:

$$G \implies \mathsf{wlp}\,[\![C]\!]\,(F) \qquad \text{iff} \qquad \mathsf{sp}\,[\![C]\!]\,(G) \implies F$$

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Our quantity transformers:

$$g \preceq \mathsf{wlp}\,[\![C]\!]\,(f) \qquad \text{iff} \qquad \mathsf{sp}\,[\![C]\!]\,(g) \preceq f$$

Galois Connections between Weakest Pre and Strongest Post

Dijkstra's classical predicate transformers:

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Our quantity transformers:

$$g \preceq \mathsf{wlp}\,[\![C]\!]\,(f) \qquad \text{iff} \qquad \mathsf{sp}\,[\![C]\!]\,(g) \preceq f$$

$$\mathsf{wp}\,[\![C]\!]\,(g) \preceq f \qquad \text{iff} \qquad g \preceq \mathsf{slp}\,[\![C]\!]\,(f)$$

Thoughts on Hoare Logic and Incorrectness Logic

Total vs. Partial Correctness Hoare Logic

Total vs. Partial Correctness Hoare Logic

Total Correctness Hoare Triples:

$$\langle\, \varphi \,\rangle\ C\ \langle\, \psi \,\rangle \text{ valid for total correctness} \qquad \text{iff} \qquad \varphi \implies \text{wp } [\![C]\!]\ (\psi)$$

Total vs. Partial Correctness Hoare Logic

Total Correctness Hoare Triples:

$$\langle\, \varphi \,\rangle\ C\ \langle\, \psi \,\rangle \text{ valid for total correctness} \qquad \text{iff} \qquad \varphi \implies \text{wp } [\![C]\!]\ (\psi)$$

Partial Correctness:

$$\langle\, \varphi \,\rangle\ C\ \langle\, \psi \,\rangle \text{ valid for partial correctness} \qquad \text{iff} \qquad \varphi \implies \text{wlp } [\![C]\!]\ (\psi)$$

Incorrectness Logic [O'Hearn'19]

Incorrectness Triples:

$$[\varphi]\ C\ [\psi] \text{ valid for incorrectness} \quad \text{iff} \quad \psi \implies \mathsf{sp}\,[\![C]\!]\,(\varphi)$$

Incorrectness Logic [O'Hearn'19]

Incorrectness Triples:

$$[\varphi]\ C\ [\psi] \text{ valid for incorrectness} \quad \text{iff} \quad \psi \implies \mathsf{sp}\,[\![C]\!]\,(\varphi)$$

Partial/Liberal/??? Incorrectness Triples:

$$[\varphi]\ C\ [\psi] \text{ valid for partial incorrectness} \quad \text{iff} \quad \psi \implies \mathsf{slp}\,[\![C]\!]\,(\varphi)$$

Claim: There are Precisely 6 Fundamentally Different Hoare Logics

	implication		**defines**
G	$\Longrightarrow$	$\mathsf{wp}\,[\![C]\!]\,(F)$	total correctness
G	$\Longrightarrow$	$\mathsf{wlp}[\![C]\!]\,(F)$	partial correctness
$\mathsf{wp}\,[\![C]\!]\,(F)$	$\Longrightarrow$	G	partial incorrectness
$\mathsf{wlp}[\![C]\!]\,(F)$	$\Longrightarrow$	G	???
F	$\Longrightarrow$	$\mathsf{sp}\,[\![C]\!]\,(G)$	(total) incorrectness
F	$\Longrightarrow$	$\mathsf{slp}[\![C]\!]\,(G)$	partial incorrectness
$\mathsf{sp}\,[\![C]\!]\,(G)$	$\Longrightarrow$	F	partial correctness
$\mathsf{slp}[\![C]\!]\,(G)$	$\Longrightarrow$	F	¿¿¿

Backup Slides

Rules for Strongest (Liberal) Post

C	$\mathbf{sp}\,[\![C]\!]\,(f)$	$\mathbf{slp}\,[\![C]\!]\,(f)$
`diverge`	$-\infty$	$+\infty$
$x := e$	$\exists \alpha\colon [x = e\,[x/\alpha]] \ \wedge\ f\,[x/\alpha]$	$\forall \alpha\colon [x \neq e\,[x/\alpha]] \ \vee\ f\,[x/\alpha]$
$C_1 \fatsemi C_2$	$\mathbf{sp}\,[\![C_2]\!]\,(\mathbf{sp}\,[\![C_1]\!]\,(f))$	$\mathbf{slp}[\![C_2]\!]\,(\mathbf{slp}[\![C_1]\!]\,(f))$
$\{\,C_1\,\} \,\square\, \{\,C_2\,\}$	$\mathbf{sp}\,[\![C_1]\!]\,(f) \ \vee\ \mathbf{sp}\,[\![C_2]\!]\,(f)$	$\mathbf{slp}[\![C_1]\!]\,(f) \ \wedge\ \mathbf{slp}[\![C_2]\!]\,(f)$
`if (`φ`) {`C_1`} else {`C_2`}`	$\mathbf{sp}\,[\![C_1]\!]\,([\varphi] \wedge f) \ \vee\ \mathbf{sp}\,[\![C_2]\!]\,([\neg\varphi] \wedge f)$	$\mathbf{slp}[\![C_1]\!]\,([\neg\varphi] \vee f) \ \wedge\ \mathbf{slp}[\![C_2]\!]\,([\varphi] \vee f)$
`while (`φ`) {`C'`}`	$[\neg\varphi] \wedge (\mathbf{lfp}\ Y.\ f \vee \mathbf{sp}\,[\![C']\!]\,([\varphi] \wedge Y))$	$[\varphi] \vee (\mathbf{gfp}\ Y.\ f \ \wedge\ \mathbf{slp}[\![C']\!]\,([\neg\varphi] \vee Y))$