On the challenges of Quantitative Reasoning

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based on joint work with

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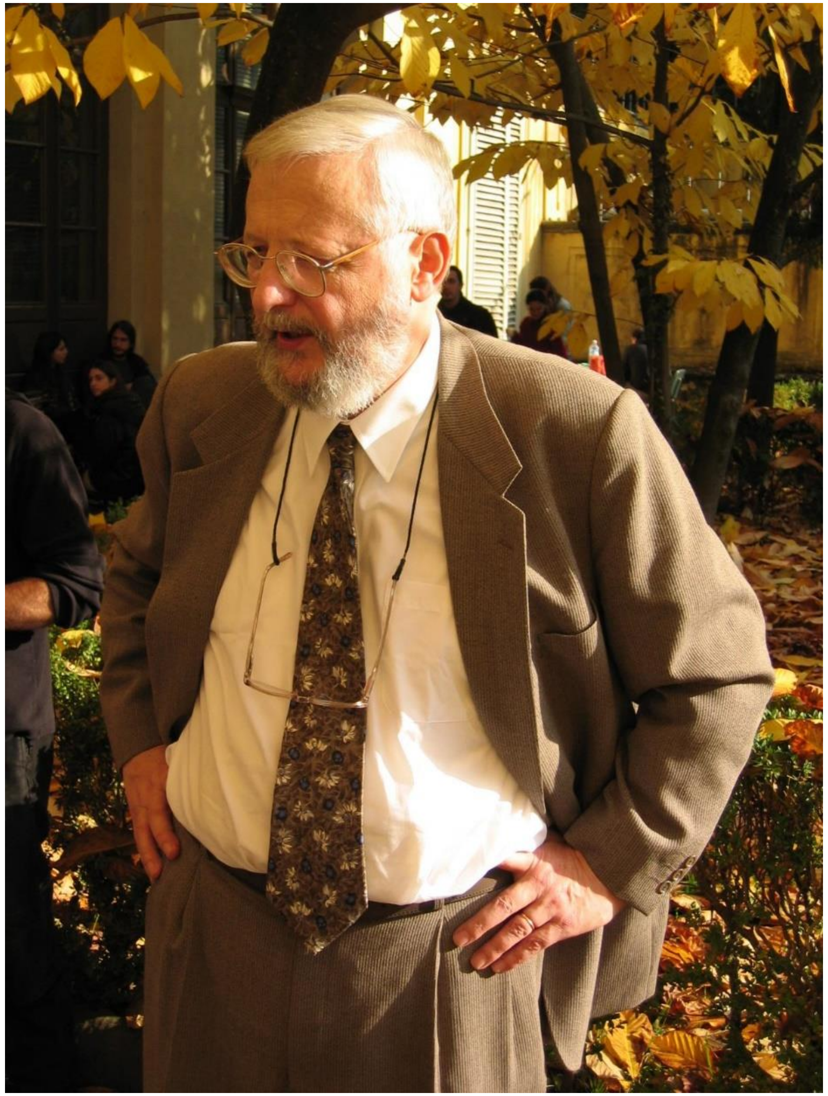
Dedicated to the memory of **F. William Lawvere**



A visionary genius and inspirational figure and mentor.

Major challenge in Computer Science

Approximation theory

exact behaviour ⟹ approximated behaviour

deciding bisimulations ⟹ measuring similarities

solving expensive problems ⟹ approximating solutions

proving properties ⟹ predicting "most probable" properties

working with complex systems (exploiting compositionality) ⟹ working with simple approximations (exploiting compositionality)

Quantitative Equational Reasoning [Mardare, Plotkin, Panangaden 2016]

A quantitative theory of computational algebraic effects

Classic case

SOS induces an algebraic theory of bisimulation

$$\frac{s = s' \qquad t = t'}{s|t = s'|t'}$$

Quantitative case

QA is the algebraic theory of a behavioural metric

$$\frac{s =_\varepsilon s' \qquad t =_\delta t'}{s|t =_{\varepsilon+\delta} s'|t'}$$

Quantitative Equational Reasoning

Classic case	Quantitative case
$s = t$ means $s \sim t$	$s =_\varepsilon t$ means $d(s,t) \leq \varepsilon$
(Refl) $\vdash s = s$	(Refl) $\vdash s =_\varepsilon s$
(Symm) $s = t \vdash t = s$	(Symm) $s =_\varepsilon t \vdash t =_\varepsilon s$

Quantitative Equational Reasoning

Classic case

$s = t$ means $s \sim t$

(Trans) $s = t, t = u \vdash s = u$

Quantitative case

$s =_\varepsilon t$ means $d(s,t) \leq \varepsilon$

(Triang) $s =_\varepsilon t, t =_\delta u \vdash s =_{\varepsilon+\delta} u$

(Continuity): for fixed $\varepsilon \geq 0$

$\{ s =_\delta t \mid \delta > \varepsilon \} \vdash s =_\varepsilon t$

Quantitative Algebra

Given a signature Ω, $\mathcal{A} = (A, \Omega, d)$

- (A, Ω) is an Ω-algebra
- (A, d) is a metric space
- all functions in Ω are non-expansive

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Quantitative algebras are models for quantitative equational theories

$\mathcal{A} \models s =_\varepsilon t$ iff for any interpretation σ of s and t in $\mathcal{A}$,

$$d(\sigma(a), \sigma(b)) \leq \varepsilon$$

Approximation theory

- it is meant to avoid infinitary reasoning: We might need an infinitary proof for

$$\vdash s =_r t$$

but hopefully we can have a finitary proof for showing that

$$\forall \varepsilon > 0 , \quad \vdash s =_{r+\varepsilon} t$$

Approximated completeness

Approximation completeness fails for general Quantitative Theories

Example: $\Omega = \{a:0,\ b:0,\ f:1\}$, $X = \{x, y\}$

$$\Pi : \left| \begin{array}{l} \vdash a =_2 b \\ x =_{1+\varepsilon} y \vdash x =_{1+\frac{\varepsilon}{2}} y \\ x =_1 y \vdash f(x) =_0 f(y) \end{array} \right.$$

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Example: $\Omega = \{a:0, b:0, f:1\}$, $X = \{x, y\}$

$$\mathbb{T}: \left|\begin{array}{l} \vdash a =_2 b \\ x =_{1+\varepsilon} y \vdash x =_{1+\frac{\varepsilon}{2}} y \\ x =_1 y \vdash f(x) =_0 f(y) \end{array}\right.$$

- In all the models of $\mathbb{T}$

$$\models f(a) =_0 f(b)$$

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- In all the models of $\mathbb{T}$

$$\models f(a) =_0 f(b)$$

- Using infinitary proofs one gets

$$\vdash f(a) =_0 f(b)$$

- Using finitary proofs it is not possible to prove that

$$\forall \varepsilon > 0, \quad \vdash f(a) =_\varepsilon f(b)$$

Approximation completeness fails for general Quantitative Theories

It seems that we need to look deeper into the structures of quantitative equations!

$$s_1 =_{\varepsilon_1} t_1, \dots, s_n =_{\varepsilon_n} t_n \vdash s =_{\varepsilon} t \qquad \varepsilon_1, \dots, \varepsilon_n, \varepsilon \in \mathbb{R}_+$$

$$\Downarrow$$

$$s_1 =_{\varepsilon_1} t_1, \dots, s_n =_{\varepsilon_n} t_n \vdash s =_{f(\varepsilon_1, \dots, \varepsilon_n)} t \qquad f : \mathbb{R}_+^n \to \mathbb{R}$$

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Or

$$\vdash s =_{\varepsilon} t \implies [s = t] \in \overline{\mathbb{R}}_+ \; \& \; [s = t] \leq \varepsilon$$

A brilliant paper by Lawvere

Metric spaces, generalized logics and closed categories [1973]

- Metric spaces (suitably generalized) are categories enriched over $[0, +\infty]$

- Lawvere quantale $[0, +\infty]$

$+$ - the monoidal product

$\wedge$ - minimum

$\vee$ - maximum

- reversed order $\begin{cases} \top & \text{is } 0 \\ \bot & \text{is } \infty \end{cases}$

$$r \dot{-} s = \begin{cases} 0 & r \leq s \\ r - s & \text{otherwise} \end{cases}$$

Lawvere Logics

$$\top \mid \bot \mid p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \otimes \psi \mid \varphi \multimap \psi \mid \mathbb{1} \mid r\varphi$$

$p \in \mathbb{P}$ - atomic propositions, $\quad r \in [0,\infty)$

<u>Semantics</u> $\quad m : \mathbb{P} \to [0,\infty]$

$m(\bot) := \infty$

$m(\top) := 0$

$m(\mathbb{1}) := \mathbb{1}$

$m(\varphi \otimes \psi) = m(\varphi) + m(\psi)$

$m(\varphi \multimap \psi) = m(\psi) \dot{-} m(\varphi)$

$m(r\varphi) = r \cdot m(\varphi)$

$m(\varphi \wedge \psi) = \max\{m(\varphi), m(\psi)\}$

$m(\varphi \vee \psi) = \min\{m(\varphi), m(\psi)\}$

Judgements of Lawvere Logics

$$\top \mid \bot \mid p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \otimes \psi \mid \varphi \multimap \psi \mid \mathbb{1} \mid r\varphi$$

$$\varphi_1, \ldots, \varphi_n \vdash \psi$$

Semantics:

$\varphi_1, \ldots, \varphi_n \models_m \psi$ iff $m(\varphi_1) + \ldots + m(\varphi_n) \geq m(\psi)$

So, instead of $s =_\varepsilon t$ we write

$\varepsilon * \mathbb{1} \vdash s = t$ with $s = t \in \mathbb{P}$

Lawvere Logics

$$\top \mid \bot \mid p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \otimes \psi \mid \varphi \multimap \psi \mid \mathbb{I} \mid r\varphi$$

derived operators:

$$\neg\varphi := \varphi \multimap \bot$$

$$m(\neg\varphi) = \infty \mathbin{\dot{-}} m(\varphi)$$

$$\varphi \multimapboth \psi := (\varphi \multimap \psi) \wedge (\psi \multimap \varphi)$$

$$m(\varphi \multimapboth \psi) = \max\{m(\varphi) \mathbin{\dot{-}} m(\psi),\ m(\psi) \mathbin{\dot{-}} m(\varphi)\}$$

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$\varphi \models_m \psi$ iff $m(\varphi) \geqslant m(\psi)$

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$\varphi \models_m \psi$ iff $m(\varphi) \geq m(\psi)$

$\models_m \varphi$ iff $\top \models_m \varphi$ iff $0 \geq m(\varphi)$ iff $m(\varphi) = 0$

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$\models_m \varphi$ iff $\top \models_m \varphi$ iff $0 \geqslant m(\varphi)$ iff $m(\varphi) = 0$

$\varphi \models_m \bot$ iff $m(\varphi) \geqslant \infty$ iff $m(\varphi) = \infty$

Judgements of Lawvere Logics

$\varphi \vDash_m \psi$ iff $m(\varphi) \geqslant m(\psi)$

$\vDash_m \varphi$ iff $\top \vDash_m \varphi$ iff $0 \geqslant m(\varphi)$ iff $m(\varphi) = 0$

$\varphi \vDash_m \bot$ iff $m(\varphi) \geqslant \infty$ iff $m(\varphi) = \infty$

$\vDash_m \varphi \multimap \psi$ iff $m(\varphi \multimap \psi) \leqslant 0$ iff $m(\psi) \dot{-} m(\varphi) = 0$ iff $m(\varphi) \geqslant m(\psi)$

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$\vDash_m \varphi \multimap \psi$ iff $m(\varphi \multimap \psi) \leqslant 0$ iff $m(\psi) \dot{-} m(\varphi) = 0$ iff $m(\varphi) \geqslant m(\psi)$

Hence,

$\varphi \vdash \psi$ iff $\vdash \varphi \multimap \psi$

$\varphi \vdash \top$

$\bot \vdash \varphi$

Judgements of Lawvere Logics

$\models_m \neg\varphi$ iff $m(\varphi \multimap \bot) \leq 0$ iff $m(\varphi) = \infty$

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$$\vdash \neg\varphi \vee \neg\neg\varphi$$

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Hence,

$$\not\vdash \neg\varphi \vee \neg\neg\varphi$$

$\models_m \varphi \vee \neg\varphi$ iff $\left.\begin{array}{l} \text{either } m(\varphi) = 0 \\ \text{or } m(\varphi) = \infty \end{array}\right\}$ φ is Boolean

Proof Rules of Lawvere Logics

(Id): $$\dfrac{}{\varphi \vdash \varphi}$$

(Cut) $$\dfrac{\Gamma \vdash \varphi \qquad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi}$$

(Weak): $$\dfrac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi}$$

(Perm) $$\dfrac{\Gamma, \varphi, \psi \vdash \theta}{\Gamma, \psi, \varphi \vdash \theta}$$

But we do not have

$$\dfrac{\Gamma, \varphi, \varphi \vdash \theta}{\Gamma, \varphi \vdash \theta}$$

Proof Rules of Lawvere Logics

$$\frac{\Gamma, \varphi, \psi \vdash \theta}{\Gamma, \varphi \otimes \psi \vdash \theta} \qquad \frac{\Gamma, \varphi \otimes \psi \vdash \theta}{\Gamma, \varphi, \psi \vdash \theta}$$

$$\frac{\Gamma, \varphi \otimes \psi \vdash \theta}{\Gamma, \varphi \vdash \psi \multimap \theta} \qquad \frac{\Gamma, \varphi \vdash \psi \multimap \theta}{\Gamma, \varphi \otimes \psi \vdash \theta}$$

$$\frac{\vdash \mathbb{1} \vee \neg \mathbb{1}}{\vdash \bot} \qquad \frac{\varphi \vdash \psi}{r\varphi \vdash r\psi} \qquad \frac{r\varphi \vdash r\psi}{\varphi \vdash \psi}\ r > 0$$

Proof Rules of Lawvere Logics

$$\frac{\Gamma, \theta \vdash \varphi \otimes \psi \qquad \vdash \neg\neg\varphi}{\Gamma, \varphi \multimap \theta \vdash \psi}$$

$$\frac{\Gamma, \varphi \wedge \psi \vdash \theta \qquad \varphi \vdash \psi}{\Gamma, \varphi \vdash \theta}$$

Supplementary Judgements of Lawvere Logics

Either $\models_m \neg\varphi$ or $\models_m \neg\neg\varphi$

Either $\varphi \models_m \psi$ or $\psi \models_m \varphi$

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Totality Lemmas

I

$$\frac{S, \vdash \neg\varphi}{\vdash \theta} \qquad \frac{S, \vdash \neg\neg\varphi}{\vdash \theta} \quad \Rightarrow \quad \frac{S}{\vdash \theta}$$

II

$$\frac{S, \varphi \vdash \psi}{\vdash \theta} \qquad \frac{S, \psi \vdash \varphi}{\vdash \theta} \quad \Rightarrow \quad \frac{S}{\vdash \theta}$$

Normal Forms

$$\varphi_1,\dots,\varphi_n \vdash \psi \quad \longrightarrow \quad T_1 \left[\begin{array}{c} \dots \\ a^1_1 p_1 \otimes \dots \otimes a^1_n p_n \otimes a^1 \mathbb{1} \vdash b^1_1 q_1 \otimes \dots \otimes b^1_m q_m \otimes b^1 \mathbb{1} \\ \dots \end{array}\right.$$

$$\varphi_1,\dots,\varphi_n \vdash \psi \quad \longrightarrow \quad T_2 \left[\begin{array}{c} \dots \\ a^2_1 p_1 \otimes \dots \otimes a^2_n p_n \otimes a^2 \mathbb{1} \vdash b^2_1 q_1 \otimes \dots \otimes b^2_m q_m \otimes b^2 \mathbb{1} \\ \dots \end{array}\right.$$

$$\dots\dots\dots$$

$$\varphi_1,\dots,\varphi_n \vdash \psi \quad \longrightarrow \quad T_k \left[\begin{array}{c} \dots \\ a^k_1 p_1 \otimes \dots \otimes a^k_n p_n \otimes a^k \mathbb{1} \vdash b^k_1 q_1 \otimes \dots \otimes b^k_m q_m \otimes b^k \mathbb{1} \\ \dots \end{array}\right.$$

For $p_i, q_j \in \mathbb{P}$ and $a^i_j, a^i, b^i_j, b_i \in [0,\infty)$

Completeness and Incompleteness of Lawvere Logics

Incompleteness Theorem Lawvere logics are incomplete:

We have $\mathcal{M}\mathcal{G}(S) \subseteq \mathcal{M}\mathcal{G}(\vdash \theta)$

But we cannot prove $\dfrac{S}{\vdash \theta}$

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Completeness Theorem Finitely axiomatized Lawvere logics are complete

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We have $\mathcal{MC}(S) \subseteq \mathcal{MC}(\vdash \theta)$

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The key:

$$a_1 p_1 \otimes \dots \otimes a_n p_n \otimes a \mathbb{1} \vdash b_1 q_1 \otimes \dots \otimes b_m q_m \otimes b \mathbb{1}$$

means:

$$a_1 x_1 + \dots + a_n x_n + a \geqslant b_1 y_1 + \dots + b_m y_m + b$$

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The key: Fourier – Motzkin elimination theorem

Motzkin transposition theorem

Farkas Lemma

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$$\frac{S, \vdash \varphi}{\vdash \theta} \text{ implies } \frac{S}{\varphi \vdash \theta} \quad \left(\text{or } \frac{S}{n\varphi \vdash \theta} \text{ for some } n>0\right)$$

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- Provability cannot be internalized

$$\frac{\vdash \varphi}{\vdash \psi} \quad \text{iff} \quad \exists \theta, \vdash \theta \quad \text{– uniqueness!}$$

A new chapter in Model Theory: rule-based theories

Quantitative Equational Logics:

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$$= \mathrm{Mod}(\vdash \psi) \cup \bigcup_{\varepsilon > 0} \mathrm{Mod}(\varphi \vdash \varepsilon)$$

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(R) defines an infinity of theories!

Completeness of rule-based Lawvere Logics

Incompleteness Theorem The rule-based Lawvere logics are incomplete

Completeness Theorem The finite rule-based Lawvere logics are complete:

$\mathfrak{J}$ - finite set of rules R - one rule

If $WG(\mathfrak{J})$ are closed under R, then R can be derived from $\mathfrak{J}$

Fourier-Motzkin

Conclusions

- One of the most powerfull logic - it can encode all the others
- A natural logic to encode Linear Arithmetic
- The proper tool for Approximation theories

- use linear approximations of real functions Stone-Weierstrass thm

- It properly encode (Quantitative) Equational Logics
  - proper for behavioural reasoning

Open questions

- First-order and higher-order extentions
- Algorithmic aspects