Approximate Bisimulations for Stochastic Processes: An Open Problem

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Stochastic Systems Modeling

Continuous Time Markov Chains and Process Algebra

- Continuous Time Markov Chains are the underlying semantics of many high-level formalisms for modeling, analysing and verifying stochastic systems, such as Stochastic Petri nets, Stochastic Automata Networks, Markovian process algebras
- High-level languages simplify the specification task thanks to compositionality and hierarchical approach
- Even very compact specifications can generate very large stochastic systems that are difficult/impossible to analyse

Specification Languages: Markovian Process Algebra

MPA	=	PA	+	CTMC
activity (α, r)		action α		rate r
		compositionality		product forms
		bisimulation		lumpability
		causal consistent reversibility		time reversibility

Specification Languages: Markovian Process Algebra

MPA	=	PA	+	CTMC
activity (α, r)		action α		rate r
		compositionality		product forms
		bisimulation		**lumpability**
		causal consistent reversibility		time reversibility

State Space Reduction Techniques

- In the non-deterministic setting bisimulation allows to quotient the state space [Milner 1989]
- On Markov Chains lumpability [Kemeny-Snell 1976] plays the same role, preserving stationary quantities [Buchholz 1994]

Problem

Lumpability is too demanding

As a consequence it usually provides poor reductions

Approximations

Quasi Lumpability and ϵ-Bisimulation

- Quasi Lumpability relates states allowing ϵ perturbations of the outgoing probabilities/rates [Franceschinis et al. 1994]
- Bounds on the stationary distributions have been proved
- Behavioural properties have been studied on ϵ-Bisimulation [Desharnais et al. 2008, Tracol et al. 2011, Abate et al. 2014, Abate et al. 2017]
- Algorithmic solutions have been proposed [Milios et al. 2012]

Unfortunately

It is not possible to exactly reconstruct the stationary distribution of the original system

Proportional Lumpability

Motivation

We aim at relaxing the conditions of lumpability while allowing to derive the exact stationary indices for the original system

Contribution

- We define the notion of Proportional Lumpability over Continuous Time Markov Chains (CTMC)
- We show that this allows to exactly derive the original stationary distribution
- We introduce the notion of Proportional Bisimulation over the stochastic process algebra PEPA and prove that it induces a proportional lumpability on the underlying semantics

Contionuous Time Markov Chains

CTMC

Let $X(t)$ with $t \in \mathbb{R}^+$ be a stochastic process taking values in a discrete space $\mathcal{S}$.

$X(t)$ is a CTMC if it is stationary and markovian

We focus on finite, time-homogeneous, ergodic Markov Chains

Infinitesimal Generator Matric

A CTMC is given as a matrix $\mathbf{Q}$ of dim. $|\mathcal{S}| \times |\mathcal{S}|$ such that:

- $q(i,j) \geq 0$ is the transition rate from i to j for $i \neq j$
- $q(i,i) = -\sum_{j \neq i} q(i,j)$

Stationary Analysis

Stationary Distribution

A distribution π over $\mathcal{S}$ such that $\pi(i)$ is the probability of being in i when time goes to ∞

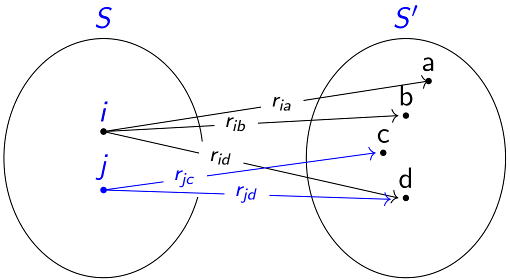
In our setting π is the unique distribution that solves

$$\pi \mathbf{Q} = 0$$

Stationary Performances Indices

Stationary performances indices, such as throughput, expected response time, resource utilization, can be computed from the steady state distribution π

Lumpability in CTMCs - Intuitively



$$r_{ia} + r_{ib} + r_{id} = r_{jc} + r_{jd}$$

Lumpability

Strong Lumpability

The strong lumpability $\sim$ is the largest equivalence over $\mathcal{S}$ such that $\forall S, S' \in \mathcal{S}/\sim$ and $\forall i, j \in S$

$$\sum_{a \in S'} q(i,a) = \sum_{a \in S'} q(j,a)$$

Properties

- There always exists a unique maximum lumpability
- The stationary distribution of the lumped chain is the aggregation of π

Quasi Lumpability

Quasi Lumpability [Franceschinis et al. '94, Milios et al. 2012]

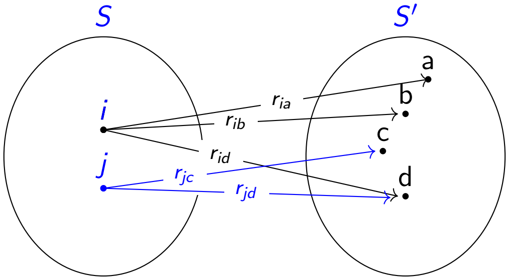
An ϵ-quasi lumpability $\mathcal{R}$ is an equivalence over $\mathcal{S}$ such that $\forall S, S' \in \mathcal{S}/\mathcal{R}$ and $\forall i, j \in S$

$$|\sum_{a \in S'} q(i,a) - \sum_{a \in S'} q(j,a)| \leq \epsilon$$

Properties

- It was originary defined splitting Q into Q^- and Q^ϵ (perturbation)
- Bounds on the exact stationary distribution (indices) can be computed
- Algorithms for approximating an optimal aggregation have been proposed

Quasi Lumpability – Example



$$r_{ia} + r_{ib} + r_{id} = 10 \qquad r_{jc} + r_{jd} = 100$$

$$\epsilon \geq 90$$

Proportional Lumpability

Proportional Lumpability

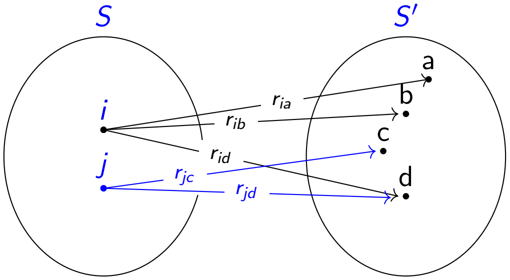
Given $\kappa : \mathcal{S} \to \mathbb{R}^+$, a κ-proportional lumpability $\mathcal{R}$ is an equivalence over $\mathcal{S}$ such that $\forall S, S' \in \mathcal{S}/\mathcal{R}$ and $\forall i, j \in S$

$$\frac{\sum_{a \in S'} q(i,a)}{\kappa(i)} = \frac{\sum_{a \in S'} q(j,a)}{\kappa(j)}$$

Properties

- There exists a unique maximum κ-proportional lumpability $\sim_\kappa$
- It allows us to derive an exact solution of the original process

Proportional Lumpability – Example



$r_{ia} + r_{ib} + r_{id} = 10 \qquad r_{jc} + r_{jd} = 100$

$\kappa(i) = 1 \qquad \kappa(j) = 10$

Perturbed System

Perturbed Systems

It is any CTMC $X'(t)$ over the state space $\mathcal{S}$ having generator Q' such that $\forall i \in \mathcal{S}$

$$q'(i,a) = \frac{q(i,a)}{\kappa(i)} \qquad \text{for any } a \neq i$$

Stationary Distributions of Perturbed Systems

Proposition

The stationary distributions of $X(t)$ and $X'(t)$ are related as follows

$$\pi(i) = \frac{\pi'(i)}{C \ \kappa(i)}$$

where the normalization factor is $C = \sum_{i \in \mathcal{S}} \pi'(i)/\kappa(i)$

Performances Evaluation Process Algebra (PEPA)

PEPA Syntax

Let $\mathcal{A}$ be a set of actions with $\tau \in \mathcal{A}$
Let $\alpha \in \mathcal{A}$, $A \subseteq \mathcal{A}$, and $r \in \mathbb{R}$

$$S ::= (\alpha, r).S \mid S + S \mid X$$
$$P ::= P \underset{A}{\bowtie} P \mid P/A \mid P \setminus A \mid S$$

Each variable X is associated to a definition $X \stackrel{\text{def}}{=} P$

PEPA Semantics

It can be defined in terms of Labeled Continuous Time Markov Chains

Performances Evaluation Process Algebra (PEPA)

$$\frac{P \xrightarrow{(\alpha,r)} P'}{P \underset{A}{\bowtie} Q \xrightarrow{(\alpha,r)} P' \underset{A}{\bowtie} Q} \ (\alpha \notin A) \qquad \frac{Q \xrightarrow{(\alpha,r)} Q'}{P \underset{A}{\bowtie} Q \xrightarrow{(\alpha,r)} P \underset{A}{\bowtie} Q'} \ (\alpha \notin A)$$

$$\frac{P \xrightarrow{(\alpha,r_1)} P' \quad Q \xrightarrow{(\alpha,r_2)} Q'}{P \underset{A}{\bowtie} Q \xrightarrow{(\alpha,R)} P' \underset{A}{\bowtie} Q'} \qquad (\alpha \in A)$$

where $R = \frac{r_1}{r_\alpha(P)} \frac{r_2}{r_\alpha(Q)} \min(r_\alpha(P), r_\alpha(Q))$

Strong Equivalence

Strong Equivalence [Hillston 1996]

A strong equivalence is an equivalence $\mathcal{R}$ such that for each action α, $\forall S, S' \in \mathcal{C}/\mathcal{R}$, and $\forall P, Q \in S$

$$\sum_{P' \in S',\ P \xrightarrow{(\alpha,r)} P'} r = \sum_{Q' \in S',\ Q \xrightarrow{(\alpha,r)} Q'} r$$

Properties

There exists a unique maximum lumpable bisimilarity $\approx_l$, it is *contextual*, *action preserving*, and induces a *lumpability*

Approximate Strong Equivalence

Approximate Strong Equivalence [Gilmore et al. 2015]

A approximate strong equivalence w.r.t. $\varepsilon \geq 0$ is an equivalence $\mathcal{R}$ such that for each action α, $\forall S, S' \in \mathcal{C}/\mathcal{R}$, and $\forall P, Q \in S$

$$\left| \sum_{P' \in S',\ P \xrightarrow{(\alpha,r)} P'} r - \sum_{Q' \in S',\ Q \xrightarrow{(\alpha,r)} Q'} r \right| \leq \varepsilon$$

Properties

It induces a *quasi lumpability*, however it is not preserved under union.

Proportional Bisimilarity

Proportional bisimilarity

Given $\kappa : \mathcal{C} \to \mathbb{R}^+$ a κ-proportional bisimilarity is an equivalence $\mathcal{R}$ such that for each action α, $\forall S, S' \in \mathcal{C}/\mathcal{R}$, and $\forall P, Q \in S$

$$\frac{\sum_{P' \in S',\ P \xrightarrow{(\alpha,r)} P'} r}{\kappa(P)} = \frac{\sum_{Q' \in S',\ Q \xrightarrow{(\alpha,r)} Q'} r}{\kappa(Q)}$$

Properties

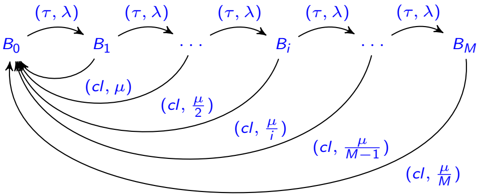
There exists a unique maximum proportional bisimilarity $\approx_l^\kappa$, it induces a *proportional lumpability*

Example - A simple buffer system

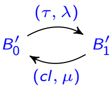
PEPA specification

$$
\begin{aligned}
B_n &= (\tau, \lambda).B_{n+1} && 0 \leq n \leq M-1 \\
B_n &= (cl, \mu/n).B_0 && 0 \leq n \leq M
\end{aligned}
$$

Original buffer system



Reduced buffer system



Conclusions

- The notion of proportional lumpability has been introduced
- It “preserves” the stationary distribution
- It can be applied for PEPA components reduction
- We are looking at its compositionality properties

Open problems

- Computing Probabilistic Bisimilarity Distances