How to Explain Probabilistic Bisimilarity Distances?

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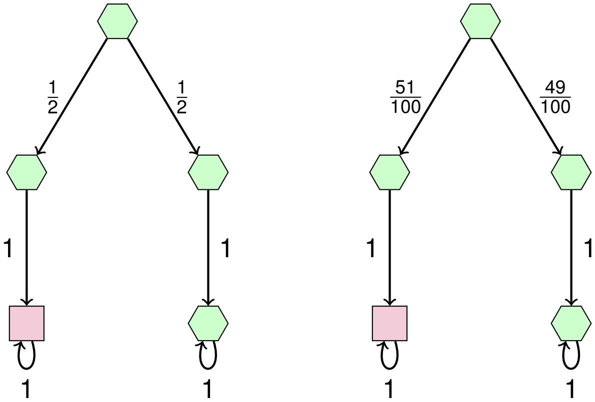
Probabilistic bisimilarity is *not* robust



Probabilistic bisimilarity is *not* robust



Giacalone, Jou, and Smolka, PROCOMET 1990.

Probabilistic bisimilarity is *not* robust

Fundamental problem

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Robust alternative

Instead of an equivalence relation

$$\sim : S \times S \to \{\text{true}, \text{false}\}$$

use distances

$$d : S \times S \to [0, 1].$$

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Robust alternative

Instead of an equivalence relation

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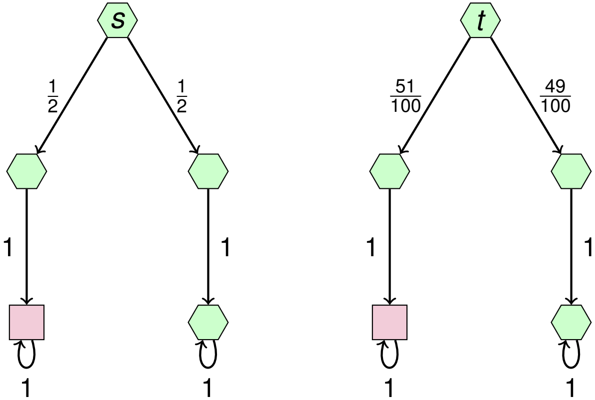
use distances

$$d : S \times S \to [0, 1].$$

Generalization

For all states s and t, $s \sim t$ iff $d(s, t) = 0$.

Probabilistic bisimilarity distances



Question

How do we explain that the distance of s and t is $\frac{1}{100}$?

Explainability

Question

How do we explain why states of a labelled transition system are *not* bisimilar?

Explainability

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How do we explain why states of a labelled transition system are *not* bisimilar?

- Game
  Ehrenfeucht and Fraïssé, 1950 and 1961.
- Logic
  Hennessy and Milner, 1980.
- ...

Bisimulation game

Theorem

States s and t of a labelled transition system are bisimilar iff duplicator has a winning strategy from (s, t).

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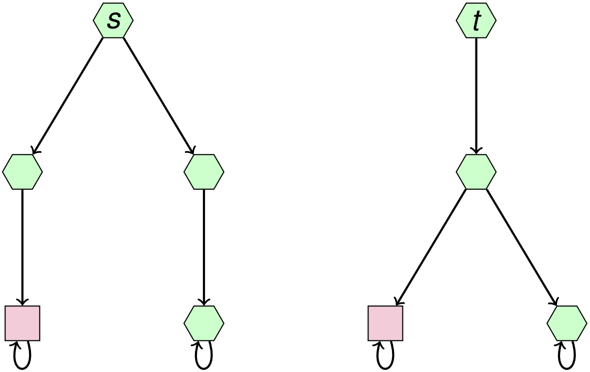
How do we explain that states of a labelled transition system are *not* bisimilar?

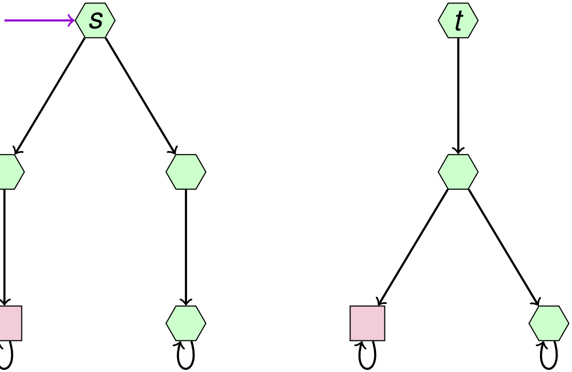
Answer

Find a strategy for spoiler that is winning from (s, t).

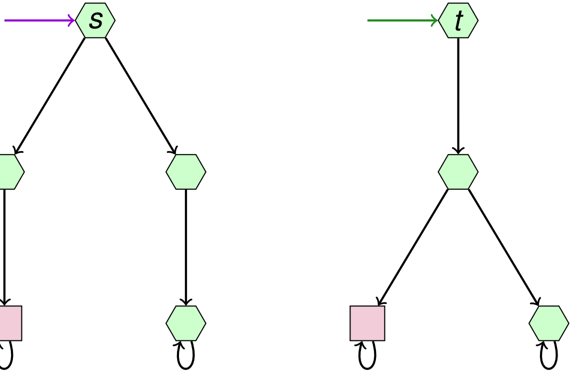
Bisimilarity



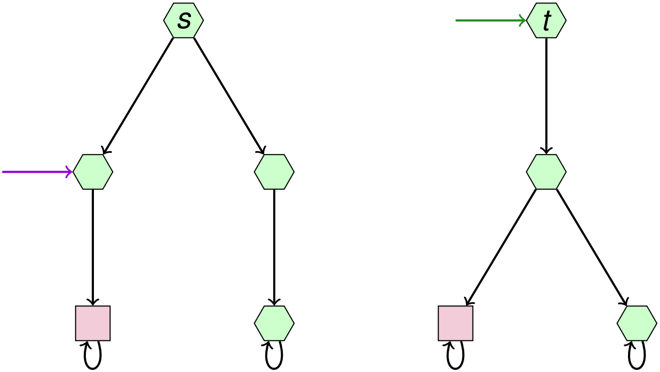
Bisimilarity



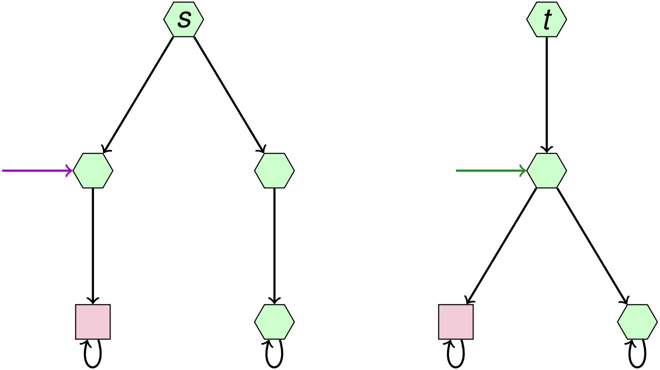
Bisimilarity



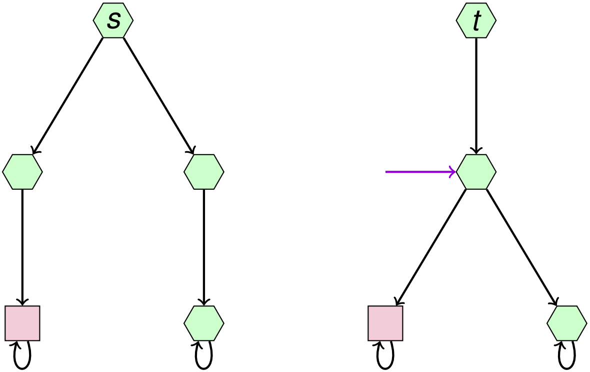
Bisimilarity



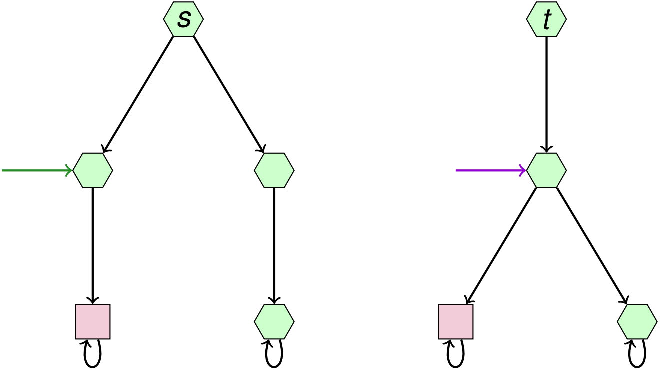
Bisimilarity



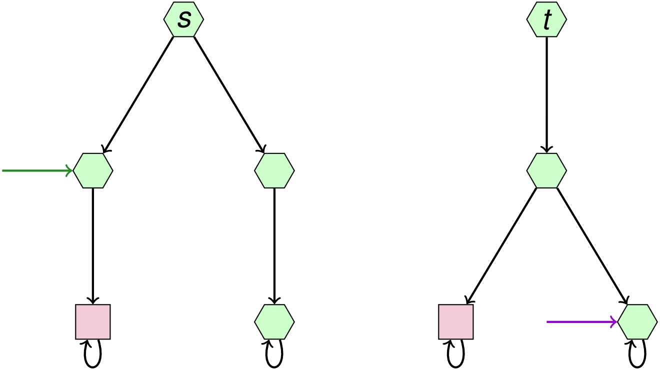
Bisimilarity



Bisimilarity



Bisimilarity



Hennessy-Milner theorem

Theorem

States of a labelled transition system are bisimilar iff they satisfy the same formulas of the Hennessy-Milner logic.

Hennessy and Milner, ICALP 1980.

Explainability

Question

How do we explain that states of a labelled transition system are *not* bisimilar?

Explainability

Question

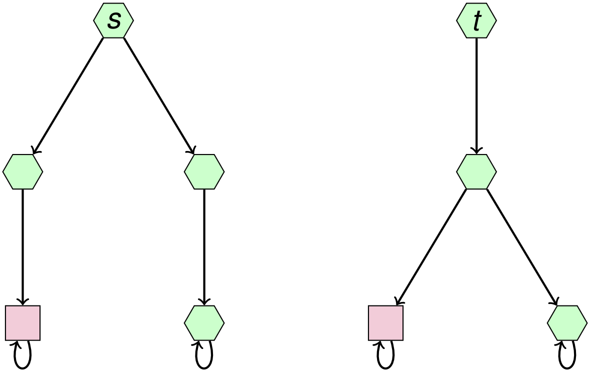
How do we explain that states of a labelled transition system are *not* bisimilar?

Answer

Find a formula of the Hennessy-Milner logic that is satisfied in one of the states but not the other.

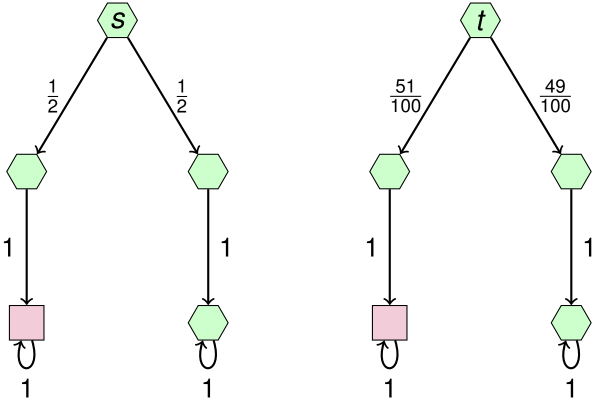
Cleaveland, CAV 1990.

Bisimilarity



State *t* satisfies $\bigcirc((\bigcirc$  $) \wedge (\bigcirc$  $))$, but state *s* does not.

Probabilistic bisimilarity distances



Question

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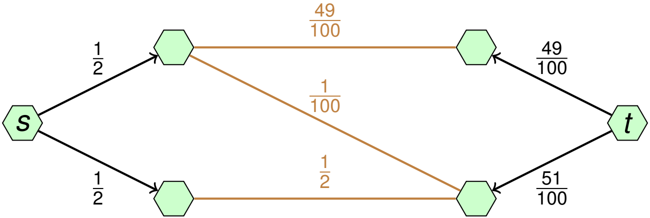
Probabilistic bisimilarity distances

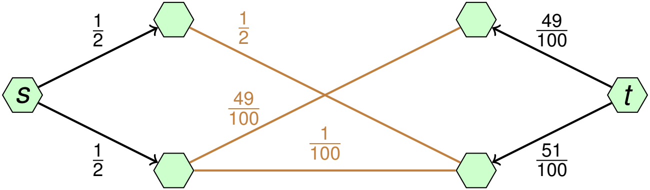
1. Game.
2. Logic.

Matching probabilities

Matching probabilities



Matching probabilities



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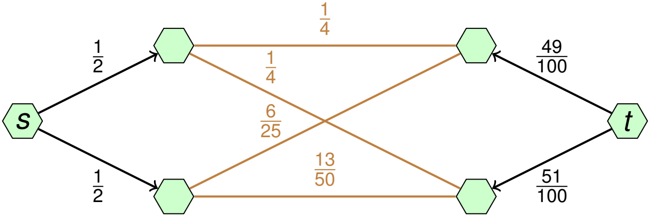
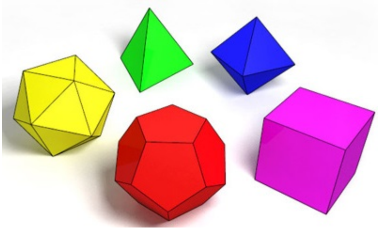


Matching probabilities = couplings

The couplings form a convex polytope



Doeblin, 1938.

A convex polytope is fully determined by its vertices

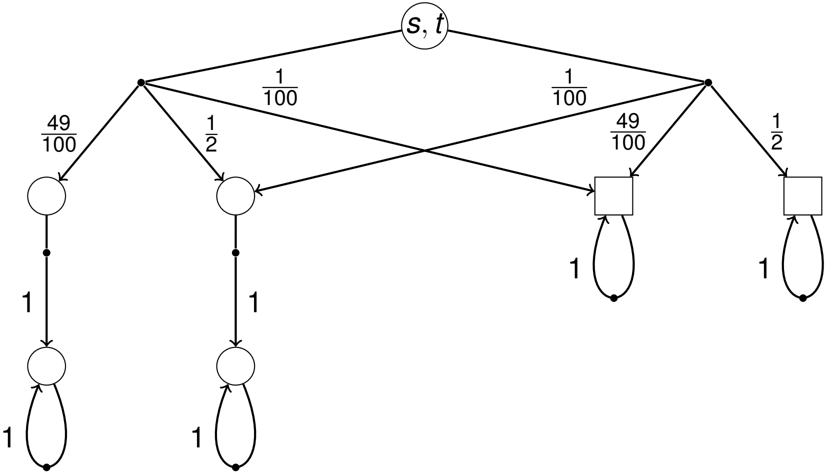
Probabilistic bisimilarity distance game

Labelled Markov chain $\langle S, \ldots \rangle$

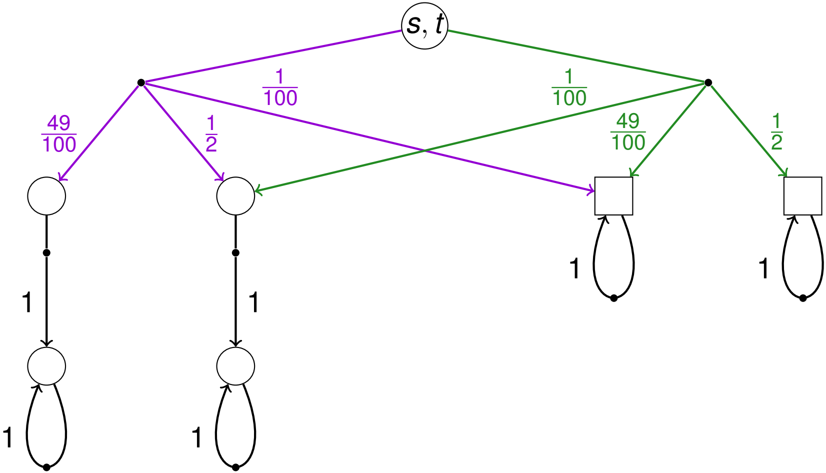
$\Downarrow$

Markov decision process $\langle S \times S, \ldots \rangle$

Markov decision process



Markov decision process



Probabilistic bisimilarity distances game

Theorem

$$d(s, t) = \min_{\text{strategy}} \text{probability of reaching } \square$$

Chen, Worrell, and vB, FoSSaCS 2012.

Explainability

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Explainability

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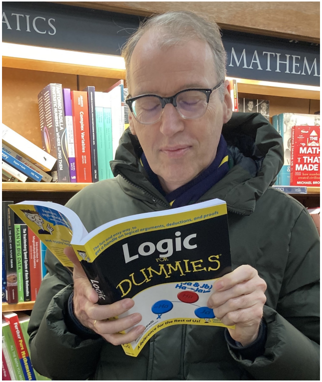
How do we explain the probabilistic bisimilarity distances?

Answer

Find a strategy that minimizes the probability of reaching $\Box$.

Probabilistic bisimilarity distances

1. Game.
2. Logic.



A quantitative Hennessy-Milner theorem

Theorem

For all states s, t,

$$d(s,t) = \sup_{f \in \mathcal{L}} [\![f]\!](s) - [\![f]\!](t).$$

A quantitative Hennessy-Milner theorem

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The logic $\mathcal{L}$ is defined by

$$f ::= a \mid \neg f \mid \bigcirc f \mid f \ominus q \mid f \wedge f$$

where a is a label and $q \in \mathbb{Q} \cap [0,1]$.

A logical characterization

Theorem

For all states s, t,

$$d(s,t) = \sup_{f \in \mathcal{L}} [\![f]\!](s) - [\![f]\!](t).$$

$$[\![a]\!](s) = \begin{cases} 1 & \text{if } a \text{ is label of } s \\ 0 & \text{otherwise} \end{cases}$$

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\end{array}$$

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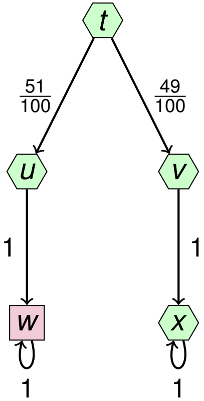
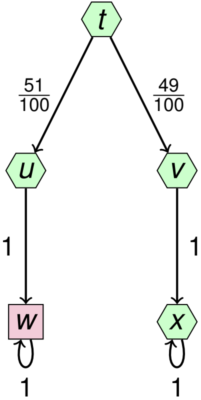
[\![f \ominus q]\!](s) & = & \max([\![f]\!](s) - q, 0) \\
[\![f \wedge g]\!](s) & = & \min([\![f]\!](s), [\![g]\!](s))
\end{array}$$

A logical characterization



$$[\![\bigcirc(⬢ \wedge \bigcirc ⬢)]\!](t)$$

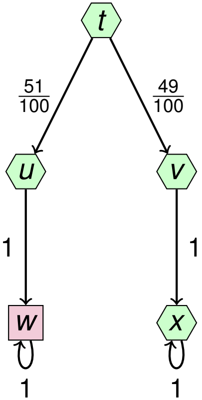
A logical characterization



$$\llbracket \bigcirc(⬡ \wedge \bigcirc ⬡) \rrbracket(t)$$

$$= \frac{51}{100} \llbracket ⬡ \wedge \bigcirc ⬡ \rrbracket(u) + \frac{49}{100} \llbracket ⬡ \wedge \bigcirc ⬡ \rrbracket$$

A logical characterization



$$[\![\bigcirc(\hexagon \wedge \bigcirc \hexagon)]\!](t)$$

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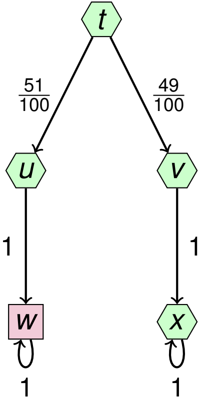

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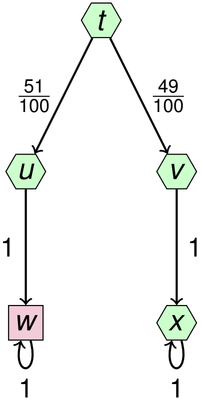

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$$= \frac{49}{100}$$

Explainability

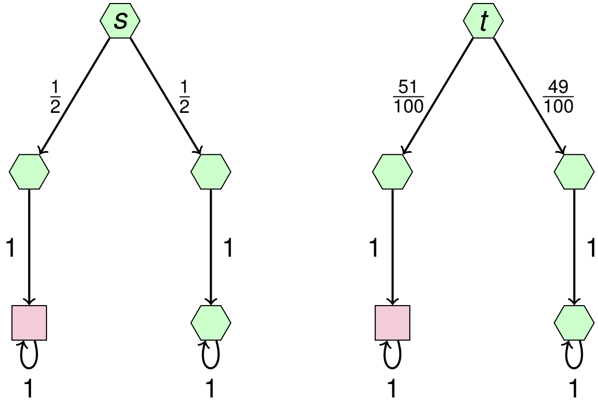
Question

How do we explain the probabilistic bisimilarity distances?

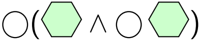
Attempt

Find a formula f of the logic $\mathcal{L}$ with $d(s, t) = [\![f]\!](s) - [\![f]\!](t)$.

Explainability



The distance of states *s* and *t* can be explained by the formula



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Attempt fails

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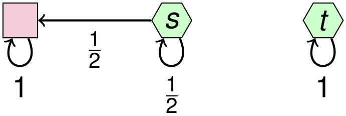


For every formula f, $[\![f]\!](s) - [\![f]\!](t) < d(s, t) = 1$.

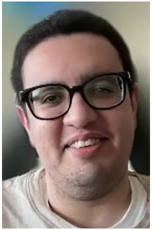
Explainability

Question

How do we explain the probabilistic bisimilarity distances?

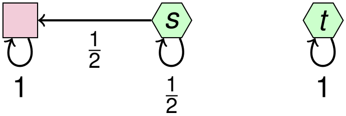
Attempt succeeds

Find a sequence $(f_n)_n$ of formulas of the logic $\mathcal{L}$ with $d(s,t) = \lim_n [\![f_n]\!](s) - [\![f_n]\!](t)$.



Rady and vB, FoSSaCS 2023.

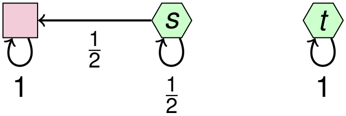
Explainability



The distance of states *s* and *t* can be explained by the sequence of formulas

- false
- $\bigcirc \square$
- $\bigcirc (\square \vee \bigcirc \square)$
- $\bigcirc (\square \vee \bigcirc (\square \vee \bigcirc \square))$
- $\ldots$

Explainability



The distance of states *t* and *s* can be explained by the sequence of formulas

- false
- $(\bigcirc ⬡) \ominus \frac{1}{2}$
- $(\bigcirc (⬡ \wedge ((\bigcirc ⬡) \ominus \frac{1}{2} \oplus \frac{1}{2}))) \ominus \frac{1}{4}$
- $(\bigcirc (⬡ \wedge ((\bigcirc (⬡ \wedge ((\bigcirc ⬡) \ominus \frac{1}{2} \oplus \frac{1}{2}))) \ominus \frac{1}{4} \oplus \frac{1}{4}))) \ominus \frac{1}{8}$
- $\ldots$

Some technical details

Objective

Construct a sequence $(f_{st}^n)_n$ of formulas of the logic $\mathcal{L}$ with $d(s,t) = \lim_n [\![f_{st}^n]\!](s) - [\![f_{st}^n]\!](t)$.

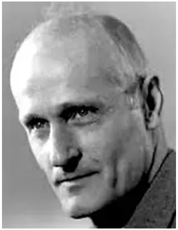
Some technical details

Objective

Construct a sequence $(f^n_{st})_n$ of formulas of the logic $\mathcal{L}$ with $d(s,t) = \lim_n [\![f^n_{st}]\!](s) - [\![f^n_{st}]\!](t)$.

Kleene fixed point theorem

$d = \lim_n d_n$.



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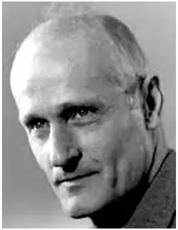


Objective

Construct a formula f_{st}^n of the logic $\mathcal{L}$ with $d_n(s,t) = [\![f_{st}^n]\!](s) - [\![f_{st}^n]\!](t)$.

Some technical details

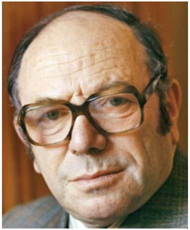
Objective

Construct a formula f^n_{st} of the logic $\mathcal{L}$ with
$d_n(s,t) = [\![f^n_{st}]\!](s) - [\![f^n_{st}]\!](t)$.

Kantorovich-Rubinstein duality theorem

There exists $F^n_{st} : S \to [0,1]$ such that

$$d_n(s,t) = \left(\sum_{u \in S} P(s,u)\, F^n_{st}(u)\right) - \left(\sum_{u \in S} P(t,u)\, F^n_{st}(u)\right).$$



Some technical details

The formula g^n_{stuv} is defined by

$$\begin{cases} \text{false} \oplus F^n_{st}(u) & \text{if } F^n_{st}(u) = F^n_{st}(v) \\ (f^n_{uv} \ominus (d_n(u,v) - (F^n_{st}(u) - F^n_{st}(v))) \oplus F^n_{st}(v) & \text{if } F^n_{st}(u) > F^n_{st}(v) \\ (f^n_{vu} \ominus (d_n(u,v) - (F^n_{st}(v) - F^n_{st}(u))) \oplus F^n_{st}(u) & \text{otherwise.} \end{cases}$$

Stone-Weierstrass approximation theorem

$$f^n_{st} = \left[\!\left[\bigwedge_{u \in S} \bigvee_{v \in S} g^n_{stuv} \right]\!\right]$$

The formula f_{st}^{n}

Some technical details

The formula f^n_{st}

If s and t have a different label then f^n_{st}= label of s.

Some technical details

The formula f^n_{st}

If s and t have a different label then f^n_{st}= label of s.

Otherwise,

$$f^n_{st} = \left(\bigcirc \bigwedge_{u \in S} \bigvee_{v \in S} g^n_{stuv} \right) \ominus \left(\sum_{u \in S} P(t,u)\, F^n_{st}(u) \right).$$

Open problems

- How to extend the logic $\mathcal{L}$ and, for states s and t, find a formula $f_{st} \in \mathcal{L}$ such that

$$d(s,t) = [\![f_{st}]\!](s) - [\![f_{st}]\!](t)?$$

- How to extend the logic $\mathcal{L}$ and, for state s, find a formula $f_s \in \mathcal{L}$ such that for all states t

$$d(s,t) = [\![f_s]\!](s) - [\![f_s]\!](t)?$$

- How to extend the logic $\mathcal{L}$ and, for state s, find a formula $f_s \in \mathcal{L}$ such that for all states t

$$d(s,t) = [\![f_s]\!](t)?$$

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