The Matrix Geometric/Analytic Methods for Structured Markov Chains

Markov chains whose transition matrices have a special block structure.

Example:

$$\begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_2 & 0 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad (1)$$

Each state can be written as $\{(\eta, k), \eta \geq 0, 1 \leq k \leq K\}$
— ordered by increasing value of η then by increasing value of k.

States are grouped into "levels" according to their η value.

The block tridiagonal effect: transitions are permitted
— between states of the same level (diagonal blocks),
— to states in the next highest level (super-diagonal blocks),
— and to states in the adjacent lower level (sub-diagonal blocks).

Called *Quasi-Birth-Death* (QBD) processes.

Example:

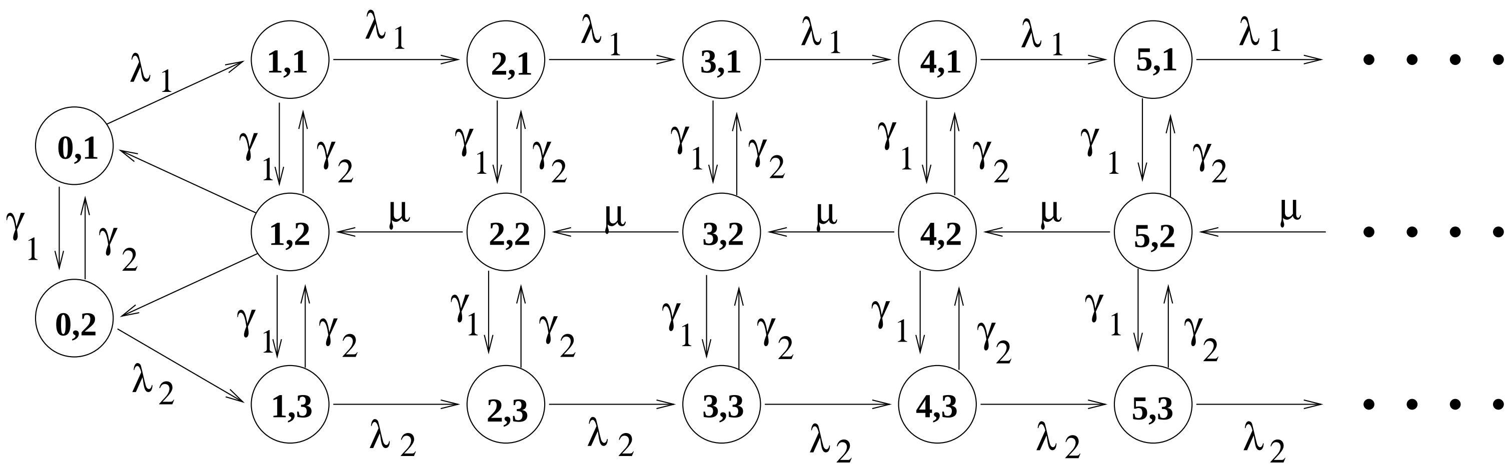


Figure 1: State transition diagram for an M/M/1-type process.

Transition rate matrix:

$$
\left(\begin{array}{cc|ccc|ccc|ccc|ccc|ccc|c}
* & \gamma_1 & \lambda_1 & & & & & & & & & & & & & & & \\
\gamma_2 & * & & & \lambda_2 & & & & & & & & & & & & & \\
\hline
 & & * & \gamma_1 & & \lambda_1 & & & & & & & & & & & & \\
\mu/2 & \mu/2 & \gamma_2 & * & \gamma_1 & & & & & & & & & & & & & \\
 & & & \gamma_2 & * & & & \lambda_2 & & & & & & & & & & \\
\hline
 & & & & & * & \gamma_1 & & \lambda_1 & & & & & & & & & \\
 & & & \mu & & \gamma_2 & * & \gamma_1 & & & & & & & & & & \\
 & & & & & & \gamma_2 & * & & & \lambda_2 & & & & & & & \\
\hline
 & & & & & & & & * & \gamma_1 & & \lambda_1 & & & & & & \\
 & & & & & & \mu & & \gamma_2 & * & \gamma_1 & & & & & & & \\
 & & & & & & & & & \gamma_2 & * & & & \lambda_2 & & & & \\
\hline
 & & & & & & & & & & & * & \gamma_1 & & \lambda_1 & & & \\
 & & & & & & & & & \mu & & \gamma_2 & * & \gamma_1 & & & & \\
 & & & & & & & & & & & & \gamma_2 & * & & & \lambda_2 & \\
\hline
 & & & & & & & & & & & & \ddots & & & \ddots & & \ddots
\end{array}\right)
$$

Block matrices:

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} -(\gamma_1 + \lambda_1) & \gamma_1 & 0 \\ \gamma_2 & -(\mu + \gamma_1 + \gamma_2) & \gamma_1 \\ 0 & \gamma_2 & -(\gamma_2 + \lambda_2) \end{pmatrix}$$

and

$$B_{00} = \begin{pmatrix} -(\gamma_1 + \lambda_1) & \gamma_1 \\ \gamma_2 & -(\gamma_2 + \lambda_2) \end{pmatrix},$$

$$B_{01} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad B_{10} = \begin{pmatrix} 0 & 0 \\ \mu/2 & \mu/2 \\ 0 & 0 \end{pmatrix}.$$

Most common extensions:

— block upper Hessenberg
($M/G/1$-type, solved using the matrix analytic approach)

— block lower Hessenberg
$GI/M/1$-type, solved using the matrix geometric approach).

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & B_{11} & A_0 & 0 & 0 & 0 & 0 & \cdots \\ B_{20} & B_{21} & A_1 & A_0 & 0 & 0 & 0 & \cdots \\ B_{30} & B_{31} & A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ B_{40} & B_{41} & A_3 & A_2 & A_1 & A_0 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

The Quasi-Birth-Death Case

When the blocks of a QBD process are reduced to single elements:

$$Q = \begin{pmatrix} -\lambda & \lambda & & & & \\ \mu & -(\lambda+\mu) & \lambda & & & \\ & \mu & -(\lambda+\mu) & \lambda & & \\ & & \mu & -(\lambda+\mu) & \lambda & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

From $\pi Q = 0$, we may write $-\lambda\pi_0 + \mu\pi_1 = 0$, $\pi_1 = (\lambda/\mu)\pi_0$

In general

$$\lambda\pi_{i-1} - (\lambda+\mu)\pi_i + \mu\pi_{i+1} = 0,$$

which gives

$$\pi_{i+1} = (\lambda/\mu)\pi_i \qquad i = 1, 2, \ldots$$

Proof by induction: Basis clause, $\pi_1 = (\lambda/\mu)\pi_0$.

From the inductive hypothesis $\pi_i = (\lambda/\mu)\pi_{i-1}$ and hence

$$\pi_{i+1} = \left(\frac{\lambda+\mu}{\mu}\right)\pi_i - \left(\frac{\lambda}{\mu}\right)\pi_{i-1} = \left(\frac{\lambda}{\mu}\right)\pi_i.$$

$$\pi_i = \left(\frac{\lambda}{\mu}\right)^i \pi_0 = \rho^i \pi_0 \quad \text{where} \;\; \rho = \lambda/\mu.$$

Once π_0 is known, the remaining values, $\pi_i, \; i = 1, 2, \ldots,$ may be determined recursively.

A similar result exists when Q is a QBD process:

— the parameter ρ becomes a square matrix R of order K

— the components π_i become subvectors of length K.

QBD process $\pi Q = 0$ with

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_2 & 0 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Let π be partitioned conformally with Q, i.e.

$$\pi = (\pi_0, \pi_1, \pi_2, \cdots)$$

where

$$\pi_i = (\pi(i,1), \pi(i,2), \cdots \pi(i,K))$$

This gives the following equations

$$
\begin{aligned}
\pi_0 B_{00} + \pi_1 B_{10} &= 0 \\
\pi_0 B_{01} + \pi_1 A_1 + \pi_2 A_0 &= 0 \\
\pi_1 A_2 + \pi_2 A_1 + \pi_3 A_0 &= 0 \\
&\vdots \\
\pi_{i-1} A_2 + \pi_i A_1 + \pi_{i+1} A_0 &= 0, \qquad i = 2, 3, \ldots
\end{aligned}
$$

In analogy with the point situation, there exists a constant matrix R s.t.

$$\pi_i = \pi_{i-1} R, \quad \text{for} \;\; i = 2, 3, \ldots \tag{2}$$

The subvectors π_i are *geometrically* related to each other since

$$\pi_i = \pi_1 R^{i-1}, \quad \text{for} \;\; i = 2, 3, \ldots \tag{3}$$

Given π_0, π_1 and R, we can find all other π_i.

Substituting from Equation (3) into

$$\pi_{i-1}A_2 + \pi_i A_1 + \pi_{i+1}A_0 = 0$$

gives

$$\pi_1 R^{i-2} A_2 + \pi_1 R^{i-1} A_1 + \pi_1 R^i A_0 = 0$$

i.e.,

$$\pi_1 R^{i-2}\left(A_2 + RA_1 + R^2 A_0\right) = 0$$

So find R from

$$\left(A_2 + RA_1 + R^2 A_0\right) = 0. \qquad (4)$$

The simplest way: successive substitution. Equation (4) gives

$$A_2 A_1^{-1} + R + R^2 A_0 A_1^{-1} = 0$$

i.e.,

$$R = -A_2 A_1^{-1} - R^2 A_0 A_1^{-1} = -V - R^2 W$$

$$R_{(0)} = 0; \quad R_{(k+1)} = -V - R_{(k)}^2 W, \quad k = 1, 2, \ldots \qquad (5)$$

Derivation of π_0 and π_1: The first two equations of $\pi Q = 0$ are

$$\begin{aligned} \pi_0 B_{00} + \pi_1 B_{10} &= 0 \\ \pi_0 B_{01} + \pi_1 A_1 + \pi_2 A_0 &= 0 \end{aligned}$$

Replacing π_2 with $\pi_1 R$

$$(\pi_0, \pi_1) \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & A_1 + RA_0 \end{pmatrix} = (0, 0) \qquad (6)$$

which can be solved for π_0 and π_1 with the condition $\pi e = 1$.

$$\begin{aligned} 1 = \pi e &= \pi_0 e + \pi_1 e + \sum_{i=2}^{\infty} \pi_i e \\ &= \pi_0 e + \pi_1 e + \sum_{i=2}^{\infty} \pi_1 R^{i-1} e \\ &= \pi_0 e + \sum_{i=1}^{\infty} \pi_1 R^{i-1} e = \pi_0 e + \sum_{i=0}^{\infty} \pi_1 R^i e. \end{aligned}$$

This implies the condition

$$\pi_0 e + \pi_1 \left(\sum_{i=0}^{\infty} R^i \right) e = 1.$$

The eigenvalues of R lie *inside* the unit circle which means that $(I - R)$ is nonsingular and hence that

$$\left(\sum_{i=0}^{\infty} R^i \right) = (I - R)^{-1}. \quad (7)$$

Normalize the vectors π_0 and π_1 by computing

$$\alpha = \pi_0 e + \pi_1 (I - R)^{-1} e$$

and dividing the computed subvectors π_0 and π_1 by α.

Ergodicity condition for QBD processes:
— the *drift* to higher numbered levels must be strictly less than the *drift* to lower levels.

Let $A = A_0 + A_1 + A_2$ and

$$\pi_A A = 0.$$

The following condition must hold for a QBD process to be ergodic

$$\pi_A A_2 e \; < \; \pi_A A_0 e \tag{8}$$

Elements of A_2 move the process up a level while those of A_0 move it down a level.

SUMMARY: Matrix geometric method:

1. Ensure that the matrix has the requisite block structure.
2. Use Equation (8) to ensure that the Markov chain is ergodic.
3. Use Equation (5) to compute the matrix R.
4. Solve the system of equations (6) for π_0 and π_1.
5. Compute the normalizing constant α and normalize π_0 and π_1.
6. Use Equation (2) to compute the remaining components of the stationary distribution vector.

For a discrete-time Markov chain, replace $-A_1^{-1}$ with $(I - A_1)^{-1}$.

Example: We use the following values of the parameters:

$$\lambda_1 = 1,\ \lambda_2 = .5,\ \mu = 4,\ \gamma_1 = 5,\ \gamma_2 = 3.$$

The infinitesimal generator is then given by

$$Q = \left(\begin{array}{cc|ccc|ccc|cc|c}
-6 & 5.0 & 1 & & & & & & & & \\
3 & -3.5 & & & .5 & & & & & & \\
\hline
 & & -6 & 5 & & 1 & & & & & \\
2 & 2 & 3 & -12 & 5.0 & & & & & & \\
 & & & 3 & -3.5 & & & .5 & & & \\
\hline
 & & & & & -6 & 5 & & 1 & & \\
 & & & 4 & & 3 & -12 & 5.0 & & & \\
 & & & & & & 3 & -3.5 & & .5 & \\
\hline
 & & & & & & \ddots & & \ddots & & \ddots
\end{array} \right)$$

1. The matrix obviously has the correct QBD structure.

2. We check that the system is stable by verifying Equation (8). The infinitesimal generator matrix

$$A = A_0 + A_1 + A_2 = \begin{pmatrix} -5 & 5 & 0 \\ 3 & -8 & 5 \\ 0 & 3 & -3 \end{pmatrix}$$

has stationary probability vector

$$\pi_A = (.1837,\ .3061,\ .5102)$$

and

$$.4388 = \pi_A A_2 e \ < \ \pi_A A_0 e = 1.2245$$

3. We now initiate the iterative procedure to compute the rate matrix R. The inverse of A_1 is

$$A_1^{-1} = \begin{pmatrix} -.2466 & -.1598 & -.2283 \\ -.0959 & -.1918 & -.2740 \\ -.0822 & -.1644 & -.5205 \end{pmatrix}$$

which allows us to compute

$$V = A_2 A_1^{-1} = \begin{pmatrix} -.2466 & -.1598 & -.2283 \\ 0 & 0 & 0 \\ -.0411 & -.0822 & -.2603 \end{pmatrix}$$

$$W = A_0 A_1^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ -.3836 & -.7671 & -1.0959 \\ 0 & 0 & 0 \end{pmatrix}.$$

Equation (5) becomes

$$R_{(k+1)} = \begin{pmatrix} .2466 & .1598 & .2283 \\ 0 & 0 & 0 \\ .0411 & .0822 & .2603 \end{pmatrix} + R_{(k)}^2 \begin{pmatrix} 0 & 0 & 0 \\ .3836 & .7671 & 1.0959 \\ 0 & 0 & 0 \end{pmatrix}$$

and iterating successively, beginning with $R_{(0)} = 0$, we find

$$R_{(1)} = \begin{pmatrix} .2466 & .1598 & .2283 \\ 0 & 0 & 0 \\ .0411 & .0822 & .2603 \end{pmatrix}, \quad R_{(2)} = \begin{pmatrix} .2689 & .2044 & .2921 \\ 0 & 0 & 0 \\ .0518 & .1036 & .2909 \end{pmatrix},$$

$$R_{(3)} = \begin{pmatrix} .2793 & .2252 & .2921 \\ 0 & 0 & 0 \\ .0567 & .1134 & .3049 \end{pmatrix}, \quad \cdots$$

Observe that the elements are non-decreasing.

After 48 iterations, successive differences are less than 10^{-12}, at which point

$$R_{(48)} = \begin{pmatrix} .2917 & .2500 & .3571 \\ 0 & 0 & 0 \\ .0625 & .1250 & .3214 \end{pmatrix}.$$

4. Proceeding to the boundary conditions:

$$(\pi_0, \pi_1) \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & A_1 + RA_0 \end{pmatrix} = (\pi_0, \pi_1) \left(\begin{array}{rr|rrr} -6 & 5.0 & 1 & 0 & 0 \\ 3 & -3.5 & 0 & 0 & .5 \\ \hline 0 & 0 & -6 & 6.0 & 0 \\ 2 & 2 & 3 & -12.0 & 5.0 \\ 0 & 0 & 0 & 3.5 & -3.5 \end{array}\right) = (0, 0)$$

Solve this by replacing the last equation with $\pi_{0_1} = 1$, i.e., set the first component of the subvector π_0 to 1.

$$(\pi_0, \pi_1) \left(\begin{array}{cc|ccc} -6 & 5.0 & 1 & 0 & 1 \\ 3 & -3.5 & 0 & 0 & 0 \\ \hline 0 & 0 & -6 & 6.0 & 0 \\ 2 & 2 & 3 & -12.0 & 0 \\ 0 & 0 & 0 & 3.5 & 0 \end{array} \right) = (0, 0 \mid 0, 0, 1)$$

with solution

$$(\pi_0, \pi_1) = (1.0,\ 1.6923,\ \mid .3974,\ .4615,\ .9011)$$

Now on to the normalization stage.

5. The normalization constant is

$$\begin{aligned}
\alpha &= \pi_0 e + \pi_1 (I - R)^{-1} e \\
&= (1.0,\ 1.6923)e + (.3974,\ .4615,\ .9011) \begin{pmatrix} 1.4805 & .4675 & .7792 \\ 0 & 1 & 0 \\ .1364 & .2273 & .15455 \end{pmatrix} e \\
&= 2.6923 + 3.2657 = 5.9580
\end{aligned}$$

which allows us to compute

$$\pi_0/\alpha = (.1678,\ .2840)$$

and

$$\pi_1/\alpha = (.0667,\ .0775,\ .1512)$$

6. Subcomponents of the stationary distribution:

— computed from $\pi_k = \pi_{k-1}R$.

$$\pi_2 = \pi_1 R = (.0667,\ .0775,\ .1512)\begin{pmatrix} .2917 & .2500 & .3571 \\ 0 & 0 & 0 \\ .0625 & .1250 & .3214 \end{pmatrix}$$

$$= (.0289,\ .0356,\ .0724)$$

and

$$\pi_3 = \pi_2 R = (.0289,\ .0356,\ .0724)\begin{pmatrix} .2917 & .2500 & .3571 \\ 0 & 0 & 0 \\ .0625 & .1250 & .3214 \end{pmatrix}$$

$$= (.0130,\ .0356,\ .0336)$$

and so on.

Block Lower-Hessenberg Markov Chains

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & B_{11} & A_0 & 0 & 0 & 0 & 0 & \cdots \\ B_{20} & B_{21} & A_1 & A_0 & 0 & 0 & 0 & \cdots \\ B_{30} & B_{31} & A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ B_{40} & B_{41} & A_3 & A_2 & A_1 & A_0 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

Transitions are now permitted from any level to any *lower* level.

Objective:compute the stationary probability vector π from $\pi Q = 0$.

π is partitioned conformally with Q, i.e. $\pi = (\pi_0, \pi_1, \pi_2, \cdots)$

— $\pi_i = (\pi(i,1), \pi(i,2), \cdots \pi(i,K))$.

A matrix geometric solution exists which mirrors that of a QBD process,.

There exists a positive matrix R such that

$$\pi_i = \pi_{i-1} R, \quad \text{for} \;\; i = 2, 3, \ldots$$

i.e., that

$$\pi_i = \pi_1 R^{i-1}, \quad \text{for} \;\; i = 2, 3, \ldots$$

From $\pi Q = 0$

$$\sum_{k=0}^{\infty} \pi_{k+j} A_k = 0, \qquad j = 1, 2, \ldots$$

and in particular,

$$\pi_1 A_0 + \pi_2 A_1 + \sum_{k=2}^{\infty} \pi_{k+1} A_k = 0$$

Substituting $\pi_i = \pi_1 R^{i-1}$ into

$$\pi_1 A_0 + \pi_1 R A_1 + \sum_{k=2}^{\infty} \pi_1 R^k A_k = 0$$

gives

$$\pi_1 \left(A_0 + R A_1 + \sum_{k=2}^{\infty} R^k A_k \right) = 0$$

So find R from

$$A_0 + R A_1 + \sum_{k=2}^{\infty} R^k A_k = 0 \qquad (9)$$

Equation (9) reduces to Equation (4) when $A_k = 0$ for $k > 2$.

Rearranging Equation (9), we find

$$R = -A_0 A_1^{-1} - \sum_{k=2}^{\infty} R^k A_k A_1^{-1}$$

$$R_{(0)} = 0; \quad R_{(l+1)} = -A_0 A_1^{-1} - \sum_{k=2}^{\infty} R_{(l)}^k A_k A_1^{-1}, \quad l = 1, 2, \ldots$$

In many cases, the structure of the infinitesimal generator is such that the blocks A_i are zero for relatively small values of i, which limits the computational effort needed in each iteration.

Derivation of the initial subvectors π_0 and π_1.

From the first equation of $\pi Q = 0$,

$$\sum_{i=0}^{\infty} \pi_i B_{i0} = 0$$

and we may write

$$\pi_0 B_{00} + \sum_{i=1}^{\infty} \pi_i B_{i0} = \pi_0 B_{00} + \sum_{i=1}^{\infty} \pi_1 R^{i-1} B_{i0} = \pi_0 B_{00} + \pi_1 \left(\sum_{i=1}^{\infty} R^{i-1} B_{i0} \right) = 0, \tag{10}$$

From the second equation of $\pi Q = 0$,

$$\pi_0 B_{01} + \sum_{i=1}^{\infty} \pi_i B_{i1} = 0, \quad \text{i.e.,} \quad \pi_0 B_{01} + \pi_1 \sum_{i=1}^{\infty} R^{i-1} B_{i1} = 0. \tag{11}$$

In matrix form, we can compute π_0 and π_1 from

$$(\pi_0, \pi_1) \begin{pmatrix} B_{00} & B_{01} \\ \\ \sum_{i=1}^{\infty} R^{i-1} B_{i0} & \sum_{i=1}^{\infty} R^{i-1} B_{i1} \end{pmatrix} = (0, 0).$$

Once found, normalize by dividing by

$$\alpha = \pi_0 e + \pi_1 \left(\sum_{i=1}^{\infty} R^{k-1} \right) e = \pi_0 e + \pi_1 (I - R)^{-1} e.$$

For discrete-time Markov chains, replace $-A_1^{-1}$ with $(I - A_1)^{-1}$.

Same example as before, but with additional transitions ($\xi_1 = .25$ and $\xi_2 = .75$) to lower non-neighboring states.

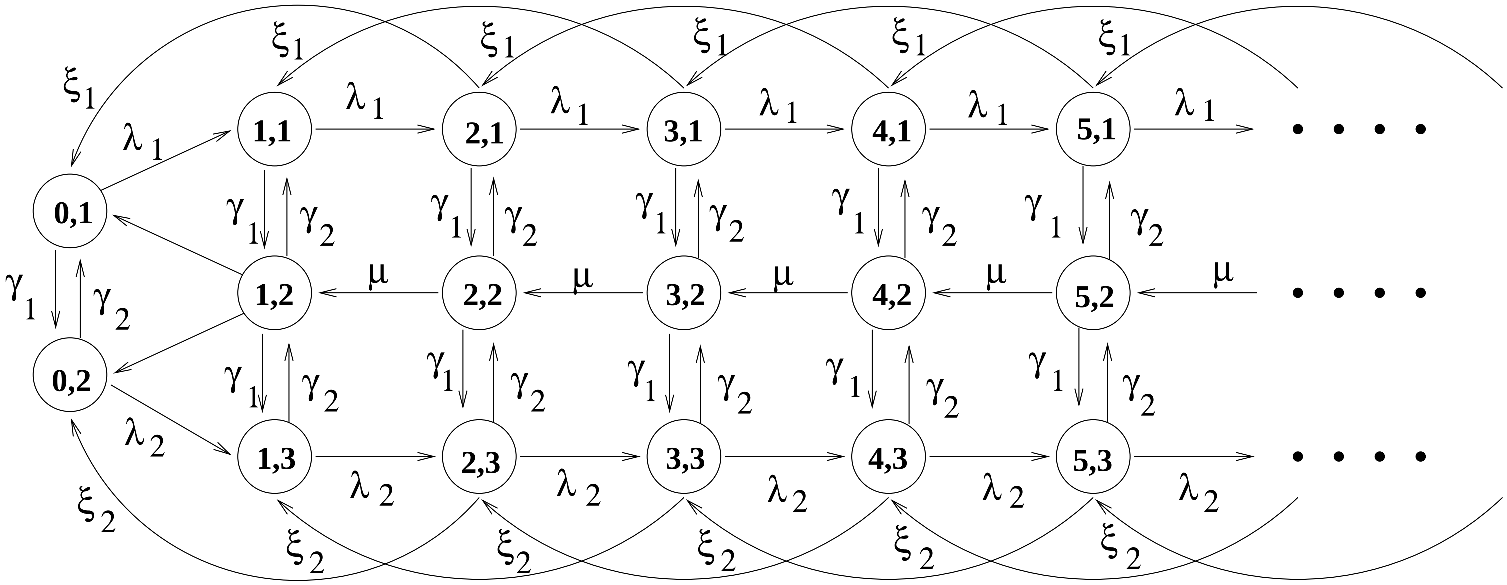


Figure 2: State transition diagram for a GI/M/1-type process.

$$
Q = \left(\begin{array}{cc|ccc|ccc|ccc|c}
-6 & 5.0 & 1 & & & & & & & & & \\
3 & -3.5 & & & .5 & & & & & & & \\
\hline
 & & -6 & 5 & & 1 & & & & & & \\
2.00 & 2.00 & 3 & -12 & 5 & & & & & & & \\
 & & & 3 & -3.5 & & & .5 & & & & \\
\hline
.25 & & & & & -6.25 & 5 & & 1 & & & \\
 & & & 4 & & 3.00 & -12 & 5.00 & & & & \\
 & .75 & & & & & 3 & -4.25 & & & .5 & \\
\hline
 & & .25 & & & & & & -6.25 & 5 & & 1 \\
 & & & & & & 4 & & 3.00 & -12 & 5.00 & \\
 & & & & .75 & & & & & 3 & -4.25 & \\
\hline
 & & & & & \ddots & & & \ddots & & & \ddots
\end{array}\right.
$$

The computation of the matrix R proceeds as previously:

$$A_1^{-1} = \begin{pmatrix} -.2233 & -.1318 & -.1550 \\ -.0791 & -.1647 & -.1938 \\ -.0558 & -.1163 & -.3721 \end{pmatrix}$$

which allows us to compute

$$A_0 A_1^{-1} = \begin{pmatrix} -.2233 & -.1318 & -.1550 \\ 0 & 0 & 0 \\ -.0279 & -.0581 & -.1860 \end{pmatrix}, \quad A_2 A_1^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ -.3163 & -.6589 & -.7752 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_3 A_1^{-1} = \begin{pmatrix} -.0558 & -.0329 & -.0388 \\ 0 & 0 & 0 \\ -.0419 & -.0872 & -.2791 \end{pmatrix},$$

The iterative process is

$$R_{(k+1)} = \begin{pmatrix} .2233 & .1318 & .1550 \\ 0 & 0 & 0 \\ .0279 & .0581 & .1860 \end{pmatrix} + R_{(k)}^2 \begin{pmatrix} 0 & 0 & 0 \\ .3163 & .6589 & .7752 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+R_{(k)}^3 \begin{pmatrix} .0558 & .0329 & .0388 \\ 0 & 0 & 0 \\ .0419 & .0872 & .2791 \end{pmatrix}$$

Iterating successively, beginning with $R_{(0)} = 0$, we find

$$R_{(1)} = \begin{pmatrix} .2233 & .1318 & .1550 \\ 0 & 0 & 0 \\ .0279 & .0581 & .1860 \end{pmatrix}, \quad R_{(2)} = \begin{pmatrix} .2370 & .1593 & .1910 \\ 0 & 0 & 0 \\ .0331 & .0686 & .1999 \end{pmatrix},$$

$$R_{(3)} = \begin{pmatrix} .2415 & .1684 & .2031 \\ 0 & 0 & 0 \\ .0347 & .0719 & .2043 \end{pmatrix}, \quad \cdots$$

After 27 iterations, successive differences are less than 10^{-12}, at which point

$$R_{(27)} = \begin{pmatrix} .2440 & .1734 & .2100 \\ 0 & 0 & 0 \\ .0356 & .0736 & .1669 \end{pmatrix}.$$

The boundary conditions are now

$$(\pi_0, \pi_1) \begin{pmatrix} B_{00} & B_{01} \\ B_{10} + RB_{20} & B_{11} + RB_{21} + R^2 B_{31} \end{pmatrix} = (0,\ 0)$$

$$= (\pi_0, \pi_1) \left(\begin{array}{cc|ccc} -6.0 & 5.0 & 1 & 0 & 0 \\ 3.0 & -3.5 & 0 & 0 & .5 \\ \hline .0610 & .1575 & -5.9832 & 5.6938 & .0710 \\ 2.0000 & 2.000 & 3.000 & -12.0000 & 5.0000 \\ .0089 & .1555 & .0040 & 3.2945 & -3.4624 \end{array} \right) = (0,\ 0).$$

Solve this by replacing the last equation with $\pi_{0_1} = 1$.

$$(\pi_0, \pi_1) \left(\begin{array}{cc|ccc} -6.0 & 5.0 & 1 & 0 & 1 \\ 3.0 & -3.5 & 0 & 0 & 0 \\ \hline .0610 & .1575 & -5.9832 & 5.6938 & 0 \\ 2.0000 & 2.000 & 3.000 & -12.0000 & 0 \\ .0089 & .1555 & .0040 & 3.2945 & 0 \end{array} \right) = (0, 0 \mid 0, 0, 1)$$

Solution

$$(\pi_0, \pi_1) = (1.0,\ 1.7169,\ \mid .3730,\ .4095,\ .8470)$$

The normalization constant is

$$\begin{aligned}\alpha &= \pi_0 e + \pi_1 (I - R)^{-1} e \\ &= (1.0,\ 1.7169)e + (.3730,\ .4095,\ .8470) \begin{pmatrix} 1.3395 & .2584 & .3546 \\ 0 & 1 & 0 \\ .0600 & .1044 & 1.2764 \end{pmatrix} e \\ &= 2.7169 + 2.3582 = 5.0751\end{aligned}$$

Thus:

$$\pi_0/\alpha = (.1970,\ .3383), \quad \text{and} \quad \pi_1/\alpha = (.0735,\ .0807,\ .1669).$$

Successive subcomponents are now computed from $\pi_k = \pi_{k-1} R$.

$$\pi_2 = \pi_1 R = (.0735,\ .0807,\ .1669) \begin{pmatrix} .2440 & .1734 & .2100 \\ 0 & 0 & 0 \\ .0356 & .0736 & .1669 \end{pmatrix}$$

$$= (.0239,\ .0250,\ .0499)$$

and

$$\pi_3 = \pi_2 R = (.0239,\ .0250,\ .0499) \begin{pmatrix} .2440 & .1734 & .2100 \\ 0 & 0 & 0 \\ .0356 & .0736 & .1669 \end{pmatrix}$$

$$= (.0076,\ .0078,\ .0135)$$

and so on.

Simplifications occur when the initial B blocks have the same dimensions as the A blocks and when

$$Q = \begin{pmatrix} B_{00} & A_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_0 & 0 & 0 & 0 & 0 & \cdots \\ B_{20} & A_2 & A_1 & A_0 & 0 & 0 & 0 & \cdots \\ B_{30} & A_3 & A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ B_{40} & A_4 & A_3 & A_2 & A_1 & A_0 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

In this case

$$\pi_i = \pi_0 R^i, \quad \text{for} \;\; i = 1, 2, \ldots,$$

$\sum_{i=0}^{\infty} R^i B_{i0}$ is an infinitesimal generator matrix

π_0 is the stationary probability vector of $\sum_{i=0}^{\infty} R^i B_{i0}$

— normalized so that $\pi_0 (I - R)^{-1} e = 1$.

Also, in some applications more than two boundary columns can occur.

$$Q = \begin{pmatrix} B_{00} & B_{01} & B_{02} & A_0 & & & & & & & & & \\ B_{10} & B_{11} & B_{12} & A_1 & A_0 & & & & & & & & \\ B_{20} & B_{21} & B_{22} & A_2 & A_1 & A_0 & & & & & & & \\ B_{30} & B_{31} & B_{32} & A_3 & A_2 & A_1 & A_0 & & & & & & \\ B_{40} & B_{41} & B_{42} & A_4 & A_3 & A_2 & A_1 & A_0 & & & & & \\ B_{50} & B_{51} & B_{52} & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & & & & \\ B_{60} & B_{61} & B_{62} & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & & & \\ B_{70} & B_{71} & B_{72} & A_7 & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & & \\ B_{80} & B_{81} & B_{82} & A_8 & A_7 & A_6 & A_5 & A_4 & 0 & 0 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

At present, this matrix is *not* block lower Hessenberg.

Restructured into the form

$$\left(\begin{array}{ccc|ccc|ccc|ccc|c}
B_{00} & B_{01} & B_{02} & A_0 & & & & & & & & & \\
B_{10} & B_{11} & B_{12} & A_1 & A_0 & & & & & & & & \\
B_{20} & B_{21} & B_{22} & A_2 & A_1 & A_0 & & & & & & & \\
\hline
B_{30} & B_{31} & B_{32} & A_3 & A_2 & A_1 & A_0 & & & & & & \\
B_{40} & B_{41} & B_{42} & A_4 & A_3 & A_2 & A_1 & A_0 & & & & & \\
B_{50} & B_{51} & B_{52} & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & & & & \\
\hline
B_{60} & B_{61} & B_{62} & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & & & \\
B_{70} & B_{71} & B_{72} & A_7 & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 & A_0 & & \\
B_{80} & B_{81} & B_{82} & A_8 & A_7 & A_6 & A_5 & A_4 & 0 & 0 & A_1 & A_0 & \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}\right)$$

$$\overline{A_0} = \begin{pmatrix} A_0 & & \\ A_1 & A_0 & \\ A_2 & A_1 & A_0 \end{pmatrix}, \quad \overline{A_1} = \begin{pmatrix} A_3 & A_2 & A_1 \\ A_4 & A_3 & A_2 \\ A_5 & A_4 & A_3 \end{pmatrix}, \quad \overline{B_{00}} = \begin{pmatrix} B_{00} & B_{01} & B_{02} \\ B_{10} & B_{11} & B_{12} \\ B_{20} & B_{21} & B_{22} \end{pmatrix}, \quad \cdots$$

Block Upper-Hessenberg Markov Chains

For QBD and GI/M/1-type processes, we posed the problem in terms of continuous-time Markov chains.
Discrete-time Markov chains can be treated if the matrix inverse A_1^{-1} is replaced with the inverse $(I - A_1)^{-1}$.

This time we shall consider the discrete-time case.

$$P = \begin{pmatrix} B_{00} & B_{01} & B_{02} & B_{03} & \cdots & B_{0j} & \cdots \\ B_{10} & A_1 & A_2 & A_3 & \cdots & A_j & \cdots \\ 0 & A_0 & A_1 & A_2 & \cdots & A_{j-1} & \cdots \\ 0 & 0 & A_0 & A_1 & \cdots & A_{j-2} & \cdots \\ 0 & 0 & 0 & A_0 & \cdots & A_{j-3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$A = \sum_{i=0}^{\infty} A_i$ is a stochastic matrix assumed to be irreducible.

$$\pi_A A = \pi_A, \quad \text{and} \quad \pi_A e = 1.$$

P is known to be positive-recurrent if

$$\pi_A \left(\sum_{i=1}^{\infty} i A_i \; e \right) \equiv \pi_A \; b \; < 1. \tag{12}$$

We seek to compute π from $\pi P = \pi$. As before, we partition π conformally with P, i.e.

$$\pi = (\pi_0, \pi_1, \pi_2, \cdots)$$

where

$$\pi_i = (\pi(i,1), \pi(i,2), \cdots \pi(i,K))$$

The analysis of M/G/1-type processes is more complicated than that of QBD or GI/M/1-type processes because the subvectors π_i no longer have a matrix geometric relationship with one another.

The key to solving upper block-Hessenberg structured Markov chains is the computation of a certain stochastic matrix G.
G_{ij} is the conditional probability that starting in state i of any level $n \geq 2$, the process enters level $n-1$ for the first time by arriving at state j of that level.

This matrix satisfies the fixed point equation

$$G = \sum_{i=0}^{\infty} A_i G^i$$

and is indeed is the minimal non-negative solution of

$$X = \sum_{i=0}^{\infty} A_i X^i.$$

It can be found by means of the iteration

$$G_{(0)} = 0; \quad G_{(k+1)} = \sum_{i=0}^{\infty} A_i G_{(k)}^i = 0, \quad k = 0, 1, \ldots$$

Once the matrix G has been computed, then successive components of π can be found. From $\pi P = \pi$ $\pi(I - P) = 0$,

$$(\pi_0,\ \pi_1,\ \cdots,\ \pi_j,\ \cdots) \left(\begin{array}{c|cccccc} I - B_{00} & -B_{01} & -B_{02} & -B_{03} & \cdots & -B_{0j} & \cdots \\ \hline -B_{10} & I - A_1 & -A_2 & -A_3 & \cdots & -A_j & \cdots \\ 0 & -A_0 & I - A_1 & -A_2 & \cdots & -A_{j-1} & \cdots \\ 0 & 0 & -A_0 & I - A_1 & \cdots & -A_{j-2} & \cdots \\ 0 & 0 & 0 & -A_0 & \cdots & -A_{j-3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right) \tag{13}$$

$$= (0,\ 0,\ \cdots\ 0,\ \cdots).$$

The submatrix in the lower right block is block Toeplitz.
There is a decomposition of this Toeplitz matrix into a block upper triangular matrix U and block lower triangular matrix L.

$$U = \begin{pmatrix} A_1^* & A_2^* & A_3^* & A_4^* & \cdots \\ 0 & A_1^* & A_2^* & A_3^* & \cdots \\ 0 & 0 & A_1^* & A_2^* & \cdots \\ 0 & 0 & 0 & A_1^* & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} I & 0 & 0 & 0 & \cdots \\ -G & I & 0 & 0 & \cdots \\ 0 & -G & I & 0 & \cdots \\ 0 & 0 & -G & I & \cdots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}.$$

Once the matrix G has been formed then L is known.

The inverse of L can be written in terms of the powers of G.

$$\begin{pmatrix} I & 0 & 0 & 0 & \cdots \\ -G & I & 0 & 0 & \cdots \\ 0 & -G & I & 0 & \cdots \\ 0 & 0 & -G & I & \cdots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 & \cdots \\ G & I & 0 & 0 & \cdots \\ G^2 & G & I & 0 & \cdots \\ G^3 & G^2 & G & I & \cdots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}.$$

From Equation (13),

$$(\pi_0,\ \pi_1,\ \cdots,\ \pi_j,\ \cdots)\left(\begin{array}{c|cccccc} I-B_{00} & -B_{01} & -B_{02} & -B_{03} & \cdots & -B_{0j} & \cdots \\ \hline -B_{10} & & & & & & \\ 0 & & & & & & \\ 0 & & & UL & & & \\ 0 & & & & & & \\ \vdots & & & & & & \end{array}\right)$$

$$= (0,\ 0,\ \cdots\ 0,\ \cdots)$$

which allows us to write

$$\pi_0\,(-B_{01},\ -B_{02},\ \cdots\ ) + (\pi_1,\ \pi_2,\ \cdots\ )\,UL = 0$$

or

$$\pi_0\,(B_{01},\ B_{02},\ \cdots\ )\,L^{-1} = (\pi_1,\ \pi_2,\ \cdots\ )\,U,$$

$$\pi_0 \left(B_{01},\ B_{02},\ \cdots \right) \begin{pmatrix} I & 0 & 0 & 0 & \cdots \\ G & I & 0 & 0 & \cdots \\ G^2 & G & I & 0 & \cdots \\ G^3 & G^2 & G & I & \cdots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix} = \left(\pi_1,\ \pi_2,\ \cdots \right) U$$

$$\pi_0 \left(B^*_{01},\ B^*_{02},\ \cdots \right) = \left(\pi_1,\ \pi_2,\ \cdots \right) U \qquad (14)$$

$$\begin{aligned} B^*_{01} &= B_{01} + B_{02}G + B_{03}G^2 + \cdots = \sum_{k=1}^{\infty} B_{0k} G^{k-1} \\ B^*_{02} &= B_{02} + B_{03}G + B_{04}G^2 + \cdots = \sum_{k=2}^{\infty} B_{0k} G^{k-2} \\ &\vdots \\ B^*_{0i} &= B_{0i} + B_{0,i+1}G + B_{0,i+2}G^2 + \cdots = \sum_{k=i}^{\infty} B_{0k} G^{k-i} \end{aligned}$$

Can compute the successive components of π once π_0 and U are known:

$$\pi_0 \left(B^*_{01},\ B^*_{02},\ \cdots \right) = \left(\pi_1,\ \pi_2,\ \cdots \right) \begin{pmatrix} A^*_1 & A^*_2 & A^*_3 & A^*_4 & \cdots \\ 0 & A^*_1 & A^*_2 & A^*_3 & \cdots \\ 0 & 0 & A^*_1 & A^*_2 & \cdots \\ 0 & 0 & 0 & A^*_1 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

Observe that

$$\begin{aligned} \pi_0 B^*_{01} &= \pi_1 A^*_1 & \Longrightarrow \quad & \pi_1 = \pi_0 B^*_{01} A^{*\,-1}_1 \\ \pi_0 B^*_{02} &= \pi_1 A^*_2 + \pi_2 A^*_1 & \Longrightarrow \quad & \pi_2 = \pi_0 B^*_{02} A^{*\,-1}_1 - \pi_1 A^*_2 A^{*\,-1}_1 \\ \pi_0 B^*_{03} &= \pi_1 A^*_3 + \pi_2 A^*_2 + \pi_3 A^*_1 & \Longrightarrow \quad & \pi_3 = \pi_0 B^*_{03} A^{*\,-1}_1 - \pi_1 A^*_3 A^{*\,-1}_1 - \pi_2 A^*_2 A^{*\,-1}_1 \\ & & \vdots & \end{aligned}$$

In general:

$$\pi_i = \left(\pi_0 B^*_{0i} - \pi_1 A^*_i - \pi_2 A^*_{i-1} - \cdots - \pi_{i-1} A^*_2\right) A_1^{*\,-1}, \quad i = 1, 2, \ldots$$

$$= \left(\pi_0 B^*_{0i} - \sum_{k=1}^{i-1} \pi_k A^*_{i-k+1}\right) A_1^{*\,-1}.$$

First subvector π_0 : $\pi_0 \left(B^*_{01},\ B^*_{02},\ \cdots\ \right) = \left(\pi_1,\ \pi_2,\ \cdots\ \right) U$

$$(\pi_0,\ \pi_1,\ \cdots,\ \pi_j,\ \cdots) \left(\begin{array}{c|cccccc} I - B_{00} & -B^*_{01} & -B^*_{02} & -B^*_{03} & \cdots & -B^*_{0j} & \cdots \\ \hline -B_{10} & A^*_1 & A^*_2 & A^*_3 & \cdots & A^*_j & \cdots \\ 0 & 0 & A^*_1 & A^*_2 & \cdots & A_{j-1} & \cdots \\ 0 & 0 & 0 & A^*_1 & \cdots & A^*_{j-2} & \cdots \\ 0 & 0 & 0 & 0 & & \vdots & \cdots \\ \vdots & & & & & & \end{array}\right)$$

$$= (0,\ 0,\ \cdots\ 0,\ \cdots)$$

First two equations:

$$\pi_0 \, (I - B_{00}) - \pi_1 B_{10} = 0, \qquad -\pi_0 B^*_{01} + \pi_1 A^*_1 = 0.$$

Second gives

$$\pi_1 = \pi_0 B^*_{01} A^{*\,-1}_1.$$

Substitute into first

$$\pi_0 \, (I - B_{00}) - \pi_0 B^*_{01} A^{*\,-1}_1 B_{10} = 0$$

or

$$\pi_0 \left(I - B_{00} - B^*_{01} A^{*\,-1}_1 B_{10} \right) = 0$$

Can now compute π_0 to a multiplicative constant.

To normalize, enforce the condition:

$$\pi_0 e + \pi_0 \left(\sum_{i=1}^{\infty} B^*_{0i} \right) \left(\sum_{i=1}^{\infty} A^*_i \right)^{-1} e = 1. \qquad (15)$$

Computation of the matrix U from

$$UL = \begin{pmatrix} I - A_1 & -A_2 & -A_3 & \cdots & -A_j & \cdots \\ -A_0 & I - A_1 & -A_2 & \cdots & -A_{j-1} & \cdots \\ 0 & -A_0 & I - A_1 & \cdots & -A_{j-2} & \cdots \\ 0 & 0 & -A_0 & \cdots & -A_{j-3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

$$\begin{pmatrix} A_1^* & A_2^* & A_3^* & A_4^* & \cdots \\ 0 & A_1^* & A_2^* & A_3^* & \cdots \\ 0 & 0 & A_1^* & A_2^* & \cdots \\ 0 & 0 & 0 & A_1^* & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} I - A_1 & -A_2 & -A_3 & \cdots & -A_j & \cdots \\ -A_0 & I - A_1 & -A_2 & \cdots & -A_{j-1} & \cdots \\ 0 & -A_0 & I - A_1 & \cdots & -A_{j-2} & \cdots \\ 0 & 0 & -A_0 & \cdots & -A_{j-3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 & \cdots \\ G & I & 0 & 0 & \cdots \\ G^2 & G & I & 0 & \cdots \\ G^3 & G^2 & G & I & \cdots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}$$

$$A_1^* = I - A_1 - A_2 G - A_3 G^2 - A_4 G^3 - \cdots \; = \; I - \sum_{k=1}^{\infty} A_k G^{k-1}$$

$$A_2^* = -A_2 - A_3 G - A_4 G^2 - A_5 G^3 - \cdots \; = \; -\sum_{k=2}^{\infty} A_k G^{k-2}$$

$$
\begin{aligned}
A_1^* &= I - A_1 - A_2 G - A_3 G^2 - A_4 G^3 - \cdots = I - \sum_{k=1}^{\infty} A_k G^{k-1} \\
A_2^* &= -A_2 - A_3 G - A_4 G^2 - A_5 G^3 - \cdots = -\sum_{k=2}^{\infty} A_k G^{k-2} \\
A_3^* &= -A_3 - A_4 G - A_5 G^2 - A_6 G^3 - \cdots = -\sum_{k=3}^{\infty} A_k G^{k-3} \\
&\vdots \\
A_i^* &= -A_i - A_{i+1} G - A_{i+2} G^2 - A_{i+3} G^3 - \cdots = -\sum_{k=i}^{\infty} A_k G^{k-i}, \quad i \geq 2.
\end{aligned}
$$

We now have all the results we need.

The basic algorithm is

- Construct the matrix G.
- Obtain π_0 by solving the system of equations
  $\pi_0 \left(I - B_{00} - B_{01}^* A_1^{*\,-1} B_{10}\right) = 0$,
  subject to the normalizing condition, Equation (15).
- Compute π_1 from $\pi_1 = \pi_0 B_{01}^* A_1^{*\,-1}$.
- Find all other required π_i from
  $\pi_i = \left(\pi_0 B_{0i}^* - \sum_{k=1}^{i-1} \pi_k A_{i-k+1}^*\right) A_1^{*\,-1}$.

where

$$B_{0i}^* = \sum_{k=i}^{\infty} B_{0k} G^{k-i}, \quad i \geq 1; \quad A_1^* = I - \sum_{k=1}^{\infty} A_k G^{k-1}$$

$$\text{and} \quad A_i^* = -\sum_{k=i}^{\infty} A_k G^{k-i}, \quad i \geq 2.$$

Computational questions:

(1) The matrix G. The iterative procedure suggested is very slow:

$$G_{(0)} = 0; \quad G_{(k+1)} = \sum_{i=0}^{\infty} A_i G_{(k)}^i, \quad k = 0, 1, \ldots$$

Faster variant from Neuts:

$$G_{(0)} = 0; \quad G_{(k+1)} = (I - A_1)^{-1} \left(A_0 + \sum_{i=2}^{\infty} A_i G_{(k)}^i \right), \quad k = 0, 1, \ldots$$

Among fixed point iterations, Bini and Meini has the fastest convergence

$$G_{(0)} = 0; \quad G_{(k+1)} = \left(I - \sum_{i=1}^{\infty} A_i G_{(k)}^{i-1} \right)^{-1} A_0, \quad k = 0, 1, \ldots$$

More advanced techniques based on cyclic reduction have been developed and converge much faster.

2) Computation of infinite summations:

Frequently the structure of the matrix is such that A_k and B_k are zero for relatively small values of k.

Since $\sum_{k=0}^{\infty} A_k$ and $\sum_{k=0}^{\infty} B_k$ are stochastic A_k and B_k are negligibly small for large values of i and can be set to zero once k exceeds some threshold k_M.

When k_M is not small, finite summations of the type $\sum_{k=i}^{k_M} Z_k G^{k-i}$ should be evaluated using Horner's rule. For example, if $k_M = 5$

$$Z_1^* = \sum_{k=1}^{5} Z_k G^{k-1} = Z_1 G^4 + Z_2 G^3 + Z_3 G^2 + Z_4 G + A_5$$

should be evaluated from the inner-most parenthesis outwards as

$$Z_1^* = \Big(\big[(Z_1 G + Z_2) G + Z_3 \big] G + Z_4 \Big) G + Z_5.$$

Example:

Same as before but with incorporates additional transitions ($\zeta_1 = 1/48$ and $\zeta_2 = 1/16$) to higher numbered non-neighboring states.

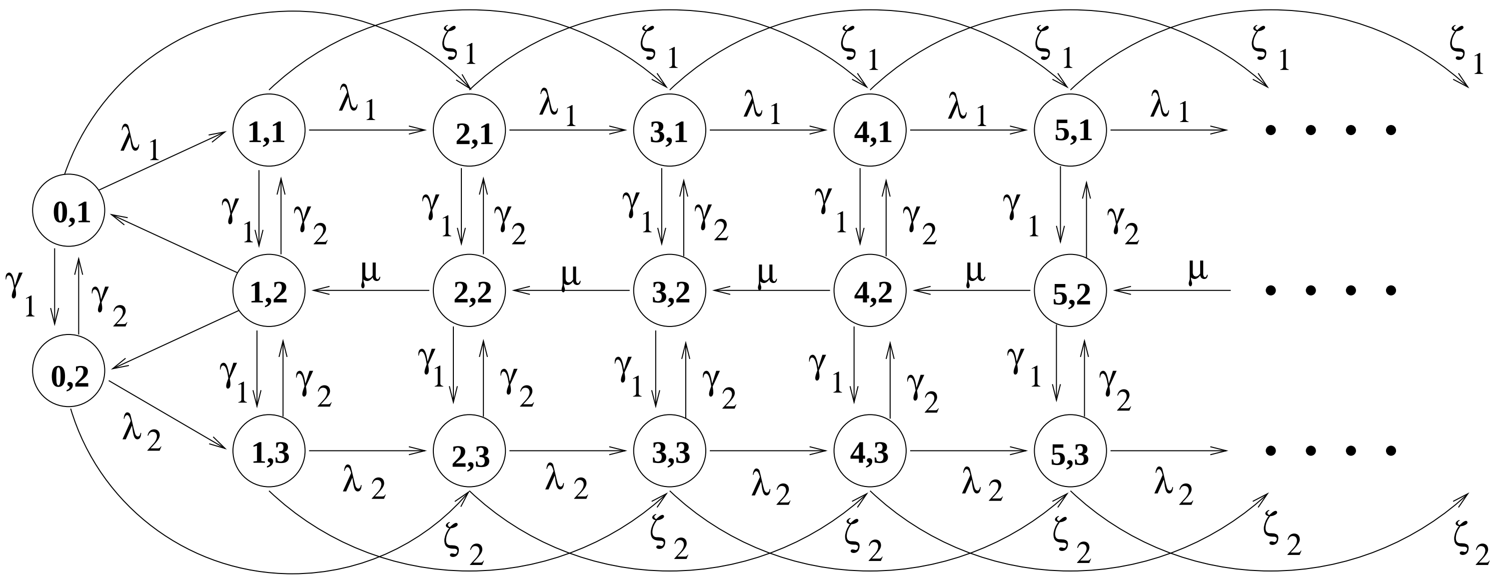


Figure 3: State transition diagram for an M/G/1-type process.

$$
P = \left(\begin{array}{cc|ccc|ccc|ccc|}
23/48 & 5/12 & 1/12 & & & 1/48 & & & & & \\
1/4 & 31/48 & & & 1/24 & & & 1/16 & & & \\
\hline
 & & 23/48 & 5/12 & & 1/12 & & & 1/48 & & \\
1/3 & 1/3 & 1/4 & & 1/12 & & & & & & \\
 & & & 1/4 & 31/48 & & & 1/24 & & & 1/16 \\
\hline
 & & & & & 23/48 & 5/12 & & 1/12 & & \\
 & & & 2/3 & & 1/4 & & 1/12 & & & \\
 & & & & & & 1/4 & 31/48 & & & 1/24 \\
\hline
 & & & & & & & & 1/2 & 5/12 & \\
 & & & & & & 2/3 & & 1/4 & & 1/12 \\
 & & & & & & & & & 1/4 & 31/48 \\
\hline
 & & & & & \ddots & & & \ddots & &
\end{array}\right.
$$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 23/48 & 5/12 & 0 \\ 1/4 & 0 & 1/12 \\ 0 & 1/4 & 31/48 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1/12 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/24 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1/48 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/16 \end{pmatrix}, \quad B_{00} = \begin{pmatrix} 23/48 & 5/12 \\ 1/4 & 31/48 \end{pmatrix}, \quad B_{01} = \begin{pmatrix} 1/12 & 0 & 0 \\ 0 & 0 & 1/24 \end{pmatrix},$$

$$B_{02} = \begin{pmatrix} 1/48 & 0 & 0 \\ 0 & 0 & 1/16 \end{pmatrix} \quad \text{and} \quad B_{10} = \begin{pmatrix} 0 & 0 \\ 1/3 & 1/3 \\ 0 & 0 \end{pmatrix}.$$

First, using Equation (12), we verify that the Markov chain with transition probability matrix P is positive-recurrent.

$$A = A_0 + A_1 + A_2 + A_3 = \begin{pmatrix} .583333 & .416667 & 0 \\ .250000 & .666667 & .083333 \\ 0 & .250000 & .750000 \end{pmatrix}.$$

$$\pi_A = (.310345,\ .517241,\ .172414).$$

Also

$$b = (A_1 + 2A_2 + 3A_3)e = \begin{pmatrix} .708333 & .416667 & 0 \\ .250000 & 0 & .083333 \\ 0 & .250000 & .916667 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.125000 \\ 0.333333 \\ 1.166667 \end{pmatrix}.$$

The Markov chain is positive-recurrent since

$$\pi_A\ b\ = (.310345,\ .517241,\ .172414) \begin{pmatrix} 1.125000 \\ 0.333333 \\ 1.166667 \end{pmatrix} = .722701 < 1$$

Computation of the matrix G:

The ij element of G is the conditional probability that starting in state i of any level $n \geq 2$, the process enters level $n-1$ for the first time by arriving at state j of that level.

For this particular example this means that the elements in column 2 of G must all be equal to 1 and all other elements must be zero —- the only transitions from any level n to level $n-1$ are from and to the second element.

Nevertheless, let see how each of the three different fixed point formula actually perform.

We take the initial value, $G_{(0)}$, to be zero.

Formula #1: $G_{(k+1)} = \sum_{i=0}^{\infty} A_i G_{(k)}^i, \quad k = 0, 1, \ldots$

$$G_{(k+1)} = A_0 + A_1 G_{(k)} + A_2 G_{(k)}^2 + A_3 G_{(k)}^3$$

After 10 iterations, the computed matrix is

$$G_{(10)} = \begin{pmatrix} 0 & .867394 & 0 \\ 0 & .937152 & 0 \\ 0 & .766886 & 0 \end{pmatrix}.$$

Formula #2:

$G_{(k+1)} = (I - A_1)^{-1} \left(A_0 + \sum_{i=2}^{\infty} A_i G_{(k)}^i \right), \quad k = 0, 1, \ldots$

$$G_{(k+1)} = (I - A_1)^{-1} \left(A_0 + A_2 G_{(k)}^2 + A_3 G_{(k)}^3 \right)$$

After 10 iterations:

$$G_{(10)} = \begin{pmatrix} 0 & .999844 & 0 \\ 0 & .999934 & 0 \\ 0 & .999677 & 0 \end{pmatrix}.$$

Formula #3: $G_{(k+1)} = \left(I - \sum_{i=1}^{\infty} A_i G_{(k)}^{i-1}\right)^{-1} A_0, \quad k = 0, 1, \ldots$

$$G_{(k+1)} = \left(I - A_1 - A_2 G_{(k)} - A_3 G_{(k)}^2\right)^{-1} A_0$$

This is the fastest of the three. After 10 iterations:

$$G_{(10)} = \begin{pmatrix} 0 & .999954 & 0 \\ 0 & .999979 & 0 \\ 0 & .999889 & 0 \end{pmatrix}.$$

We continue with the algorithm using the exact value of G.

In preparation, we compute the following quantities, using the fact that $A_k = 0$ for $k > 3$ and $B_{0k} = 0$ for $k > 2$.

$$A_1^* = I - \sum_{k=1}^{\infty} A_k G^{k-1} = I - A_1 - A_2 G - A_3 G^2 = \begin{pmatrix} .520833 & -.520833 & 0 \\ -.250000 & 1 & -.083333 \\ 0 & -.354167 & .354167 \end{pmatrix}$$

$$A_2^* = -\sum_{k=2}^{\infty} A_k G^{k-2} = -(A_2 + A_3 G) = \begin{pmatrix} -.083333 & -.020833 & 0 \\ 0 & 0 & 0 \\ 0 & -.062500 & -.041667 \end{pmatrix}$$

$$A_3^* = -\sum_{k=3}^{\infty} A_k G^{k-3} = -A_3 = \begin{pmatrix} -.020833 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -.062500 \end{pmatrix}$$

$$B_{01}^* = \sum_{k=1}^{\infty} B_{0k} G^{k-1} = B_{01} + B_{02}G = \begin{pmatrix} .083333 & .020833 & 0 \\ 0 & .062500 & .041667 \end{pmatrix}$$

$$B_{02}^* = \sum_{k=2}^{\infty} B_{0k} G^{k-2} = B_{02} = \begin{pmatrix} .020833 & 0 & 0 \\ 0 & 0 & .062500 \end{pmatrix}$$

$$A_1^{*\,-1} = \begin{pmatrix} 2.640 & 1.50 & .352941 \\ .720 & 1.50 & .352941 \\ .720 & 1.50 & 3.176470 \end{pmatrix}$$

Now compute the initial subvector, π_0, from

$$0 = \pi_0 \left(I - B_{00} - B_{01}^* A_1^{*\,-1} B_{10} \right) = \pi_0 \begin{pmatrix} .468750 & -.468750 \\ -.302083 & .302083 \end{pmatrix}$$

gives (un-normalized)

$$\pi_0 = (.541701,\ .840571).$$

Normalization:

$$\pi_0 e + \pi_0 \left(\sum_{i=1}^{\infty} B_{0i}^* \right) \left(\sum_{i=1}^{\infty} A_i^* \right)^{-1} e = 1.$$

i.e.,

$$\pi_0 e + \pi_0 \left(B_{01}^* + B_{02}^* \right) \left(A_1^* + A_2^* + A_3^* \right)^{-1} e = 1.$$

Evaluating

$$\left(B_{01}^* + B_{02}^* \right) \left(A_1^* + A_2^* + A_3^* \right)^{-1}$$

$$= \begin{pmatrix} .104167 & .020833 & 0 \\ 0 & .062500 & .104167 \end{pmatrix} \begin{pmatrix} .416667 & -.541667 & 0 \\ -.250000 & 1 & -.083333 \\ 0 & -.416667 & .250000 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} .424870 & .291451 & .097150 \\ .264249 & .440415 & .563472 \end{pmatrix}$$

$$(.541701,\ .840571)\begin{pmatrix}1\\1\end{pmatrix}+(.541701,\ .840571)\begin{pmatrix}.424870 & .291451 & .097150\\ .264249 & .440415 & .563472\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= 2.888888$$

Finally, initial subvector is

$$\pi_0 = (.541701,\ .840571)/2.888888 = (.187512,\ .290967)$$

We can now find π_1 from the relationship $\pi_1 = \pi_0 B^*_{01} A^{*\,-1}_1 =$

$$(.187512,\ .290967)\begin{pmatrix}.083333 & .020833 & 0\\ 0 & .062500 & .041667\end{pmatrix}\begin{pmatrix}2.640 & 1.50 & .352941\\ .720 & 1.50 & .352941\\ .720 & 1.50 & 3.176470\end{pmatrix}$$

$$= (.065888,\ .074762,\ .0518225).$$

Finally, all needed remaining subcomponents of π can be found from

$$\pi_i = \left(\pi_0 B^*_{0i} - \sum_{k=1}^{i-1} \pi_k A^*_{i-k+1} \right) A_1^{*\,-1}$$

$$\begin{aligned}
\pi_2 &= \left(\pi_0 B^*_{02} - \pi_1 A^*_2\right) A_1^{*\,-1} \\
&= (.042777,\ .051530,\ .069569) \\
\pi_3 &= \left(\pi_0 B^*_{03} - \pi_1 A^*_3 - \pi_2 A^*_2\right) A_1^{*\,-1} = \left(-\pi_1 A^*_3 - \pi_2 A^*_2\right) A_1^{*\,-1} \\
&= (.0212261,\ .024471,\ .023088) \\
\pi_4 &= \left(\pi_0 B^*_{04} - \pi_1 A^*_4 - \pi_2 A^*_3 - \pi_3 A^*_2\right) A_1^{*\,-1} = \left(-\pi_2 A^*_3 - \pi_3 A^*_2\right) A_1^{*\,-1} \\
&= (.012203,\ .014783,\ .018471) \\
&\vdots
\end{aligned}$$

The probability that the Markov chain is in any level i is given by $\|\pi_i\|_1$.

Thus the probabilities of this Markov chain being in the first 5 levels

$$\|\pi_0\|_1 = .478479, \quad \|\pi_1\|_1 = .192473, \quad \|\pi_2\|_1 = .163876,$$

$$\|\pi_3\|_1 = .068785, \quad \|\pi_4\|_1 = .045457$$

The sum of these five probabilities is 0.949070.

Phase Type Distributions

Goals:

(1) Find ways to model general distributions while maintaining the tractability of the exponential.

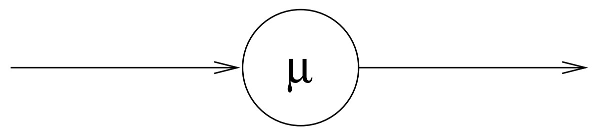
(2) Find way to form a distribution having some given expectation and variance.

Phase-type distributions are represented as the passage through a succession of exponential phases or stages (and hence the name).

The Exponential Distribution

— consists of a single exponential phase.

Random variable X is exponentially distributed with parameter $\mu > 0$.



The diagram represents customers entering the phase from the left, spending an amount of time that is exponentially distributed with parameter μ within the phase and then exiting to the right.

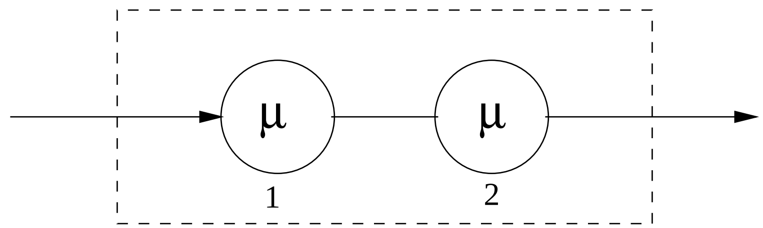
Exponential density function:

$$f_X(x) \equiv \frac{dF(x)}{dx} = \mu e^{-\mu x}, \quad x \geq 0$$

Expectation and variance, $E[X] = 1/\mu; \quad \sigma_X^2 = 1/\mu^2$.

The Erlang-2 Distribution

Service provided to a customer is expressed as one exponential phase followed by a second exponential phase.



Although both service phases are exponentially distributed with the same parameter, they are completely independent — the servicing process does *not* contain two independent servers.

Probability density function of each of the phases:

$$f_Y(y) = \mu e^{-\mu y}, \quad y \geq 0$$

Expectation and variance, $E[Y] = 1/\mu; \quad \sigma_Y^2 = 1/\mu^2$.

Total time in service is the sum of two independent and identically distributed exponential random variables. $X = Y + Y$

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_Y(y) f_Y(x-y)dy \\ &= \int_0^y \mu e^{-\mu y} \mu e^{-\mu(x-y)} dy \\ &= \mu^2 e^{-\mu x} \int_0^x dy = \mu^2 x e^{-\mu x}, \quad x \geq 0, \end{aligned}$$

and is equal to zero for $x \leq 0$ — the Erlang-2 distribution: E_2

The corresponding cumulative distribution function is given by

$$F_X(x) = 1 - e^{-\mu x} - \mu x e^{-\mu x} = 1 - e^{-\mu x}(1 + \mu x), \quad x \geq 0.$$

Laplace transform of the overall service time distribution:

$$\mathcal{L}_X(s) \equiv \int_0^\infty e^{-sx} f_X(x)dx$$

Laplace transform of each of the exponential phases:

$$\mathcal{L}_Y(s) \equiv \int_0^\infty e^{-sy} f_Y(y)dy.$$

Then

$$\mathcal{L}_X(s) = E[e^{-sx}] = E[e^{-s(y_1+y_2)}] = E[e^{-sy_1}]E[e^{-sy_2}] = \mathcal{L}_Y(s)\mathcal{L}_Y(s)$$

$$= \left(\frac{\mu}{s+\mu}\right)^2,$$

To invert, look up tables of transform pairs.

$$\frac{1}{(s+a)^{r+1}} \iff \frac{x^r}{r!}e^{-ax}. \tag{16}$$

Setting $a = \mu$ and $r = 1$ allows us to invert $\mathcal{L}_X(s)$ to obtain

$$f_X(x) = \mu^2 x e^{-\mu x} = \mu(\mu x)e^{-\mu x}, \quad x \geq 0$$

Moments may be found from the Laplace transform as

$$E[X^k] = (-1)^k \left. \frac{d^k}{ds^k}\mathcal{L}_X(s)\right|_{s=0} \quad \text{for } k = 1, 2, \ldots$$

$$E[X] = -\left.\frac{d}{ds}\mathcal{L}_X(s)\right|_{s=0} = -\mu^2 \left.\frac{d}{ds}(s+\mu)^{-2}\right|_{s=0} = \mu^2 \; 2(s+\mu)^{-3}\big|_{s=0} = \frac{2}{\mu}.$$

Time spent in service is the sum of two iid random variables:

$$E[X] = E[Y] + E[Y] = 1/\mu + 1/\mu = 2/\mu$$

$$\sigma_X^2 = \sigma_Y^2 + \sigma_Y^2 = \left(\frac{1}{\mu}\right)^2 + \left(\frac{1}{\mu}\right)^2 = \frac{2}{\mu^2}.$$

Example:

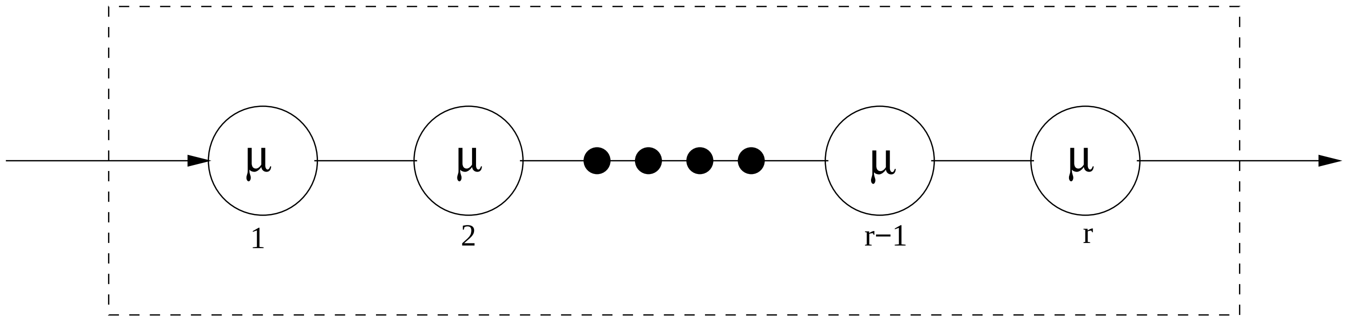
Exponential random variable with parameter μ;
Two phase Erlang-2 random variable, each phase having parameter 2μ.

	Mean	Variance
Exponential	$1/\mu$	$1/\mu^2$
Erlang-2	$1/\mu$	$1/2\mu^2$

An Erlang-2 random variable has less variability than an exponentially distributed random variable with the same mean.

The Erlang-r Distribution

A succession of r identical, but independent, exponential phases with parameter μ.



Probability density function at phase i:

$$f_Y(y) = \mu e^{-\mu y}; \quad y \geq 0$$

Expectation and variance per phase:

$$E[Y] = 1/\mu, \quad \text{and} \quad \sigma_Y^2 = 1/\mu^2 \quad \text{respectively.}$$

Distribution of total time spent is the sum of r iid random variables.

$$E[X] = r\left(\frac{1}{\mu}\right) = \frac{r}{\mu}; \qquad \sigma_X^2 = r\left(\frac{1}{\mu}\right)^2 = \frac{r}{\mu^2}.$$

$$\text{Laplace transform of the service time:} \quad \mathcal{L}_X(s) = \left(\frac{\mu}{s+\mu}\right)^r$$

$$\text{Using the transform pair:} \quad \frac{1}{(s+a)^{r+1}} \quad \Longleftrightarrow \quad \frac{x^r}{r!}e^{-ax} \quad \text{with } a = \mu$$

$$f_X(x) = \frac{\mu(\mu x)^{r-1}e^{-\mu x}}{(r-1)!}, \quad x \geq 0. \tag{17}$$

This is the Erlang-r probability density function.

The corresponding cumulative distribution function is given by

$$F_X(x) = 1 - e^{-\mu x}\sum_{i=0}^{r-1}\frac{(\mu x)^i}{i!}, \quad x \geq 0 \text{ and } r = 1, 2, \ldots \tag{18}$$

Differentiating $F_X(x)$ with respect to x shows that (18) is the distribution function with corresponding density function (17).

$$
\begin{aligned}
f_X(x) = \frac{d}{dx}F_X(x) &= \mu e^{-\mu x}\sum_{k=0}^{r-1}\frac{(\mu x)^k}{k!} - e^{-\mu x}\sum_{k=0}^{r-1}\frac{k(\mu x)^{k-1}\mu}{k!} \\
&= \mu e^{-\mu x} + \mu e^{-\mu x}\sum_{k=1}^{r-1}\frac{(\mu x)^k}{k!} - e^{-\mu x}\sum_{k=1}^{r-1}\frac{k(\mu x)^{k-1}\mu}{k!} \\
&= \mu e^{-\mu x} - \mu e^{-\mu x}\sum_{k=1}^{r-1}\left(\frac{k(\mu x)^{k-1}}{k!} - \frac{(\mu x)^k}{k!}\right) \\
&= \mu e^{-\mu x}\left\{1 - \sum_{k=1}^{r-1}\left(\frac{(\mu x)^{k-1}}{(k-1)!} - \frac{(\mu x)^k}{k!}\right)\right\} \\
&= \mu e^{-\mu x}\left\{1 - \left(1 - \frac{(\mu x)^{r-1}}{(r-1)!}\right)\right\} = \frac{\mu(\mu x)^{r-1}}{(r-1)!}e^{-\mu x}.
\end{aligned}
$$

The area under this density curve is equal to one. Let

$$I_r = \int_0^\infty \frac{\mu^r x^{r-1}\, e^{-\mu x}}{(r-1)!} dx, \quad r = 1, 2, \ldots$$

$I_1 = 1$ is the area under the exponential density curve.

Using integration by parts:
$\left(\int udv = uv - \int vdu\right.$ with $u = \mu^{r-1}x^{r-1}/(r-1)!$ and $\left. dv = \mu e^{-\mu x}dx\right)$

$$\begin{aligned} I_r &= \int_0^\infty \frac{\mu^{r-1}x^{r-1}\,\mu e^{-\mu x}}{(r-1)!} dx \\ &= \left. \frac{\mu^{r-1}x^{r-1}}{(r-1)!} e^{-\mu x} \right|_0^\infty + \int_0^\infty \frac{\mu^{r-1}x^{r-2}}{(r-2)!} e^{-\mu x} dx = 0 + I_{r-1} \end{aligned}$$

It follows that $I_r = 1$ for all $r \geq 1$.

Squared coefficient of variation, C_X^2, for the family of Erlang-r distributions.

$$C_X^2 = \frac{r/\mu^2}{(r/\mu)^2} = \frac{1}{r} < 1, \qquad \text{for } r \geq 2.$$

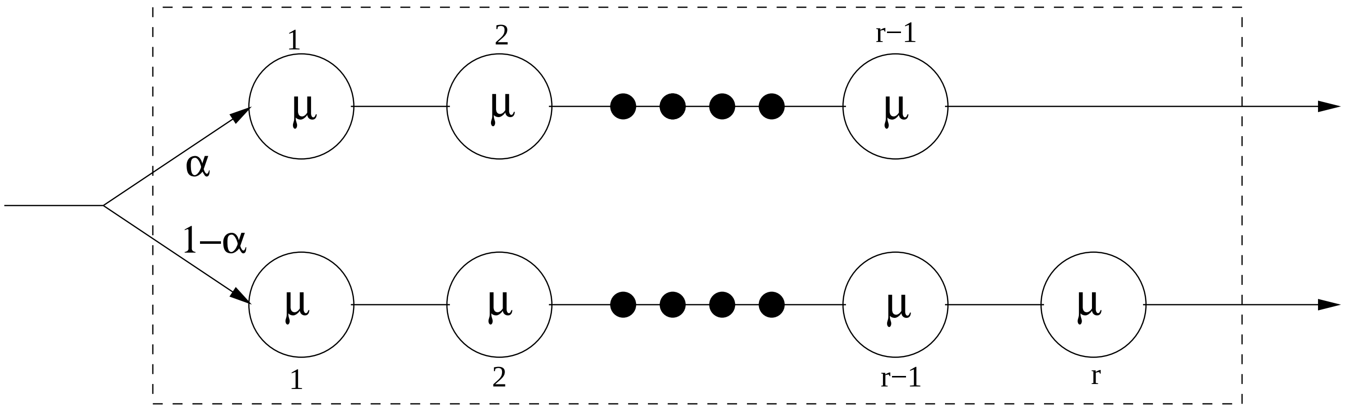
"More regular" than exponential random variables.

Possible values:

$$\frac{1}{2}, \; \frac{1}{3}, \; \frac{1}{4}, \ldots$$

Allows us to approximate a *constant* distribution.

Mixing an Erlang-$(r-1)$ distribution with an Erlang-r distribution gives a distribution with $1/r \le C_X^2 \le 1/(r-1)$.

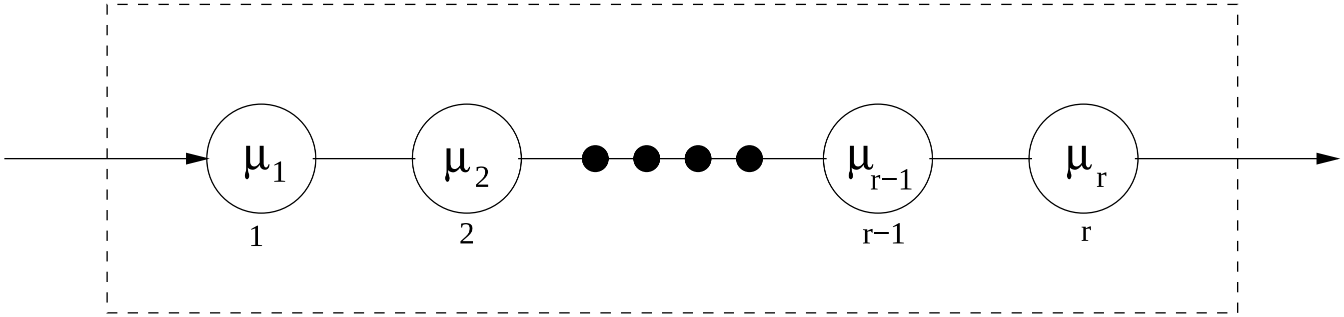


$\alpha = 1 \Rightarrow C_X^2 = 1/(r-1); \qquad \alpha = 0 \Rightarrow C_X^2 = 1/r.$

For a given $E[X]$ and $C_X^2 \in [1/r, 1/(r-1)]$ choose

$$\alpha = \frac{1}{1+C_X^2}\left(rC_X^2 - \sqrt{r(1+C_X^2) - r^2C_X^2}\right) \quad \text{and} \quad \mu = \frac{r-\alpha}{E[X]} \quad (19)$$

The Hypoexponential Distribution



Two phases: exponentially distributed RVs, Y_1 and Y_2: $\quad X = Y_1 + Y_2$.

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{Y_1}(y) f_{Y_2}(x-y)\,dy \\
&= \int_0^x \mu_1 e^{-\mu_1 y}\, \mu_2 e^{-\mu_2(x-y)}\,dy \\
&= \mu_1\mu_2 e^{-\mu_2 x} \int_0^x e^{-(\mu_1-\mu_2)y}\,dy \\
&= \frac{\mu_1\mu_2}{\mu_1-\mu_2}\left(e^{-\mu_2 x} - e^{-\mu_1 x}\right); \qquad x \geq 0.
\end{aligned}
$$

Corresponding cumulative distribution function is given by

$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1} e^{-\mu_2 x}, \quad x \geq 0.$$

Expectation, variance and squared coefficient of variation:

$$E[X] = \frac{1}{\mu_1} + \frac{1}{\mu_2}, \quad Var[X] = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}, \quad \text{and} \quad C_X^2 = \frac{\sqrt{\mu_1^2 + \mu_2^2}}{\mu_1 + \mu_2} < 1,$$

Laplace transform

$$\mathcal{L}_X(s) = \left(\frac{\mu_1}{s + \mu_1}\right)\left(\frac{\mu_2}{s + \mu_2}\right).$$

The Laplace transform for an r phase hypoexponential random variable:

$$\mathcal{L}_X(s) = \left(\frac{\mu_1}{s + \mu_1}\right)\left(\frac{\mu_2}{s + \mu_2}\right) \cdots \left(\frac{\mu_r}{s + \mu_r}\right).$$

The density function, $f_X(x)$, is the convolution of r exponential densities each with its own parameter μ_i and is given by

$$f_X(x) = \sum_{i=1}^{r} \alpha_i \mu_i e^{-\mu_i x}, \quad x > 0 \quad \text{where} \quad \alpha_i = \prod_{j=1,\, j\neq i}^{r} \frac{\mu_i}{\mu_j - \mu_i},$$

Expectation, variance and squared coefficient of variation:

$$E[X] = \sum_{i=1}^{r} \frac{1}{\mu_i}, \quad Var[X] = \sum_{i=1}^{r} \frac{1}{\mu_i^2} \quad \text{and} \quad C_X^2 = \frac{\sum_i 1/\mu_i^2}{\left(\sum_i 1/\mu_i\right)^2} \leq 1.$$

Observe that C_X^2 cannot exceed 1.

Example:

Three exponential phases with parameters $\mu_1 = 1$, $\mu_2 = 2$ and $\mu_3 = 3$.

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{3} \frac{1}{\mu_i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \\
Var[X] &= \sum_{i=1}^{3} \frac{1}{\mu_i^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} = \frac{49}{36} \\
C_X^2 &= \frac{49/36}{121/36} = \frac{36}{121} = 0.2975.
\end{aligned}
$$

Probability density function of X .

$$f_X(x) = \sum_{i=1}^{r} \alpha_i \mu_i e^{-\mu_i x}, \quad x > 0 \quad \text{where} \quad \alpha_i = \prod_{j=1,\, j\neq i}^{r} \frac{\mu_i}{\mu_j - \mu_i},$$

$$\alpha_1 = \prod_{j=1,\, j\neq i}^{r} \frac{\mu_1}{\mu_j - \mu_1} = \frac{\mu_1}{\mu_2 - \mu_1} \times \frac{\mu_1}{\mu_3 - \mu_1} = \frac{1}{1} \times \frac{1}{2} = \frac{1}{2}$$

$$\alpha_2 = \prod_{j=1,\, j\neq i}^{r} \frac{\mu_2}{\mu_j - \mu_2} = \frac{\mu_2}{\mu_1 - \mu_2} \times \frac{\mu_2}{\mu_3 - \mu_2} = \frac{2}{-1} \times \frac{2}{1} = -4$$

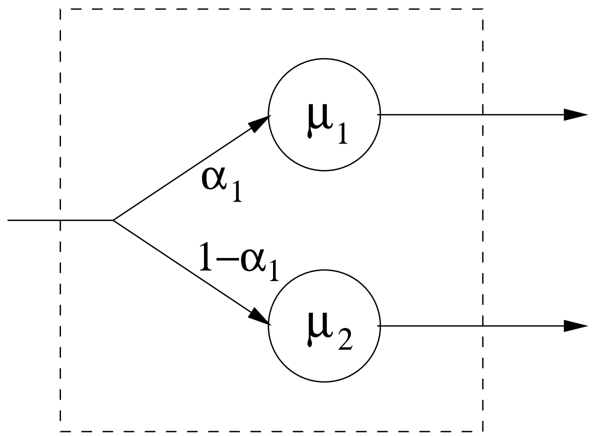
$$\alpha_3 = \prod_{j=1,\, j\neq i}^{r} \frac{\mu_3}{\mu_j - \mu_3} = \frac{\mu_3}{\mu_3 - \mu_1} \times \frac{\mu_3}{\mu_3 - \mu_2} = \frac{3}{-2} \times \frac{3}{-1} = \frac{9}{2}$$

It follows then that

$$f_X(x) = \sum_{i=1}^{3} \alpha_i \mu_i e^{-\mu_i x} = (0.5)e^{-x} + 8e^{-2x} + (13.5)e^{-3x}, \qquad x > 0$$

The Hyperexponential Distribution

Our goal now is to find a phase-type arrangement that gives larger coefficients of variation than the exponential.



The density function:

$$f_X(x) = \alpha_1 \mu_1 e^{-\mu_1 x} + \alpha_2 \mu_2 e^{-\mu_2 x}, \quad x \geq 0$$

Cumulative distribution function:

$$F_X(x) = \alpha_1 (1 - e^{-\mu_1 x}) + \alpha_2 (1 - e^{-\mu_2 x}), \quad x \geq 0.$$

Laplace transform:

$$\mathcal{L}_X(s) = \alpha_1 \frac{\mu_1}{s+\mu_1} + \alpha_2 \frac{\mu_2}{s+\mu_2}.$$

First and second moments:

$$E[X] = \frac{\alpha_1}{\mu_1} + \frac{\alpha_2}{\mu_2} \quad \text{and} \quad E[X^2] = \frac{2\alpha_1}{\mu_1^2} + \frac{2\alpha_2}{\mu_2^2}.$$

Variance:

$$Var[X] = E[X^2] - (E[X])^2.$$

Squared coefficient of variation:

$$C_X^2 = \frac{E[X^2] - (E[X])^2}{(E[X])^2} = \frac{E[X^2]}{(E[X])^2} - 1 = \frac{2\alpha_1/\mu_1^2 + 2\alpha_2/\mu_2^2}{(\alpha_1/\mu_1 + \alpha_2/\mu_2)^2} - 1 \geq 1.$$

Example:

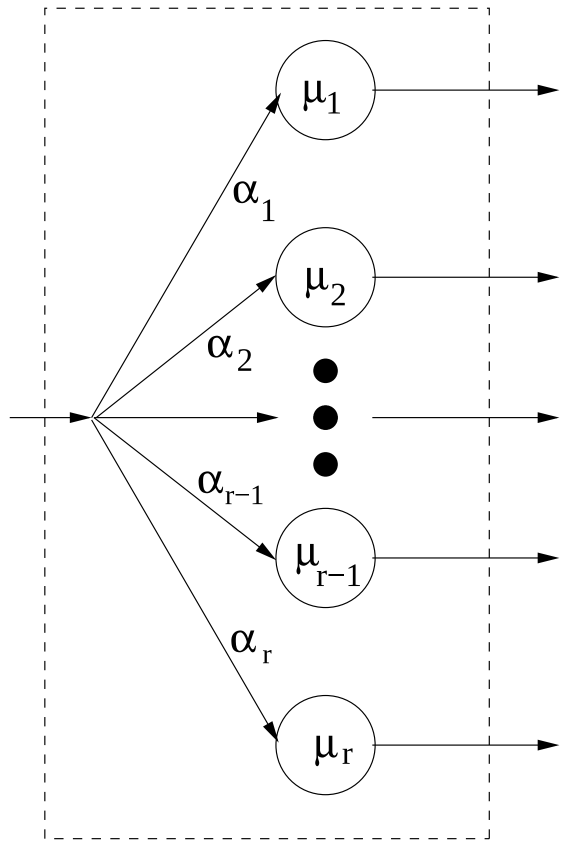
Given $\alpha_1 = 0.4$, $\mu_1 = 2$ and $\mu_2 = 1/2$.

$$E[X] = \frac{0.4}{2} + \frac{0.6}{0.5} = 1.40 \qquad E[X^2] = \frac{0.8}{4} + \frac{1.2}{0.25} = 5$$

$$\sigma_X = \sqrt{5 - 1.4^2} = \sqrt{3.04} = 1.7436$$

$$C_X^2 = \frac{5}{1.4^2} - 1 = 2.5510 - 1.0 = 1.5510$$

With r parallel phases and branching probabilities $\sum_{i=1}^{r} \alpha_i = 1$:



$$f_X(x) = \sum_{i=1}^{r} \alpha_i \mu_i e^{-\mu_i x}, \quad x \geq 0; \qquad \mathcal{L}_X(s) = \sum_{i=1}^{r} \alpha_i \frac{\mu_i}{s + \mu_i}.$$

$$E[X] = \sum_{i=1}^{r} \frac{\alpha_i}{\mu_i} \quad \text{and} \quad E[X^2] = 2\sum_{i=1}^{r} \frac{\alpha_i}{\mu_i^2}$$

$$C_X^2 = \frac{E[X^2]}{(E[X])^2} - 1 = \frac{2\sum_{i=1}^{r} \alpha_i/\mu_i^2}{\left(\sum_{i=1}^{r} \alpha_i/\mu_i\right)^2} - 1.$$

To show that this squared coefficient of variation is greater than or equal to one, it suffices to show that

$$\left(\sum_{i=1}^{r} \alpha_i/\mu_i\right)^2 \leq \sum_{i=1}^{r} \alpha_i/\mu_i^2.$$

Use the Cauchy-Schwartz inequality: for real a_i and b_i

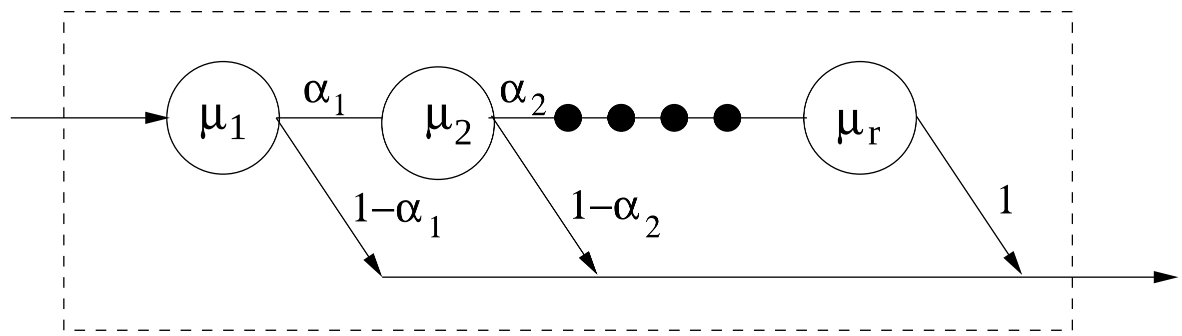
$$\left(\sum_i a_i b_i\right)^2 \leq \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right).$$

Substituting $a_i = \sqrt{\alpha_i}$ and $b_i = \sqrt{\alpha_i}/\mu_i$ implies that

$$\begin{aligned}
\left(\sum_i \frac{\alpha_i}{\mu_i}\right)^2 &= \left(\sum_i \sqrt{\alpha_i}\frac{\sqrt{\alpha_i}}{\mu_i}\right)^2 \\
&\le \sum_i \sqrt{\alpha_i}^2 \sum_i \left(\frac{\sqrt{\alpha_i}}{\mu_i}\right)^2, \qquad \text{using Cauchy} - \text{Schwartz} \\
&= \left(\sum_i \alpha_i\right)\left(\sum_i \frac{\alpha_i}{\mu_i^2}\right) = \sum_i \frac{\alpha_i}{\mu_i^2}, \qquad \text{since } \sum_i \alpha_i = 1.
\end{aligned}$$

Therefore $C_X^2 \ge 1$.

The Coxian Distribution

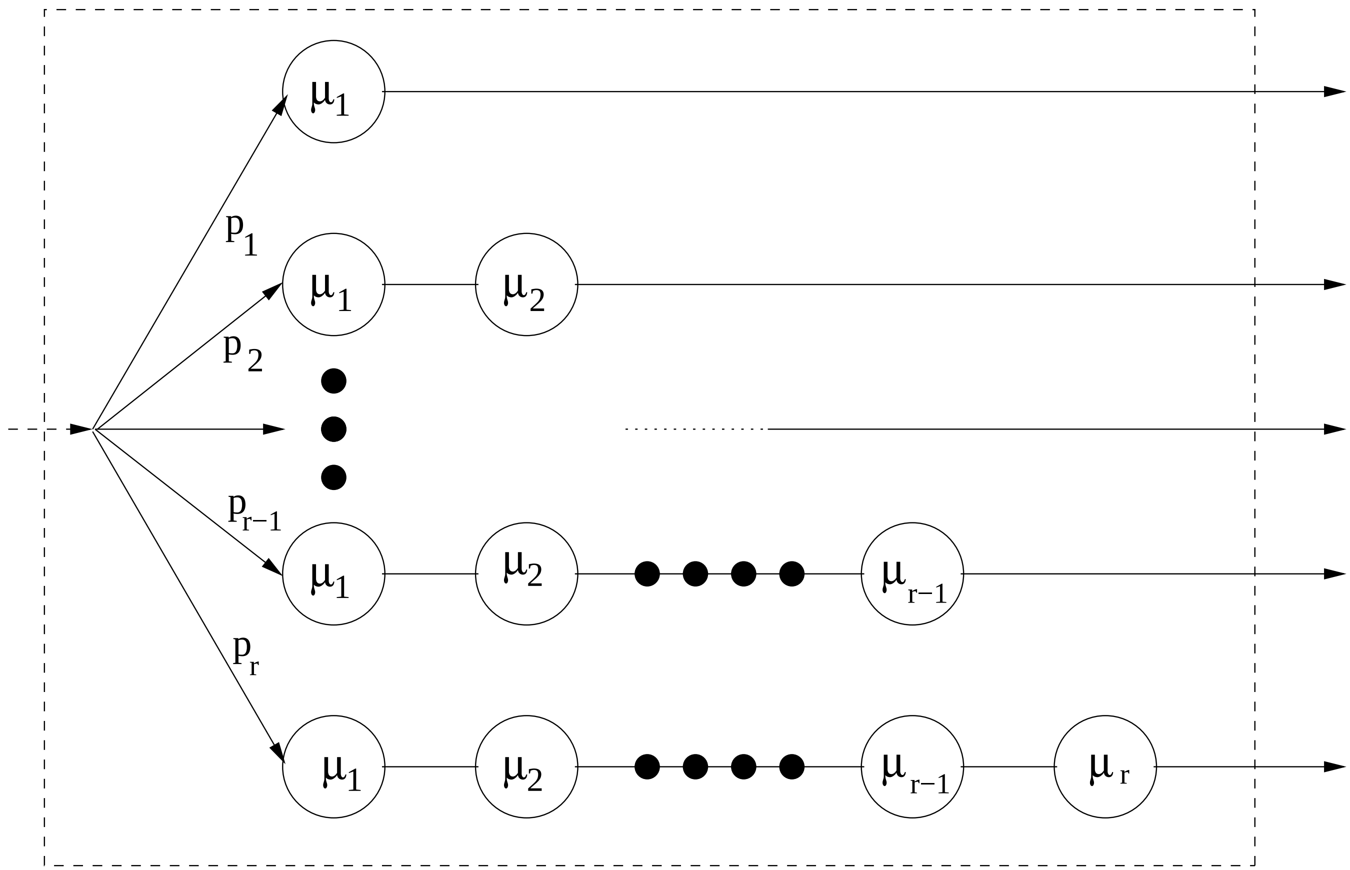


With probability $p_1 = 1 - \alpha_1$, process terminates after phase 1.

With probability $p_2 = \alpha_1(1 - \alpha_2)$, it terminates after phase 2.

With probability $p_k = (1 - \alpha_k) \prod_{i=1}^{k-1} \alpha_i$, it terminates after phase k.

A Coxian distribution may be represented as a probabilistic choice from among r *hypoexponential* distributions:


μ1
p1
μ1
μ2
p 2
pr−1
μ1
μ2
μr−1
pr
μ1
μ2
μr−1
μ r

Phase 1 is always executed and has expectation $E[X_1] = 1/\mu_1$.

Phase 2 is executed with probability α_1 and has $E[X_2] = 1/\mu_2$.

Phase $k > 1$ is executed with probability $\prod_{j=1}^{k-1} \alpha_j$ and $E[X_k] = 1/\mu_k$.

Since the expectation of a sum is equal to the sum of the expectations:

$$E[X] = \frac{1}{\mu_1} + \frac{\alpha_1}{\mu_2} + \frac{\alpha_1\,\alpha_2}{\mu_3} + \cdots + \frac{\alpha_1\,\alpha_2\,\cdots\,\alpha_{r-1}}{\mu_r} = \sum_{k=1}^{r} \frac{A_k}{\mu_k},$$

where $A_1 = 1$ and, for $k > 1$, $A_k = \prod_{j=1}^{k-1} \alpha_j$.

The case of a Cox-2 random variable is especially important.

$$E[X] = \frac{1}{\mu_1} + \alpha\frac{1}{\mu_2} = \frac{\mu_2 + \alpha\mu_1}{\mu_1\mu_2}, \qquad (20)$$

Laplace transform of a Cox-2

$$\mathcal{L}_X(s) = (1-\alpha)\frac{\mu_1}{s+\mu_1} + \alpha\frac{\mu_1}{s+\mu_1}\frac{\mu_2}{s+\mu_2}$$

$$\begin{aligned}
E[X^2] &= (-1)^2\frac{d^2}{ds^2}\left(\frac{(1-\alpha)\mu_1}{s+\mu_1} + \frac{\alpha\mu_1\mu_2}{(s+\mu_1)(s+\mu_2)}\right)\Bigg|_{s=0} \\
&= \frac{d}{ds}\left(\frac{-(1-\alpha)\mu_1}{(s+\mu_1)^2} + \alpha_1\mu_1\mu_2\left[\frac{-1}{(s+\mu_1)(s+\mu_2)^2} + \frac{-1}{(s+\mu_1)^2(s+\mu_2)}\right]\right) \\
&= \frac{2(1-\alpha)\mu_1}{(s+\mu_1)^3} + \frac{2\alpha\mu_1\mu_2}{(s+\mu_1)(s+\mu_2)^3} + \frac{\alpha\mu_1\mu_2}{(s+\mu_1)^2(s+\mu_2)^2} \\
&\qquad + \frac{2\alpha\mu_1\mu_2}{(s+\mu_1)^3(s+\mu_2)} + \frac{\alpha\mu_1\mu_2}{(s+\mu_1)^2(s+\mu_2)^2}\Bigg|_{s=0}
\end{aligned}$$

$$= \frac{2(1-\alpha)}{\mu_1^2} + \frac{2\alpha}{\mu_2^2} + \frac{\alpha}{\mu_1\mu_2} + \frac{2\alpha}{\mu_1^2} + \frac{\alpha}{\mu_1\mu_2}$$

$$= \frac{2}{\mu_1^2} + \frac{2\alpha}{\mu_2^2} + \frac{2\alpha}{\mu_1\mu_2}$$

$$Var[X] = \left(\frac{2}{\mu_1^2} + \frac{2\alpha}{\mu_2^2} + \frac{2\alpha}{\mu_1\mu_2}\right) - \left(\frac{1}{\mu_1} + \frac{\alpha}{\mu_2}\right)^2$$

$$= \frac{2\mu_1^2 + 2\alpha\mu_1^2 + 2\alpha\mu_1\mu_2}{\mu_1^2\mu_2^2} - \frac{(\mu_2 + \alpha\mu_1)^2}{\mu_1^2\mu_2^2}$$

$$= \frac{\mu_2^2 + 2\alpha\mu_1^2 - \alpha^2\mu_1^2}{\mu_1^2\mu_2^2} = \frac{\mu_2^2 + \alpha\mu_1^2(2-\alpha)}{\mu_1^2\mu_2^2}$$

$$C_X^2 = \frac{Var[X]}{E[X]^2} = \frac{\mu_2^2 + \alpha\mu_1^2(2-\alpha)}{\mu_1^2\mu_2^2} \times \frac{\mu_1^2\mu_2^2}{(\mu_2 + \alpha\mu_1)^2} = \frac{\mu_2^2 + \alpha\mu_1^2(2-\alpha)}{(\mu_2 + \alpha\mu_1)^2}. \tag{21}$$

Example:

Coxian-2 RV with parameters $\mu_1 = 2$, $\mu_2 = 0.5$ and $\alpha = 0.25$,

$$E[X] = \frac{1}{\mu_1} + \frac{\alpha}{\mu_2} = \frac{1}{2} + \frac{1/4}{1/2} = 1$$

$$E[X^2] = \frac{2}{\mu_1^2} + \frac{2\alpha}{\mu_2^2} + \frac{2\alpha}{\mu_1\mu_2} = \frac{2}{4} + \frac{1/2}{1/4} + \frac{1/2}{1} = 3$$

$$Var[X] = E[X^2] - E[X]^2 = 3 - 1 = 2$$

$$C_X^2 = Var[X]/E[X]^2 = 2$$

General Phase Type Distributions

Phase type distributions need not be restricted to linear arrangements.

Define a phase type distribution on k phases with parameters μ_i:
— distribution of the total time spent moving in some probabilistic fashion among the k different phases.

It suffices to specify:

- the initial probability distribution: $\sigma_i,\ i = 1, 2, \ldots, k$
  $\sum_{i=1}^{k} \sigma_i = 1$
- the *routing* probabilities $r_{ij},\ i, j = 1, 2, \ldots, k;\ j \neq i$
  — $\sum_{j=1}^{k} r_{ij} < 1$.
- the terminal probability distribution: $\eta_i,\ i = 1, 2, \ldots, k$
  — for all $i = 1, 2, \ldots, k, \quad \eta_i + \sum_{j=1}^{k} r_{ij} = 1$:

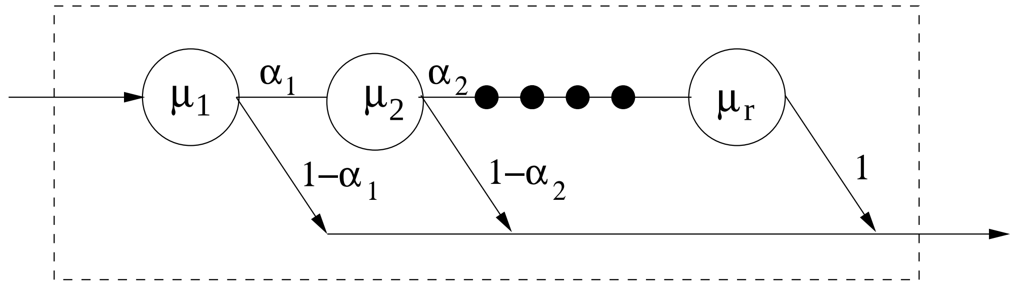


Figure 4: The Coxian Distribution, again.

Example: Coxian distribution:

Initial distribution: $\sigma = (1, 0, 0, \ldots, 0)$.

Terminal distribution: $\eta = (1 - \alpha_1, 1 - \alpha_2, \ \ldots, \ 1 - \alpha_{k-1}, 1)$.

Probabilities r_{ij}:

$$R = \begin{pmatrix} 0 & \alpha_1 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & \alpha_{k-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

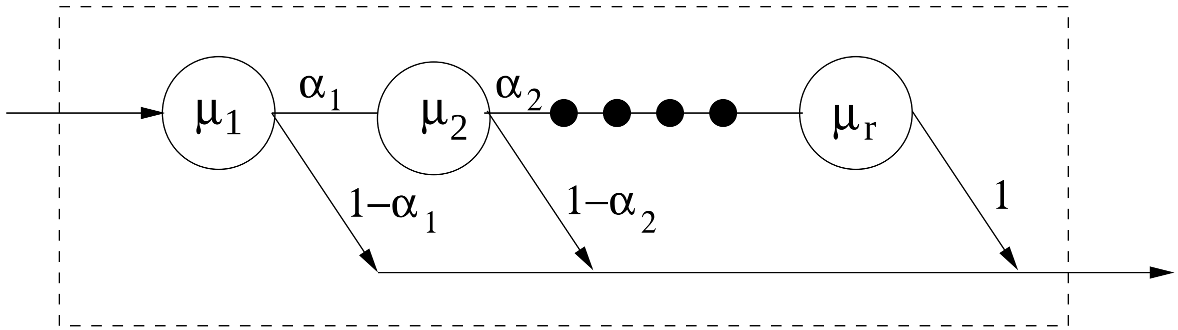


Figure 5: A General Phase Type Distribution.

Example: General phase type distribution:

Initial distribution: $\sigma = (0, .4, 0, .6)$

Terminal distribution: $\eta = (0, 0, 1, 0)$

Routing probability matrix:

$$R = \begin{pmatrix} 0 & .5 & .5 & 0 \\ 0 & 0 & 1 & 0 \\ .2 & 0 & 0 & .7 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Appended an extra phase to represent the exterior
— called a sink or an absorbing phase

Now combine the parameters of the exponential distributions of the phases and the routing probabilities into a single matrix Q
— q_{ij} is the rate of transition (on exiting phase i) from phase i to some other phase j, i.e., $q_{ij} = \mu_i r_{ij}$.

Associated Markov chain has a single absorbing state and an initial probability vector.

Example: Coxian distribution:

Initial distribution: $(1, 0, 0, \ldots, 0 \mid 0) = (\sigma \mid 0)$.

$$Q = \left(\begin{array}{ccccc|c} 0 & \mu_1\alpha_1 & 0 & \cdots & 0 & \mu_1(1-\alpha_1) \\ 0 & 0 & \mu_2\alpha_2 & \cdots & 0 & \mu_2(1-\alpha_2) \\ \vdots & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & & \mu_{k-1}\alpha_{k-1} & \mu_{k-1}(1-\alpha_{k-1}) \\ 0 & 0 & 0 & \cdots & 0 & \mu_k \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right).$$

General phase type distribution:

Initial distribution: $(0, .4, 0, .6 \mid 0) = (\sigma \mid 0)$.

$$Q = \left(\begin{array}{cccc|c} 0 & .5\mu_1 & .5\mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & 0 \\ .2\mu_3 & 0 & 0 & .7\mu_3 & .1\mu_3 \\ \mu_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Fitting Phase Distributions to Means and Variances

Use Coxian distributions.

One criterion: use the smallest number of phases possible.

We differentiate between $C_X \leq 1$ and $C_X > 1$, when constructing Coxian distributions to match a given expectation $E[X]$ and a given C_X^2.

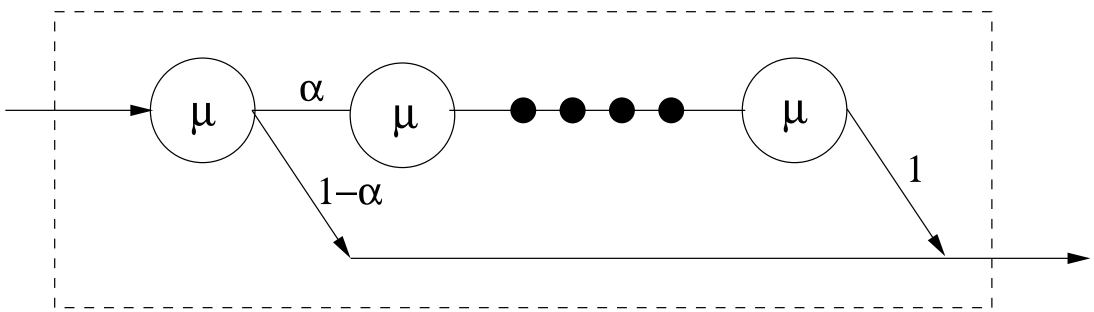


Figure 6: Suggestion Coxian for $C_X^2 < 1$

What values do we assign to μ and to α?

Laplace transform:

$$\mathcal{L}_X(s) = (1-\alpha)\frac{\mu}{s+\mu} + \alpha\prod_{i=1}^{r}\frac{\mu}{s+\mu} = (1-\alpha)\frac{\mu}{s+\mu} + \alpha\frac{\mu^r}{(s+\mu)^r}.$$

Then

$$\begin{aligned}
E[X] &= -\frac{d}{ds}\left((1-\alpha)\frac{\mu}{s+\mu} + \alpha\frac{\mu^r}{(s+\mu)^r}\right)\Bigg|_{s=0} \\
&= \left((1-\alpha)\frac{\mu}{(s+\mu)^2} + \alpha\frac{\mu^r r}{(s+\mu)^{r+1}}\right)\Bigg|_{s=0} \\
&= (1-\alpha)\frac{1}{\mu} + \alpha\frac{r}{\mu} \qquad (22)
\end{aligned}$$

$$
\begin{aligned}
E[X^2] &= \frac{d^2}{ds^2}\left((1-\alpha)\frac{\mu}{s+\mu}+\alpha\frac{\mu^r}{(s+\mu)^r}\right)\Bigg|_{s=0} \\
&= \frac{d}{ds}\left(-(1-\alpha)\frac{\mu}{(s+\mu)^2}-\alpha\frac{\mu^r r}{(s+\mu)^{r+1}}\right)\Bigg|_{s=0} \\
&= \left((1-\alpha)\frac{2\mu}{(s+\mu)^3}+\alpha\frac{\mu^r r(r+1)}{(s+\mu)^{r+2}}\right)\Bigg|_{s=0} \\
&= (1-\alpha)\frac{2}{\mu^2}+\alpha\frac{r(r+1)}{\mu^2} \qquad (23)
\end{aligned}
$$

$$
Var[X] = E[X^2]-E[X]^2 = \frac{2(1-\alpha)+\alpha r(r+1)-(1-\alpha+\alpha r)^2}{\mu^2}
$$

$$
C_X^2 = \frac{Var[X]}{E[X]^2} = \frac{2(1-\alpha)+\alpha r(r+1)-(1-\alpha+\alpha r)^2}{(1-\alpha+\alpha r)^2}. \qquad (24)
$$

We choose r, α and μ to satisfy (22) and (24).

Also choose r to be greater than $1/C_X^2$:

$$r = \left\lceil \frac{1}{C_X^2} \right\rceil .$$

Now use Equation (24) (which involves only r, C_X^2 and α) to find α.

$$\alpha = \frac{r - 2C_X^2 + \sqrt{r^2 + 4 - 4rC_X^2}}{2(C_X^2 + 1)(r - 1)} .$$

Finally, compute μ from Equation (22):

$$\mu = \frac{1 + \alpha(r - 1)}{E[X]} .$$

Example: Phase-type distribution with$E[X]=4$ and $Var[X]=5$.

Then $C_X^2 = 5/16 = 0.3125 < 1$.

$$r = \left\lceil \frac{1}{C_X^2} \right\rceil = \left\lceil \frac{1}{0.3125} \right\rceil = \lceil 3.2 \rceil$$
$$= 4.$$

$$\alpha = \frac{r - 2C_X^2 + \sqrt{r^2 + 4 - 4rC_X^2}}{2(C_X^2+1)(r-1)} = \frac{4 - 2(0.3125) + \sqrt{16 + 4 - 16(0.3125)}}{2(0.3125+1)(3)}$$
$$= 0.9204.$$

$$\mu = \frac{1 + \alpha(r-1)}{E[X]} = \frac{1 + 3(0.9204)}{4} = 0.9403$$

Check:

$$E[X] = (1-\alpha)\frac{1}{\mu} + \alpha\frac{r}{\mu} = (0.0796)\frac{1}{0.9403} + (0.9204)\frac{4}{0.9403} = 0.0847 + 3.9153$$

$$= 4.0.$$

$$\begin{aligned} Var[X] &= \frac{2(1-\alpha) + \alpha r(r+1) - (1-\alpha+\alpha r)^2}{\mu^2} \\ &= \frac{2(0.0796) + (0.9204)20 - [0.0796 + 4(0.9204)]^2}{(0.9403)^2} = \frac{4.4212}{0.8841} \end{aligned}$$

$$= 5.0.$$

A two-phase Coxian is sufficient. for $C_X^2 > 1$.

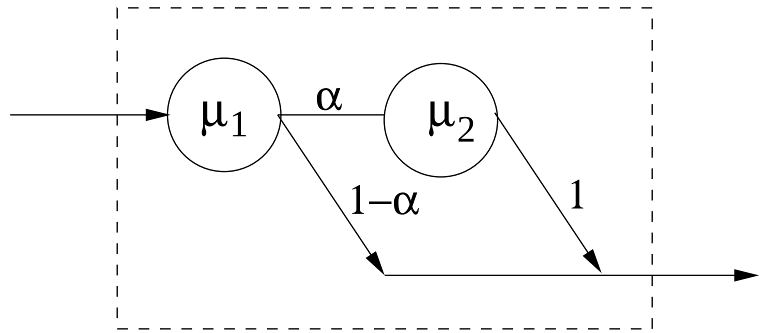


Figure 7: Suggested Coxian for $C_X^2 \geq 0.5$

Need to find μ_1, μ_2 and α from $E[X]$ and C_X^2 where

$$E[X] = \frac{\mu_2 + \alpha\mu_1}{\mu_1\mu_2}$$

$$C_X^2 = \frac{\mu_2^2 + \alpha\mu_1^2(2-\alpha)}{(\mu_2 + \alpha\mu_1)^2}$$

Infinite number of solutions possible:

The following yields particularly simple forms.

$$\mu_1 = \frac{2}{E[X]}, \quad \alpha = \frac{1}{2C_X^2} \quad \text{and} \quad \mu_2 = \frac{1}{E[X]\ C_X^2}.$$

— valid for values of C_X^2 that satisfy $C_X^2 \geq 0.5$.

Example: $E[X] = 3$ and $\sigma_X = 4$. This means that $C_X^2 = 16/9$ so

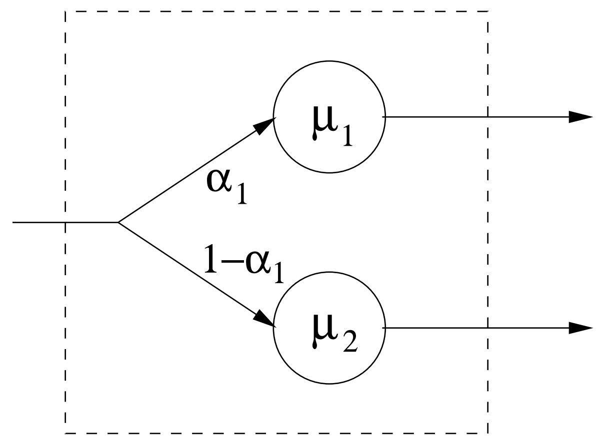
$$\mu_1 = \frac{2}{E[X]} = \frac{2}{3}, \quad \alpha = \frac{1}{2C_X^2} = \frac{9}{32} \quad \text{and} \quad \mu_2 = \frac{1}{E[X]\ C_X^2} = \frac{3}{16}.$$

Check:

$$\frac{\mu_2 + \alpha\mu_1}{\mu_1\mu_2} = \frac{\frac{3}{16} + \frac{9}{32}\frac{2}{3}}{\frac{2}{3}\frac{3}{16}} = \frac{\frac{6}{16}}{\frac{1}{8}} = 3$$

$$\frac{\mu_2^2 + \alpha\mu_1^2(2-\alpha)}{(\mu_2 + \alpha\mu_1)^2} = \frac{\frac{9}{256} + \frac{9}{32}\frac{4}{9}\frac{55}{32}}{\left(\frac{3}{16} + \frac{9}{32}\frac{2}{3}\right)^2} = \frac{0.25}{0.1406} = 1.7778 = \frac{16}{9}.$$

Alternative: a two-phase hyperexponential distribution.



Add an additional balance condition:

$$\frac{\alpha}{\mu_1} = \frac{1-\alpha}{\mu_2}.$$

This leads to the formulae

$$\alpha = \frac{1}{2}\left(1 + \sqrt{\frac{C_X^2 - 1}{C_X^2 + 1}}\right), \quad \mu_1 = \frac{2\alpha}{E[X]} \quad \text{and} \quad \mu_2 = \frac{2(1-\alpha)}{E[X]}.$$

Example: $E[X] = 3$ and $C_X^2 = 16/9$:

$$\alpha = \frac{1}{2}\left(1 + \sqrt{\frac{C-1}{C+1}}\right) = \frac{1}{2}\left(1 + \sqrt{\frac{7/9}{25/9}}\right) = 0.7646.$$

$$\mu_1 = \frac{2\alpha}{E} = \frac{1.5292}{3} = 0.5097 \quad \text{and} \quad \mu_2 = \frac{2(1-\alpha)}{E} = \frac{0.4709}{3} = 0.1570$$

Check:

$$E[X] = \frac{\alpha}{\mu_1} + \frac{1-\alpha}{\mu_2} = \frac{0.7646}{0.5097} + \frac{0.2354}{0.1570} = 1.50 + 1.50 = 3.0.$$

$$C_X^2 = \frac{2\alpha/\mu_1^2 + 2(1-\alpha)/\mu_2^2}{(\alpha/\mu_1 + (1-\alpha)/\mu_2)^2} - 1 = \frac{1.5292/0.2598 + 0.4708/0.0246}{(0.7646/0.5097 + 0.2354/0.1570)^2} - 1$$

$$= \frac{25}{9} - 1 = \frac{16}{9}.$$

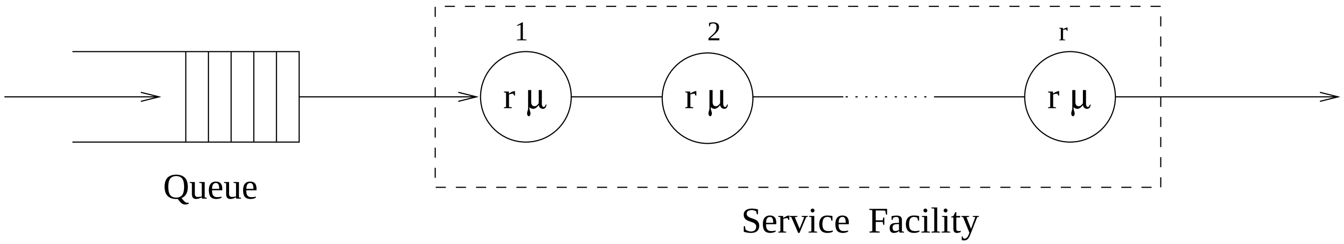
Queues with Phase-Type Laws: Neuts' Matrix-Geometric Method

Beyond *Birth-Death* processes and tridiagonal transition matrices.

Phase-type arrival or service mechanisms have *block* tridiagonal transition matrices

— *Quasi*-Birth-Death (QBD) processes.

The Erlang-r Service Model — The $M/E_r/1$ Queue



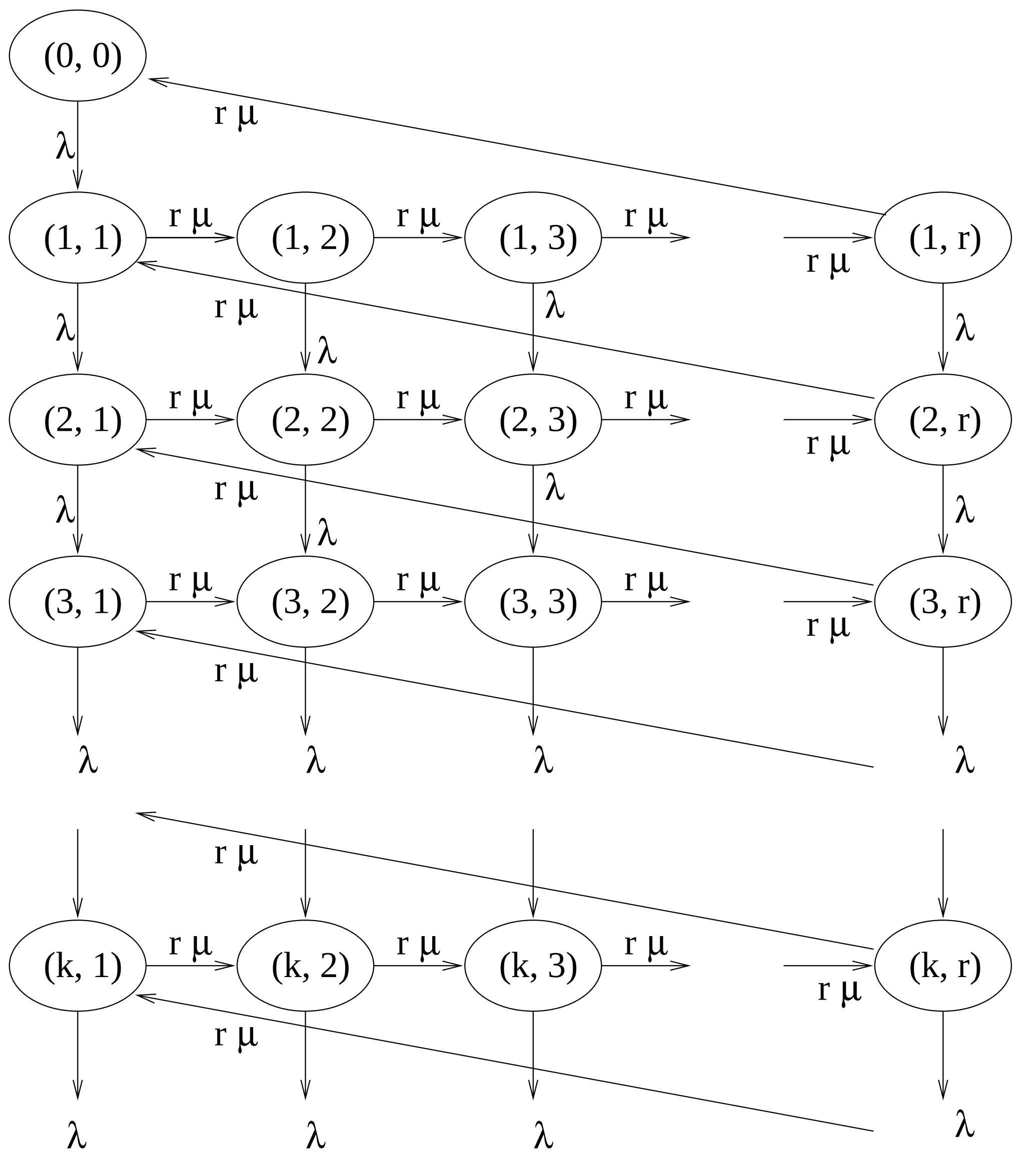
$$a(t) = \lambda e^{-\lambda t}, \quad t \geq 0$$

$$b(x) = \frac{r\mu(r\mu x)^{r-1}e^{-r\mu x}}{(r-1)!}, \quad x \geq 0.$$

State descriptor: $(k,\ i)$

— k $(k \geq 0)$, is the number of customers in the system,

— i $(1 \leq i \leq r)$, denotes the current phase of service.


(0, 0)
(1, 1)
(1, 2)
(1, 3)
(1, r)
(2, 1)
(2, 2)
(2, 3)
(2, r)
(3, 1)
(3, 2)
(3, 3)
(3, r)
(k, 1)
(k, 2)
(k, 3)
(k, r)
λ
r μ

States that have exactly k customers constitute level k.

Transition rate matrix has the typical block-tridiagonal (QBD) form:

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & \cdots \\ 0 & 0 & 0 & A_0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Matrices A_0 represent service completions at rate $r\mu$

Matrices A_2 represent arrivals at rate λ.

Super-diagonal elements A_1 represent service completion at rate $r\mu$.

The matrices B represent initial conditions.

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r\mu & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_2 = \lambda I \quad \text{and}$$

$$A_1 = \begin{pmatrix} -\lambda - r\mu & r\mu & 0 & 0 & \cdots & 0 \\ 0 & -\lambda - r\mu & r\mu & 0 & \cdots & 0 \\ 0 & 0 & -\lambda - r\mu & r\mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & r\mu \\ 0 & 0 & 0 & 0 & \cdots & -\lambda - r\mu \end{pmatrix}.$$

Example: The $M/E_r/1$ queue with $\lambda = 1, \mu = 1.5$ and $r = 3$.

$$Q =$$

$$\left(\begin{array}{c|ccc|ccc|ccc|c}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\hline
0 & -5.5 & 4.5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & -5.5 & 4.5 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
4.5 & 0 & 0 & -5.5 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
\hline
0 & 0 & 0 & 0 & -5.5 & 4.5 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & -5.5 & 4.5 & 0 & 1 & 0 & \cdots \\
0 & 4.5 & 0 & 0 & 0 & 0 & -5.5 & 0 & 0 & 1 & \cdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 4.5 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5.5 & 4.5 & \cdots \\
0 & 0 & 0 & 0 & 4.5 & 0 & 0 & 0 & 0 & -5.5 & \cdots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4.5 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -5.5 & 4.5 & 0 \\ 0 & -5.5 & 4.5 \\ 0 & 0 & -5.5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$B_{00} = -1, \quad B_{01} = (1, 0, 0), \quad B_{10} = \begin{pmatrix} 0 \\ 0 \\ 4.5 \end{pmatrix}.$$

We seek π from $\pi Q = 0$ with $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_i, \ldots)$.

Successive subvectors of π satisfy $\pi_{i+1} = \pi_i R$ for $i = 1, 2, \ldots$.

Compute R from

$$R_{l+1} = -(V + R_l^2 W)$$

with $V = A_2 A_1^{-1}$, $W = A_0 A_1^{-1}$ and $R_0 = 0$.

The ij element of the inverse of M:

$$M = \begin{pmatrix} d & a & 0 & 0 & \cdots & 0 \\ 0 & d & a & 0 & \cdots & 0 \\ 0 & 0 & d & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & a \\ 0 & 0 & 0 & 0 & \cdots & d \end{pmatrix},$$

$$M^{-1}_{ij} = (-1)^{j-i}\frac{1}{d}\left(\frac{a}{d}\right)^{j-i}, \quad \text{if} \;\; i \le j \le r; \qquad 0 \text{ otherwise} \qquad (25)$$

$M/E_r/1$ queue: $d = -(\lambda + r\mu)$ and $a = r\mu$.

$W = A_0 A_1^{-1}$ and A_0 has a single nonzero element $r\mu$ in position $r1$.
$\Rightarrow W$ has only one nonzero row, the last, with elements given by
$r\mu \times$ first row of A_1^{-1}.

$$W_{ri} = -\left(\frac{r\mu}{\lambda + r\mu}\right)^i \quad \text{for} \ \ 1 \leq i \leq r; \qquad \text{otherwise} \ \ 0.$$

$V = A_2 A_1^{-1}$ and $A_2 = \lambda I \Rightarrow$ multiply each element of A_1^{-1} by λ.

Example: $M/E_3/1$ queue continued

$$A_1^{-1} = \begin{pmatrix} -2/11 & -18/121 & -162/1331 \\ 0 & -2/11 & -18/121 \\ 0 & 0 & -2/11 \end{pmatrix}$$

$$V = \begin{pmatrix} -2/11 & -18/121 & -162/1331 \\ 0 & -2/11 & -18/121 \\ 0 & 0 & -2/11 \end{pmatrix}. \quad W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -9/11 & -81/121 & -729/1331 \end{pmatrix}$$

Begin iterating with

$$R_{l+1} = -V - R_l^2 W.$$

$R_0 = 0,$

$$R_1 = \begin{pmatrix} 2/11 & 18/121 & 162/1331 \\ 0 & 2/11 & 18/121 \\ 0 & 0 & 2/11 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0.236136 & 0.193202 & 0.158075 \\ 0.044259 & 0.218030 & 0.178388 \\ 0.027047 & 0.022130 & 0.199924 \end{pmatrix},$$

$\ldots$

$$R_{50} = \begin{pmatrix} 0.331961 & 0.271605 & 0.222222 \\ 0.109739 & 0.271605 & 0.222222 \\ 0.060357 & 0.049383 & 0.222222 \end{pmatrix} = R.$$

Next step: computation of initial vectors for $\pi_{i+1} = \pi_i R, \; i = 1, 2, \ldots$

$$(\pi_0, \pi_1, \pi_2, \ldots, \pi_i, \ldots) \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & \cdots \\ 0 & 0 & 0 & A_0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (0, 0, 0, \ldots, 0, \ldots)$$

$$\begin{aligned} \pi_0 B_{00} + \pi_1 B_{10} &= 0 \\ \pi_0 B_{01} + \pi_1 A_1 + \pi_2 A_0 &= 0 \end{aligned}$$

Writing π_2 as $\pi_1 R$, we obtain

$$(\pi_0, \pi_1) \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & A_1 + R A_0 \end{pmatrix} = (0, 0) \qquad (26)$$

There is no unique solution so the computed π must be normalized.

$$1 = \pi_0 + \sum_{k=1}^{\infty} \pi_k e = \pi_0 + \sum_{k=0}^{\infty} \pi_1 R^k e = \pi_0 + \pi_1 (I - R)^{-1} e$$

Thereafter:

$$\pi_{k+1} = \pi_k R.$$

<u>Example, continued:</u>

To find π_0 and π_1: Observe that $4.5 \times 0.222222 = 1$ and so

$$RA_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_1 + RA_0 = \begin{pmatrix} -4.5 & 4.5 & 0 \\ 1 & -5.5 & 4.5 \\ 1 & 0 & -5.5 \end{pmatrix}.$$

$$(\pi_0, \pi_1)\left(\begin{array}{c|ccc} -1 & 1 & 0 & 0 \\ \hline 0 & -4.5 & 4.5 & 0 \\ 0 & 1 & -5.5 & 4.5 \\ 4.5 & 1 & 0 & -5.5 \end{array}\right) = (0, 0).$$

Coefficient matrix has rank 3, so arbitrarily setting $\pi_0 = 1$:

$$(\pi_0, \pi_{1_1}, \pi_{1_2}, \pi_{1_3})\left(\begin{array}{c|ccc} -1 & 1 & 0 & 1 \\ \hline 0 & -4.5 & 4.5 & 0 \\ 0 & 1 & -5.5 & 0 \\ 4.5 & 1 & 0 & 0 \end{array}\right) = (0, 0, 0, 1).$$

Solution

$$(\pi_0, \pi_{1_1}, \pi_{1_2}, \pi_{1_3}) = (1, 0.331962, 0.271605, 0.222222).$$

This solution needs to be normalized so that

$$\pi_0 + \pi_1 (I - R)^{-1} e = 1.$$

Substituting, we obtain

$$1 + (0.331962, 0.271605, 0.222222) \begin{pmatrix} 1.666666 & 0.666666 & 0.666666 \\ 0.296296 & 1.518518 & 0.518518 \\ 0.148148 & 0.148148 & 1.370370 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= 3.$$

Thus, the normalized solution is given as

$$\begin{aligned} (\pi_0, \pi_{1_1}, \pi_{1_2}, \pi_{1_3}) &= (1/3, 0.331962/3, 0.271605/3, 0.222222/3) \\ &= (1/3,\ 0.110654,\ 0.090535,\ 0.0740741) \end{aligned}$$

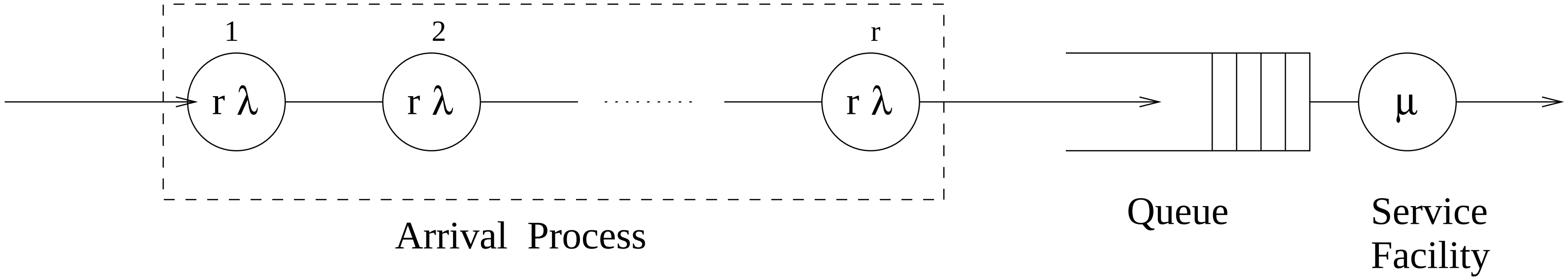
Additional probabilities may now be computed from $\pi_{k+1} = \pi_k R$.

$$\begin{aligned}
\pi_2 = \pi_1 R &= (0.051139, 0.058302, 0.061170) \\
\pi_3 = \pi_2 R &= (0.027067, 0.032745, 0.037913) \\
\pi_4 = \pi_3 R &= (0.014867, 0.018117, 0.021717) \\
\pi_5 = \pi_4 R &= (0.008234, 0.010031, 0.012156)
\end{aligned}$$

The probability of having $0, 1, 2, \ldots$ customers is found by adding the components of these subvectors. We have

$$p_0 = 1/3, \quad p_1 = 0.275263, \quad p_2 = 0.170610, \quad p_3 = 0.097725, \quad \ldots$$

The Erlang-r Arrival Model — The $E_r/M/1$ Queue



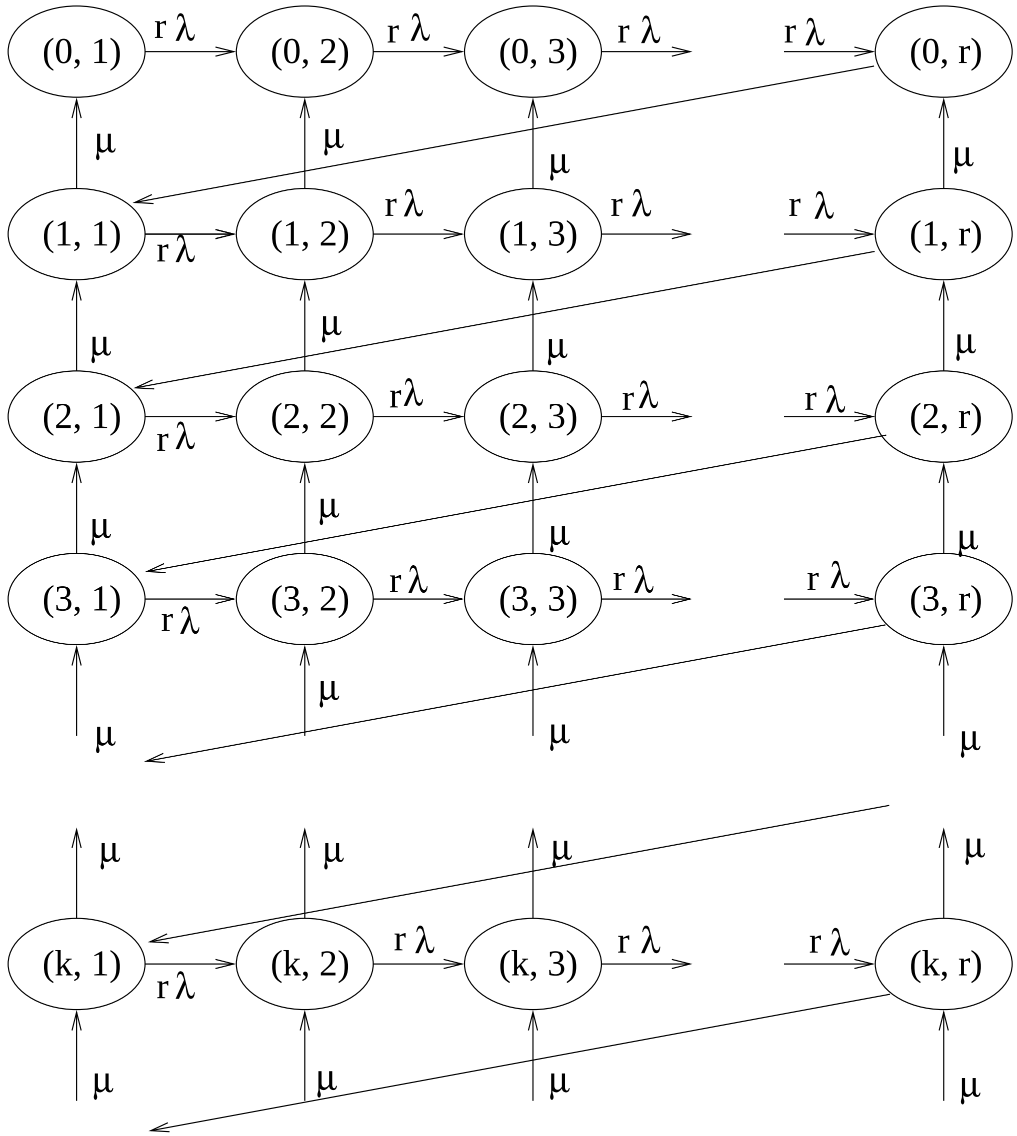
$$a(t) = \frac{r\lambda(r\lambda t)^{r-1}e^{-r\lambda t}}{(r-1)!}, \quad t \geq 0,$$

$$b(t) = \mu e^{-\mu x}, \quad x \geq 0.$$

Before actually appearing in the queue proper, an arriving customer must pass through r exponential phases each with parameter $r\lambda$.

State descriptor: $(k,\ i)$

— arranged into levels according to the number of customers present.


(0, 1)
(0, 2)
(0, 3)
(0, r)
(1, 1)
(1, 2)
(1, 3)
(1, r)
(2, 1)
(2, 2)
(2, 3)
(2, r)
(3, 1)
(3, 2)
(3, 3)
(3, r)
(k, 1)
(k, 2)
(k, 3)
(k, r)
r λ
μ

The transition rate matrix:

$$Q = \begin{pmatrix} B_{00} & A_2 & 0 & 0 & 0 & 0 & \cdots \\ A_0 & A_1 & A_2 & 0 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & 0 & \cdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \end{pmatrix}$$

$$A_0 = \mu I, \quad A_1 = \begin{pmatrix} -\mu - r\lambda & r\lambda & 0 & 0 & \cdots & 0 \\ 0 & -\mu - r\lambda & r\lambda & 0 & \cdots & 0 \\ 0 & 0 & -\mu - r\lambda & r\lambda & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & r\lambda \\ 0 & 0 & 0 & 0 & \cdots & -\mu - r\lambda \end{pmatrix},$$

$$\text{and} \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r\lambda & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Matrices A_0 represent service completions at rate μ,

Matrices A_2 represent an actual arrival to the queue.

Super-diagonal elements of the matrices A_1 represent the completion of one arrival phase $i < r$, at rate $r\lambda$.

Example: An $E_r/M/1$ queue with parameters $\lambda = 1.0,\ \mu = 1.5,\ r = 3.$

$$Q = \left(\begin{array}{ccc|ccc|ccc|c} -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline 1.5 & 0 & 0 & -4.5 & 3 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1.5 & 0 & 0 & -4.5 & 3 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1.5 & 0 & 0 & -4.5 & 3 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & 1.5 & 0 & 0 & -4.5 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1.5 & 0 & 0 & -4.5 & 3 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1.5 & 0 & 0 & -4.5 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

$$A_0 = \begin{pmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1.5 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -4.5 & 3 & 0 \\ 0 & -4.5 & 3 \\ 0 & 0 & -4.5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}.$$

Find π from $\pi Q = 0$, $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_k, \ldots)$.

Successive subvectors of π satisfy $\pi_{i+1} = \pi_i R$ for $i = 1, 2, \ldots$.

R is obtained from

$$R_{l+1} = -(V + R_l^2 W)$$

with $V = A_2 A_1^{-1}$ and $W = A_0 A_1^{-1}$.

$$V_{ri} = -\left(\frac{r\lambda}{\mu + r\lambda}\right)^i \quad \text{for} \;\; 1 \le i \le r; \qquad \text{otherwise} \;\; 0.$$

Since $A_0 = \mu I$, $W = A_0 A_1^{-1}$ is easy to find:
— multiply each element of A_1^{-1} by μ.

Example continued:

$$A_1^{-1} = \begin{pmatrix} -2/9 & -4/27 & -8/81 \\ 0 & -2/9 & -4/27 \\ 0 & 0 & -2/9 \end{pmatrix}$$

$$W = A_0 A_1^{-1} = \begin{pmatrix} -1/3 & -2/9 & -4/27 \\ 0 & -1/3 & -2/9 \\ 0 & 0 & -1/3 \end{pmatrix},$$

$$V = A_2 A_1^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2/3 & -4/9 & -8/27 \end{pmatrix}$$

$R_{l+1} = -V - R_l^2 W$:

$$R_0 = 0,$$

$$R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2/3 & 4/9 & 8/27 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.732510 & 0.532236 & 0.3840878 \end{pmatrix},$$

$$\ldots$$

$$R_{50} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.810536 & 0.656968 & 0.532496 \end{pmatrix} = R.$$

The boundary equations are different in the $E_r/M/1$ queue from those in the $M/E_r/1$ queue.

Only a single subvector, π_0, needs to be found

$$\pi_{i+1} = \pi_i R = \pi_0 R^{i+1} \quad \text{for} \quad i = 0, 1, 2, \ldots$$

From

$$(\pi_0, \pi_1, \pi_2, \ldots, \pi_i, \ldots) \begin{pmatrix} B_{00} & A_2 & 0 & 0 & 0 & \cdots \\ A_0 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & \cdots \\ 0 & 0 & 0 & A_0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (0, 0, 0, \ldots, 0, \ldots),$$

$$\pi_0 B_{00} + \pi_1 A_0 = \pi_0 B_{00} + \pi_0 R A_0 = \pi_0 (B_{00} + R A_0) = 0.$$

A unique solution is found by enforcing the constraint

$$1 = \sum_{k=0}^{\infty} \pi_k e = \sum_{k=0}^{\infty} \pi_0 R^k e = \pi_0 (I - R)^{-1} e$$

Example, continued:

$$\pi_0(B_{00} + RA_0) = (\pi_{01}, \pi_{02}, \pi_{03}) \begin{pmatrix} -3 & 3 & 1 \\ 0 & -3 & 0 \\ 1.215803 & 0.98545 & 0 \end{pmatrix} = (0, 0, 0).$$

Solution: $\pi_0 = (1,\ 1.810536,\ 2.467504).$

Now normalize so that $\pi_0(I - R)^{-1} e = 1.$

$$(1,\ 1.810536,\ 2.467504) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.810536 & -0.656968 & 0.467504 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 15.834116$$

Divide each component of π_0 by 15.834116:

$$\pi_0 = (0.063155,\ 0.114344,\ 0.155835).$$

The remaining subvectors of π found from $\pi_k = \pi_{k-1} R = \pi_0 R^k$.

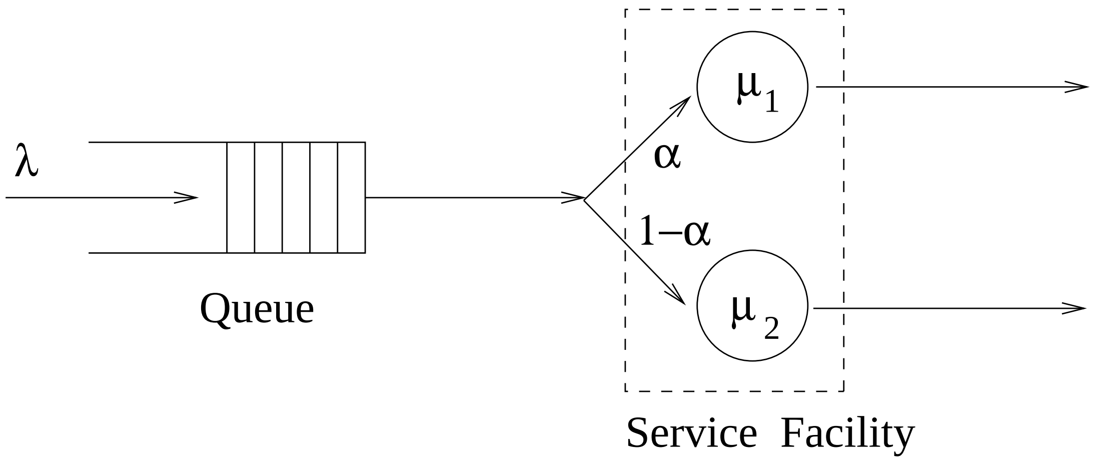
$$\begin{aligned}
\pi_1 = \pi_0 R &= (0.126310,\ 0.102378,\ 0.082981) \\
\pi_2 = \pi_1 R &= (0.067259,\ 0.054516,\ 0.044187) \\
\pi_3 = \pi_2 R &= (0.035815,\ 0.029030,\ 0.023530) \\
\pi_4 = \pi_3 R &= (0.019072,\ 0.015458,\ 0.012529) \\
&\text{etc.}
\end{aligned}$$

Probability of having $0, 1, 2, \ldots$ customers:

$$p_0 = 1/3, \quad p_1 = 0.311669, \quad p_2 = 0.165963, \quad p_3 = 0.088374, \quad \ldots$$

The $M/H_2/1$ and $H_2/M/1$ Queues

The $M/H_2/1$ queue.

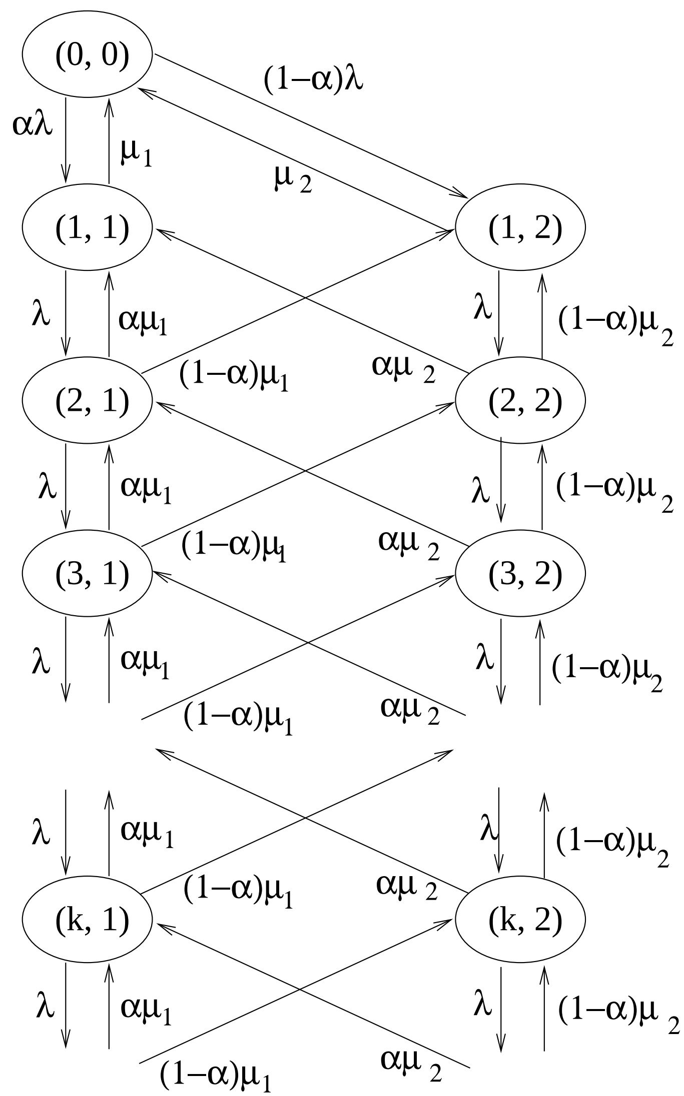


Arrivals are Poisson at rate λ.

With probability α, a customer receives service at rate μ_1.

With probability $1 - \alpha$, this customer receives service at rate μ_2.

Transition rate diagram for the $M/H_2/1$ queue:



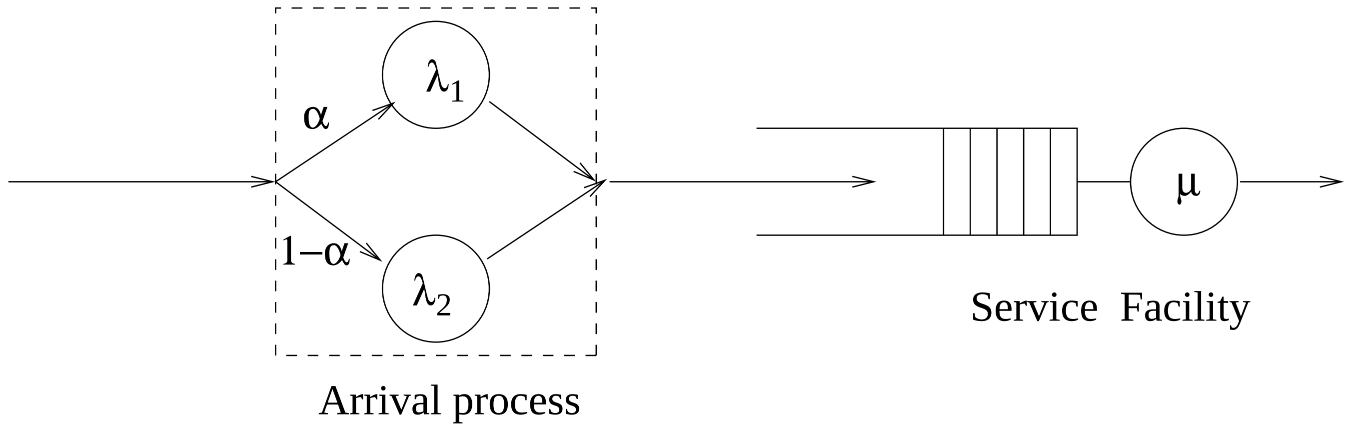
The transition rate matrix for the $M/H_2/1$:

$$\left(\begin{array}{c|cc|cc|cc|c}
-\lambda & \alpha\lambda & (1-\alpha)\lambda & 0 & 0 & 0 & 0 & \cdots \\
\hline
\mu_1 & -(\lambda+\mu_1) & 0 & \lambda & 0 & 0 & 0 & \cdots \\
\mu_2 & 0 & -(\lambda+\mu_2) & 0 & \lambda & 0 & 0 & \cdots \\
\hline
0 & \alpha\mu_1 & (1-\alpha)\mu_1 & -(\lambda+\mu_1) & 0 & \lambda & 0 & \\
0 & \alpha\mu_2 & (1-\alpha)\mu_2 & 0 & -(\lambda+\mu_2) & 0 & \lambda & \\
\hline
0 & 0 & 0 & \alpha\mu_1 & (1-\alpha)\mu_1 & -(\lambda+\mu_1) & 0 & \ddots \\
0 & 0 & 0 & \alpha\mu_2 & (1-\alpha)\mu_2 & 0 & -(\lambda+\mu_2) & \ddots \\
\hline
\vdots & \vdots & \vdots & & & \ddots & \ddots & \ddots
\end{array}\right)$$

$$A_0 = \begin{pmatrix} \alpha\mu_1 & (1-\alpha)\mu_1 \\ \alpha\mu_2 & (1-\alpha)\mu_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -(\lambda+\mu_1) & 0 \\ 0 & -(\lambda+\mu_2) \end{pmatrix}, \quad A_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

$$B_{00} = \begin{pmatrix} -\lambda \end{pmatrix}, \quad B_{01} = \begin{pmatrix} \alpha\lambda & (1-\alpha)\lambda \end{pmatrix}, \quad B_{10} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.$$

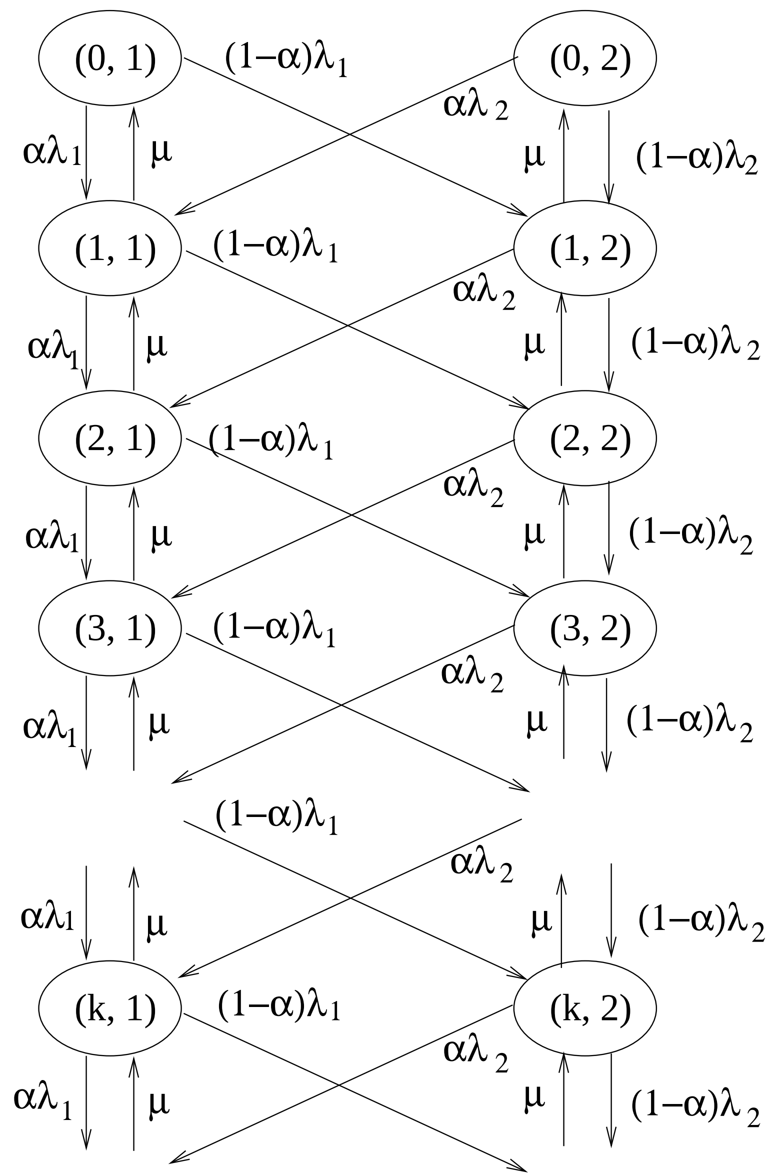
The $H_2/M/1$ queue:



The instant a customer enters the queue, a new customer immediately initiates its arrival process.

With probability α this exponentially distribution has rate λ_1,
— while with probability $1 - \alpha$ it has rate λ_2.

Service is exponentially distributed with rate μ.


(0, 1)
(0, 2)
(1, 1)
(1, 2)
(2, 1)
(2, 2)
(3, 1)
(3, 2)
(k, 1)
(k, 2)
(1−α)λ1
αλ2
αλ1
μ
(1−α)λ2

Transition rate matrix:

$$
\left(\begin{array}{cc|cc|cc|cc|c}
-\lambda_1 & 0 & \alpha\lambda_1 & (1-\alpha)\lambda_1 & 0 & 0 & 0 & 0 & \cdots \\
0 & -\lambda_2 & \alpha\lambda_2 & (1-\alpha)\lambda_2 & 0 & 0 & 0 & 0 & \cdots \\
\hline
\mu & 0 & -(\lambda_1+\mu) & 0 & \alpha\lambda_1 & (1-\alpha)\lambda_1 & 0 & 0 & \cdots \\
0 & \mu & 0 & -(\lambda_2+\mu) & \alpha\lambda_2 & (1-\alpha)\lambda_2 & 0 & 0 & \cdots \\
\hline
0 & 0 & \mu & 0 & -(\lambda_1+\mu) & 0 & \alpha\lambda_1 & (1-\alpha)\lambda_1 & \\
0 & 0 & 0 & \mu & 0 & -(\lambda_2+\mu) & \alpha\lambda_2 & (1-\alpha)\lambda_2 & \\
\hline
0 & 0 & 0 & 0 & \mu & 0 & -(\lambda_1+\mu) & 0 & \\
0 & 0 & 0 & 0 & 0 & \mu & 0 & -(\lambda_2+\mu) & \\
\hline
\vdots & \vdots & \vdots & \vdots & & & \ddots & \ddots & \ddots
\end{array}\right)
$$

$$
A_0 = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}, \;\; A_1 = \begin{pmatrix} -(\lambda_1+\mu) & 0 \\ 0 & -(\lambda_2+\mu) \end{pmatrix}, \;\; A_2 = \begin{pmatrix} \alpha\lambda_1 & (1-\alpha)\lambda_1 \\ \alpha\lambda_2 & (1-\alpha)\lambda_2 \end{pmatrix},
$$

$$
B_{00} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \;\; B_{01} = \begin{pmatrix} \alpha\lambda_1 & (1-\alpha)\lambda_1 \\ \alpha\lambda_2 & (1-\alpha)\lambda_2 \end{pmatrix} = A_2, \;\; B_{10} = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} = A_0.
$$

Automating the Analysis of Single Server Phase-Type Queues

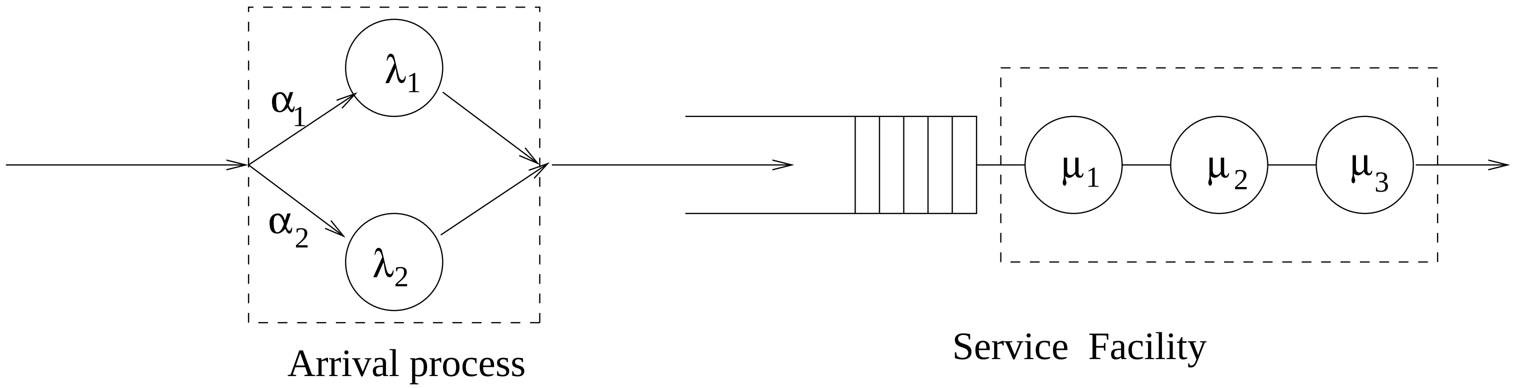
The procedure for solving phase-type queueing system by means of the matrix-geometric approach has four steps, namely

1. Construct the block submatrices
2. Form Neuts' R matrix
3. Solve the boundary equations
4. Generate successive components of the solution

Possible to write (Matlab) code for each of these four steps separately;
— complete program obtained by concatenating these.

In moving from one phase-type queueing system to another only the first of these sections should change.

The $H_2/E_3/1$ Queue and General Ph/Ph/1 Queues



State descriptor needs 3 parameters:

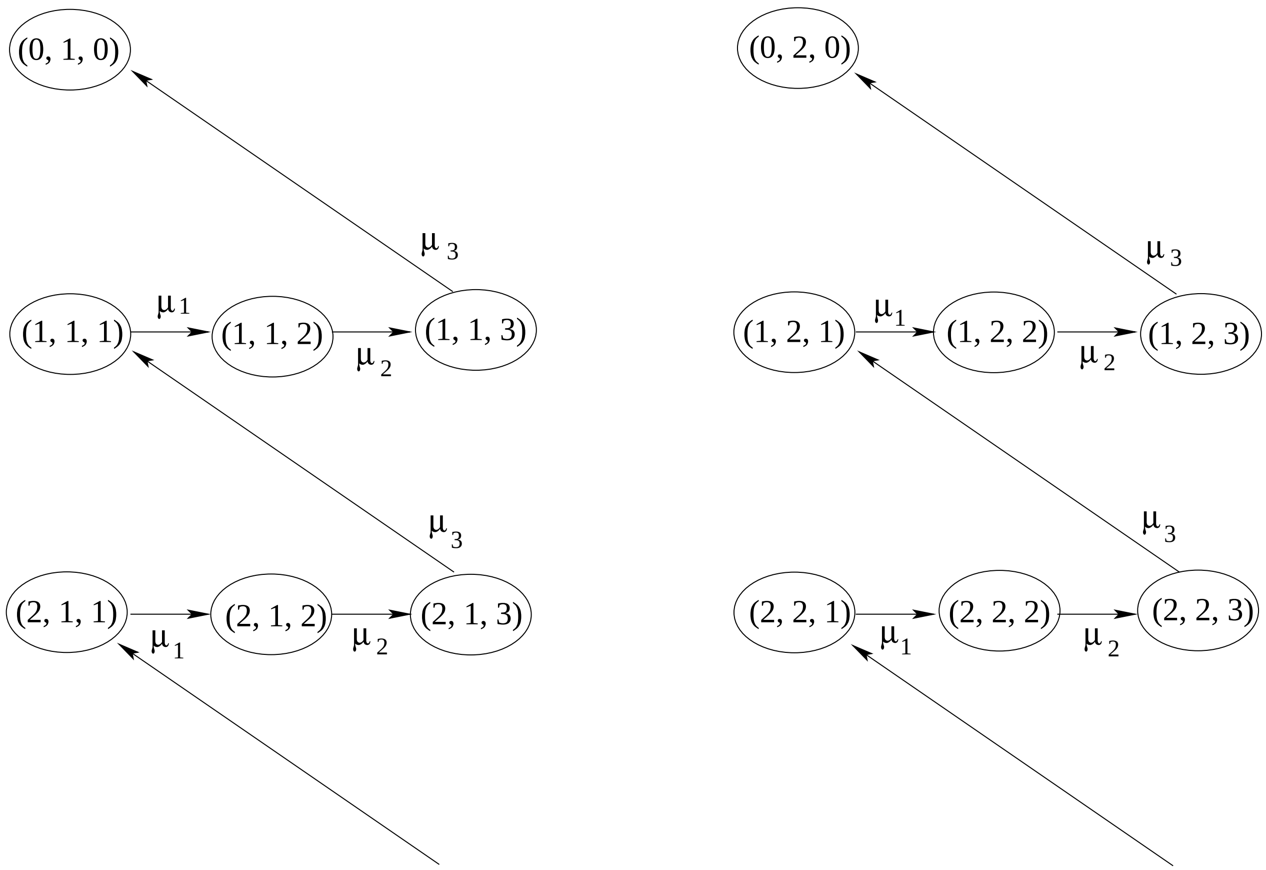
— k, the number of customers actually present,

— a, the arrival phase of the "arriving" customer,

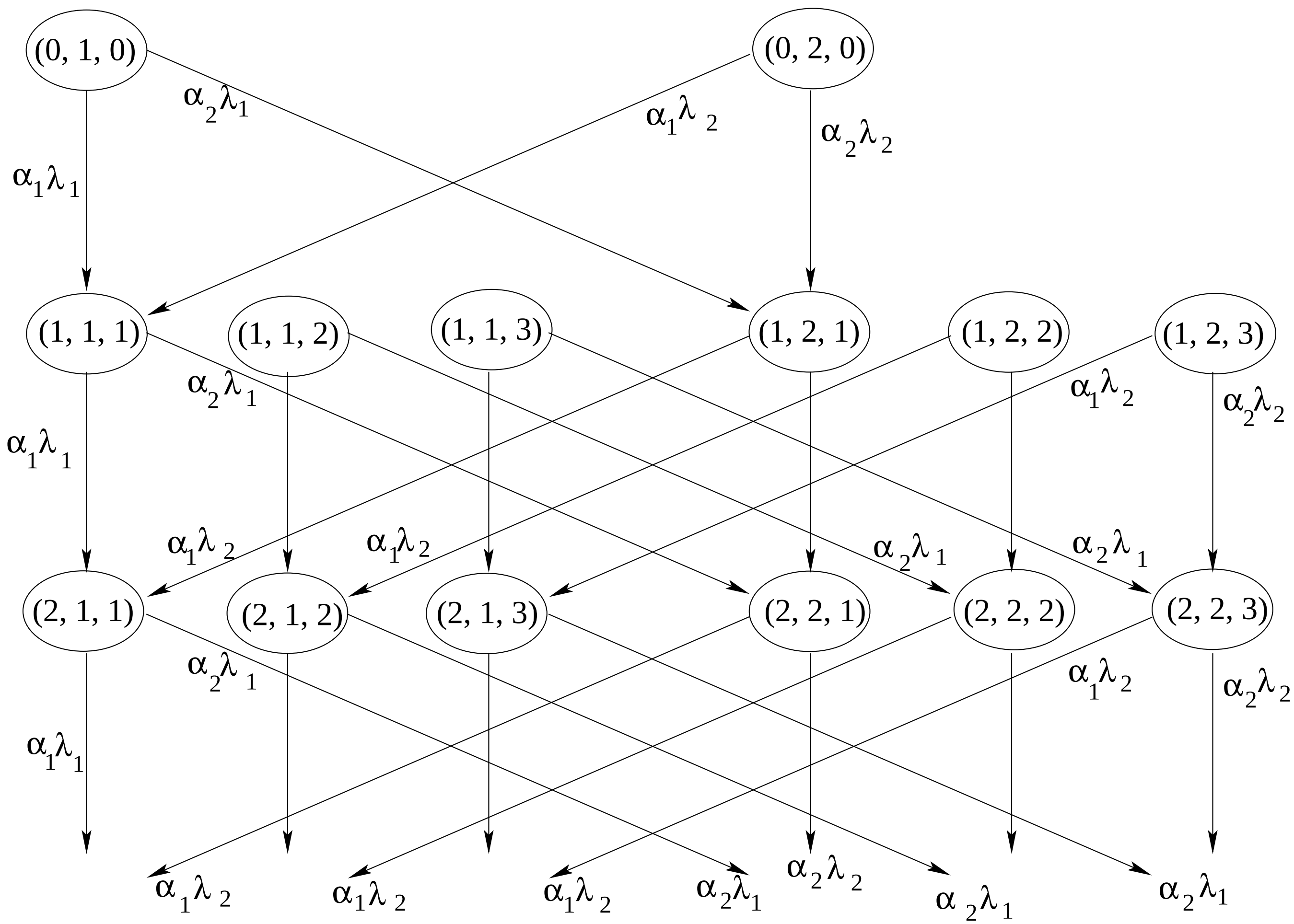
— s, the current phase of service.

States first ordered according to the number of customers present. Within each level, k, states are ordered first according to the arrival phase and secondly according to the service phase (k, a, s).

Transitions generated by arrivals:



Transitions generated by service completions:



$$
\left(\begin{array}{cc|cccccc|cccccc}
-\lambda_1 & 0 & \alpha_1\lambda_1 & 0 & 0 & \alpha_2\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\lambda_2 & \alpha_1\lambda_2 & 0 & 0 & \alpha_2\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & * & \mu_1 & 0 & 0 & 0 & 0 & \alpha_1\lambda_1 & 0 & 0 & \alpha_2\lambda_1 & 0 & 0 \\
0 & 0 & 0 & * & \mu_2 & 0 & 0 & 0 & 0 & \alpha_1\lambda_1 & 0 & 0 & \alpha_2\lambda_1 & 0 \\
\mu_3 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & \alpha_1\lambda_1 & 0 & 0 & \alpha_2\lambda_1 \\
0 & 0 & 0 & 0 & 0 & * & \mu_1 & 0 & \alpha_1\lambda_2 & 0 & 0 & \alpha_2\lambda_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & \mu_2 & 0 & \alpha_1\lambda_2 & 0 & 0 & \alpha_2\lambda_2 & 0 \\
0 & \mu_3 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & \alpha_1\lambda_2 & 0 & 0 & \alpha_2\lambda_2 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & \mu_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & \mu_2 & 0 & 0 & 0 \\
0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & \mu_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & \mu_2 \\
0 & 0 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right)
$$

Can construct the block submatrices A_0, A_1, A_2, B_{00}, B_{01} and B_{10} from the diagrams and then apply the matrix-geometric approach.

However, it is evident that this can become quite messy.

An arbitrary Markov chain with a single absorbing state and an initial probability distribution contains the essence of a phase-type distribution.

A phase-type distribution is defined as the distribution of the time to absorption into the single absorbing state when the Markov chain is started with the given initial probability distribution.

Examples:

Three stage hypoexponential distribution with parameters μ_1, μ_2 and μ_3:

$$S' = \left(\begin{array}{ccc|c} -\mu_1 & \mu_1 & 0 & 0 \\ 0 & -\mu_2 & \mu_2 & 0 \\ 0 & 0 & -\mu_3 & \mu_3 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} S & S^0 \\ 0 & 0 \end{pmatrix},$$

$$\sigma' = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} \sigma & 0 \end{pmatrix}.$$

Two stage hyperexponential distribution with branching probabilities α_1 and α_2 $(= 1 - \alpha_1)$ and exponential phases with rates λ_1 and λ_2:

$$T' = \left(\begin{array}{cc|c} -\lambda_1 & 0 & \lambda_1 \\ 0 & -\lambda_2 & \lambda_2 \\ \hline 0 & 0 & 0 \end{array}\right) = \begin{pmatrix} T & T^0 \\ 0 & 0 \end{pmatrix},$$

$$\xi' = \left(\begin{array}{cc|c} \alpha_1 & \alpha_2 & 0 \end{array}\right) = \begin{pmatrix} \xi & 0 \end{pmatrix}.$$

A *Ph/Ph/1* queue with r_a phases in the description of the arrival process and r_s phases in the description of the service process:

$$A_0 = I_{r_a} \otimes (S^0 \cdot \sigma), \quad A_1 = T \otimes I_{r_s} + I_{r_a} \otimes S \quad \text{and} \quad A_2 = (T^0 \cdot \xi) \otimes I_{r_s}$$

$$B_{00} = T, \quad B_{01} = (T^0 \cdot \xi) \otimes \sigma \quad \text{and} \quad B_{10} = I_{r_a} \otimes S^0$$

I_n is the identity matrix of order n.

The symbol $\otimes$ denotes the Kronecker (or tensor) product.

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & a_{13}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & a_{23}B & \cdots & a_{2n}B \\ a_{31}B & a_{32}B & a_{33}B & \cdots & a_{3n}B \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & a_{m3}B & \cdots & a_{mn}B \end{pmatrix}.$$

For example, the Kronecker product of

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{is}$$

$$A \otimes B = \begin{pmatrix} aB & bB & cB \\ dB & eB & fB \end{pmatrix} = \left(\begin{array}{cc|cc|cc} a\alpha & a\beta & b\alpha & b\beta & c\alpha & c\beta \\ a\gamma & a\delta & b\gamma & b\delta & c\gamma & c\delta \\ \hline d\alpha & d\beta & e\alpha & e\beta & f\alpha & f\beta \\ d\gamma & d\delta & e\gamma & e\delta & f\gamma & f\delta \end{array}\right).$$

Block submatrices for the $H_2/E_3/1$ queue :

$$A_0 = I_2 \otimes (S^0 \cdot \sigma) = I_2 \otimes \begin{pmatrix} 0 \\ 0 \\ \mu_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = I_2 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu_3 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_3 & 0 & 0 \end{pmatrix},$$

$$A_1 = T \otimes I_3 + I_2 \otimes S = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \otimes I_3 + I_2 \otimes \begin{pmatrix} -\mu_1 & \mu_1 & 0 \\ 0 & -\mu_2 & \mu_2 \\ 0 & 0 & -\mu_3 \end{pmatrix}$$

$$= \left(\begin{array}{ccc|ccc} -\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_2 \end{array}\right) + \left(\begin{array}{ccc|ccc} -\mu_1 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & -\mu_2 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & -\mu_3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\mu_1 & \mu_1 & 0 \\ 0 & 0 & 0 & 0 & -\mu_2 & \mu_2 \\ 0 & 0 & 0 & 0 & 0 & -\mu_3 \end{array}\right)$$

$$A_2 = (T^0 \cdot \xi) \otimes I_3 = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \otimes I_3 = \begin{pmatrix} \alpha_1\lambda_1 & \alpha_2\lambda_1 \\ \alpha_1\lambda_2 & \alpha_2\lambda_2 \end{pmatrix} \otimes I_3$$

$$= \left(\begin{array}{ccc|ccc} \alpha_1\lambda_1 & 0 & 0 & \alpha_2\lambda_1 & 0 & 0 \\ 0 & \alpha_1\lambda_1 & 0 & 0 & \alpha_2\lambda_1 & 0 \\ 0 & 0 & \alpha_1\lambda_1 & 0 & 0 & \alpha_2\lambda_1 \\ \hline \alpha_1\lambda_2 & 0 & 0 & \alpha_2\lambda_2 & 0 & 0 \\ 0 & \alpha_1\lambda_2 & 0 & 0 & \alpha_2\lambda_2 & 0 \\ 0 & 0 & \alpha_1\lambda_2 & 0 & 0 & \alpha_2\lambda_2 \end{array}\right)$$

$$B_{00} = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix},$$

$$B_{01} = (T^0 \cdot \xi) \otimes \sigma = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1\lambda_1 & \alpha_2\lambda_1 \\ \alpha_1\lambda_2 & \alpha_2\lambda_2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha_1\lambda_1 & 0 & 0 & \alpha_2\lambda_1 & 0 & 0 \\ \alpha_1\lambda_2 & 0 & 0 & \alpha_2\lambda_2 & 0 & 0 \end{pmatrix},$$

$$B_{10} = I_2 \otimes S^0 = I_2 \otimes \begin{pmatrix} 0 \\ 0 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \mu_3 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \mu_3 \end{pmatrix}.$$

```
%%%   H_2 Arrival Process:
      alpha1 = 0.4; alpha2 = 0.6; lambda1 = 1.9; lambda2 = 2;
      T = [-lambda1, 0 ; 0, -lambda2];
      T0 = [lambda1;lambda2];
      xi = [alpha1, alpha2];

%%%   E_3 Service Process:
      mu1 = 4; mu2 = 8; mu3 = 8;
      S = [-mu1, mu1, 0; 0, -mu2, mu2; 0,0, -mu3];
      S0 = [0;0;mu3];
      sigma = [1,0,0];

%%% Block Submatrices for all types of queues:
      ra = size(T,2);  rs = size(S,2);
      A0 = kron(eye(ra), S0*sigma);
      A1 = kron(T, eye(rs)) + kron(eye(ra), S);
      A2 = kron(T0*xi, eye(rs));
      B00 = T;
      B01 = kron(T0*xi,sigma);
      B10 = kron(eye(ra),S0);
      l = size(B00,2);  r = size(A0,2);
```

Stability Results for Ph/Ph/1 Queues.

Stability condition for M/M/1 queue: $\lambda < \mu$.

$$\frac{1}{E[A]} < \frac{1}{E[S]} \quad \text{or} \quad E[S] < E[A]$$

A similar condition holds for other *Ph/Ph/1* queues.

Example: The expectation of a two-phase hyperexponential:
$E[A] = \alpha_1/\lambda_1 + \alpha_2/\lambda_2$.

Expectation of a three-phase Erlang: $E[S] = 1/\mu_1 + 1/\mu_2 + 1/\mu_3$.
($\alpha_1 = 0.4, \ \alpha_2 = 0.6, \ \lambda_1 = 1, \ \lambda_2 = 2, \ \mu_1 = 4, \ \mu_2 = 8$ and $\mu_3 = 8$)

$$E[S] = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 0.5 \quad < \quad \frac{0.4}{1} + \frac{0.6}{2} = 0.7 = E[A].$$

For a general phase-type distribution, with

$$Z' = \begin{pmatrix} Z & Z_0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \zeta' = (\zeta, 0)$$

Expected time to absorption:

$$E[A] = \| - \zeta Z^{-1} \|_1.$$

Example: Average interarrival time in the $H_2/E_3/1$ queue:

$$E[A] = \left\| -(\alpha_1, \alpha_2) \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \right\|_1 = \left\| -(0.4,\ 0.6) \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}^{-1} \right\|_1$$

$$= \left\| \begin{pmatrix} 0.4 \\ 0.3 \end{pmatrix} \right\|_1 = 0.7$$

The same stability condition may be derived from $A_0,\ A_1$ and A_2.

$A = A_0 + A_1 + A_2$ is an infinitesimal generator matrix

$$\gamma A = \gamma(A_0 + A_1 + A_2) = 0.$$

Non-zero elements of A_0 move the system down a level
— relates to service completions in a *Ph/Ph/1* queue.

Non-zero elements of A_2 move the system up a level l
— the number of customers in the queue increases by one.

For stability, the effect of A_2 must be less than the effect of the A_0.

The condition for stability becomes

$$\|\gamma A_2\|_1 < \|\gamma A_0\|_1.$$

Example:

Same $H_2/E_3/1$ queue:

$$A = \begin{pmatrix} -4.6 & 4.0 & 0 & 0.6 & 0 & 0 \\ 0 & -8.6 & 8.0 & 0 & 0.6 & 0 \\ 8.0 & 0 & -8.6 & 0 & 0 & 0.6 \\ 0.8 & 0 & 0 & -4.8 & 4.0 & 0 \\ 0 & 0.8 & 0 & 0 & -8.8 & 8.0 \\ 0 & 0 & 0.8 & 8.0 & 0 & -8.8 \end{pmatrix}$$

Stationary probability vector, obtained by solving $\gamma A = 0$ with $\|\gamma\|_1 = 1$:

$$\gamma = (0.285714,\ 0.142857,\ 0.142857,\ 0.214286,\ 0.107143,\ 0.107143).$$

Computing $\|\gamma A_2\|_1$ and $\|\gamma A_0\|_1$:

$$\lambda = \|\gamma A_2\|_1 = \left\| \gamma \begin{pmatrix} 0.4 & 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0.4 & 0 & 0 & 0.6 \\ 0.8 & 0 & 0 & 1.2 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 1.2 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 1.2 \end{pmatrix} \right\|_1$$

$$= \|(0.285714,\ 0.142857,\ 0.142857,\ 0.428571,\ 0.214286,\ 0.214286)\|_1 = 1.428571$$

$$\mu = \|\gamma A_0\|_1 = \left\| \gamma \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \end{pmatrix} \right\|_1$$

$$= \|(1.142857,\ 0,\ 0,\ 0.857143,\ 0,\ 0)\|_1 = 2.0$$

λ_1	ρ	SS	LR
0.1	0.1163	28	5
0.5	0.4545	50	6
1.0	0.7143	98	6
1.5	0.8824	237	8
1.6	0.9091	303	8
1.7	0.9341	412	8
1.8	0.9574	620	9
1.9	0.9794	1197	10
1.95	0.9898	2234	11
2.0	1.0	∞	∞

Table 1: Effect of varying λ_1 on ρ and convergence to R.

Performance Measures for Ph/Ph/1 Queues

(1) Probability that there are k customers present:

$$p_k = \|\pi_k\|_1 = \|\pi_0 R^k\|_1.$$

(2) Probability that the system is empty $p_0 = \|\pi_0\|_1$.

(3) Probability that the system is busy is $1 - p_0$.

(4) Probability that there are k or more customers present :

$$\text{Prob}\{N \geq k\} = \sum_{j=k}^{\infty} \|\pi_j\|_1 = \left\| \pi_1 \sum_{j=k}^{\infty} R^{j-1} \right\|_1 = \left\| \pi_1 R^{k-1} \sum_{j=0} \infty R^j \right\|_1$$

$$= \left\| \pi_1 R^{k-1} (I - R)^{-1} \right\|_1 .$$

Mean number of customers in a *Ph/Ph/1* queue:

$$E[N] = \sum_{k=1}^{\infty} k \, \|\pi_k\|_1 = \sum_{k=1}^{\infty} k \, \|\pi_1 R^{k-1}\|_1 = \left\| \pi_1 \sum_{k=1}^{\infty} \frac{d}{dR} R^k \right\|_1$$

$$= \left\| \pi_1 \frac{d}{dR} \left(\sum_{k=1}^{\infty} R^k \right) \right\|_1 = \left\| \pi_1 \frac{d}{dR} \left((I-R)^{-1} - I \right) \right\|_1 = \left\| \pi_1 (I-R)^{-2} \right\|_1 .$$

— mean number of customers waiting in the queue, $E[N_q]$;

— average response time, $E[R]$;

— average time spent waiting in the queue, $E[W_q]$

can now be obtained from the standard formulae.

$$\begin{aligned} E[N_q] &= E[N] - \lambda/\mu \\ E[R] &= E[N]/\lambda \\ E[W_q] &= E[N_q]/\lambda \end{aligned}$$

Matlab code for Ph/Ph/1 Queues

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%  Example 1:  M/E_4/1 Queue
%%%  Exponential arrival:
%    lambda = 4;
%    T = [-lambda]; T0=[lambda]; xi = [1];

%%%   Erlang-4 Service (use mu_i = r*mu per phase)
%     mu1 = 20; mu2 = 20; mu3 = 20; mu4 = 20;
%     S = [-mu1, mu1, 0,0; 0, -mu2, mu2,0; 0,0 -mu3,mu3;0,0,0, -mu4];
%     S0 = [0;0;0;mu4];
%     sigma = [1,0,0,0];

%%%   Example 2: H_2/Ph/1 queue:
%%%   H_2 Arrival Process:
      alpha1 = 0.4; alpha2 = 0.6; lambda1 = 1.9; lambda2 = 2;
      T = [-lambda1, 0 ; 0, -lambda2];
      T0 = [lambda1;lambda2];
```

```
      xi = [alpha1, alpha2];

%%%   Hypo-exponential-3 Service Process:
      mu1 = 4; mu2 = 8; mu3 = 8;
      S = [-mu1, mu1, 0; 0, -mu2, mu2; 0,0, -mu3];
      S0 = [0;0;mu3];
      sigma = [1,0,0];


%%%%%%%%%  Block Submatrices for all types of queues:  %%%%%%
      ra = size(T,2);  rs = size(S,2);
      A0 = kron(eye(ra), S0*sigma);
      A1 = kron(T, eye(rs)) + kron(eye(ra), S);
      A2 = kron(T0*xi, eye(rs));
      B00 = T;
      B01 = kron(T0*xi,sigma);
      B10 = kron(eye(ra),S0);
      l = size(B00,2);  r = size(A0,2);
```

```
%%%%%%%%%  Check stability  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

      meanLambda = 1/norm(-xi* inv(T),1);
      meanMu = 1/norm(-sigma * inv(S),1);
      rho = meanLambda/meanMu

%%%%%%%%%%%  Alternatively:   %%%%%%%%%
      A = A0+A1+A2;
      for k=1:r
          A(k,r) = 1;
      end
      rhs = zeros(1,r); rhs(r)= 1;
      ss = rhs*inv(A);
      rho = norm(ss*A2,1)/norm(ss*A0,1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

      if  rho >=0.999999
          error('Unstable System');
      else
          disp('Stable system')
      end
```

```
%%%%%%%%%%    Form Neuts' R matrix   %%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%           by              %%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%  Successive Substitution  %%%%%%%%%%%%%%%%%%%%%%%%%

     V = A2 * inv(A1);  W = A0 * inv(A1);

     R = -V;      Rbis = -V - R*R * W;
     iter = 1;
     while (norm(R-Rbis,1)> 1.0e-10 & iter<100000)
        R = Rbis;  Rbis = -V - R*R * W;
        iter = iter+1;
     end
     iter
     R = Rbis;

%%%%%%%%%%%%         or  by           %%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%   Logarithmic Reduction %%%%%%%%%%%%%%%%%%%%%%%%%

%    Bz = -inv(A1)*A2;  Bt = -inv(A1)*A0;
%    T = Bz;  S = Bt;
%    iter = 1;
```

```
%     while (norm(ones(r,1)-S*ones(r,1) ,1)> 1.0e-10 & iter<100000)
%        D = Bz*Bt + Bt*Bz;
%        Bz = inv(eye(r)-D) *Bz*Bz;
%        Bt = inv(eye(r)-D) *Bt*Bt;
%        S = S + T*Bt;
%        T = T*Bz;
%        iter = iter+1;
%     end
%     iter
%     U = A1 + A2*S;
%     R = -A2 * inv(U)

%%%%%%%%%%%%%%  Solve boundary equations  %%%%%%%%%%%%%%%%%%%%
      N = [B00,B01;B10,A1+R*A0];  % Set up boundary equations

      N(1,r+l) = 1;               % Set first component equal to 1
      for k=2:r+l
         N(k,r+l) = 0;
      end
      rhs = zeros(1,r+l); rhs(r+l)= 1;
      soln = rhs * inv(N);         % Un-normalized pi_0 and pi_1
```

```
        pi0 = zeros(1,l);  pi1 = zeros(1,r);
        for k=1:l
          pi0(k) = soln(k);         % Extract pi_0
        end
        for k=1:r
          pi1(k) = soln(k+l);       % Extract pi_1
        end
        e = ones(r,1);
        sum = norm(pi0,1) +  pi1 * inv(eye(r)-R) * e;    % Normalize solution
        pi0 = pi0/sum; pi1 = pi1/sum;


%%%%%%%%%%%%%%%%%  Print results %%%%%%%%%%%%%%%%%%%%%%%%%%%%
        max = 10;     %  maximum population requested
        pop = zeros(max+1,1);
        pop(1) =  norm(pi0,1);
        for k=1:max
          pi = pi1 * R^(k-1);          % Get successive components of pi
          pop(k+1) = norm(pi,1);
        end
        pop
```

```
%%%%%%%%%%%%%  Measures of Effectiveness  %%%%%%%%%%%%%%%%%%%%
     EN = norm(pi1*inv(eye(r)-R)^2,1)
%    ENq = EN-meanLambda/meanMu
%    ER =  EN/meanLambda
%    EWq = ENq/meanLambda

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```