Spatial Representations and Analysis Techniques
Part I: Representation

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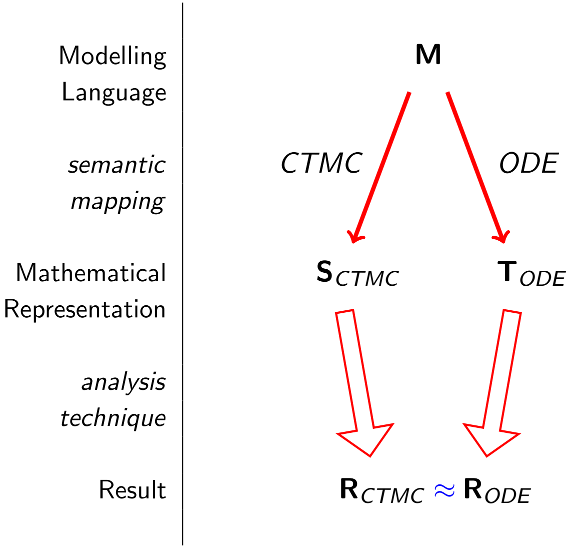
Outline

1. Overview of modelling
2. Modelling without space
3. Addition of space
4. Discrete space
5. Continuous space
6. Classification of space
7. Assessment for CAS
8. Examples of spatial models

Overview of modelling

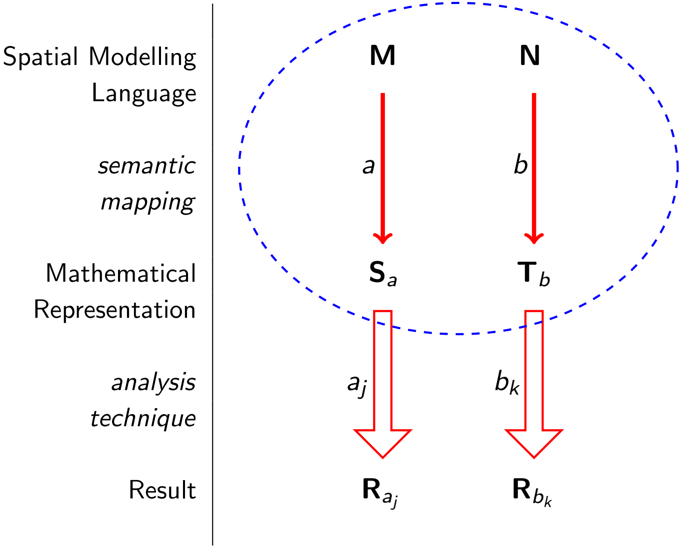
Fluid/mean-field approximation



Modelling of space

$$\begin{tikzcd}
\text{Spatial Modelling Language} & \mathbf{M} \arrow[dd, "a"'] & \color{blue}{?} & \mathbf{N} \arrow[dd, "b"'] \\
\textit{semantic mapping} & & & \\
\text{Mathematical Representation} & \mathbf{S}_a \arrow[dd, Rightarrow, "a_j"'] & \color{blue}{?} & \mathbf{T}_b \arrow[dd, Rightarrow, "b_k"'] \\
\textit{analysis technique} & & & \\
\text{Result} & \mathbf{R}_{a_j} & \color{blue}{?} & \mathbf{R}_{b_k}
\end{tikzcd}$$

Language-based modelling of space



Spatial Modelling Language: **M** **N**

semantic mapping: a b

Mathematical Representation: $\mathbf{S}_a$ $\mathbf{T}_b$

analysis technique: a_j b_k

Result: $\mathbf{R}_{a_j}$ $\mathbf{R}_{b_k}$

Languages for space

process algebra with implicit location

stochastic π-calculus

process algebra with explicit location or space

non-quantitative

CCS with locations sccp *LSCRP*
process algebra for wireless mesh networks
plain CHOCS *π-calculus extensions*
CCS WITH GRADED SPATIAL ACTIONS

quantitative

PEPA with locations PALOMA
stochastic HYPE STOKLAIM *Ambients*
STOCS **real space process algebra**
spatial stochastic process algebra

process algebra for biology

3π *Bio-Ambients* *Imperative Pi* **BioSpace^L** MELA
Bio-PEPA with compartments **Shape Calculus** **Sπ@**
Space Pi **Attributed Pi with priorities** **spatial π-calculus**

spatial programming languages

TOTA Proto
Protelis

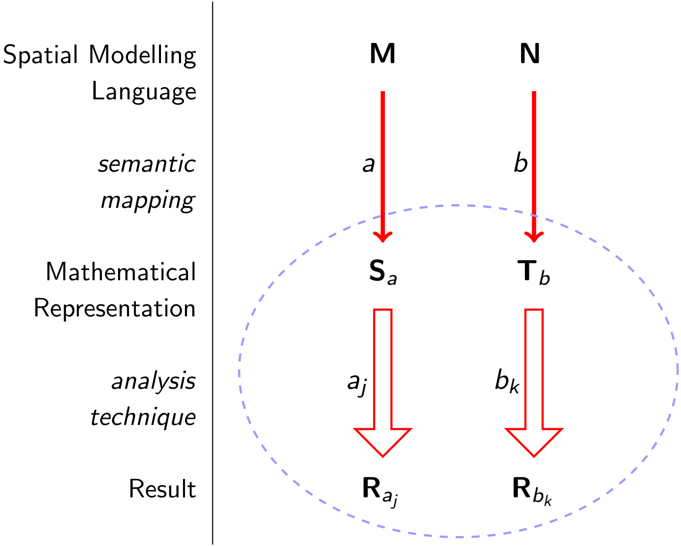
graphical approaches

Petri nets with explicit locations
bigraphs PEPA nets

rule-based approaches

ML-SPACE Kappa with space *ML-DEVS*
spatial calculus of looping sequences
Kappa with geometry ML-Rules

Mathematical modelling of space



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Modelling without space

Without space

- often space is not taken into account in quantitative modelling
- not relevant to the modelling question under consideration
  - in biological modelling, mass action assumes a spatial homogeneity
  - individuals and population are not located because space plays no role in their behaviour

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  - passage of time
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- overview of nonspatial representations

Non-spatial dimensions

TIME	discrete	continuous

Non-spatial dimensions

TIME	discr	cont

Non-spatial dimensions

TIME	discr	cont
	probabilistic	stochastic

Non-spatial dimensions

TIME	discr	cont

probabilistic

stochastic

non-determinism
fixed delay

Non-spatial dimensions

TIME	discr	cont

probabilistic

stochastic

non-determinism
fixed delay

non-negative, strictly increasing, infinite

Non-spatial dimensions

TIME	discr	cont
STATE	discr	cont

Non-spatial dimensions

TIME	discr	cont
STATE	discr	cont
	often finite	infinite

Non-spatial dimensions

TIME	discr	cont
STATE	discr	cont

Non-spatial dimensions

TIME	discr	cont
STATE	discr	cont
AGGR	none	state-based

Non-spatial dimensions

TIME	discr	cont
STATE	discr	cont
AGGR	none	state

Non-spatial dimensions

TIME	discr	cont
STATE	discr	cont
AGGR	none	state

individual
current state

Non-spatial dimensions

TIME	discr	cont
STATE	discr	cont
AGGR	none	state
	individual current state	populations state frequency data numerical vector form counting abstraction occupancy measure

Non-spatial dimensions

TIME	discr	cont
STATE	discr	cont
AGGR	none	state

Non-spatial dimensions

TIME discr — hybrid — cont

STATE discr — hybrid — cont

AGGR none — hybrid — state

Non-spatial dimensions

TIME	discr	cont
STATE	discr	cont
AGGR	none	state

Non-spatial dimensions

Time	discr	cont
Aggr	none	state
State	discr	cont

Non-spatial dimensions

TIME	discr	cont
AGGR	none	state

STATE	discr	cont

Non-spatial dimensions

Time	discr		cont	
Aggr	none	state	none	state

State	discr	cont

Non-spatial dimensions

TIME	discr		cont	
AGGR	none	state	none	state
STATE	discr		cont	

Non-spatial dimensions

Time	discr				cont			
Aggr	none		state		none		state	
State	discr	cont	discr	cont	discr	cont	discr	cont

Non-spatial dimensions

TIME	discr				cont			
AGGR	none		state		none		state	
STATE	discr	cont	discr	cont	discr	cont	discr	cont

DTMC

DTMC discrete-time Markov chain

Non-spatial dimensions

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AGGR	none		state		none		state	
STATE	discr	cont	discr	cont	discr	cont	discr	cont

DTMC ?

DTMC discrete-time Markov chain

Non-spatial dimensions

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AGGR	none		state		none		state	
STATE	discr	cont	discr	cont	discr	cont	discr	cont
	DTMC	?	popn DTMC					

DTMC discrete-time Markov chain

Non-spatial dimensions

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AGGR	none		state		none		state	
STATE	discr	cont	discr	cont	discr	cont	discr	cont
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Non-spatial dimensions

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STATE	discr	cont	discr	cont	discr	cont	discr	cont
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DTMC	discrete-time Markov chain
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Non-spatial dimensions

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DTMC	discrete-time Markov chain
CTMC	continuous-time Markov chain
LMP	labelled Markov process
popn	population

Non-spatial dimensions

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Non-spatial dimensions

<table>
<tr><td>TIME</td><td colspan="4">cont</td></tr>
<tr><td>AGGR</td><td colspan="2">none</td><td colspan="2">state</td></tr>
<tr><td>STATE</td><td>discr</td><td>cont</td><td>discr</td><td>cont</td></tr>
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CTMC</td><td>ODE</td></tr>
</table>

Non-spatial dimensions

TIME	cont			
AGGR	none		state	
STATE	discr	cont	discr	cont

		LMP		
			popn	
	CTMC		CTMC	ODE

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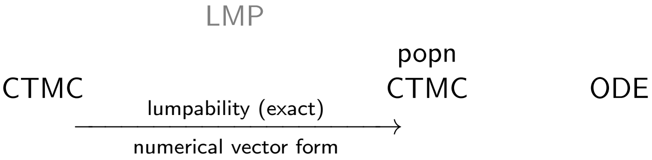

LMP

CTMC

popn CTMC

ODE

CTMC → popn CTMC: lumpability (exact) / numerical vector form

Non-spatial dimensions

TIME	cont			
AGGR	none		state	
STATE	discr	cont	discr	cont

LMP

$$\begin{tikzcd} & \text{popn} & \\ \text{CTMC} \arrow[r, "\text{lumpability (exact)}", "\text{numerical vector form}"'] & \text{CTMC} \arrow[r, "\text{fluid}", "\text{approx}"'] & \text{ODE} \end{tikzcd}$$

Addition of space

Scalable analysis applied to space

Current approach for scalable modelling

Scalable analysis applied to space

Current approach for scalable modelling

State-space explosion $\xrightarrow{\text{fluid approximation}}$ ODEs

Scalable analysis applied to space

Current approach for scalable modelling

State-space explosion $\xrightarrow{\text{fluid approximation}}$ ODEs

Does this work with discrete space?

Scalable analysis applied to space

Current approach for scalable modelling

State-space explosion $\xrightarrow{\text{fluid approximation}}$ ODEs

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Add space

Scalable analysis applied to space

Current approach for scalable modelling

State-space explosion $\xrightarrow{\text{fluid approximation}}$ ODEs

Does this work with discrete space?

Bigger state-space explosion $\xrightarrow{\text{fluid approximation}}$ PDEs

Scalable analysis applied to space

Current approach for scalable modelling

State-space explosion $\xrightarrow{\text{fluid approximation}}$ ODEs

Does this work with discrete space?

Bigger state-space explosion $\xrightarrow{\text{fluid approximation}}$ PDEs

Maybe but not the only approach

Quantitative modelling with space

- what is the main objective here?

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- modelling collective adaptive systems (CAS)
    - quantitative: behaviour over time is important
    - aggregation: many agents lead to state space explosion
    - space: behaviour with respect to space relevant to many CAS

Quantitative modelling with space

- what is the main objective here?

- modelling collective adaptive systems (CAS)
  - quantitative: behaviour over time is important
  - aggregation: many agents lead to state space explosion
  - space: behaviour with respect to space relevant to many CAS

- consider mathematical representations of space and movement
  - results obtained by analysis techniques
  - semantic target for CAS modelling languages
  - understand relationships between representations

Spatial aspects of representations

Space	discr	cont

Spatial aspects of representations

SPACE	discr	cont
	grid, lattice, regular neighbourhood single individual at each node multiple individuals at each node	

Spatial aspects of representations

SPACE	discr	cont

grid, lattice, regular
neighbourhood
single individual at each node
multiple individuals at each node

patches, irregular
explicit adjacency relationship
regions of cont space
multiple individuals in each patch

Spatial aspects of representations

SPACE	discr	cont
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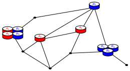
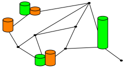
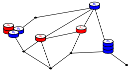
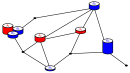
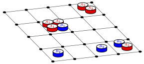
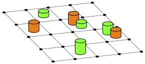
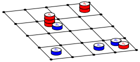
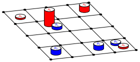
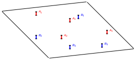
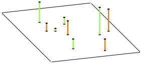
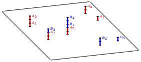
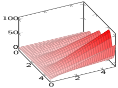
typically 2-dimensional or 3-dimensional

Spatial aspects of representations

SPACE	discr	cont
	grid, lattice, regular neighbourhood single individual at each node multiple individuals at each node	infinite location usually changes smoothly
	patches, irregular explicit adjacency relationship regions of cont space multiple individuals in each patch	Another approach: topological space will covered tomorrow in spatio-temporal logic

typically 2-dimensional or 3-dimensional

Spatial classification

Time	continuous			
Aggr	none		state and/or space	
State	discrete	continuous	discrete	continuous
Space				
discrete				
regular				
continuous				

Discrete space

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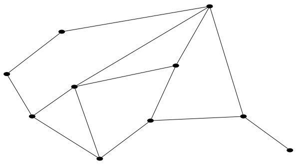
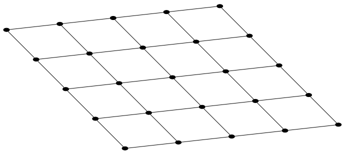


discrete



regular discrete

assumption that vertices and edges are static but parameters associated with edges (movement) and vertices (local interaction) may be time-dependent

Discrete space

- set of locations: $\mathcal{L}$
- undirected graph over locations: $(\mathcal{L}, E_{\mathcal{L}})$
- edges are two element sets: $\{\ell_1, \ell_2\} \in \mathcal{P}_2(\mathcal{L})$
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  - $\lambda(\ell)$ for all locations $\ell \in \mathcal{L}$, and
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- locations
  - points in space: $\mathcal{L}$
  - regions in space: $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathcal{L}$

Connectivity in discrete space

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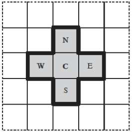
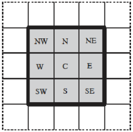
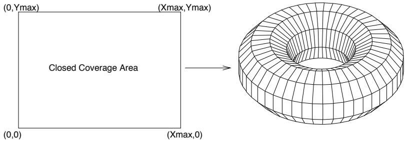
  - one-hop neighbour: traverse a single edge
  - *n*-hop neighbour: traverse *n* edges
- neighbourhood: spatially regular graph
  - von Neumann: N, E, S, W
  - Moore: N, NE, E, SE, S, SW, W, NW





Aspects of discrete space

- boundary conditions
  - avoid: graph defined over torus or sphere



  - include: can be an accurate model of reality

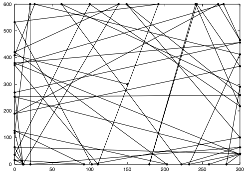


Figure 6: Traveling pattern of an MN using the Random Direction Mobility Model.

Space and homogeneity

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- **spatially homogeneous**:
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- **spatially homogeneous**:
  parameter homogeneous and complete location graph (every location neighbours every other location)
- spatial homogeneity may lead to analytic solutions rather than simulation of differential equations

Space and regularity

- **spatially regular**
  - maybe be parameter homogeneous
  - not spatially homogeneous
  - not easy to define from a graph but obvious to identify
- two dimensions: triangles, rectangles, hexagons
- one dimension: path
- other possibilities
- characterised by regular way to define neighbours

Spatial classification

TIME	continuous			
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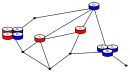
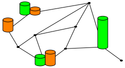
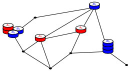
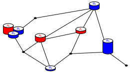
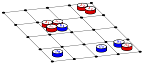
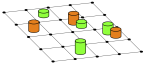
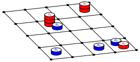
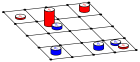
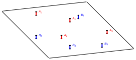
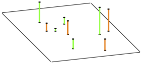
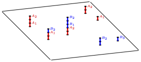
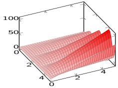
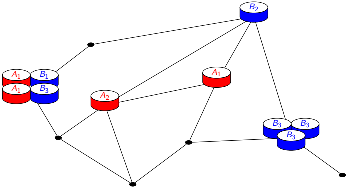
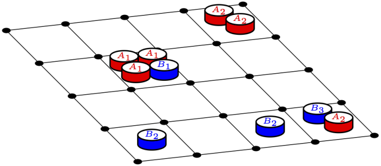
STATE	discrete	continuous	discrete	continuous
SPACE				
discrete				
regular				
continuous				

Discrete space

discrete state without aggregation





Discrete space: no state aggregation

- no state aggregation: modelling individuals

Discrete space: no state aggregation

- no state aggregation: modelling individuals
- consider J, named individual
    - $\text{loc}(J, t) \in \mathcal{L}$, location at time t
    - $\text{state}(J, t) = A_i$ or $\text{state}(J, t) = Y$
    - set of rules to describe behaviour

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- discrete state, discrete space and rates:
  CTMC with states of the form
  $\big((\text{loc}(J_1, t), \text{state}(J_1, t)), \ldots, (\text{loc}(J_N, t), \text{state}(J_N, t))\big)$

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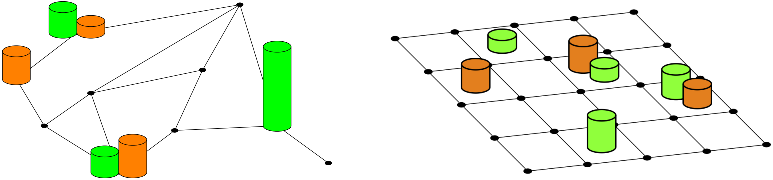
- potential for $(n \times p)^N$ states in CTMC where p is number of locations, n is number of states and N is number of individuals

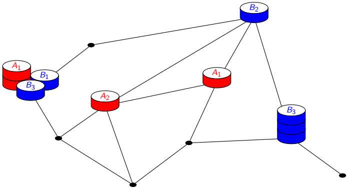
Discrete space

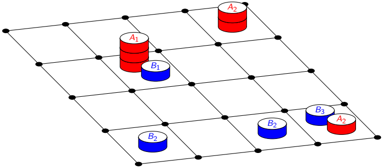
continuous state without aggregation



Discrete space

discrete state with aggregation





Notation

A subpopulation is the subset of a population that is in a given state.

Assuming n subpopulations and p locations, then $X_i^{(j)}(t)$ is the size of the subpopulation i at location j at time t.

$$\begin{aligned} \mathbf{X}_i &= (X_i^{(1)}, \ldots, X_i^{(p)}) \\ \mathbf{X}^{(j)} &= (X_1^{(j)}, \ldots, X_n^{(j)}) \\ \mathbf{X} &= (\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(n)}) \end{aligned} \qquad \begin{aligned} X_i &= \textstyle\sum_{j=1}^{p} X_i^{(j)} \\ X^{(j)} &= \textstyle\sum_{i=1}^{n} X_i^{(j)} \\ X &= \textstyle\sum_{i=1}^{n} \sum_{j=1}^{p} X_i^{(j)} \\ &= \textstyle\sum_{j=1}^{p} \sum_{i=1}^{n} X_i^{(j)} \end{aligned}$$

The size of subpopulation $X_i^{(j)}$ at time t is $N_i^{(j)}(t)$.
The total size of subpopulation X_i at time t is $N_i(t)$.

Discrete space: state aggregation

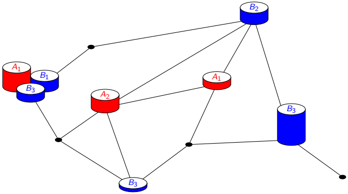
- discrete space, discrete aggregated state:

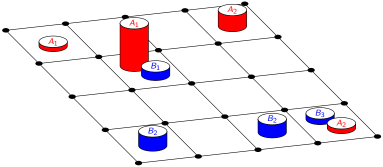
  population CTMC with states of the form

  $$\left(X_1^{(1)},\ldots,X_n^{(1)},\ldots,X_1^{(k)},\ldots,X_n^{(k)},\ldots,X_1^{(p)},\ldots,X_n^{(p)}\right)$$
- this CTMC is much smaller than that for discrete space and discrete state without aggregation
- analysis provides same results at population level
- potential for $(M+1)^{n\times p}$ states in CTMC where p is number of locations, n is number of states and M is the maximum subpopulation size

Discrete space

continuous state with aggregation





Discrete space: state aggregation

- continuous aggregated state gives population ODEs that approximate CTMC results (under certain conditions)

$$\begin{aligned}\frac{dX_i^{(j)}}{dt} &= f_{i,j}\big(X_1^{(j)},\ldots,X_n^{(j)}\big) + \\ &\sum_{k=1,k\neq j}^{p} \big(g_{i,j,k}(X_1^{(k)},\ldots,X_n^{(k)}) - h_{i,j,k}(X_1^{(j)},\ldots,X_n^{(j)})\big)\end{aligned}$$

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- a simpler form with parameter homogeneity

$$\frac{dX_i^{(j)}}{dt} = f\left(X_1^{(j)}, \ldots, X_n^{(j)}\right) + \sum_{j=k, j\neq j}^{p} \left(g(X_1^{(j)}) - h(X_1^{(k)})\right)$$

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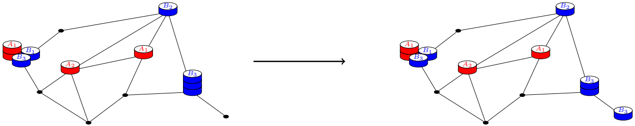
- $n \times p$ ODEs where p is number of locations and n is number of states

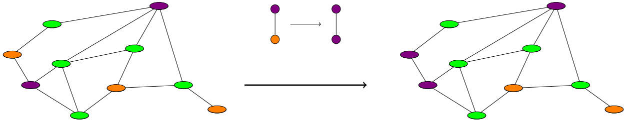
Movement in discrete space

Discrete space and movement



- discrete space with state-based aggregation
- interaction between and within populations at locations
- movement between locations
- patch population models
- population CTMCs and ODEs with locations

Discrete space and movement

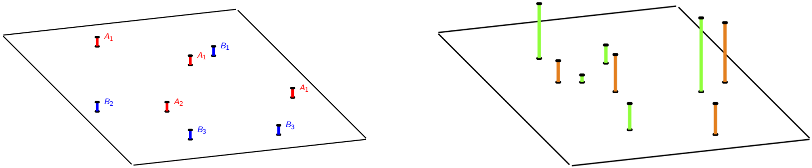


- single occupancy of locations
- discrete space without state-based aggregation
- graph transformation rules, change at a location
- cellular automata
- interacting particle systems

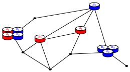
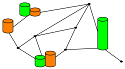
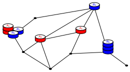
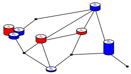
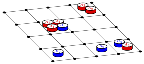
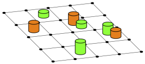
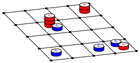
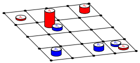
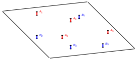
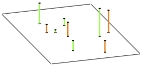
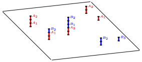
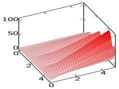
Continuous space

Continuous space

without aggregation



Spatial classification

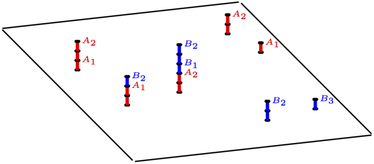
TIME	continuous			
AGGR	none		state and/or space	
STATE	discrete	continuous	discrete	continuous
SPACE				
discrete				
regular				
continuous				

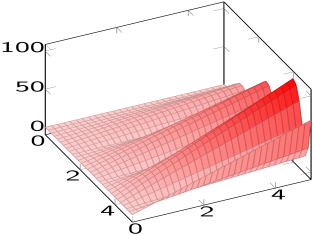
Continuous space

- $\mathbb{R} \times \mathbb{R}$ or contiguous subset
- radius to define neighbourhood
- boundary conditions: similar to discrete space
- no aggregation, consider individual J
    - $\text{loc}(J, t) = (x, y)$ which is its location at time t
    - $\text{state}(J, t) = A_i$ or $\text{state}(J, t) = Y$
    - interaction rules
    - movement description: random walk, etc
- agent model
- continuous space and state: continuous-time Markov processes
- continuous space and discrete state: hybrid

Continuous space

with aggregation





Continuous space

- discrete aggregated state: spatio-temporal point processes
  $X_i((x, y), t) \in \mathbb{N}$ and $\lambda((x, y), t)$ describes behaviour

  $\frac{dE[X_i]}{dt}$ describes change in average density, global measure

Continuous space

- discrete aggregated state: spatio-temporal point processes
  $X_i((x,y),t) \in \mathbb{N}$ and $\lambda((x,y),t)$ describes behaviour

  $\frac{dE[X_i]}{dt}$ describes change in average density, global measure
- continuous aggregated state: partial differential equations

$$\begin{aligned}\frac{\partial X_i}{\partial t} &= f_i(X_1,\ldots,X_n^{()}) \\ &+ \frac{\partial}{\partial x}\left(D(X_i,(x,y))\frac{\partial X_i}{\partial x}\right) \\ &+ \frac{\partial}{\partial y}\left(D(X_i,(x,y))\frac{\partial X_i}{\partial y}\right)\end{aligned}$$

D can depend on more than current population size and location

Movement in continuous space

Movement

- relationship with parameters and model abstraction

Movement

- relationship with parameters and model abstraction
- movement in continuous space
  - probability
  - speed
  - direction
  - individual/group/concentration
  - boundaries: reflection, absorption, none

Mobility models from networking

- survey articles [Camp *et al* 2002, Musolesi and Mascolo 2009]
- distributions often unclear

Mobility models from networking

- survey articles [Camp *et al* 2002, Musolesi and Mascolo 2009]
- distributions often unclear
- mobility models for single node/mobile entity
    - random walk: random direction and speed, reflect
    - random way-point: random destination and speed, pause, reflect
    - boundless simulation area:
    - Gauss-Markov: normally distributed random variables used to update speed and direction from current speed and direction, parameter to tune randomness
    - probabilistic random walk: probability matrix to determine new direction (if any) and position, fixed step size

Mobility models from networking

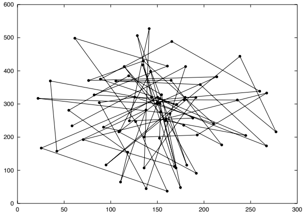


Figure 1: Traveling pattern of an MN using the 2-D Random Walk Mobility Model (time).

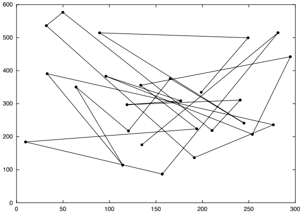


Figure 3: Traveling pattern of an MN using the Random Waypoint Mobility Model.

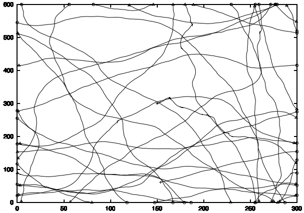


Figure 8: Traveling pattern of an MN using the Boundless Simulation Area Mobility Model.

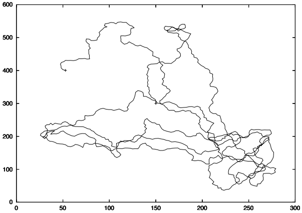


Figure 10: Traveling pattern of an MN using the Gauss-Markov Mobility Model.

Mobility models from networking

- group mobility models
  - reference point group: subsumes earlier models
    each node has a reference point, relative position of reference points are fixed, each node moves randomly around its reference point, references points move as a group
  - introduction of barriers, use of Voronoi graphs
- use of data from real logs to generate synthetic data
- connectivity models
  - dynamic graphs
  - parameter identification: contact duration, time between contacts

Mobility models: networking

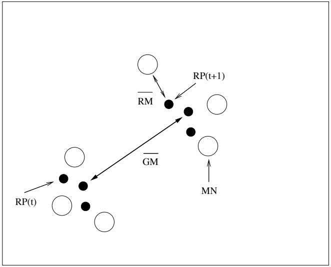


Figure 18: Movements of three MNs using the RPGM model.

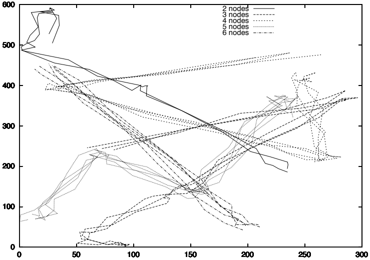
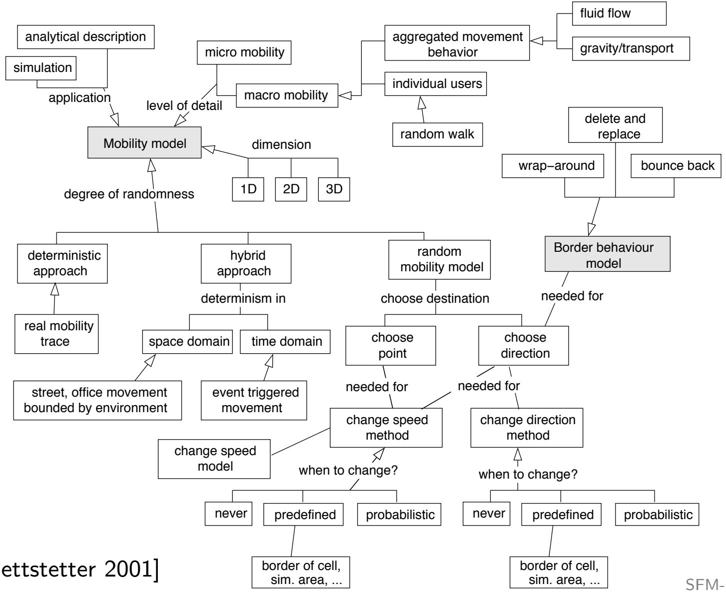


Figure 20: Traveling pattern of five groups using the RPGM model.

Mobility models: networking



[Bettstetter 2001]

Movement models: biology/ecology

- survey articles [Codling *et al* 2012, Holmes *et al* 1994]
- focus on deterministic models with continuous space, PDEs
- assume a density function $X_1(x, y, t)$ over 2-dimensional space

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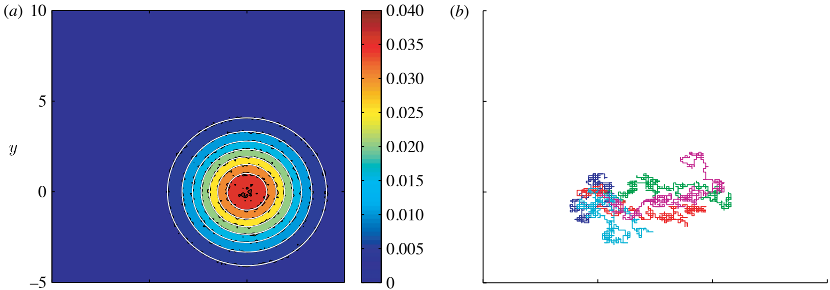
- focus on deterministic models with continuous space, PDEs
- assume a density function $X_1(x, y, t)$ over 2-dimensional space
- Brownian random motion/random walk

$$\frac{\partial X_1}{\partial t} = D\left(\frac{\partial^2 X_1}{\partial x^2} + \frac{\partial^2 X_1}{\partial y^2}\right) = D \triangle X_1$$

  - diffusion constant: D
  - suits homogeneous space and uniform movement rates
  - allows unbounded movement

Movement models: biology/ecology



[Codling *et al* 2012]

Movement models: biology/ecology

- Brownian motion with drift/biased random walk

$$\frac{\partial X_1}{\partial t} = D\left(\frac{\partial^2 X_1}{\partial x^2} + \frac{\partial^2 X_1}{\partial y^2}\right) - w_x \frac{\partial X_1}{\partial x} - w_y \frac{\partial X_1}{\partial y}$$

  - w_x and w_y are drift velocities
  - models external stimuli affecting movement
  - zig-zag motion

Movement models: biology/ecology

- Brownian motion with drift/biased random walk

$$\frac{\partial X_1}{\partial t} = D\left(\frac{\partial^2 X_1}{\partial x^2} + \frac{\partial^2 X_1}{\partial y^2}\right) - w_x \frac{\partial X_1}{\partial x} - w_y \frac{\partial X_1}{\partial y}$$

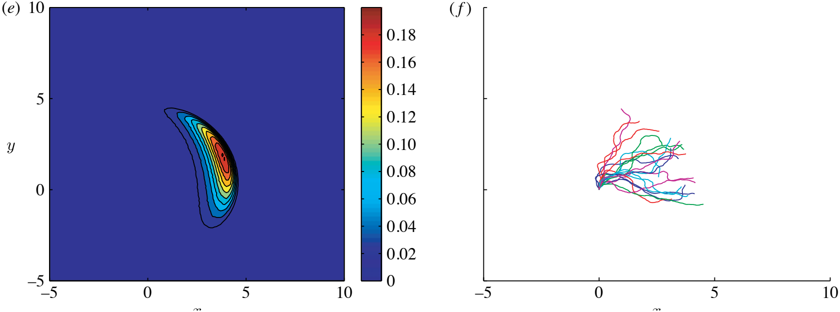
  - w_x and w_y are drift velocities
  - models external stimuli affecting movement
  - zig-zag motion

- correlated random walk, telegraph equation

$$\frac{\partial X_1}{\partial t} = v^2\left(\frac{\partial^2 X_1}{\partial x^2} + \frac{\partial^2 X_1}{\partial y^2}\right) - 2\lambda \frac{\partial^2 X_1}{\partial t^2}$$

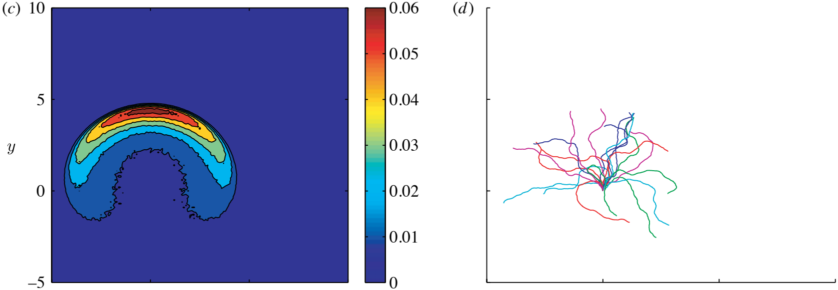
  - velocity of organisms: v
  - bounded distribution, no inconsistent movement

Movement models: biology/ecology



[Codling *et al* 2012]

Movement models: biology/ecology



[Codling *et al* 2012]

Movement models: biology/ecology

- biased movement relative to other animals, $k \in \mathbb{R}$

$$\frac{\partial X_1}{\partial t} = D\left(\frac{\partial^2 X_1}{\partial x^2} + \frac{\partial^2 X_1}{\partial y^2}\right) + \frac{\partial}{\partial x}\left(kX_1 \frac{\partial X_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(kX_1 \frac{\partial X_1}{\partial y}\right)$$

$k > 0$ towards others, $k < 0$ away from others

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  $k > 0$ towards others, $k < 0$ away from others

- density-dependent movement, density function $\psi(u)$

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 X_1}{\partial x^2} + \frac{\partial^2 X_1}{\partial y^2}\right) + \frac{\partial^2 \psi(X_1)}{\partial x^2} + \frac{\partial^2 \psi(X_1)}{\partial y^2}$$

  ψ is negative at low density and positive at high density

Movement models: biology/ecology

- diffusion and reaction with two species, pairwise interaction

$$\begin{aligned}\frac{\partial X_1}{\partial t} &= D_1\left(\frac{\partial^2 X_1}{\partial x^2}+\frac{\partial^2 X_1}{\partial y^2}\right)+(r_1-\alpha_{11}X_1-\alpha_{1v}X_2)X_1\\ \frac{\partial X_2}{\partial t} &= D_2\left(\frac{\partial^2 X_2}{\partial x^2}+\frac{\partial^2 X_2}{\partial y^2}\right)+(r_2-\alpha_{22}X_2-\alpha_{21}X_1)X_2\end{aligned}$$

  - growth terms: r_1, r_2
  - effect on own species: α_{11}, α_{22}
  - effect on other species: α_{12}, α_{21}

Classification and assessment

Classification of representations

Time	continuous			
Aggr	none		state	
State	discrete	continuous	discrete	continuous
Space				
discrete	CTMC, cellular automata, IPS	TDSHA, piecewise deterministic Markov process (PDMP)	population CMTCs with locations	population ODEs with locations
continuous	agents, molecular dynamics	continuous-time Markov process (CTMP)	spatio-temporal point process (STPP)	partial differential equation (PDE)

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Approximation techniques

TIME	continuous			
AGGR	none		state	
STATE	discrete	continuous	discrete	continuous
SPACE				
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continuous				partial differential equation (PDE)

Approximation techniques

Time	continuous			
Aggr	none		state	
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Space			fluid approx →	
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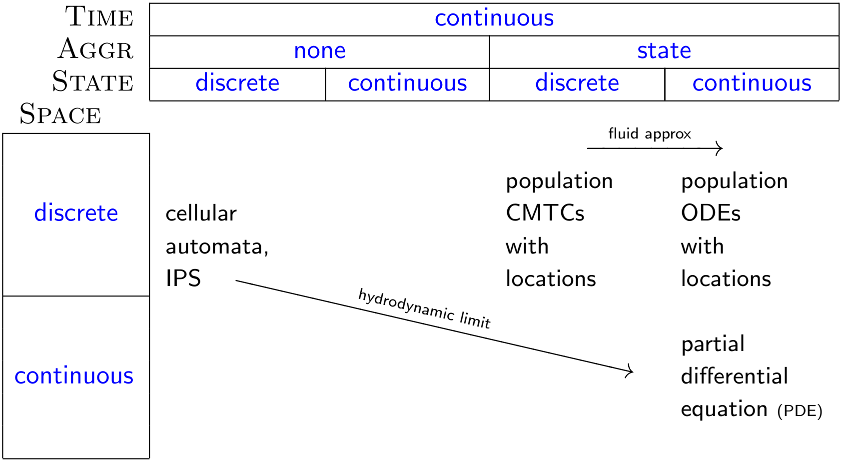
Approximation techniques



TIME	continuous			
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SPACE				
discrete	cellular automata, IPS		population CMTCs with locations	population ODEs with locations
continuous				partial differential equation (PDE)

Assessment for CAS

Observations and guidelines

Discrete space with aggregation

Existing Approaches

general	population CTMCs patch models reaction-dispersal networks metapopulation models compartments
regular	lattice/grid models coupled-map lattices subvolumes cellular Potts models pattern formation models

Observations and guidelines

Discrete space with aggregation

Assessment

- This type of approach has been used for smart transport case studies.
- Many use approximations of stochastic models by ODEs that are typically easy to solve numerically.
- Some focus on average or global behaviour. We want to consider local behaviour as well.
- Some modelling approaches use features of the modelling scenario to construct useful approximations like differences in rates.
- Sufficient subpopulations per location are needed when using ODEs.

Observations and guidelines

Continuous space with aggregation

Existing Approaches

partial differential equations (PDEs): These describe continuous aggregation over continuous space, and are typically solved by discretization techniques. They can be obtained by the hydrodynamic limit of individuals in regular discrete space.

spatio-temporal point processes (STPPs): These describe discrete aggregation over continuous space. STPPs are typically analysed by finding average measures of density as ODEs hence they are global in nature

Observations and guidelines

Continuous space with aggregation

Assessment

- These approaches appear to have a limited match with smart transport but are suitable for density-related aspects.
- Discretisation and the associated solutions of continuous space models may provide approaches to analysing discrete space models.

Observations and guidelines

Discrete/continuous space without aggregation

Existing Approaches

discrete	CTMCs
regular discrete	interacting particle systems cellular automata contact processes Markov random fields Gibbs states
continuous	labelled Markov processes transition-driven stochastic hybrid automata piecewise deterministic Markov processes particle space agent modelling

Observations and guidelines

Discrete/continuous space without aggregation

Assessment

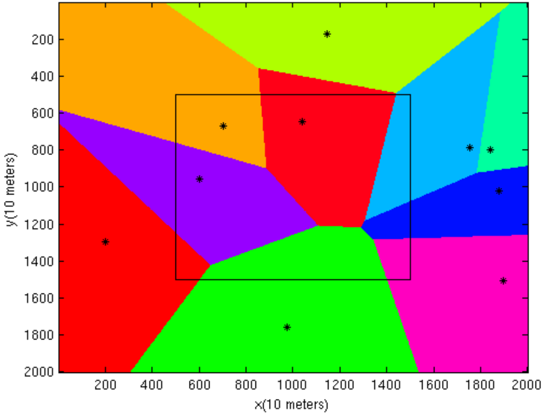
- These approaches involve modelling each individual and its location.
- They will be useful for types of smart transport modelling involving individual entities, such as some bus modelling.
- Continuous space with individuals provides a starting point for transforming continuous space to discrete space, resulting in aggregation of both state and space.

Examples of spatial models

ZebraNet

- model of ad hoc network using wildlife
- original model is continuous space and not aggregated [Juang *et al*, 2002; Feng, 2014]
- transformation to a discrete space, aggregated model [Feng, 2014]
- use of simulation from full model to obtain movement parameters
- patches identified by Voronoi tessellation based on waterholes
- results are good approximation to full model
- much larger models can be considered

ZebraNet



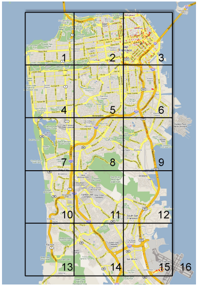
y(10 meters)
200
400
600
800
1000
1200
1400
1600
1800
2000
x(10 meters)

Modelling movement

- Age of Gossip (Chaintreau *et al* 2009)
- aim: build a meanfield model of data exchange and ageing for taxis in San Francisco Bay area
- GPS traces: location and time stamp
- division of area into equal regions
- generation of contact traces
    - *meeting* defined by radio range and time in range
- parameter extraction from contact trace
    - counts of vehicles and meetings
    - movement rates, contact rates (3 different types)
- parameters used in stochastic and meanfield simulations
- comparison of contact traces and both simulations

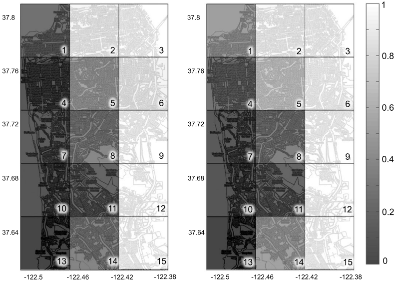
Modelling movement

Locations/Patches



Trace

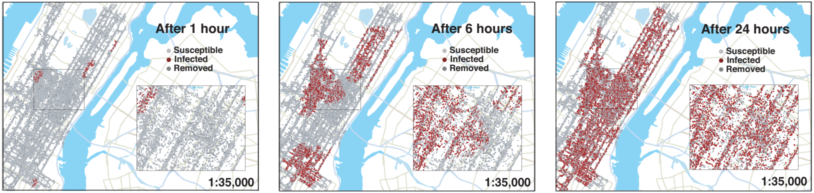
Mean field



Proportion of mobile nodes with age $z \leq 20$ at time $t = 300$ min

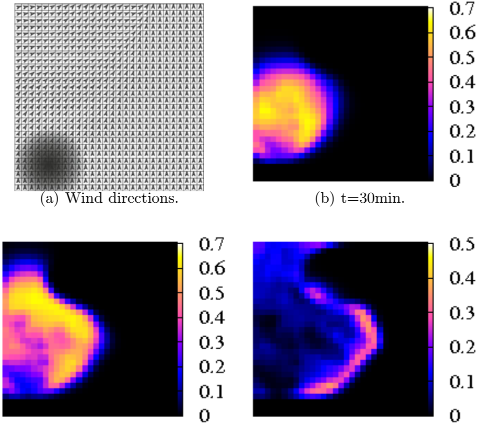
[Chaintreau, Le Boudec and Ristanovic, 2009]

Wireless virus spread



[Hu *et al*, 2009]

Fire propagation



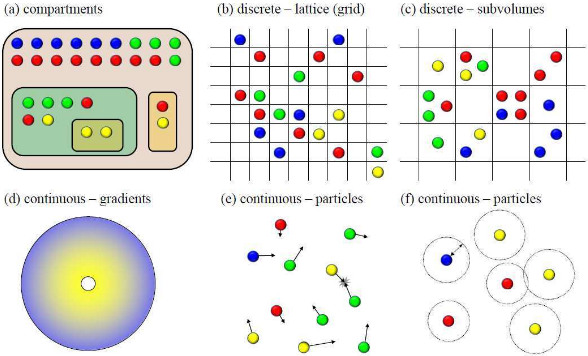
(a) Wind directions. (b) t=30min.

(c) t=40min. (d) t=90min.

Fig. 9. Fire Propagation with a spatial-dependent wind

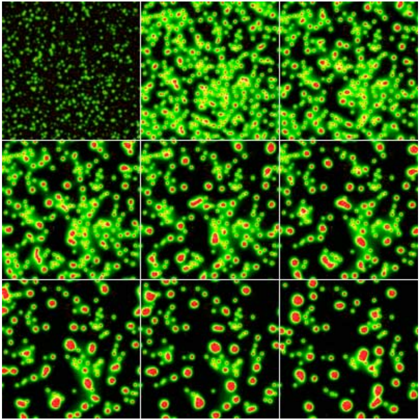
[Cerotti *et al*, 2009]

Space modelling in biology



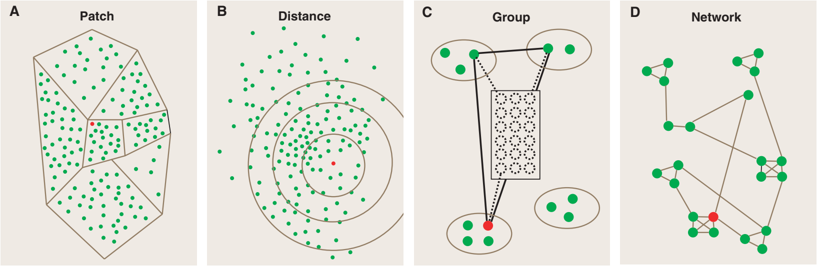
[Bittig and Uhrmacher, 2001]

Cellular automata model



[Matthews, http://www.generation5.org, 2004]

Spread of disease



[Riley, 2007]

End of Part I