Spatial Logic and Spatial Model Checking for Closure Spaces

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Introduction

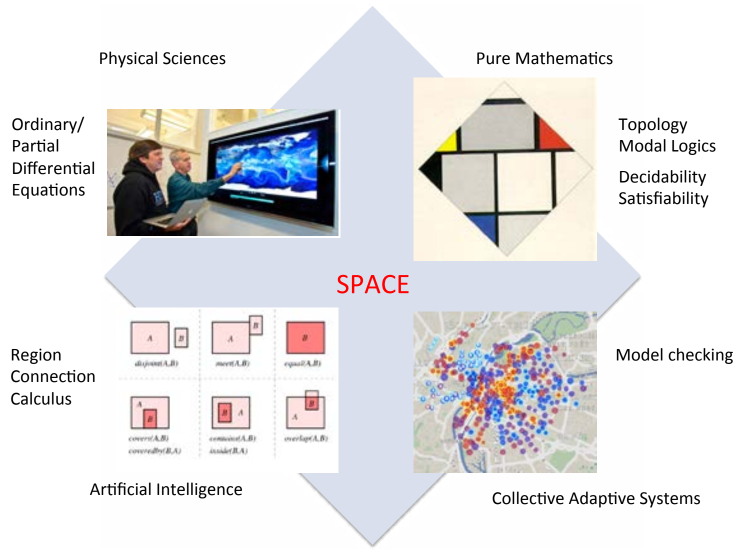
Origins of Spatial Reasoning

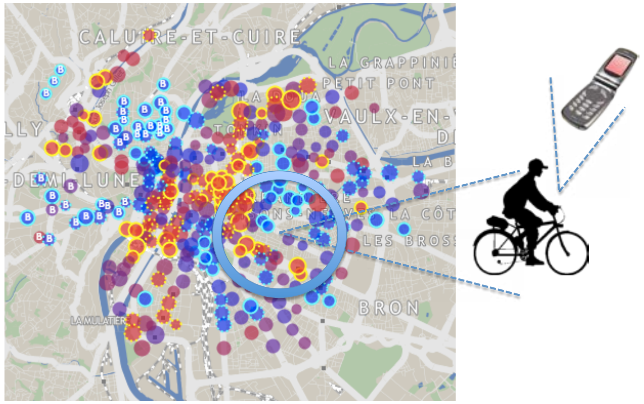
"Space, like time, is one of the most fundamental categories of human cognition.

It structures all our activities and relationships with the external world.

It also structures many of our reasoning capabilities: it serves as the basis for many metaphors, including temporal, and gave rise to mathematics itself, geometry being the first formal system known."

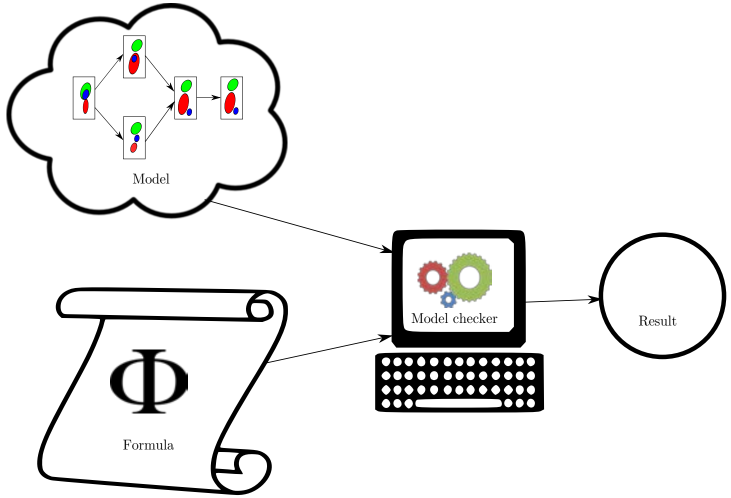
(Laure Vieu)





Continuous space, discrete regular grid, graph of stations, street map

Spatio-temporal model checking



Unified Framework for Spatial Model Checking?

- Generalising some topological notions
- Bridging the gap between continuous and discrete space
- Spatial Logics for Model Checking

Bringing us to explore

Closure Spaces and Quasi-discrete Closure Spaces

following up on work by, a.o., A. Galton and M. B. Smyth et al.

PART I

Logics and Space

Topological Space

A pair (X, O) where

- $X \neq \emptyset$ is a set
- O is a collection of open sets $O \subseteq \mathcal{P}(X)$

such that

- $\emptyset, X \in O$
- O is closed under arbitrary unions and finite intersections

O is called the collection of *open sets* of the topological space

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Example: Euclidian space (2D)

open set

closed set

- open balls (in $\mathbb{R}^n$) are open sets
- an open set containing $x \in X$ is called an *open neighbourhood* of x
- x is an *interior point* of $S \subseteq X$ iff $\exists$ open neighbourhood U of x such that $U \subseteq S$

- $\mathcal{I}^T(S)$ is the *largest open set* contained in S
- $\mathcal{C}^T(S)$ is the *smallest closed set* containing S

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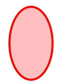

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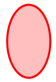

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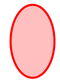

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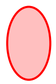

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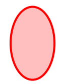

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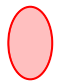

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Alternative characterisation of Topological Space

[Kuratowski]

A topological space is a pair $(X, \mathcal{C}^T)$ with $\mathcal{C}^T : 2^X \to 2^X$ such that

for each $A, B \subseteq X$:

- $\mathcal{C}^T(\emptyset) = \emptyset$
- $\mathcal{C}^T(A \cup B) = \mathcal{C}^T(A) \cup \mathcal{C}^T(B)$
- $A \subseteq \mathcal{C}^T(A)$
- $\mathcal{C}^T(\mathcal{C}^T(A)) = \mathcal{C}^T(A)$

Interior and closure are duals:

- $\mathcal{I}^T(A) = \overline{\mathcal{C}^T(\overline{A})}$
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Modal Logic

$$\Phi ::= p \mid \top \mid \bot \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Box\Phi \mid \Diamond\Phi$$

A *(Kripke) frame* (X, R)

- *X a set*
- *$R \subseteq X \times X$ an accessibility relation*

A *model* $\mathcal{M} = ((X, R), \mathcal{V})$

- (X, R) a frame
- $\mathcal{V} : P \to \mathcal{P}(X)$ a *valuation function*

$\mathcal{V}$ assigns to each atomic proposition the set of points (also called 'possible worlds') that satisfy it.

$$\begin{array}{lcl}
\mathcal{M}, x \models \top & \iff & \textit{true} \\
\mathcal{M}, x \models p & \iff & x \in \mathcal{V}(p) \\
\mathcal{M}, x \models \neg\phi & \iff & \texttt{not}\ \mathcal{M}, x \models \phi \\
\mathcal{M}, x \models \phi \wedge \psi & \iff & \mathcal{M}, x \models \phi\ \texttt{and}\ \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models \Box\phi & \iff & \forall y \in X.(x, y) \in R \implies \mathcal{M}, y \models \phi \\
\mathcal{M}, x \models \Diamond\phi & \iff & \exists y \in X.(x, y) \in R \wedge \mathcal{M}, y \models \phi
\end{array}$$

Modal Logic of Space [Tarski]

$$\Phi ::= p \mid \top \mid \bot \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Box\Phi \mid \Diamond\Phi$$

A topological space (X, O)

- X a set of points
- O the set of open sets of X

A *model* $\mathcal{M} = ((X, O), \mathcal{V})$

- (X, O) a topological space
- $\mathcal{V} : P \to \mathcal{P}(X)$ a valuation function

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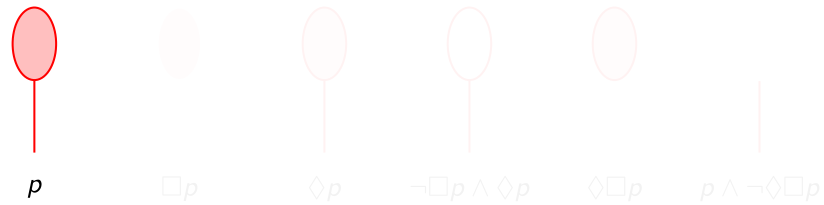
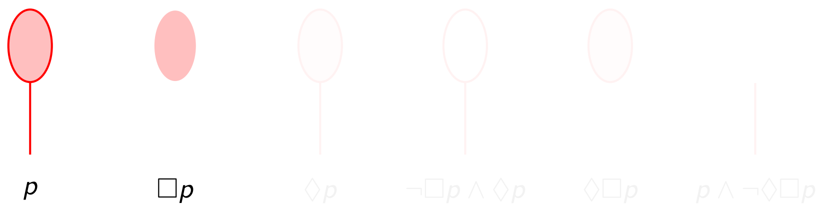
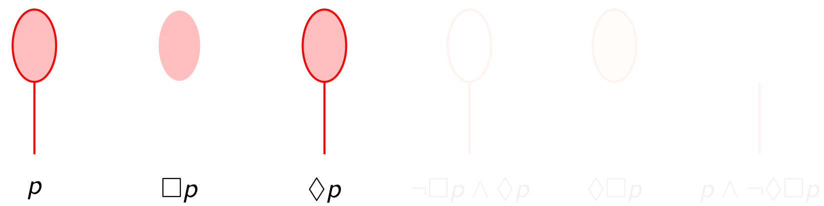
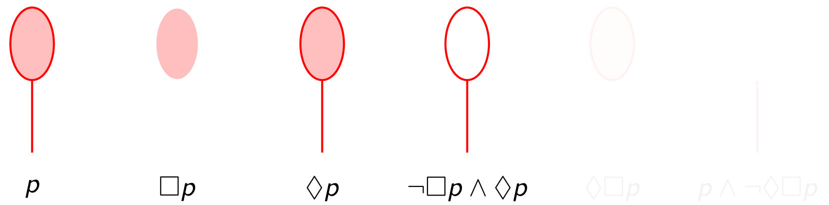
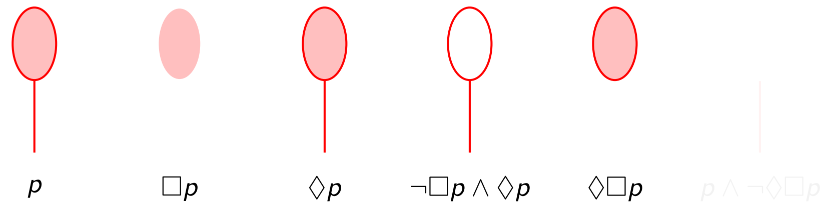
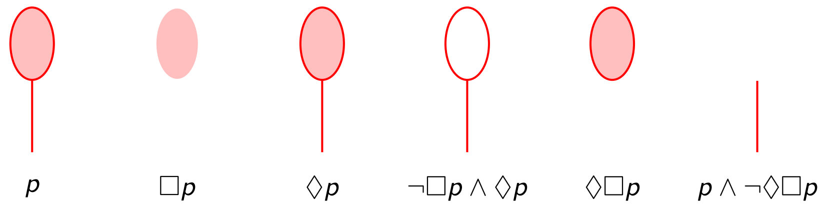












Axiomatic Aspects and Relational Semantics

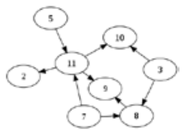
- Modal logic interpreted on transitive and reflexive Kripke frames
- Spatial modal logic interpreted on topological spaces

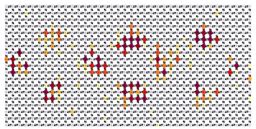
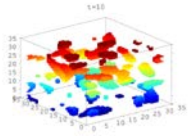
are both characterised by the same set of axioms $\mathcal{S}4$:

$\Box(p \to q) \to (\Box p \to \Box q)$	(K)	distributivity
$\Box p \to \Box\Box p$	(4)	transitivity
$\Box p \to p$	(T)	reflexivity
$\frac{\phi,\ \phi\to\psi}{\psi}$		modus ponens
$\frac{\phi}{\Box\phi}$		necessitation

where $\Box\phi = \neg\Diamond\neg\phi$

What about *Discrete* Spatial Structures?

Čech Spaces or Closure Spaces

A *closure space* is a pair $(X, \mathcal{C})$ with $\mathcal{C} : 2^X \to 2^X$ such that

for each $A, B \subseteq X$:

- $\mathcal{C}(\emptyset) = \emptyset$
- $\mathcal{C}(A \cup B) = \mathcal{C}(A) \cup \mathcal{C}(B)$
- $A \subseteq \mathcal{C}(A)$
- $\cancel{\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)}$

Define:

- $\mathcal{I}(A) = \overline{\mathcal{C}(\overline{A})}$
- A is *open* iff $A = \mathcal{I}(A)$
- A is *closed* iff $A = \mathcal{C}(A)$
- A is a *neighbourhood* of $x \in X$ iff $x \in \mathcal{I}(A)$

Interior and closure are duals:

- $\mathcal{C}(A) = \overline{\mathcal{I}(\overline{A})}$

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Analogy with Topological Spaces [Galton03]

Properties of a closure space $(X, \mathcal{C})$ with $A, B \subseteq X$:

1. A is open iff $\overline{A}$ is closed
2. $A \subseteq B$ then $\mathcal{C}(A) \subseteq \mathcal{C}(B)$
3. $\mathcal{C}(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} \mathcal{C}(A_i)$
4. - $\emptyset$ and X are open
   - arbitrary unions and finite intersections of open sets are open

The *topological* closure $\mathcal{C}^T : 2^X \to 2^X$ is defined as:

$$\mathcal{C}^T(A) = \bigcap \{B \subseteq X | A \subseteq B \wedge \mathcal{C}(B) = B\}$$

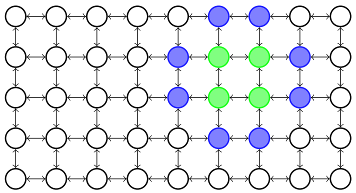
… and satisfies the four basic axioms including idempotence

Graphs as Closure Spaces [Galton03]

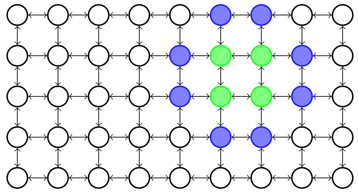
A graph is a set of nodes X and a binary relation $R \subseteq X \times X$

$$\mathcal{C}_R(A) = A \cup \{x \in X | \exists a \in A.(x,a) \in R\}$$

The pair $(X, \mathcal{C}_R)$ is a closure space



Boundary in Closure Spaces



- $A = \{\bullet\}$
- $\mathcal{I}(A) = \emptyset$ and $\mathcal{C}(A) = \{\bullet, \bullet\}$
- $\mathcal{B}(A) = \mathcal{C}(A) \setminus \mathcal{I}(A) = \{\bullet, \bullet\}$
- $\mathcal{B}^-(A) = A \setminus \mathcal{I}(A) = \{\bullet\}$
- $\mathcal{B}^+(A) = \mathcal{C}(A) \setminus A = \{\bullet\}$

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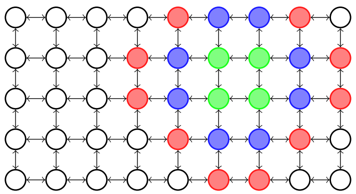

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Quasi-discrete Closure Spaces

A closure space $(X, \mathcal{C})$ is *quasi-discrete* if and only if one of the following holds:

- each $x \in X$ has a *minimal neighbourhood* N_x
- for each $A \subseteq X$, $\mathcal{C}(A) = \bigcup_{a \in A} \mathcal{C}(\{a\})$

A is a neighbourhood of $x \in X$ iff $x \in \mathcal{I}(A)$

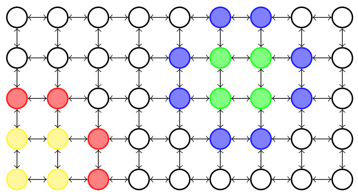
Theorem

$(X, \mathcal{C})$ is quasi-discrete iff there is $R \subseteq X \times X$ such that $\mathcal{C} = \mathcal{C}_R$

Lemma

$\mathcal{C}_R$ is idempotent iff the reflexive closure $R^=$ of R is transitive

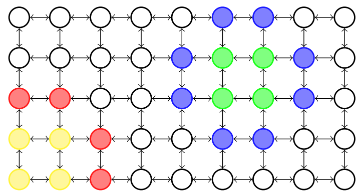
Graph inducing a Quasi-discrete Closure Space



- $\mathcal{I}(\bullet)$?
- $\mathcal{C}_R(\bullet)$?
- $\mathcal{B}^+(\bullet)$?
- $\mathcal{B}^-(\bullet)$?
- $\mathcal{B}^-(\bullet \cup \bullet)$?

But also graphs with an uncountable set of nodes/points such as $(\mathbb{R}, \leq)$ are quasi-discrete closure spaces

Graph inducing a Quasi-discrete Closure Space



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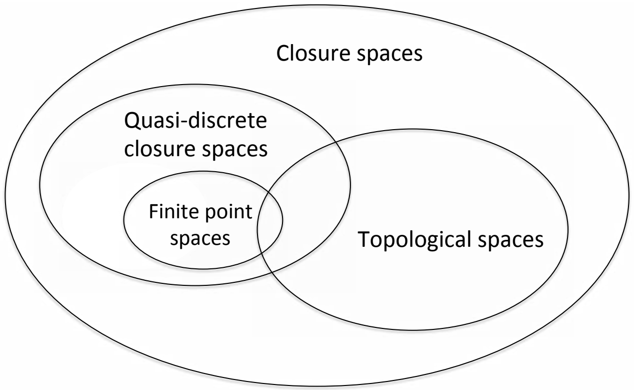
Hierarchy of Closure Spaces



PART II

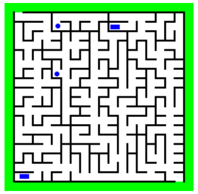
Spatial Logic for Closure Spaces

Spatial Logic for Closure Spaces (SLCS)

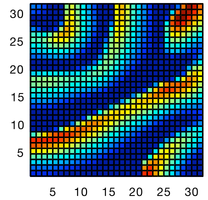
Clusters of full stations

Near to a bus stop

Spatial reachability



Areas with low concentration



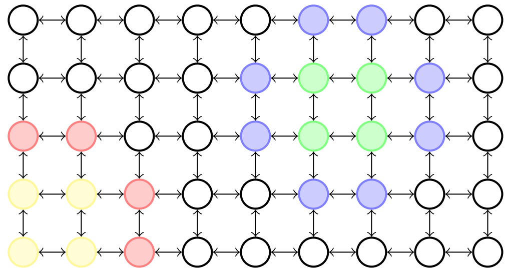
Spatial Logic for Closure Spaces (SLCS)

..... a little alchemy ...

SLCS syntax

$$
\begin{array}{llll}
\Phi & ::= & p & [\textsc{Atomic proposition}] \\
 & | & \top & [\textsc{True}] \\
 & | & \neg\Phi & [\textsc{Not}] \\
 & | & \Phi \wedge \Phi & [\textsc{And}] \\
 & | & \mathcal{N}\Phi & [\textsc{Near}] \\
 & | & \Phi\,\mathcal{S}\,\Phi & [\textsc{Surrounded}]
\end{array}
$$

Spatial operators: intuition



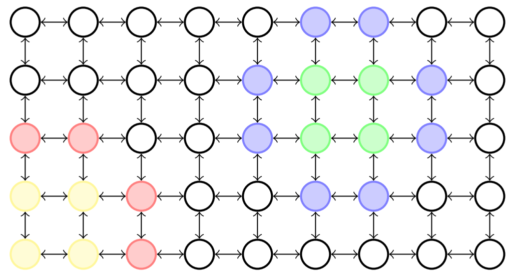
All red and yellow points satisfy $\mathcal{N}$*yellow*
One yellow point satisfies $\mathcal{I}$*yellow*
No points satisfy $\mathcal{I}$*green*
Green points satisfy *green* $\mathcal{S}$ *blue*
Yellow points satisfy *yellow* $\mathcal{S}$ *red*

Semantics of SLCS

Satisfaction $\mathcal{M}, x \models \phi$ of formula ϕ at point x in quasi-discrete closure model $\mathcal{M} = ((X, \mathcal{C}), \mathcal{V})$ is defined, by induction on terms, as follows:

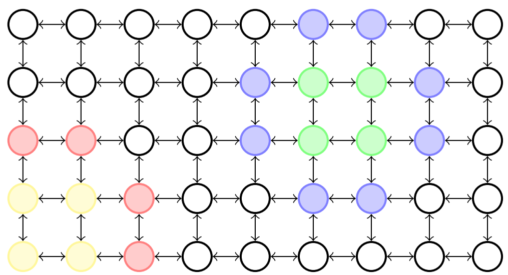
$$
\begin{array}{lcllcl}
\mathcal{M}, x & \models & p & \iff & x \in \mathcal{V}(p) \\
\mathcal{M}, x & \models & \top & \iff & \mathit{true} \\
\mathcal{M}, x & \models & \neg\phi & \iff & \texttt{not}\ \mathcal{M}, x \models \phi \\
\mathcal{M}, x & \models & \phi \wedge \psi & \iff & \mathcal{M}, x \models \phi\ \texttt{and}\ \mathcal{M}, x \models \psi \\
\mathcal{M}, x & \models & \mathcal{N}\phi & \iff & x \in \mathcal{C}(\{y \in X | \mathcal{M}, y \models \phi\}) \\
\mathcal{M}, x & \models & \phi\,\mathcal{S}\,\psi & \iff & \exists A \subseteq X. x \in A \wedge \forall y \in A. \mathcal{M}, y \models \phi \wedge \\
 & & & & \quad \forall z \in \mathcal{B}^+(A). \mathcal{M}, z \models \psi
\end{array}
$$

Derived operators



$$
\begin{array}{rcll}
\mathcal{I}\phi & \triangleq & \neg(\mathcal{N}\neg\phi) & \text{(interior)} \\
\delta\phi & \triangleq & (\mathcal{N}\phi) \wedge (\neg\mathcal{I}\phi) & \text{(boundary)} \\
\delta^-\phi & \triangleq & \phi \wedge (\neg\mathcal{I}\phi) & \text{(internal/interior boundary)} \\
\delta^+\phi & \triangleq & (\mathcal{N}\phi) \wedge (\neg\phi) & \text{(external/closure boundary)}
\end{array}
$$

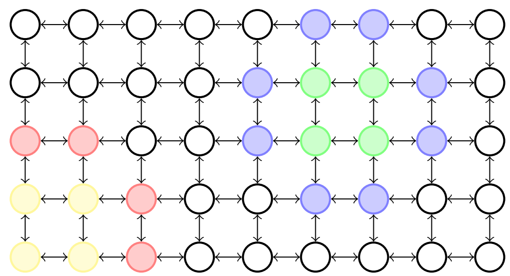
Derived operators



$$
\begin{array}{rcll}
\mathcal{E}\phi & \triangleq & \phi\,\mathcal{S}\,\bot & \text{(everywhere)} \\
\mathcal{F}\phi & \triangleq & \neg\mathcal{E}(\neg\phi) & \text{(somewhere)}
\end{array}
$$

Derived operators

$$
\begin{array}{rcll}
\phi\,\mathcal{R}\psi & \triangleq & \neg((\neg\psi)\,\mathcal{S}(\neg\phi)) & \text{(reachability)} \\
\phi\,\mathcal{T}\,\psi & \triangleq & \phi \wedge ((\phi \vee \psi)\,\mathcal{R}\psi) & \text{(from-to)}
\end{array}
$$



$\phi\,\mathcal{R}\psi$: either ψ holds in x, or there exists a sequence of points after x, all satisfying ϕ leading to a point satisfying both ϕ and ψ

$(\mathit{white} \vee \mathit{blue})\,\mathcal{R}\mathit{blue}$ satisfied by $\{\bullet, \bullet, \circ, \bullet\}$
$\mathit{white}\,\mathcal{T}\,\mathit{blue}$ satisfied by $\{\circ\}$

PART III

Model Checking Spatial Logics

Model checking (finite models)

Model checking in quasi-discrete closure spaces is analysis of a graph

Efficient algorithm $O(\textit{nodes} + \textit{arcs})$ for checking $\phi \,\mathcal{S}\, \psi$

Implemented as a "flooding" algorithm

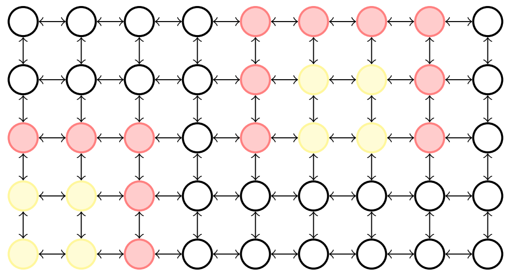
Efficient algorithm

The algorithm identifies *"bad"* areas, where $\neg\phi$ can be reached *without* passing by points satisfying ψ

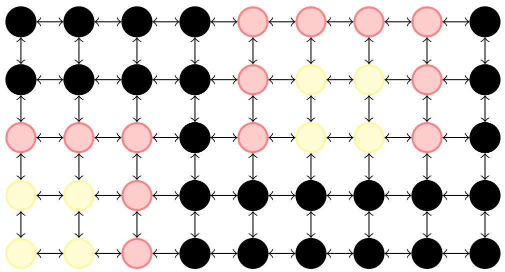
Implemented recursively as an operator that enlarges the set of "bad" points at each application

Upon fixed point: the points where ϕ holds, that are not "bad", satisfy $\phi \,\mathcal{S}\, \psi$.

yellow $\mathcal{S}$ *red*

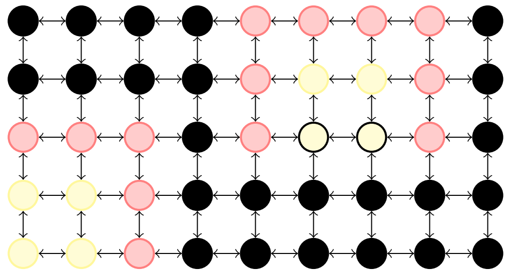


Find points satisfying *yellow* $\mathcal{S}$ *red* *yellow* $\mathcal{S}$ *red*



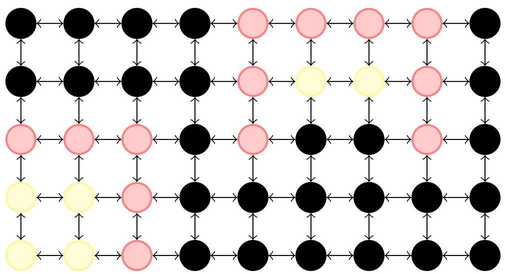
1) Find points satisfying neither *yellow* nor *red* and make them black

yellow $\mathcal{S}$ *red*



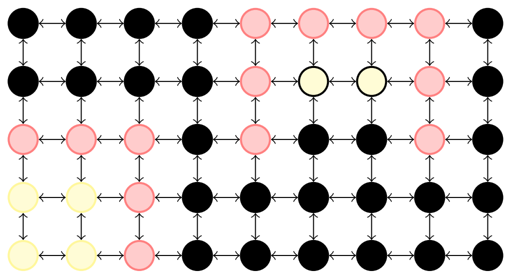
2) Identify yellow points in $\mathcal{C}(\mathit{black})$ . . .

yellow $\mathcal{S}$ *red*



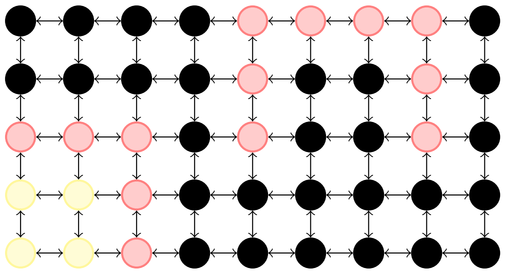
3) . . . and make them black

yellow $\mathcal{S}$ *red*



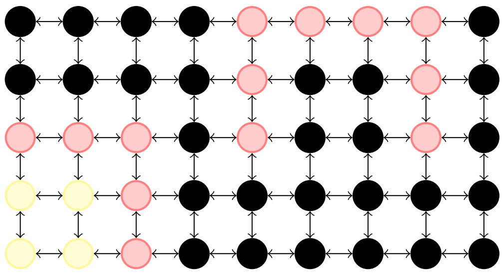
4) Identify yellow points in $\mathcal{C}(\mathit{black})$ . . .

yellow $\mathcal{S}$ *red*



5) . . . and make them black

yellow $\mathcal{S}$ *red*



Fixed point reached, the yellow points satisfy *yellow* $\mathcal{S}$ *red*

Model Checking Algorithm

Function $\texttt{Sat}(\mathcal{M}, \phi)$
  Input: Finite, quasi-discrete closure model
    $\mathcal{M} = ((X, \mathcal{C}), \mathcal{V})$, formula ϕ
  Output: Set of points $\{x \in X \mid \mathcal{M}, x \models \phi\}$
  Match ϕ
    case $\top$: **return** X
    case p : **return** $\mathcal{V}(p)$
    case $\neg\phi_1$:
      let $P = \texttt{Sat}(\mathcal{M}, \phi_1)$
      return $X \setminus P$
    case $\phi_1 \wedge \phi_2$:
      let $P = \texttt{Sat}(\mathcal{M}, \phi_1)$
      let $Q = \texttt{Sat}(\mathcal{M}, \phi_2)$
      return $P \cap Q$
    case $\mathcal{N}\phi_1$:
      let $P = \texttt{Sat}(\mathcal{M}, \phi_1)$
      return $\mathcal{C}(P)$
    case $\phi_1 \,\mathcal{S}\, \phi_2$:
      return $\texttt{CheckSurr}\ (\mathcal{M},\phi_1,\phi_2)$

Function $\texttt{CheckSurr}\ (\mathcal{M},\phi_1,\phi_2)$
  Input: Finite, quasi-discrete closure model
    $\mathcal{M} = ((X, \mathcal{C}), \mathcal{V})$, formulas ϕ_1, ϕ_2
  Output: Set of points $\{x \in X \mid \mathcal{M}, x \models \phi_1 \,\mathcal{S}\, \phi_2\}$
  var $V := \texttt{Sat}(\mathcal{M}, \phi_1)$
  let $Q = \texttt{Sat}(\mathcal{M}, \phi_2)$
  var $T := \mathcal{B}^+(V \cup Q)$
  while $T \neq \emptyset$ **do**
    var $T' := \emptyset$
    for $x \in T$ **do**
      let $N = pre(x) \cap V$
      $V := V \setminus N$
      $T' := T' \cup (N \setminus Q)$
    $T := T'$;
  return V

Correctness and Complexity

Theorem

For any finite quasi-discrete closure model $\mathcal{M} = ((X, \mathcal{C}), \mathcal{V})$ and SLCS formula ϕ, $x \in \texttt{Sat}(\mathcal{M}, \phi)$ if and only if $\mathcal{M}, x \models \phi$

Proposition

For any finite quasi-discrete model $\mathcal{M} = ((X, \mathcal{C}_R), \mathcal{V})$ and SLCS formula ϕ of size k, function $\texttt{Sat}(\mathcal{M}, \phi)$ terminates in $\mathcal{O}(k \cdot (|X| + |R|))$ steps

Implementation

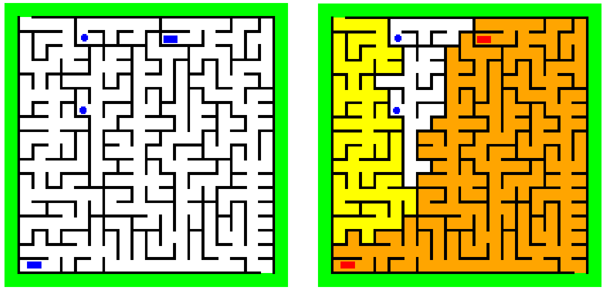
Prototype available on github
`www.github.com/vincenzoml/topochecker`

A few hundreds of lines of OCaml code

Digital images

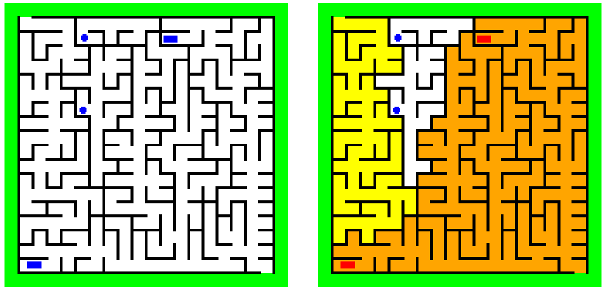
Any digital image can be treated as a finite, thus quasi discrete, closure space

```
toExit = [white] T [green] {●,●}
```



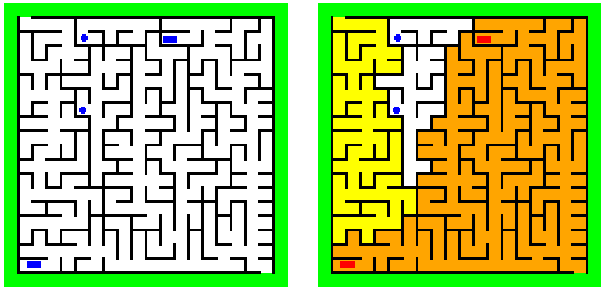
Digital images

```
toExit = [white] T [green] {●,●}
fromStartToExit = toExit & ([white] T [blue]) {●}
```



Digital images

```
toExit = [white] T [green] {●,●}
fromStartToExit = toExit & ([white] T [blue]) {●}
startCanExit = [blue] T fromStartToExit {●}
```



Turing Patterns

The Chemical Basis of Morphogenesis (1952)

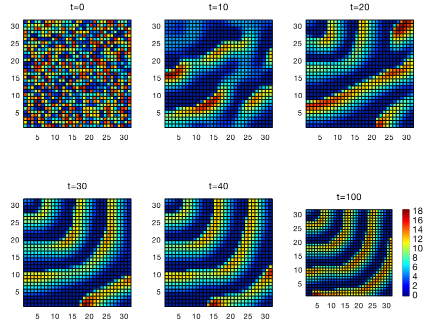
How the leopard got its spots?

Stripes

Spots

Alan Turing

Turing Patterns



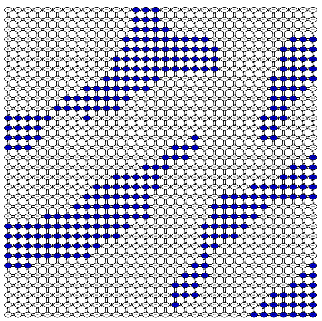
Two chemical substances A and B in a $K \times K$ grid

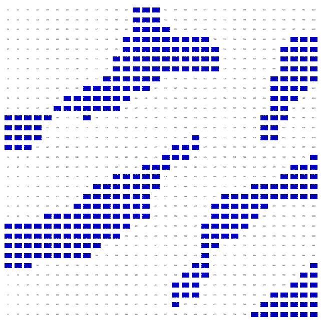
$$\begin{cases} \frac{dx^A_{i,j}}{dt} = R_1 x^A_{i,j} x^B_{i,j} - x^A_{i,j} + R_2 + D_1(\mu^A_{i,j} - x^A_{i,j}) \\ \frac{dx^B_{i,j}}{dt} = R_3 x^A_{i,j} x^B_{i,j} + R_4 + D_2(\mu^B_{i,j} - x^B_{i,j}) \end{cases}$$

[Gol,Bartocci,Belta, Conference on Decision and Control, 2014]

Areas of low concentration of A at time 10

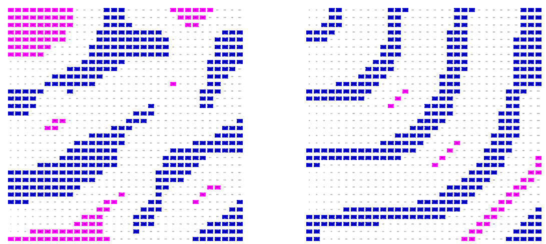
$\texttt{pattern} = [a < 2] \texttt{S} \; [a > 2]$





Areas of low concentration of A

$\texttt{far\_from\_pattern} = !(\texttt{N}(\texttt{N}(\texttt{N}(\texttt{N}(\texttt{pattern})))))$

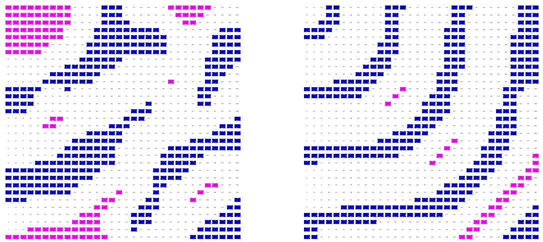


t=10 and t=100

$\neg\mathcal{F}\texttt{far\_from\_pattern}$

Areas of low concentration of A

`far_from_pattern` =!(N (N (N (N (pattern)))))

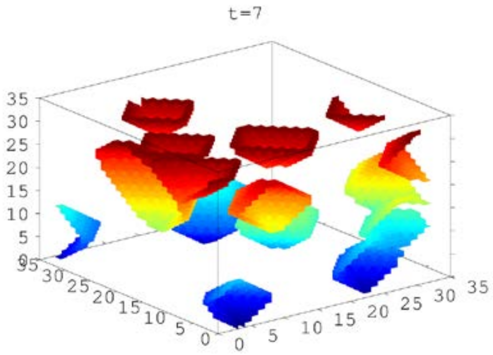


t=10 and t=100

$\neg\mathcal{F}$`far_from_pattern`

Areas of low concentration of A in 3D

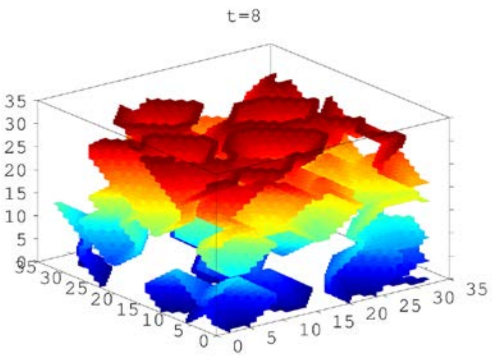
`far_from_pattern` =!(N (N (N (N (pattern)))))



Model checking time: 0.31277 seconds

Areas of low concentration of A in 3D

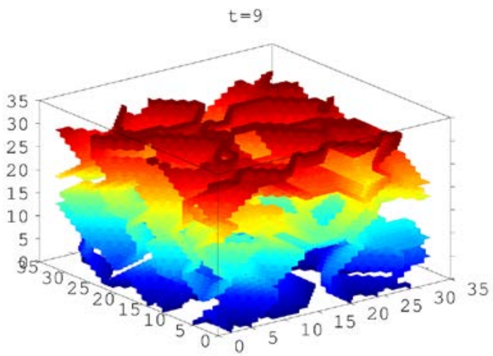
`far_from_pattern` =!(N (N (N (N (pattern)))))



Model checking time: 0.31277 seconds

Areas of low concentration of A in 3D

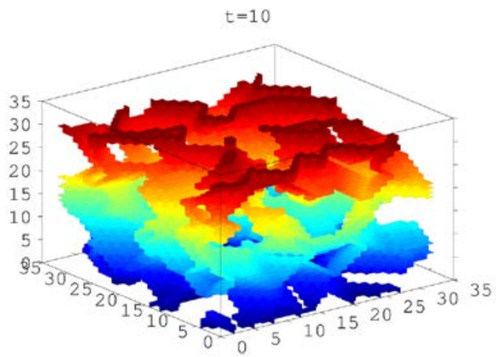
`far_from_pattern` =!(N (N (N (N (pattern)))))



Model checking time: 0.31277 seconds

Areas of low concentration of A in 3D

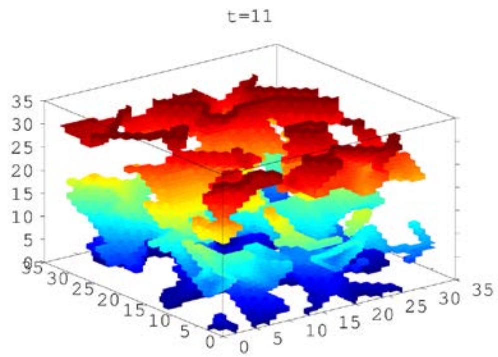
`far_from_pattern` $=!($`N(N(N(N(pattern))))`$)$



Model checking time: 0.31277 seconds

Areas of low concentration of A in 3D

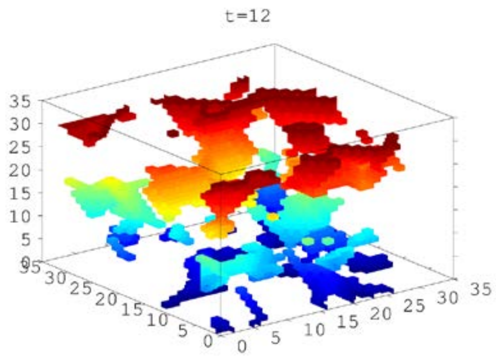
`far_from_pattern` $=!($`N(N(N(N(pattern))))`$)$



Model checking time: 0.31277 seconds

Areas of low concentration of A in 3D

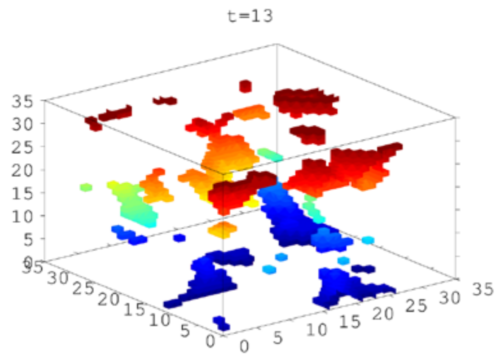
`far_from_pattern` $=!($`N(N(N(N(pattern))))`$)$



Model checking time: 0.31277 seconds

Areas of low concentration of A in 3D

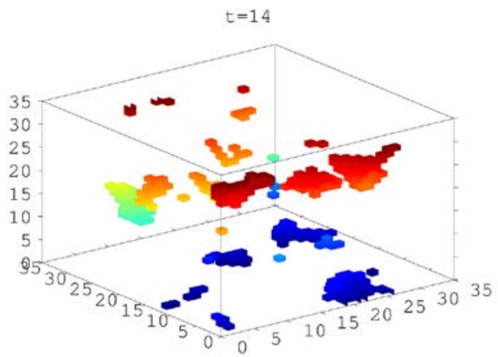
`far_from_pattern` $=!($`N(N(N(N(pattern))))`$)$



Model checking time: 0.31277 seconds

Areas of low concentration of *A* in 3D

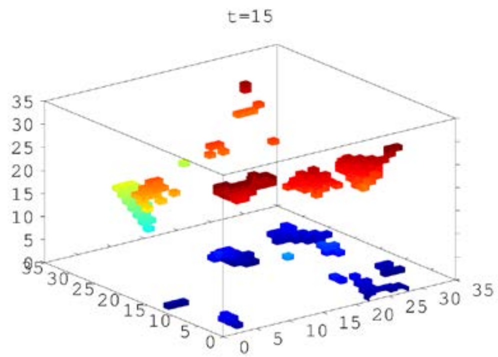
`far_from_pattern` =!(`N(N(N(N(pattern))))`)



Model checking time: 0.31277 seconds

Areas of low concentration of *A* in 3D

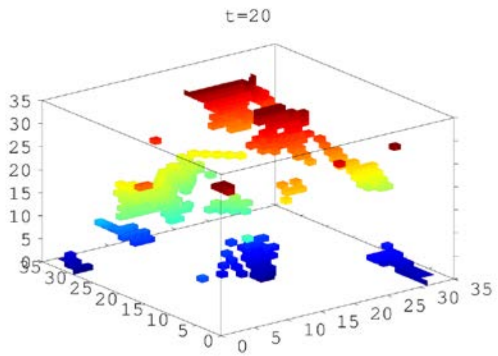
`far_from_pattern` =!(`N(N(N(N(pattern))))`)



Model checking time: 0.31277 seconds

Areas of low concentration of *A* in 3D

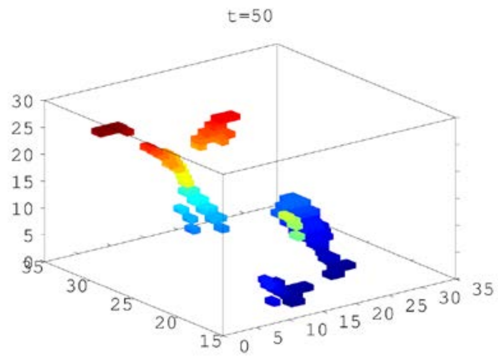
`far_from_pattern` =!(`N(N(N(N(pattern))))`)



Model checking time: 0.31277 seconds

Areas of low concentration of *A* in 3D

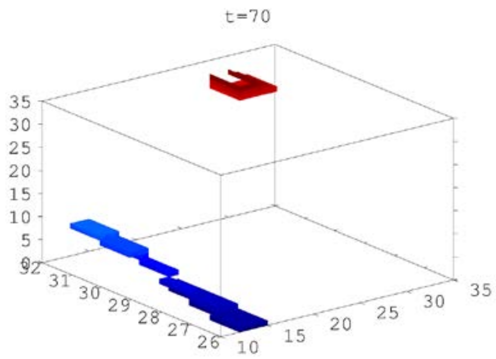
`far_from_pattern` =!(`N(N(N(N(pattern))))`)



Model checking time: 0.31277 seconds

Areas of low concentration of A in 3D

`far_from_pattern` $=$`!(N(N(N(N(pattern)))))`



Model checking time: 0.31277 seconds

Areas of low concentration of A in 3D over time

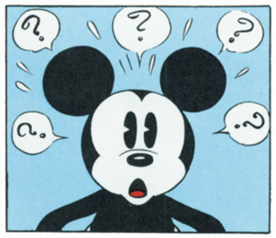
Points where pattern persists for at least 9 steps

Areas of low concentration of A in 3D over time

Points where pattern persists for at least 9 steps

...... but first we need to add time ....

QUESTIONS?



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