A nonlinear Fokker-Planck equation driven by a nonlocal drift field: uniqueness & stability

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The subject of our studies is the following Cauchy problem

$$\begin{cases} \partial_t \mu_t = -v \cdot \nabla_x \mu_t + \sigma \Delta_v \mu_t - \mathrm{div}_v(\mathfrak{v}[t,\mu](z)\mu_t) & (t,x,v) \in (0,T] \times \mathbb{R}^{2d}, \\ \mu_0 = \bar{\mu} & (x,v) \in \mathbb{R}^{2d}, \end{cases} \tag{1}$$

where $\sigma \in \mathbb{R}$ and the drift field $\mathfrak{v}$ is allowed to depend explicitly on a solution μ of the problem. Hence, the above is a nonlinear Fokker-Planck equation driven by a nonlocal drift field $\mathfrak{v}[t,\mu]$ depending on the state of the system, a typical example being the choice

$$\mathfrak{v}[t,\mu](z) = H(t,z) * \mu_t\,.$$

Furthermore, (1) is the natural mean field counterpart of second order multi-agents systems with additive noise, in which case a drift field of the form $\mathfrak{v}[t,\mu]$ naturally arises if one assumes that the natural dynamic is driven by mutual interactions.

Our aim is to show uniqueness and stability issues for (1). In doing so, we consider a set of assumptions on the drift term $\mathfrak{v}[t,\mu]$ which is as general as possible and allows for possibly unbounded non-globally Lipschitz drift fields provided some Hölder continuity combined with a dissipativity condition are satisfied. Despite the presence of a nonlinearity in the structure of (1), the results we show, that were proved in [2], strongly rely on well-posedness results for a linear counterpart, that has been analyzed in [1].

[1] F. Anceschi, G. Ascione, D. Castorina, F. Solombrino. *Well-posedness of Kolmogorov-Fokker-Planck equations with unbounded drift*, J. Math. Anal. Appl. 543 (2025) 128909

[2] F. Anceschi, G. Ascione, D. Castorina, F. Solombrino. *Optimal control problems driven by nonlinear degenerate Fokker-Planck equations*, arXiv: 2410.24000 (2024)