On the harmonic characterization of the spheres: a stability inequality and some of its consequences

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Let D be a bounded open set of $\mathbb{R}^n$ whose boundary has with finite (n-1) - Hausdorff measure and let x_0 be a point of D. Assume that $u(x_0)$ equals the average of u on ∂D for every harmonic functions u in D continuous up to the boundary. In this case one says that D is a pseudosfere centered at x_0.

In general the pseudosferes are not spheres: in 1937 Keldysch and Lavrentieff constructed a pseudosphere in $\mathbb{R}^2$ which is not a circle. As a consequence, the following problem naturally arose: when a pseudosphere is a sphere? Or, roughly speaking: is it possible to characterize the Euclidean spheres via the Gauss mean value property for harmonic function? The answer is yes.

The most general result in this direction was obtained by Lewis and Vogel in 2002: they proved that a pseudosphere is a sphere if the surface measure of its boundary has at most an Euclidean growth. Preiss and Toro, in 2007, proved the stability of Lewis and Vogel's result. Namely: a bounded domain D, whose boundary has the Lewis and Vogel regularity property, is geometrically close to a sphere centered at x_0 if the Poisson kernel of D with pole at x_0 is almost constant.

In collaboration with Giovanni Cupini we proved that analogous rigidity and stability results hold true if the domain D has finite (n-1)-Hausdorff measure and only satisfies the following property: in at least one point of ∂D nearest to x_0 the boundary of D is Lyapunov-Dini regular. To this end we have introduced and studied what we called the Kuran gap of ∂D with respect to x_0.