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On semilinear $\boldsymbol{\Delta}_{\boldsymbol{\lambda}}$-Laplace equation

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ABSTRACT

We prove some existence, nonexistence and regularity results for the boundary value problem

$$\Delta_\lambda u + f(u) = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0,$$

where Ω is a bounded subset of $\mathbb{R}^N$, $N \geq 2$, and Δ_λ is a Δ_λ*-Laplacian*, i.e. a "degenerate" elliptic operator of the kind

$$\Delta_\lambda := \sum_{i=1}^{N} \partial_{x_i}(\lambda_i^2(x)\partial_{x_i}), \quad \lambda = (\lambda_1, \ldots, \lambda_N).$$

Together with some assumptions made in Franchi and Lanconelli (1984) [1], the family λ is supposed to verify a condition making Δ_λ homogeneous of degree two with respect to a group of dilations in $\mathbb{R}^N$.

1. Introduction

1.1. Homogeneous Δ_λ-Laplacians. The semilinear Dirichlet problem

In recent years a certain number of papers have been devoted to "degenerate" elliptic operators whose simplest prototype in $\mathbb{R}^N$ is the following one.

$$\Delta_\alpha := \Delta_{(1)} + |x^{(1)}|^{2\alpha}\Delta_{(2)}, \quad \alpha > 0. \tag{1.1}$$

Here $x = (x^{(1)}, x^{(2)})$, $x^{(i)} \in \mathbb{R}^{N_i}$, $i = 1, 2$ denotes the point of $\mathbb{R}^N$, $N = N_1 + N_2$, and $\Delta_{(i)}$ stands for the classical Laplacian in $\mathbb{R}^{N_i}$.

Nowadays, Δ_α in (1.1) is usually quoted in the literature as Grushin's operator. However, if α is a nonnegative integer, Δ_α falls into the general class of Hörmander's operators, sum of squares of vector fields generating a Lie algebra of maximum rank at any point. If α is not an integer, then Δ_α is contained in a family of operators of the kind

$$\Delta_\lambda := \sum_{i=1}^{N} \partial_{x_i}(\lambda_i^2\partial_{x_i}), \quad \partial_{x_i} = \frac{\partial}{\partial x_i} \tag{1.2}$$

first studied in [1–3] with a geometrical technique taking into account the property of the *control* (or *Carnot-Carathéodory*) metric $d := d_\lambda$ generated by the vector fields

$$X_i := \lambda_i\partial_{x_i}, \quad i = 1, \ldots, N. \tag{1.3}$$

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The aim of this paper is to establish existence, nonexistence and regularity results for the problem

$$\begin{cases} \Delta_\lambda u + f(u) = 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.4}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and Δ_λ is the operator in (1.2) related to a N-tuple $\lambda = (\lambda_1, \dots, \lambda_N)$ of continuous functions in $\mathbb{R}^N$ verifying, together with the assumption in [1] (which will be recalled, for reader convenience, in Section 1.2), the following one.

(H1) There exists a group of dilations $(\delta_t)_{t>0}$,

$$\delta_t : \mathbb{R}^N \longrightarrow \mathbb{R}, \quad \delta_t(x) = \delta_t(x_1, \dots, x_N) = (t^{\varepsilon_1} x_1, \dots, t^{\varepsilon_N} x_N)$$

with $1 = \varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_N$, such that λ_i is δ_t homogeneous of degree $\varepsilon_i - 1$, i.e.,

$$\lambda_i(\delta_t(x)) = t^{\varepsilon_i - 1}\lambda_i(x), \quad \forall\, x \in \mathbb{R}^N,\ \forall\, t > 0,\ i = 1, \dots, N. \tag{1.5}$$

The number

$$Q := \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_N \tag{1.6}$$

is the *homogeneous dimension* of $\mathbb{R}^N$ with respect to $(\delta_\lambda)_{t>0}$. It will play a crucial rôle both in the geometry and the functional setting naturally associated to Δ_λ. We explicitly remark that (1.5) is equivalent to the δ_t-*homogeneity* of *degree one* of the vector field X_i, that is, to the property

$$X_i(u(\delta_t(x))) = t(Xu)(\delta_t(x)), \quad \forall\, x \in \mathbb{R}^N,\ \forall\, t > 0 \tag{1.7}$$

and for every $u \in C^\infty(\mathbb{R}^N)$.

Thus, (H1) implies that Δ_λ is δ_t-*homogeneous* of *degree two*. When the family λ satisfies all the hypotheses mentioned above, we call Δ_λ-*Laplacian* the operator in (1.2).

1.2. Hypotheses on $\lambda = (\lambda_1, \dots, \lambda_N)$

The function λ_i's are continuous in $\mathbb{R}^N$, different from zero and of class C^1 in $\mathbb{R}^N \setminus \Pi$ where

$$\Pi = \left\{ (x_1, \dots, x_N) \in \mathbb{R}^N : \prod_{i=1}^N x_i = 0 \right\}.$$

Moreover, together with (H1), we assume the following properties.

(H2) $\lambda_1 = 1,\ \lambda_i(x) = \lambda_i(x_1, \dots, x_{i-1}),\ i = 2, \dots, N.$

(H3) There exists a constant $\rho \ge 0$ such that

$$0 \le x_k \partial_{x_k} \lambda_i(x) \le \rho \lambda_i(x) \quad \forall\, k \in \{1, \dots, i-1\},\ \forall\, i = 2, \dots, N$$

and for every $x \in \mathbb{R}^N_+ := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_i \ge 0\ \forall\, i = 1, \dots, N\}$.

(H4) For every $x \in \mathbb{R}^N$, $\lambda_i(x) = \lambda_i(x^*)$ where

$$x^* = (|x_1|, \dots, |x_N|) \quad \text{if } x = (x_1, \dots, x_N).$$

Some remarks are in order.

Remark 1.1. If $\varepsilon_i = 1$ then $\lambda_i(x) = \lambda_i(0) > 0$ for every $x \in \mathbb{R}^N$. Indeed, if $\varepsilon_i = 1$ then λ_i is δ_t-homogeneous of degree zero and its continuity implies

$$\lambda_i(x) = \lambda_i(\delta_t(x)) = \lim_{t \searrow 0} \lambda_i(\delta_t(x)) = \lambda_i(0).$$

Moreover, $\lambda_i(0)$ has to be strictly positive since $\lambda_i > 0$ in $\mathbb{R}^N_+ \setminus \Pi$. We want to stress that, vice versa, if $\lambda_i(0) > 0$ from (1.5) we get $\varepsilon_i = 1$ and $\lambda_i(x) = \lambda_i(0)$ for every $x \in \mathbb{R}^N$.

Note 1. Throughout the paper, without loss of generality, we assume $\lambda_i \equiv 1$ if $\varepsilon_i = 1$.

Remark 1.2. By condition (H2) the operator Δ_λ can be written as follows

$$\Delta_\lambda = \sum_{i=1}^N \lambda_i^2 \partial_{x_i}^2 = \sum_{i=1}^N X_i^2.$$

Remark 1.3. If the λ_i's are smooth, then (H1) and (H2) imply the *hypoellipticity* of Δ_λ, i.e. the smoothness of the distributional solutions to $\Delta_\lambda u = f$ when f is smooth.

Indeed, since λ_i is smooth, δ_t-homogeneous and everywhere different from zero in $\mathbb{R}^N \setminus \Pi$, λ_i is a nonvanishing polynomial function (see, e.g., [4, Proposition 1.3.4]). Then, for every fixed $x \in \mathbb{R}^N$ there exists a multi-index $\alpha^{(i)} = (\alpha_1^{(i)}, \dots, \alpha_{i-1}^{(i)})$ such that

$$D^{\alpha^{(i)}} \lambda_i(x) \neq 0.$$

Using this property it is easy to recognize that

$$\text{Lie}\,\{X_1, \dots, X_N\}\,(x) \supseteq \left\{\partial_{x_1}, \dots, \partial_{x_N}\right\}.$$

Therefore

$$\text{rank Lie}\,\{X_1, \dots, X_N\} = N, \quad \forall\, x \in \mathbb{R}^N,$$

and, by the Hörmander Theorem in [5] and Remark 1.2, Δ_λ is hypoelliptic.

On the other hand, apart from the obvious case in which every λ_i is constant, Δ_λ is not elliptic at every point of Π (see Remark 1.1). Thus, if the λ_i's are not constant

$$\dim\left(\text{span}\left\{\lambda_i(0)\partial_{x_i} : i = 1, \dots, N\right\}\right) < N = \dim\left(\text{span}\left\{\lambda_i(x)\partial_{x_i} : i = 1, \dots, N\right\}\right) \quad \forall\, x \notin \Pi.$$

As a consequence: there is no composition law $\circ$ in $\mathbb{R}^N$ making Δ_λ left translation invariant (see [4, Proposition 1.2.13]). So that, if the λ_i's are not constant, there is no Lie group $\mathbb{G}$ in $\mathbb{R}^N$ making Δ_λ a sub-Laplacian on $\mathbb{G}$.

1.3. The Δ_λ-metric space

An absolutely continuous path $\gamma : [0, T] \longrightarrow \mathbb{R}^N$ is called λ-*subunit* if there exist measurable functions $c_1, \dots, c_N : [0, T] \longrightarrow \mathbb{R}$ such that, almost everywhere in $[0, T]$ one has

$$\dot{\gamma}(t) = \sum_{i=1}^{N} c_i(t) X_i(\gamma(t)), \qquad \sum_{i=1}^{N} c_i^2(t) \leq 1.$$

Hereafter we agree to identify the vector fields

$$X_i(x) = \lambda_i(x)\partial_{x_i} \quad \text{with the function } \lambda_i(x)e_i,\ e_i = \Big(0, \dots, \underset{i}{1}, \dots, 0\Big).$$

Assumption (H2) and the positivity of the λ_i's in $\mathbb{R}^N \setminus \Pi$ imply the λ-*connectivity* of $\mathbb{R}^N$. Precisely, for every $x, y \in \mathbb{R}^N$ there exists a λ-subunit path $\gamma : [0, T] \longrightarrow \mathbb{R}^N$ such that $\gamma(0) = x$ and $\gamma(T) = y$. We denote by $\Lambda(x, y)$ the family of the λ-subunit paths connecting x and y. Finally, if the subunit path γ is defined in the interval $[0, T]$, we let $l(\gamma) := T$. Then d_λ, what we call the λ-distance, is defined as follows: if $x, y \in \mathbb{R}^N$,

$$d_\lambda(x, y) := \inf\{l(\gamma) : \gamma \in \Lambda(x, y)\}.$$

It is quite trivial to verify that $(\mathbb{R}^N, d_\lambda)$ is a metric space. Sometime, in what follows, we will write d instead of d_λ. The d-ball of center $x \in \mathbb{R}^N$ and radius $r > 0$ will be denoted by $B_d(x, r)$. Hence

$$B_d(x, r) := \{y \in \mathbb{R}^N \mid d(x, y) < r\}.$$

A precise estimate of the d_λ-distance, and of the Lebesgue measure of the d_λ-balls, come from (H1)–(H4). Define a N-tuple of functions $F_1, \dots, F_N$ on $\mathbb{R}^N_+ \times [0, \infty[$ with the following recurrence law

$$\begin{cases} F_1(x, \tau) = \tau \\ F_i(x, \tau) = \tau\lambda_i(x_1 + F_1(x, \tau), \dots, x_{i-1} + F_{i-1}(x, \tau)), \quad i = 2, \dots, N. \end{cases} \tag{1.8}$$

Since λ_i is monotone increasing with respect to $x_j \in [0, \infty[$, $j = 1, \dots, i-1$, and strictly positive in $\mathbb{R}^N_+ \setminus \Pi$ the function $\tau \mapsto F_i(x, \tau)$ is strictly increasing in $[0, \infty[$, for every fixed $x \in \mathbb{R}^N_+$. We let

$$\phi_i(x, \cdot) = (F_i(x, \cdot))^{-1}, \quad i = 1, 2, \dots, N. \tag{1.9}$$

Then, by Theorems 2.6 and 2.7 in [1] we have the following. There exist two strictly positive constants c_1 and c_2 such that

$$c_1 \leq \frac{d(x, y)}{\sum_{i=1}^{N} \phi_i(x^*, |x_i - y_i|)} \leq c_2 \tag{1.10}$$

and

$$c_1 \leq \frac{|B_d(x, r)|}{\prod_{i=1}^{N} F_i(x^*, r)} \leq c_2 \tag{1.11}$$

for every $x = (x_1, \dots, x_N)$, $y = (y_1, \dots, y_N) \in \mathbb{R}^N$ and for every $r > 0$. $|B_d(x, r)|$ denotes the Lebesgue measure of $B_d(x, r)$.

From (1.10) and (1.11), and from the properties of the functions F_i's showed in the Appendix, we obtain some crucial result on the matric d_λ and on the measure of the d_λ-balls. Precisely:

(d1) there exist positive constants $c_1^*, \ldots, c_N^*$ such that

$$d_\lambda(x, y) \leq \sum_{i=1}^{N} c_i^* |x_i - y_i|^{\frac{1}{\varepsilon_i}} \tag{1.12}$$

for every $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$ in $\mathbb{R}^N$.
(See Corollary A.3.)

(d2) For every fixed compact subset $K \subseteq \mathbb{R}^N$ there exists $d^* = d_K^* > 0$ such that

$$d_\lambda(x, y) \geq d^* |x - y|, \quad \forall\, x, y \in K \tag{1.13}$$

(See (A.4).)

(d3) There exists a positive constant c_d such that

$$|B(x, 2r)| \leq c_d |B(x, r)| \quad \forall\, x \in \mathbb{R}^N,\ \forall\, r > 0, \tag{1.14}$$

where $c_d = \frac{c_2}{c_1}$. (See (A.1) and (1.11).)

Moreover, from (A.2),

$$|B(x, R)| \leq c_d 2^Q \left(\frac{R}{r}\right)^Q |B(x, r)| \tag{1.15}$$

for every $x \in \mathbb{R}^N$ and $0 < r < R$. Q is the number defined in (1.6), the homogeneous dimension of $\mathbb{R}^N$ with respect to $(\delta_t)_{t>0}$.

1.4. The Δ_λ-functional setting

For a function u of class C^1 we let

$$|\nabla_\lambda u|^2 := \sum_{i=1}^{N} |\lambda_i \partial_{x_i} u|^2. \tag{1.16}$$

Given a bounded open set $\Omega \subseteq \mathbb{R}^N$, for every $p \in]1, \infty[$ we denote by

$$\mathring{W}_\lambda^{1,p}(\Omega)$$

the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|u\|_{1,p} := \left(\int_\Omega |\nabla_\lambda u|^p \, dx\right)^{\frac{1}{p}}. \tag{1.17}$$

In Section 3 we will directly recognize the continuous embedding

$$\mathring{W}_\lambda^{1,p}(\Omega) \hookrightarrow L^{p_\lambda^*}(\Omega) \tag{1.18}$$

for every $p \in]1, Q[$ and

$$p_\lambda^* := \frac{pQ}{Q - p}. \tag{1.19}$$

As before, Q is the homogeneous dimension of $\mathbb{R}^N$ with respect to $(\delta_t)_{t>0}$.

We would like to stress that the embedding (1.18) also follows from generalized doubling property (1.15) and the following *Poincaré inequality*

$$\int_{B_d(x,r)} |u - u_r| \, dy \leq Cr \int_{B_d(x,\theta r)} |\nabla_\lambda u| \, dy \quad \forall\, u \in C^1(\overline{B_d(x, \theta r)}) \tag{1.20}$$

where $C > 0$ and $\theta > 1$ are suitable constants independent of u, x and r (see [3] and [6, Section 3]) and u_r stands for the average of u on $B_d(x, r)$, i.e.,

$$u_r = \frac{1}{B_d(x, r)} \int_{B_d(x,r)} u(y) \, dy.$$

Indeed, nowadays it is well known that (1.15) and (1.20) imply the embedding (1.18); some important references are [7–10].

1.5. Examples of Δ_λ operators

Example 1.4. Let us split $\mathbb{R}^N$ as follows

$$\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$$

and denote by $x = (x^{(1)}, \ldots, x^{(r)})$, $x^{(i)} \in \mathbb{R}^{N_i}$, $i = 1, \ldots, r$, the point of $\mathbb{R}^N$. Let $\Delta_{(i)}$ be the classical Laplace operator in $\mathbb{R}^{N_i}$. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_{r-1})$ $\alpha_j \geq 1, j = 1, \ldots, r-1$, define

$$\Delta_\alpha := \Delta_{(1)} + |x^{(1)}|^{2\alpha_1}\Delta_{(2)} + \cdots + |x^{(r-1)}|^{2\alpha_{r-1}}\Delta_{(r)}. \tag{1.21}$$

Then $\Delta_\alpha = \Delta_\lambda$ with $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ and $\lambda^{(i)} = |x^{(i-1)}|^{\alpha_{i-1}} 1^{(i)}$, $i = 1, \ldots, r$. Here we agree to let $|x^{(0)}|^{\alpha_0} = 1$ and $1^{(i)} = (\underset{1}{1}, \ldots, \underset{i}{1})$. A group of dilations for which λ satisfies (H1) is given by

$$\delta_t : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad \delta_t(x^{(1)}, \ldots, x^{(r)}) = (t^{\varepsilon_1}x^{(1)}, \ldots, t^{\varepsilon_r}x^{(r)}) \tag{1.22}$$

with $\varepsilon_1 = 1$ and $\varepsilon_i = \alpha_{i-1}\varepsilon_{i-1} + 1$, $i = 2, \ldots, r$. In particular, if $\alpha_1 = \cdots = \alpha_{r-1} = 1$, the operator (1.21) and the dilation (1.22) becomes, respectively

$$\Delta_{(1)} + |x^{(1)}|^2\Delta_{(2)} + \cdots + |x^{(r-1)}|^2\Delta_{(r)}$$

and

$$\delta_t(x^{(1)}, \ldots, x^{(r)}) = (tx^{(1)}, t^2x^{(2)}, \ldots, t^r x^{(r)}).$$

Example 1.5. Let $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and let $\mu : \mathbb{R}^{N_1} \to \mathbb{R}$ be continuous in its domain and of class C^1 and strictly positive outside the coordinate axis. Moreover, assume that $\mu(tx^{(1)}) = t^\alpha(x^{(1)})$, for a suitable $\alpha > 0$, and for every $x^{(1)} \in \mathbb{R}^{N_1}$ and $t > 0$. Then, if we let $\lambda = (1^{(1)}, \mu 1^{(2)})$, we have

$$\Delta_\lambda = \Delta_{(1)} + \left(\mu\left(x^{(1)}\right)\right)^2 \Delta_{(2)}. \tag{1.23}$$

This operator satisfies (H1) with respect to the dilations

$$\delta_t : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad \delta_t(x^{(1)}, x^{(2)}) = (tx^{(1)}, t^{\alpha+1}x^{(2)}).$$

The class of the operators in (1.23) obviously contains Δ_α in (1.1), as well as

$$\Delta_z + |x|^{2\alpha_1}|y|^{2\alpha_2}\Delta_t, \qquad z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^n, \quad t \in \mathbb{R}^{N_2}.$$

Note 2. When $\mu(x^{(1)}) = \frac{1}{4}|x^{(1)}|$ the operator Δ_λ in (1.23) takes the form

$$\Delta_{x^{(1)}} + \frac{1}{4}|x^{(1)}|^2\Delta_{x^{(2)}}.$$

On the other hand, if the dimension N_1 and N_2 verify the inequality $N_2 < \rho(N_1)$, where ρ is the so called Hurwitz–Radon function, then there exists a composition law $\circ$ in $\mathbb{R}^N$ making $\mathbb{H}_N := (\mathbb{R}^N, \circ, \delta_\lambda)$ a group of Heisenberg type (see [4, Remark 3.6.7]). Then if $\Delta_{\mathbb{H}_N}$ denotes the canonical sub-Laplacian on $\mathbb{H}_N$, for every smooth function $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ which is radially symmetric in the variable $x^{(1)}$, one has

$$\left(\Delta_{x^{(1)}} + \frac{1}{4}|x^{(1)}|^2\Delta_{x^{(2)}}\right)u = \Delta_{\mathbb{H}_N}u,$$

see [4, page 251].

2. Some integral identities for Δ_λ: nonexistence results

In this section we prove some integral identities extending to the Δ_λ setting the classical Pohozaev identity for semilinear Poisson equation [11]. Pohozaev identity has been extended by several authors to general elliptic equations and systems, both in Riemannian and sub-Riemannian context, see, e.g., [12–14] and the references therein. To prove our identities we closely follow the original procedure of Pohozaev, just replacing the vector field $P = \sum_{i=1}^N x_i\partial_{x_i}$ in [11, page 1410], by

$$T := \sum_{i=1}^N \varepsilon_i x_i \partial_{x_i}, \tag{2.1}$$

the *generator of the group of dilation* $(\delta_t)_{t\geq 0}$ in (H1) (we say that T generates $(\delta_t)_{t\geq 0}$ since a function u is δ_t-homogeneous of degree m if and only if $Tu = mu$).

Throughout this section the λ_j's are *not supposed* to verify assumptions (H3) and (H4) and $\Omega \subseteq \mathbb{R}^N$ will be assumed to be open, bounded, with C^1-boundary and such that $\Omega = \operatorname{int}(\overline{\Omega})$. We will denote by $\Lambda^2(\overline{\Omega})$ the linear space of the function $u \in C(\overline{\Omega})$ such that

$$X_j u, \qquad X_j^2 u, \quad j = 1, \dots, N$$

exist in the weak sense of distributions in Ω and can be continuously extended to $\overline{\Omega}$ (as above, $X_j := \lambda_j \partial_{x_j}$).

Our first integral identity is the following one.

Theorem 2.1. *For every $u \in \Lambda^2(\overline{\Omega})$ we have*

$$\int_\Omega T(u)\Delta_\lambda u\, dx = \int_{\partial\Omega} \langle \nabla_\lambda u, \nu_\lambda\rangle T(u)\, ds - \frac{1}{2}\int_{\partial\Omega} |\nabla_\lambda u|^2 \langle T, \nu\rangle\, ds + \left(\frac{Q}{2} - 1\right)\int_\Omega |\nabla_\lambda u|^2\, dx, \tag{2.2}$$

where T is the vector field (2.1), $\langle\cdot,\cdot\rangle$ stands for the Euclidean inner product, ν is the outward normal to Ω, $\nu_\lambda = (\lambda_1\nu_1, \dots, \lambda_N\nu_N)$ and

$$\nabla_\lambda u = (\lambda_1 \partial_{x_1} u, \dots, \lambda_N \partial_{x_N} u).$$

Proof. We first prove (2.2) assuming $u \in C^2(\overline{\Omega})$. An integration by parts gives[1]

$$\int_\Omega T(u)\Delta_\lambda u\, dx = \int_{\partial\Omega} \varepsilon_i x_i \partial_{x_i} u \partial_{x_j} u \lambda_j^2 \nu_j\, ds - \int_\Omega \lambda_j^2 \partial_{x_j} u \partial_{x_j}(\varepsilon_i x_i \partial_{x_i} u)\, dx =: I_1 + I_2. \tag{2.3}$$

It is easily seen that

$$I_1 = \int_{\partial\Omega} T(u)\langle \nabla_\lambda u_\lambda, \nu_\lambda\rangle\, ds, \tag{2.4}$$

while I_2 can be handled as follows

$$I_2 = -\int_\Omega \lambda_j^2 \partial_{x_j} u \delta_{ij}\varepsilon_i \partial_{x_i} u\, dx - \int_\Omega \lambda_j^2 \partial_{x_j} u \varepsilon_i x_i \partial_{x_j x_i} u\, dx =: I_{2,1} + I_{2,2}. \tag{2.5}$$

Obviously

$$I_{2,1} = -\int_\Omega \lambda_j^2 (\partial_{x_j u})^2 \varepsilon_j\, dx.$$

Moreover, an integration by parts in $I_{2,2}$ gives

$$\begin{aligned} I_{2,2} &= -\int_{\partial\Omega} \lambda_j^2 \partial_{x_j} u \varepsilon_j x_i \partial_{x_j} u \nu_i\, ds + \int_\Omega \partial_{x_j} u \partial_{x_i}(\lambda_j^2 \partial_{x_j} u \varepsilon_i x_i)\, dx \\ &= -\int_{\partial\Omega} |\nabla_\lambda u|^2 \langle T, \nu\rangle\, ds + \int_\Omega |\nabla_\lambda u|^2 (\operatorname{div} T)\, dx + \int_\Omega \lambda_j^2 \partial_{x_j} u \varepsilon_i x_i \partial_{x_i x_j} u\, dx + \int_\Omega \left(\partial_{x_j u}\right)^2 T\lambda_j^2\, dx \\ &= (\text{keeping in mind that } \operatorname{div} T = Q \text{ and that } \lambda_j \text{ is } \delta_t\text{-homogeneous} \\ &\quad \text{of degree } \varepsilon_j - 1, \text{ hence } T\lambda_j^2 = 2\lambda_j \Pi \lambda_j = 2(\varepsilon_j - 1)\lambda_j^2) \\ &\quad -\int_{\partial\Omega} |\nabla_\lambda u|^2 \langle T, \nu\rangle\, ds + Q\int_\Omega |\nabla_\lambda u|^2\, dx - I_{2,2} + 2\int_\Omega (\varepsilon_j - 1)(\lambda_j \partial_{x_j} u)^2\, dx. \end{aligned}$$

Therefore,

$$I_{2,2} = -\frac{1}{2}\int_{\partial\Omega} |\nabla_\lambda u|^2 \langle T, \nu\rangle\, ds + \left(\frac{Q}{2} - 1\right)\int_\Omega |\nabla_\lambda u|^2\, dx - I_{2,1}$$

hence, by (2.5),

$$I_2 = -\frac{1}{2}\int_{\partial\Omega} |\nabla_\lambda u|^2 \langle T, \nu\rangle\, ds + \left(\frac{Q}{2} - 1\right)\int_\Omega |\nabla_\lambda u|^2\, dx.$$

---

[1] Repeated indices i and j are understood to be summed from 1 to N.

This identity, together with (2.3) and (2.4), implies (2.2) under the assumption $u \in C^2(\overline{\Omega})$. To complete the proof when $u \in \Lambda^2(\overline{\Omega})$ we argue as in [13, page 77]. Let $(\Omega_k)_{k\in\mathbb{N}}$ be a sequence of open set with smooth boundary such that

$$\overline{\Omega}_k \subseteq \Omega_{k+1} \quad \text{for every } k \in \mathbb{N} \text{ and } \cup_{k\in\mathbb{N}} \Omega_k = \Omega.$$

On every Ω_k we can approximate the function u with a sequence of $C^2(\overline{\Omega}_k)$ functions u_j for each of which identity (2.3) holds on Ω_k. Letting j to ∞ we obtain (2.3) for u in Ω_k. Finally, as $k \to \infty$ we get our identity for $u \in \Lambda^2(\overline{\Omega})$. □

From the previous theorem we easily obtain an integral identity for $\Lambda^2(\overline{\Omega})$-solutions of the equation

$$\Delta_\lambda u + f(u) = 0, \tag{2.6}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. In what follows we let $F(u) := \int_0^u f(t)dt$ and we agree to call $\Lambda^2(\overline{\Omega})$-solution of (2.6) every function $u \in \Lambda^2(\overline{\Omega})$ satisfying

$$\sum_{j=1}^N \left(X_j^2 u\right)(x) + f(u)(x) = 0 \quad \forall x \in \Omega.$$

Theorem 2.2. *Let $u \in \Lambda^2(\overline{\Omega})$ be a solution of (2.6). Then*

$$\int_\Omega \left(F(u) + \left(\frac{1}{Q} - \frac{1}{2}\right) uf(u)\right) dx = \frac{1}{Q} \int_{\partial\Omega} \left(\langle T, \nu\rangle \left(F(u) - \frac{1}{2}|\nabla_\lambda u|^2\right)\right.$$
$$\left. + \ \langle \nabla_\lambda u, \nu\rangle \left(T(u) + \left(\frac{Q}{2} - 1\right) u\right)\right) ds. \tag{2.7}$$

Moreover, if $u = 0$ on $\partial\Omega$

$$\int_\Omega \left(F(u) + \left(\frac{1}{Q} - \frac{1}{2}\right) uf(u)\right) dx = \frac{1}{2Q} \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 |\nu_\lambda|^2 \langle T, \nu\rangle \, ds. \tag{2.8}$$

Proof. Since $Q = \operatorname{div} T$, we have

$$Q \int_\Omega F(u)\, dx = \int_\Omega \operatorname{div}(T) F(u)\, dx.$$

On the other hand, if $u \in C^2(\overline{\Omega})$, an integration by parts gives

$$\int_\Omega \operatorname{div}(T) F(u)\, dx = \int_{\partial\Omega} \langle T, \nu\rangle F(u)\, ds - \int_\Omega T(u) f(u)\, dx$$

so that

$$Q \int_{\partial\Omega} F(u)\, dx = \int_{\partial\Omega} \langle T, \nu\rangle F(u)\, ds + \int_\Omega T(u) \Delta_\lambda u\, dx. \tag{2.9}$$

On the other hand,

$$\begin{aligned} \int_\Omega uf(u)\, dx &= -\int_\Omega \left(\lambda_i^2 \partial_{x_i}^2 u\right) u\, dx \\ &= -\int_{\partial\Omega} \lambda_i^2 \partial_{x_i} u \nu_i u\, ds + \int_\Omega \lambda_i^2 (\partial_{x_i} u)^2\, dx, \end{aligned}$$

that is

$$\int_\Omega uf(u)\, dx = -\int_{\partial\Omega} u \langle \nabla_\lambda u, \nu_\lambda\rangle\, ds + \int_\Omega |\nabla_\lambda u|^2\, dx. \tag{2.10}$$

Identities (2.9) and (2.10) can be extended to functions $u \in \Lambda^2(\overline{\Omega})$ by using an approximation argument as in the proof of the previous theorem. Using (2.9) and (2.10) in (2.2) one gets (2.7). When $u = 0$ on $\partial\Omega$ (2.7) becomes

$$\int \left(F(u) + \left(\frac{1}{Q} - \frac{1}{2}\right) uf(u)\right) dx = \frac{1}{Q} \int_{\partial\Omega} \left(-\frac{1}{2}\langle T, \nu\rangle |\nabla_\lambda u|^2 + T(u) \langle \nabla_\lambda u, \nu_\lambda\rangle\right) ds. \tag{2.11}$$

We now remark that $u \in C^1(\overline{\Omega} \setminus \Pi)$, since $\lambda_j \partial_{x_j} u \in C^1(\overline{\Omega})$ and λ_j is C^1 and different from zero in $\mathbb{R}^N \setminus \Pi$. Thus, the condition $u = 0$ on $\partial\Omega$ implies

$$\partial_{x_j} u = \left(\frac{\partial u}{\partial \nu}\right) \nu_j \quad \text{at any point of } \partial\Omega \setminus \Pi$$

for every $j \in \{1, \dots, N\}$. As a consequence, on $\partial\Omega \setminus \Pi$ we have

$$\begin{aligned} -\frac{1}{2}\langle T, \nu\rangle |\nabla_\lambda u|^2 + T(u)\langle \nabla_\lambda u, \nu_\lambda\rangle &= -\frac{1}{2}\varepsilon_i x_i \nu_i \lambda_i^2 (\partial_{x_j} u)^2 + \varepsilon_i x_i \partial_{x_i} u \lambda_j^2 \partial_{x_j} u \nu_j \\ &= -\frac{1}{2}\varepsilon_i x_i \nu_i \lambda_j^2 \left(\frac{\partial u}{\partial \nu}\right)^2 \nu_j^2 + \varepsilon_i x_i \left(\frac{\partial u}{\partial \nu}\right)^2 \nu_i \lambda_j^2 \nu_j^2 \\ &= \frac{1}{2}\langle T, \nu\rangle |\nu_\lambda|^2 \left(\frac{\partial u}{\partial \nu}\right)^2. \end{aligned}$$

Replacing this identity in (2.11) we obtain (2.8). □

Our nonexistence results follow from the previous theorem. To proceed, we need the definition of δ_t-starshaped domain.

Definition 2.3. We say that Ω is δ_t-starshaped with respect to the origin if $0 \in \Omega$ and

$$\langle T, \nu\rangle \geq 0 \quad \text{at every point of } \partial\Omega.$$

Theorem 2.4. *Let Ω be δ_t-starshaped with respect to the origin. Then the problem*

$$\Delta_\lambda u + f(u) = 0 \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0 \tag{2.12}$$

has no solution $u \in \Lambda^2(\overline{\Omega})$, $u \not\equiv 0$, if

$$F(s) + \left(\frac{1}{Q} - \frac{1}{2}\right) sf(s) < 0 \tag{2.13}$$

for every $s \neq 0$. If (2.13) holds for every $s > 0$ then (2.12) has no nonnegative solution $u \not\equiv 0$.

Proof. Let $u \in \Lambda^2(\overline{\Omega})$ be a solution to (2.12). Since $\langle T, \nu\rangle \geq 0$ at any point of $\partial\Omega$ identity (2.8) implies

$$\int_\Omega \left(F(u) + \left(\frac{1}{Q} - \frac{1}{2}\right) uf(u)\right) dx \geq 0.$$

Using condition (2.13) in this inequality, one gets $u = 0$ a.e. in Ω, completing the proof of the theorem. □

Condition (2.13) can be weakened to obtain a nonexistence result of nonnegative nontrivial solutions to (2.12). For this we need a *unique continuation theorem* that seems to have an interest in its own right.

Proposition 2.5. *Let $u \in \mathring{W}_\lambda^{1,2}(\Omega)$ be a weak nonnegative solution of $\Delta_\lambda u + cu = 0$ in Ω, with $c \in L_{\mathrm{loc}}^{\frac{Q}{2}}(\Omega)$.*[2] *If there exists $x_0 \in \Omega$ such that*

$$\int_{B_d(x_0,r)} u(x)\, dx = O(r^k) \quad \text{as } r \to 0,\ \text{for every } k \in \mathbb{N},$$

then $u \equiv 0$ in the connected component of Ω containing x_0.

Proof. The proof of this proposition follows exactly the same lines as the one of Corollary A.1 in [13]. We only stress that Jerison's Poincaré inequality in [15], used in page 96 in [13], has to be replaced by the one related to our vector fields $\lambda_1\partial_{x_1}, \dots, \lambda_N\partial_{x_N}$. We omit any further details. □

We are ready to state our second nonexistence theorem.

Theorem 2.6. *Let Ω be connected and δ_t-starshaped with respect to the origin. Then problem (2.12) has no nonnegative solution $u \in \Lambda^2(\overline{\Omega})$, $u \not\equiv 0$, if f is locally Lipschitz continuous, $f(0) = 0$ and*

$$F(s) + \left(\frac{1}{Q} - \frac{1}{2}\right) sf(s) \leq 0 \quad \text{for every } s > 0. \tag{2.14}$$

[2] We refer to the next section for the definition of weak solutions.

Proof. Suppose $u \in \Lambda^2(\overline{\Omega})$, $u \geq 0$, is a solution of (2.12). Then (2.14) and identity (2.8) imply

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 |\nu_\lambda|^2 \langle T, \nu\rangle \, ds \leq 0.$$

The assumption $\langle T, \nu\rangle \geq 0$ on $\partial\Omega$ implies

$$\left(\frac{\partial u}{\partial \nu}\right)^2 \langle T, \nu\rangle = 0 \quad \text{at any point of } \partial\Omega.$$

On the other hand, since Ω is bounded, it must be $\langle T, \nu\rangle > 0$ on $V \cap (\partial\Omega \setminus \Pi)$, for a suitable bounded and connected open set $V \subseteq \mathbb{R}^N \setminus \Pi$. Thus

$$Du = \frac{\partial u}{\partial \nu} \equiv 0 \quad \text{in } V \cap (\partial\Omega \setminus \Pi).$$

Setting $u \equiv 0$ in $(\mathbb{R}^N \setminus \Omega) \cap V$ we then obtain a weak solution to

$$\Delta_\lambda u + cu = 0 \quad \text{in } V,$$

where $c = \frac{f(u)}{u}$ where $u \neq 0$ and $c = 0$ otherwise. Thus, since f is locally Lipschitz, $c \in L^\infty(V)$ and, by Proposition 2.5 $u \equiv 0$ in V. A connectedness argument yields $u \equiv 0$ in Ω. $\square$

3. Existence and regularity results

In this section we aim to prove the existence of weak solutions to problem (1.4) under natural assumptions on the nonlinearity f. The second aim of the section is to provide some regularity results for the weak solutions.

Our existence result is the following one.

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set and let $f : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying*

$$f(x, u) = o(u) \quad \text{as } u \longrightarrow 0 \quad \text{and} \quad f(x, u) = o\left(|u|^{\frac{Q+2}{Q-2}}\right) \quad \text{as } |u| \longrightarrow \infty \tag{3.1}$$

uniformly with respect to $x \in \overline{\Omega}$. Let

$$F(x, u) := \int_0^u f(x, s)\, ds$$

and assume the existence of two constants $\mu > 2$ and $\mu_0 > 0$ such that

$$uf(x, u) \geq \mu F(x, u) \quad \text{for } |u| > u_0 \quad \text{and} \quad F(x, u) > 0 \quad \text{for } u > u_0. \tag{3.2}$$

Then, there exists a nonnegative weak solution $u \in \mathring{W}^{1,2}_\lambda(\Omega)$, $u \neq 0$ to the equation

$$\Delta_\lambda u - \eta u + f(x, u) = 0 \tag{3.3}$$

for every $\eta \geq 0$.

In (3.1) the number Q stands for the homogeneous dimension of $\mathbb{R}^N$ with respect to $(\delta_t)_{t>0}$, see (1.6). By a weak solution to (3.3) we mean a *critical point* of the functional

$$J(u) = \int_\Omega \left(|\nabla_\lambda u|^2 + \eta u^2 - F(x, u)\right) dx, \quad u \in \mathring{W}^{1,2}_\lambda(\Omega). \tag{3.4}$$

The proof of Theorem 3.1 is a standard application of Ambrosetti–Rabinowitz's Mountain Pass theorem and of the following compact embedding result.

Proposition 3.2. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set. Then the embedding*

$$\mathring{W}^{1,2}_\lambda(\Omega) \hookrightarrow L^p(\Omega)$$

is compact for every $p \in [1, 2^*_\lambda[$, $2^*_\lambda = \frac{2Q}{Q-2}$.

This proposition is a consequence of the Poincaré inequality (1.20), the doubling property of the d_λ-balls and a general result from analysis in metric spaces, see, e.g., the survey paper [16]. However, it also easily follows from next embedding theorem, which is of interest in its own right. Indeed, it can be seen as a regularity result, in terms of classical Sobolev spaces, for the functions $u \in \mathring{W}^{1,p}_\lambda(\Omega)$.

Theorem 3.3. *Let Ω be a bounded open subset of $\mathbb{R}^N$. Then, for every $p \in]1, \infty[$, we have*

$$\mathring{W}^p_\lambda(\Omega) \hookrightarrow \mathring{W}^{\frac{1}{\varepsilon_1}, \dots, \frac{1}{\varepsilon_N}, p}(\Omega). \tag{3.5}$$

We denote by $\mathring{W}^{\frac{1}{\varepsilon_1}, \dots, \frac{1}{\varepsilon_N}, p}(\Omega)$ the closure of $C^1_0(\Omega)$ with respect to the norm

$$\|u\|_{\frac{1}{\varepsilon}, p} := \sum_{j=1}^N \|u\|_{\frac{1}{\varepsilon_j}, p}$$

where

$$\|u\|_{\frac{1}{\varepsilon_j}, p} := \int_{\mathbb{R}^N} |\partial_{x_j} u|^p \, dx \quad \text{if } \varepsilon_j = 1$$

while

$$\|u\|_{\frac{1}{\varepsilon_j}, p} := \int_0^1 \left(\int_{\mathbb{R}^N} \frac{|u(x + se_j) - u(x)|^p}{s^{1+\frac{p}{\varepsilon_j}}} \, dx \right) ds \quad \text{if } \varepsilon_j > 1,$$

where, as usual, $e_j = (0, \dots, \underset{j}{1}, \dots, 0)$. From an embedding theorem for classical anisotropic Sobolev-type spaces of fractional orders, we know that

$$\mathring{W}^{\frac{1}{\varepsilon_1}, \dots, \frac{1}{\varepsilon_N}, p}(\Omega) \hookrightarrow L^q(\Omega) \tag{3.6}$$

if

$$1 < p < \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_N = Q, \qquad \frac{1}{p} - \frac{1}{q} = \frac{1}{Q} \quad \text{i.e., } q = \frac{pQ}{Q - p} := p^*_\lambda,$$

and the embedding is compact if

$$q < p^*_\lambda,$$

see, e.g., [17]. As a consequence, by Theorem 3.3, Proposition 3.2 immediately follows.

Proof of Theorem 3.3. To prove embedding (3.5) we use a family of inequalities proved in [2, Theorem 2.6]. To begin with, we remark that embedding (3.5) is equivalent to the inequality

$$\|u\|^p_{\frac{1}{\varepsilon_j}, p} \le C \sum_{i=1}^N \int_\Omega |\lambda_i \partial_{x_i} u|^p \, dx, \quad j = 1, 2, \dots, p, \tag{3.7}$$

for every $u \in C^\infty_0(\Omega)$ and $C = C(\Omega, \lambda, p) > 0$. If $\varepsilon_j = 1$ inequality (3.7) is trivial because, in this case, $\lambda_j \equiv 1$ (see Remark 1.1) so that

$$\|u\|^p_{\frac{1}{\varepsilon_j}, p} = \int_\Omega |\lambda_j \partial_{x_j} u|^p \, dx.$$

Then, we can only consider the case $\varepsilon_j > 1$ and use inequality (2.6.b), page 1242, in [2]. From that inequality we obtain, for a suitable positive constant $C = C(\Omega, \lambda, p)$,

$$\int_0^1 \left(\int_{\mathbb{R}^N} \frac{|u(x + se_j) - u(x)|^p}{s(\varphi_j(s))^p} \, dx \right) ds \le C \|u\|_{\lambda, p}, \quad \forall\, u \in C^\infty_0(\Omega), \tag{3.8}$$

where $\varphi_j(s) = \phi_j(0, s)$ and ϕ_j is defined in (1.9).[3] Thus, to complete the proof of the theorem, it is enough to show that

$$\varphi_j(s) = c_j s^{\frac{1}{\varepsilon_j}} \tag{3.9}$$

for a suitable constant $c_j > 0$, and for every $j = 1, \dots, N$. Indeed, by Proposition A.2 in the Appendix we have

$$F_j(0, r) = a_j r^{\varepsilon_j}, \quad j = 1, \dots, N$$

[3] We would like to explicitly remark that the assumption (2.6.a) of Theorem 2.6 in [2] is satisfied. Indeed, due to the δ_λ-homogeneity of λ_j, we have

$$(\varepsilon_j - 1)\lambda_j = T(\lambda_j) \le \left(\max_{1 \le i \le N} \varepsilon_i \right) \langle \nabla \lambda_j, x \rangle.$$

for a suitable constant $a_i > 0$. Therefore

$$\varphi_j(s) = \left(\frac{1}{a_j}s\right)^{\frac{1}{\varepsilon_j}}, \quad j \in \{1, \ldots, N\}$$

and the proof is complete. □

Proof of Theorem 3.1. Using assumption (3.1) and Proposition 3.2 we can show that the functional J satisfies the Palais–Smale condition. With an equally standard argument we can show, using (3.1) and (3.2) that J satisfies all the other assumptions of the Mountain Pass Theorem, see e.g. [18,19]. We explicitly remark that the argument based on the Maximum Principle for Elliptic Equation used in [18,19] in order to get nonnegative solutions, also works in the present context. Indeed, Δ_λ satisfies a weak and a strong maximum principle, thanks to Theorem 3.1 and the nonhomogeneous Harnack inequality of Theorem 5.5 in [20]. □

We close this section by stating a regularity theorem which can be proved verbatim as Theorem 4.1 in [13].

Theorem 3.4. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set and let $u \in \mathring{W}^{1,2}_\lambda(\Omega)$ be a weak solution to (3.3), with f satisfying assumption (3.1). Then $u \in L^p(\Omega)$ for every $p \in [2, \infty[$.*

This theorem, together with the nonhomogeneous Harnack inequality of Theorem 5.5 in [20], leads to the following corollary.

Corollary 3.5. *Let the assumptions of Theorem 3.4 be satisfied and let $u \in \mathring{W}^{1,2}_\lambda(\Omega)$ be a weak solution to (3.3). Then u is locally Hölder continuous in Ω. Precisely, $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and there exists $\theta \in]0, 1[$ and $C > 0$ such that*

$$|u(x) - u(y)| \le C \sup_{B_d(x_0,2r)} |u| \, (d(x, y))^\theta$$

for every λ-ball $B_d(x_0, 2r) \subseteq \Omega$ and for every $x, y \in B_d(x_0, r)$. The constants θ and C are independent of u and of $B_d(x_0, 2r)$.

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Appendix

In this Appendix we prove some propositions regarding the functions F_i's and ϕ_i's from which important properties of d_λ and of $|B_d(x, r)|$ easily follow. First of all, we prove the following.

Proposition A.1. *For every $x \in \mathbb{R}^N_+$ and for every $r > 0$ we have*

$$F_i(x, 2r) \le 2^{\varepsilon_i} F_i(x, r), \quad i = 1, \ldots, N. \tag{A.1}$$

Proof. We argue by induction on the index $i \in \{1, \ldots, N\}$. Inequality (A.1) is trivial if $i = 1$. Assume it holds for $j \le i$ and prove it holds for $j = i + 1$. Indeed, since λ_{i+1} is increasing with respect to its argument

$$\begin{aligned}
F_{i+1}(x, 2r) &= 2r\lambda_{i+1}(x_1 + F_1(x, 2r), \ldots, x_i + F_i(x, 2r)) \\
&\le 2r\lambda_{i+1}(x_1 + 2r, \ldots, x_i + 2^{\varepsilon_i} F_i(x, r)) \\
&\le 2r\lambda_{i+1}(2(x_1 + r), \ldots, 2^{\varepsilon_i}(x_i + F_i(x, r))) \\
&= (\text{by the } \delta_t\text{-homogeneity of } \lambda_{i+1}) 2^{\varepsilon_{i+1}} r\lambda_{i+1}(x_1 + F_1(x, r), \ldots, x_i + F_i(x, r)) \\
&= 2^{\varepsilon_{i+1}} F_{i+1}(x, r).
\end{aligned}$$

This completes the proof. □

From (A.1), with a standard argument, we get

$$F_i(x, R) \le 2^{\varepsilon_i} \left(\frac{R}{r}\right)^{\varepsilon_i} F_i(x, r) \tag{A.2}$$

for $0 < r < R$.

The functions F_i's take an explicit form at $x = 0$. Indeed, the following proposition holds.

Proposition A.2. *For every $r > 0$ we have*

$$F_i(0, r) = a_i r^{\varepsilon_i}, \quad i = 1, \ldots, N$$

for suitable constant $a_i > 0$.

Proof. We know that $F_1(0,\tau)=\tau$, $F_j(0,\tau)=\tau\lambda_j(F_1(0,\tau),\dots,F_{j-1}(0,\tau))$ for $2\le j\le N$. Using the δ_t-homogeneity of the λ_j's, we obtain

$$F_2(0,\tau)=\tau\lambda_2(F_1(0,\tau))=\tau\lambda_2(\tau)=\tau^{\varepsilon_2}\lambda_2(1)=:a_2\tau^{\varepsilon_2}.$$

Moreover

$$F_3(0,\tau)=\tau\lambda_3(F_1(0,\tau),F_2(0,\tau))=\tau\lambda_3(\tau,a_2\tau^{\varepsilon_2})=\tau^{\varepsilon_3}\lambda_3(1,a_2)=:a_3\tau^{\varepsilon_3}.$$

An easy iteration argument shows that

$$F_j(0,\tau)=a_j\tau^{\varepsilon_j}\quad \text{for every } j\in\{1,\dots,N\}$$

with the a_j's given by the recurrence formula:

$$a_1=1,\qquad a_j=\lambda_j(a_1,\dots,a_{j-1}),\quad j\in\{2,\dots,N\}.\quad \square$$

From the previous proposition and inequalities (1.10) we immediately get the following corollary.

Corollary A.3. *For every $x,y\in\mathbb{R}^N$ we have*

$$d_\lambda(x,y)\le\sum_{i=1}^N c_i^*|x_i-y_i|,\quad c_i^*=c_2\left(\frac{1}{a_i}\right)^{\frac{1}{\varepsilon_i}},\ i=1,\dots,N$$

for every $x=(x_1,\dots,x_N)$ and $y=(y_1,\dots,y_N)\in\mathbb{R}^N$.

To obtain a local lower estimate of d_λ we need one more proposition.

Proposition A.4. *For every compact set $K\subseteq\mathbb{R}^N$ there exists a constant $d_i=d_i(K)>0$ such that*

$$F_i(x,r)\le d_i r\quad \forall\, x\in K,\forall\, r\in[0,1],\ i=1,\dots,N. \tag{A.3}$$

Proof. The inequality (A.3) is obvious if $i=1$. Suppose (A.3) is satisfied for $i\le j$. Then, for $0<r<1$,

$$\begin{aligned} F_{j+1}(x,r) &= r\lambda_{j+1}(x_1+F_1(x,r),\dots,x_j+F_j(x,r))\\ &\le r\lambda_{j+1}(x_1+d_1r,\dots,x_j+d_jr)\\ &\le r\sup_{x\in K}\lambda_{j+1}(x_1+d_1,\dots,x_j+d_j)=:d_{j+1}r. \end{aligned}$$

This completes the proof. $\square$

From (A.3) we get $F_i\left(x,\frac{r}{d_i}\right)\le r$ for $r\le d_i$. Hence $\frac{r}{d_i}\le\phi_i(x,r)$ for $0<r<d_i$. As a consequence, by (1.10),

$$d_\lambda(x,y)\ge c_1\sum_{i=1}^N\phi_i(x^*,|x_i-y_i|)\ge c_1\sum_{i=1}^N\frac{|x_i-y_i|}{d_i}.$$

Therefore, for a suitable constant d^*, we have

$$d_\lambda(x,y)\ge d^*|x-y|,\quad \forall\, x\in K,\ \forall\, y\in\mathbb{R}^N: |x-y|\le d^*. \tag{A.4}$$

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