On the Dirichlet problem in cylindrical domains for evolution Oleĭnik–Radkevič PDE's: A Tikhonov-type theorem

Alessia E. Kogoj

Dipartimento di Scienze Pure e Applicate (DiSPeA), Università degli Studi di Urbino Carlo Bo, Piazza della Repubblica, 13, IT-61029, Urbino (PU), Italy

Abstract

We consider the linear second order PDO's

$$\mathscr{L} = \mathscr{L}_0 - \partial_t := \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}\partial_{x_j}) - \sum_{j=i}^{N} b_j \partial_{x_j} - \partial_t,$$

and assume that $\mathscr{L}_0$ has nonnegative characteristic form and satisfies the Oleĭnik–Radkevič rank hypoellipticity condition. These hypotheses allow the construction of Perron-Wiener solutions of the Dirichlet problems for $\mathscr{L}$ and $\mathscr{L}_0$ on bounded open subsets of $\mathbb{R}^{N+1}$ and of $\mathbb{R}^N$, respectively.

Our main result is the following Tikhonov-type theorem:
Let $\mathcal{O} := \Omega \times]0, T[$ be a bounded cylindrical domain of $\mathbb{R}^{N+1}$, $\Omega \subset \mathbb{R}^N$, $x_0 \in \partial\Omega$ and $0 < t_0 < T$. Then $z_0 = (x_0, t_0) \in \partial\mathcal{O}$ is $\mathscr{L}$-regular for $\mathcal{O}$ if and only if x_0 is $\mathscr{L}_0$-regular for Ω.

As an application, we derive a boundary regularity criterion for degenerate Ornstein–Uhlenbeck operators.

E-mail address: alessia.kogoj@uniurb.it.

1. Introduction

We consider linear second order partial differential operators of the type

$$\mathscr{L}_0 := \sum_{i,j=1}^{N} \partial_{x_i}\left(a_{ij}\partial_{x_j}\right) + \sum_{j=1}^{N} b_j \partial_{x_j} \tag{1}$$

in an open set X of $\mathbb{R}^N$, $N \geq 2$, and their "evolution" counterpart in $X \times \mathbb{R}$

$$\mathscr{L} = \mathscr{L}_0 - \partial_t. \tag{2}$$

We assume $\mathscr{L}_0$ in (1) to be of non totally degenerate Oleǐnik and Radkevič type, i.e., we assume

(H1) $a_{ij} = a_{ji}, b_i \in C^\infty(X, \mathbb{R})$ and

$$A(x) := (a_{ij}(x))_{i,j=1,\dots,N} \geq 0 \qquad \forall x \in X.$$

Moreover

$$\inf_X a_{11} =: \alpha > 0.$$

(H2) rank Lie$\{X_1, \dots, X_N, X_0\}(x) = N \ \forall x \in X$, where,

$$X_i = \sum_{j=1}^{N} a_{ij}\partial_{x_j}, \ i=1,\dots,N, \quad \text{and} \quad X_0 = \sum_{j=1}^{N} b_j\partial_{x_j}.$$

Hypotheses (H1) and (H2) imply that $\mathscr{L}_0$ is hypoelliptic in X (see [20]), that is:

$$\Omega \text{ open subset of } X, \ u \in \mathcal{D}'(\Omega), \ \mathscr{L}_0 u \in C^\infty(\Omega, \mathbb{R}) \implies u \in C^\infty(\Omega, \mathbb{R}).$$

The same assumptions (H1) and (H2) also imply that $\mathscr{L}_0 - \partial_t$ is hypoelliptic in $X \times \mathbb{R}$.

We will show in Section 2 that $\mathscr{L}_0$ and $\mathscr{L}_0 - \partial_t$ endow X and $X \times \mathbb{R}$, respectively, with a local structure of σ^*harmonic space, in the sense of [3], Chapter 6. As a consequence, in particular, the Dirichlet problems

$$\begin{cases} \mathscr{L}_0 u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = \varphi, \end{cases} \quad \text{and} \quad \begin{cases} (\mathscr{L}_0 - \partial_t)v = 0 \text{ in } \mathcal{O} := \Omega \times]0, T[, \\ v|_{\partial\mathcal{O}} = \psi, \end{cases}$$

have a generalized solution in the sense of Perron–Wiener, for every bounded open set $\Omega \subset\subset X$, for every $T > 0$, and for every $\varphi \in C(\partial\Omega, \mathbb{R})$ and $\psi \in C(\partial\mathcal{O}, \mathbb{R})$. We will denote such generalized solutions by, respectively,

$$H_\varphi^\Omega \quad \text{and} \quad K_\psi^\mathcal{O}.$$

As usual, a point $x_0 \in \partial\Omega$ ($(x_0, t_0) \in \partial\mathcal{O}$) is called $\mathscr{L}_0$-regular for Ω ($\mathscr{L}$-regular for $\mathcal{O}$) if

$$\lim_{x \to x_0} H_\varphi^\Omega(x) = \varphi(x_0) \qquad \forall \varphi \in C(\partial\Omega, \mathbb{R})$$

$$\left(\lim_{(x,t) \to (x_0, t_0)} K_\psi^\mathcal{O}(x, t) = \psi(x_0, t_0) \qquad \forall \psi \in C(\partial\mathcal{O}, \mathbb{R}) \right).$$

The aim of this paper is to prove the following theorem:

Theorem 1.1. *Let Ω be a bounded open set with $\overline{\Omega} \subseteq X$, and let $x_0 \in \partial\Omega$ and $t_0 \in]0, T[$. Then, x_0 is $\mathscr{L}_0$-regular for Ω if and only if (x_0, t_0) is $\mathscr{L}_0 - \partial_t$-regular for $\mathcal{O} := \Omega \times]0, T[$.*

When $\mathscr{L} = \Delta - \partial_t$ is the classical heat operator, our result re-establishes a theorem proved by Tikhonov in 1938 [22]. Other proofs of the Tikhonov Theorem were given by Fulks in 1956 and in 1957 [8,9] and by Babuška and Výborný in 1962 [5]. Chan and Young extended the Tikhonov Theorem to parabolic operators with Hölder continuous coefficients in 1977 [7], and Arendt to parabolic operators with bounded measurable coefficients in 2000 [1]. The corresponding version for p-Laplacian-type evolution operators has been proved by Kilpeläinen and Lindqvist in 1996 [10] and by Banerjee and Garofalo in 2015 [3].

To the best of our knowledge, the only Tikhonov-type theorem for second order "evolution" sub-Riemannian PDO's appearing in the literature is the result by Negrini [19] in abstract β-harmonic spaces.[1]

The present paper is organized as follows. In Section 2, all the notions and results from Potential Theory that we need are briefly recalled. In particular, we recall the notion of σ^*-harmonic space and then we prove that $\mathscr{L}_0$ and $\mathscr{L}$ endow X and $X \times \mathbb{R}$, respectively, with a local structure of σ^*-harmonic space. In this way, we derive the existence of a generalized solution in the sense of Perron–Wiener in both our settings. Section 3 is devoted to two key results for the proof of the main theorem (Theorem 1.1), which is the content of Section 4. Finally, combining our Tikhonov-type theorem with a corollary of a Wiener–Landis-type criterion for Kolmogorov-type operators [11], we establish a geometric boundary regularity criterion for degenerate Ornstein–Uhlenbeck operators.

2. $\mathscr{L}_0$-harmonic and $\mathscr{L}$-harmonic spaces

2.1. The σ^*-harmonic space

For the readers' convenience we recall the definition of σ^*-harmonic space supported on an open set $E \subseteq \mathbb{R}^p$, $p \geq 2$, and refer to Chapter 6 of the monograph [4] for details.

Let $\mathcal{H}$ be a sheaf of functions in E such that $\mathcal{H}(V)$ is a linear subspace of $C(V, \mathbb{R})$, for every open set $V \subseteq E$. The functions in $\mathcal{H}(V)$ are called $\mathcal{H}$-*harmonic in* V. The open set V is called $\mathcal{H}$-*regular* if

   (i) $\overline{V} \subseteq E$ is compact;
  (ii) for every $\varphi \in C(\partial V, \mathbb{R})$ there exists a unique function such that

---

[1] For a definition of β-harmonic spaces see [6].

$$h_\varphi^V(x) \to \varphi(\xi) \text{ as } x \to \xi, \text{ for every } \xi \in \partial V;$$

(iii) $h_\varphi^V \geq 0$ if $\varphi \geq 0$.

A lower semicontinuous function $u : W \longrightarrow]-\infty, \infty]$, $W \subseteq E$ open, is called $\mathcal{H}$-*superharmonic* if

(i) $u \geq h_\varphi^V$ in V for every $\mathcal{H}$-regular open set V with $\overline{V} \subseteq W$ and for every $\varphi \in C(\partial V, \mathbb{R})$ with $\varphi \leq u|_{\partial V}$;
(ii) $\{x \in W \mid u(x) < \infty\}$ is dense in W.

A function $v : W \longrightarrow [-\infty, \infty[$, is called $\mathcal{H}$-*subharmonic* if $-v$ is $\mathcal{H}$-superharmonic. We denote by $\overline{\mathcal{H}}(W)$ ($\underline{\mathcal{H}}(W)$) the cone of the $\mathcal{H}$-superharmonic ($\mathcal{H}$-subharmonic) functions in W.

The couple $(E, \mathcal{H})$ is called a σ^*harmonic space if the following axioms hold:

(A1) There exists a function $h \in \mathcal{H}(E)$ such that $\inf h > 0$.
(A2) If $(u_n)_{n \in \mathbb{N}}$ is a monotone increasing sequence of $\mathcal{H}$-harmonic functions in an open set $V \subseteq E$ such that

$$\{x \in V \mid \sup_{n \in \mathbb{N}} u_n(x) < \infty\}$$

is dense in Ω, then

$$u := \sup_V u_n \quad \text{is } \mathcal{H}\text{-harmonic in } V.$$

(A3) The family of the $\mathcal{H}$-regular open sets is a basis of the Euclidean topology on E.
(A4) For every $x, y \in E$, $x \neq y$, there exist two nonnegative $\mathcal{H}$-superharmonic and continuous functions u, v in E such that

$$u(x)v(y) \neq u(y)v(x).$$

(A5) For every $x_0 \in E$ there exists a nonnegative $\mathcal{H}$-subharmonic and continuous function S_{x_0} in E, such that $S_{x_0}(x_0) = 0$ and

$$\inf_{E \smallsetminus V} S_{x_0} > 0$$

for every neighborhood V of x_0.

We now recall some crucial results in σ^*harmonic space theory; first of all the definition of Perron–Wiener solution to the Dirichlet problem.

Let V be a bounded open set with $\overline{V} \subseteq E$, and let $\varphi : \partial V \longrightarrow \mathbb{R}$ be a bounded lower semicontinuous or upper semicontinuous function. Define

$$\overline{\mathcal{U}}_\varphi^V := \{u \in \overline{\mathcal{H}}(V) \mid \liminf_{x \longrightarrow \xi} u(x) \geq \varphi(\xi) \quad \forall \xi \in \partial V\}$$

and

$$H_\varphi^V := \inf \overline{\mathcal{U}}_\varphi^V. \tag{3}$$

Then H_φ^V is $\mathcal{H}$-harmonic in Ω. It is called the *generalized Perron–Wiener solution* to the Dirichlet problem

$$\begin{cases} u \in \mathcal{H}(V), \\ u|_{\partial V} = \varphi. \end{cases}$$

We also have

$$H_\varphi^V := \sup \underline{\mathcal{U}}_\varphi^V, \tag{4}$$

where,

$$\underline{\mathcal{U}}_\varphi^V := \{ v \in \underline{\mathcal{H}}(V) \mid \limsup_{x \longrightarrow \xi} v(x) \le \varphi(\xi) \quad \forall \xi \in \partial V \}.$$

We say that a point $y \in \partial V$ is $\mathcal{H}$-*regular* for V if

$$\lim_{x \to y} H_\varphi^V(x) = \varphi(y) \qquad \forall \varphi \in C(\partial V, \mathbb{R}).$$

On the σ^*harmonic space the Bouligand Theorem holds. Indeed: *a point $y \in \partial V$ is $\mathcal{H}$-regular for V if and only if there exists a $\mathcal{H}$-barrier for V at y*, i.e., if there exists a function b, defined in $V \cap W$, where W is a suitable neighborhood of y, such that

  (*i*) b is $\mathcal{H}$-superharmonic;
 (*ii*) $b(x) > 0 \; \forall x \in V \cap W$ and $b(x) \longrightarrow 0$ as $x \longrightarrow y$.

For our purposes it is important to recall that if $y \in \partial V$ is $\mathcal{H}$-regular for V there exists a barrier function for V at y which is defined and $\mathcal{H}$-harmonic all over V.

Finally, we recall the *minimum (maximum) principle* for $\mathcal{H}$-superharmonic ($\mathcal{H}$-subharmonic) functions.

Let V be a bounded open set with $\overline{V} \subseteq E$ and let $u \in \overline{\mathcal{H}}(V)$ ($u \in \underline{\mathcal{H}}(V)$). If

$$\liminf_{x \longrightarrow y} u(x) \ge 0 \quad (\limsup_{x \longrightarrow y} u(x) \le 0) \quad \forall y \in \partial V,$$

then $u \ge 0$ ($u \le 0$) in V.

2.2. The $\mathcal{L}_0$-harmonic space

Let E be a bounded open subset of X such that $\overline{E} \subseteq X$. For every open set $V \subseteq E$ we let

$$\mathcal{H}(V) = \{u \in C^\infty(V, \mathbb{R}) \mid \mathcal{L}_0 u = 0 \text{ in } V\}.$$

Then, $V \longmapsto \mathcal{H}(V)$ is a sheaf of functions such that $\mathcal{H}(V)$ is a linear subspace of $C(V, \mathbb{R})$.
If $u \in \mathcal{H}(V)$ we will say that u is $\mathcal{H}$-harmonic or $\mathcal{L}_0$-harmonic in V.
We have that

$$(E, \mathcal{H}) \text{ is a } \sigma^*\text{-harmonic space.} \tag{5}$$

Before showing this statement we remark that a C^2-function u in a open set V is $\mathcal{H}$-superharmonic if and only if $\mathcal{L}_0 u \leq 0$ in V. This is an easy consequence of Picone's maximum principle (see e.g. [14], page 547). Now we are ready to prove (5).

(A1) is satisfied since the constant functions are $\mathcal{L}_0$-harmonic.

(A2) - (A4) are proved in [14]. We would like to stress that our operators $\mathcal{L}_0$ are contained in the class considered in [14] since the rank condition (H2) implies that both $\mathcal{L}_0$ and $\mathcal{L}_0 - \beta$, for every $\beta \geq 0$, are hypoelliptic.

The axiom (A5) follows from the following Lemma which seems to have an interest per se.

Lemma 2.1. *Let us consider a linear second order PDO of the kind*

$$\mathcal{L} := \sum_{i,j=1}^N a_{ij} \partial_{x_i x_j} + \sum_{j=1}^N b_j \partial_{x_j},$$

where $a_{ij} = a_{ji}, b_j$ are continuous functions in $\overline{Y}$, where Y is a bounded open subset of $\mathbb{R}^N$. Suppose[2]

$$\inf_Y a_{11} := \alpha > 0 \quad and \quad \sum_{j=1}^N a_{jj} > 0 \text{ in } Y.$$

Then, for every $x_0 \in Y$ there exists a function $h \in C^\infty(Y, \mathbb{R})$ such that

(i) $h(x_0) = 0$ and $h(x) > 0$ for every $x \neq x_0$;
(ii) $\mathcal{L}h > 0$ in X.

Proof. For the sake of simplicity we assume $x_0 = 0$. We define

$$h(x) = E(\lambda x_1) + (x_2^2 + \cdots + x_N^2), \quad x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N,$$

where $\lambda > 0$ will be fixed in the sequel. Moreover,

---

[2] We don't require $(a_{ij})_{i,j=1,\ldots,N}$ to be nonnegative definite.

$$E(s) = \exp(\phi(s)) - \exp(\phi(0))$$

and

$$\phi(s) = \sqrt{1 + s^2}, \quad s \in \mathbb{R}.$$

We have:

$$\phi(0) = 1, \quad \phi(s) > 1 \quad \forall s \neq 0, \quad E(s) > 0 \quad \forall s \neq 0, \quad E(0) = 0,$$

$$\phi'(s) = \frac{s}{\sqrt{1 + s^2}}, \quad \phi''(s) = \frac{1}{(1 + s^2)^{\frac{3}{2}}}.$$

Hence

$$\left(\phi'^2 + \phi''\right)(s) = \frac{s^2}{1 + s^2} + \frac{1}{(1 + s^2)^{\frac{3}{2}}} \geq \frac{1}{2\sqrt{2}} \quad \forall s \in \mathbb{R}.$$

On the other hand

$$E' = \exp(\phi)\phi', \quad E'' = \exp(\phi)(\phi'^2 + \phi'').$$

Therefore, letting

$$\beta := \sup_X \sum_{j=1}^{N} |b_j| \quad (< \infty) \quad \text{and} \quad \lambda = \sup_{x \in \overline{X}} |x|,$$

we get

$$\mathcal{L}h(x) = \lambda^2 E''(\lambda x_1) a_{11}(x) + \lambda E'(\lambda x_1) b_1 + 2 \sum_{j=2}^{N} (a_{jj}(x) + b_j(x)x_j)$$

$$\geq \exp(\phi(\lambda x_1)) \left(\frac{a_{11}(x)}{2\sqrt{2}} \lambda^2 - \lambda|b_1| \right) - 2 \sum_{j=2}^{N} |b_j||x_j|$$

$$\geq \lambda^2 \left(\frac{\alpha}{2\sqrt{2}} - \frac{|b_1|}{\lambda} \right) - 2\beta\lambda$$

$$\geq \lambda^2 \left(\frac{\alpha}{2\sqrt{2}} - \frac{\beta}{\lambda} \right) - 2\beta\lambda.$$

If λ is big enough, this implies

$$\mathcal{L}h > 0 \text{ in } X.$$

Moreover

$$h(0) = E(0) = 0, \quad h(x) > 0 \quad \text{if } x > 0.$$

The proof is complete. □

2.3. The $\mathscr{L}$-harmonic space

Let $\widehat{E}$ be a bounded open subset of $X \times \mathbb{R}$ such that $\overline{\overline{E}} \subseteq X \times \mathbb{R}$. For every open set $V \subseteq \widehat{E}$ we let

$$\mathcal{K}(V) = \{u \in C^\infty(V, \mathbb{R}) \mid \mathscr{L}u = 0 \text{ in } V\}.$$

Then, $V \longmapsto \mathcal{K}(V)$ is a sheaf of functions making

$$(\widehat{E}, \mathcal{K}) \text{ a } \sigma^*\text{-harmonic space.}$$

This can be proved just by proceeding as in subsection 2.2. We call $\mathcal{K}$-harmonic or $\mathscr{L}$-harmonic in a open set V the solutions to $\mathscr{L}u = 0$ in V.

Here we prove some typical results of the present $\mathcal{K}$-harmonic space, that we will need in the proof of the main theorem of this paper. We first show a "parabolic" minimum principle for $\mathscr{L}$-subharmonic functions in cylindrical domains.

Proposition 2.2. *Let Ω be a bounded open subset of X such that $\overline{\Omega} \subseteq X$ and let $T > 0$. Consider the cylindrical domain $\mathcal{O} := \Omega \times]0, T[$ and define the "parabolic boundary" of $\mathcal{O}$ as follows*

$$\partial_p \mathcal{O} := (\Omega \times \{0\}) \times (\partial\Omega \times]0, T[).$$

Then, if $u \in \overline{\mathcal{K}}(\mathcal{O})$ is such that

$$\liminf_{z \longrightarrow \zeta} u(z) \geq 0 \quad \forall \zeta \in \partial_p \mathcal{O},$$

we have $u \geq 0$ in $\mathcal{O}$.

Proof. For every arbitrarily fixed $\widehat{T} \in]0, T[$ we let $\widehat{\mathcal{O}} = \Omega \times]0, \widehat{T}[$. We will prove that $u \geq 0$ in $\widehat{\mathcal{O}}$. Since $\widehat{T}$ is arbitrarily fixed in $]0, T[$, this will guarantee the proof of our lemma. To this end, given any $\varepsilon > 0$, we define

$$u_\varepsilon(z) = u_\varepsilon(x, t) := u(x, t) + \frac{\varepsilon}{\widehat{T} - t}, \quad z \in \widehat{\mathcal{O}}.$$

Since u is $\mathcal{K}$-superharmonic in $\mathcal{O}$ and

$$\mathscr{L}\frac{\varepsilon}{\widehat{T} - t} = -\varepsilon \partial_t \frac{1}{\widehat{T} - t} = -\frac{\varepsilon}{(\widehat{T} - t)^2} < 0 \text{ in } \widehat{\mathcal{O}},$$

then u_ε is $\mathcal{K}$-superharmonic in $\mathcal{O}$. Moreover

$$\liminf_{z \longrightarrow \zeta} u_\varepsilon(z) \geq 0 \quad \forall \zeta \in \partial_p \widehat{\mathcal{O}},$$

and, for every $\xi \in \Omega$,

$$\liminf_{z \longrightarrow (\varepsilon, \widehat{T})} u_\varepsilon(z) \geq u(\varepsilon, \widehat{T}) + \liminf_{t \nearrow \widehat{T}} \frac{\varepsilon}{\widehat{T} - t} = \infty.$$

By the minimum principle recalled in subsection 2.1, we have $u_\varepsilon \geq 0$ in $\widehat{\mathcal{O}}$. Letting ε go to zero we have $u_\varepsilon \geq 0$ in $\widehat{\mathcal{O}}$, thus completing the proof. □

Proposition 2.3. *Let $\Omega \subseteq X$ be open and let T_0 and $T \in \mathbb{R}$, such that $0 < T_0 < T$. Let $\mathcal{O} := \Omega \times]0, T[$ and $u : \mathcal{O} \longrightarrow \mathbb{R}$ be such that the restrictions $u|_{\Omega \times]0, T_0[}$ and $u|_{\Omega \times]T_0, T[}$ are $\mathcal{K}$-superharmonic. Then, if*

$$\liminf_{\substack{z \longrightarrow (\xi, T_0) \\ (x,t) \in \mathcal{O}}} u(x,t) = \liminf_{\substack{z \longrightarrow (\xi, T_0) \\ t < T_0 \\ (x,t) \in \mathcal{O}}} u(x,t) = u(\xi, T_0) \quad \forall \xi \in \Omega, \tag{6}$$

the function u is $\mathcal{K}$-superharmonic in $\Omega \times]0, T[$.

Proof. Since u is lower semicontinuous in $\Omega \times]0, T_0[$ and in $\Omega \times]T_0, T[$, the assumption (6) implies that u is lower semicontinuous in $\mathcal{O} = \Omega \times]0, T[$.

To prove that u is $\mathcal{K}$-harmonic in $\mathcal{O}$ we will show the following claim.

Claim. *For every $z \in \mathcal{O}$ there exists a basis B_z of $\mathcal{K}$-regular neighborhoods V of z such that*

$$u(z) \geq K_\varphi^V(z) \qquad \forall \varphi \in C(\partial V, \mathbb{R}), u|_{\partial V} \geq \varphi.$$

Here K_φ^V denotes the unique $\mathcal{K}$-harmonic function in V, continuous up to ∂V and such that $K_\varphi^V|_{\partial V} = \varphi$.

From this Claim our assertion follows thanks to Corollary 6.4.9 in [4].

If $z \in \Omega \times]0, T_0[$ or if $z \in \Omega \times]0, T[$, the Claim is satisfied since u is $\mathcal{K}$-superharmonic both in $\Omega \times]0, T_0[$ and in $\Omega \times]0, T[$. Then it remains to prove the Claim for every point $\zeta = (\xi, T_0), \xi \in \Omega$. Let $B_\zeta = (V)$ be a basis of $\mathcal{K}$-regular neighborhoods of ζ such that $\overline{V} \subseteq \mathcal{O}$. Let $\varphi \in C(\partial V, \mathbb{R}), \varphi \leq u|_{\partial V}$. Then $u - K_\varphi^V$ is $\mathcal{K}$-superharmonic in $\Omega \times]0, T_0[$ and

$$\liminf_{z \longrightarrow z'} u(z) \geq u(z') - u(z') \geq 0 \qquad \forall z' \in \partial_p \Omega \times]0, T_0[.$$

Therefore, by Proposition 2.2,

$$u - K_\varphi^V \geq 0 \text{ in } V \cap \{t < T_0\}.$$

As a consequence, keeping in mind assumption (6), we have

$$u(\xi, T_0) = \liminf_{\substack{(x,t) \to (\xi, \tau) \\ t < T_0}} u(x,t) \geq \liminf_{\substack{(x,t) \to (\xi, T_0) \\ t < T_0}} K_\varphi^V(x,t) = K_\varphi^V(\xi, T_0),$$

that is,

$$u(\xi, T_0) \geq K_\varphi^V(\xi, T_0).$$

This completes the proof. $\quad \square$

3. Some preliminary results

The proof of our main theorem rests on the following two lemmata.

Lemma 3.1. *Let Ω be a bounded open set such that $\overline{\Omega} \subseteq X$, and let $\mathcal{O} := \Omega \times]0, T[$, $T \in \mathbb{R}$, $T > 0$. Let $\varphi : \partial \mathcal{O} \longrightarrow \mathbb{R}$ be upper semicontinuous and such that $t \longmapsto \varphi(x, t)$ is monotone decreasing, $\forall x \in \partial \Omega$ and*

$$\varphi(x, 0) = M = \sup_{\partial \mathcal{O}} \varphi \qquad (M \in \mathbb{R}).$$

Then, the Perron solution $K_\varphi^{\mathcal{O}}$ is monotone decreasing w.r.t. the variable t: more precisely

$$t \longmapsto K_\varphi^{\mathcal{O}}(x, t) \text{ is monotone decreasing for every fixed } x \in \Omega.$$

Proof. For every fixed $\delta \in]0, T[$ let us define

$$h(x, t) = K_\varphi^{\mathcal{O}}(x, t) - K_\varphi^{\mathcal{O}}(x, t + \delta), \; x \in \Omega, 0 < t < T - \delta.$$

It is enough to prove that $h \geq 0$ in $\mathcal{O}_\delta := \Omega \times]0, T - \delta[$. To this end we show that, for every $u \in \overline{\mathcal{U}}_\varphi^{\mathcal{O}}$ and $v \in \underline{\mathcal{U}}_\varphi^{\mathcal{O}}$, the function

$$w(x, t) = u(x, t) - v(x, t + \delta)$$

is nonnegative in $\mathcal{O}_\delta$. Now, we have:

(a) w is $\mathcal{K}$-superharmonic in $\mathcal{O}_\delta$, since $u \in \overline{\mathcal{K}}(\mathcal{O})$ and $(x, t) \longmapsto v(x, t + \delta)$ is $\mathcal{K}$-subharmonic in $\mathcal{O}_\delta$ being $v \in \underline{\mathcal{K}}(\mathcal{O})$ and $\mathscr{L}$ translation invariant in the variable t.

(b) For every $\overline{x} \in \Omega$,

$$\liminf_{(x,t) \to (\overline{x}, 0)} w(x, t) \geq \liminf_{(x,t) \to (\overline{x}, 0)} u(x, t) - \liminf_{(x,t) \to (\overline{x}, 0)} v(x, t + \delta)$$

$$\geq \varphi(\overline{x}, 0) - v(\overline{x}, \delta)$$

$$= M - v(\overline{x}, \delta) \geq 0.$$

We remark that $v \leq M$ in $\mathcal{O}$ since v is $\mathcal{K}$-subharmonic and

$$\limsup_{z \longrightarrow \zeta} v(z) \leq \varphi(\zeta) \leq M \quad \forall \zeta \in \partial \mathcal{O}.$$

Here we use the maximum principle for subharmonic functions.

(*c*) For every $\zeta = (\xi, \tau), \xi \in \partial\Omega, 0 < \tau < T - \delta$,

$$\liminf_{(x,t)\to(\xi,\tau)} w(x,t) \geq \varphi(\xi, \tau) - \varphi(\xi, \tau + \delta) \geq 0,$$

by hypothesis.

From (*a*), (*b*) and (*c*) and the minimum principle for superharmonic functions we get

$$w \geq 0 \text{ in } \mathcal{O}_\delta.$$

This completes the proof. □

With Lemma 3.1 at hand we can easily prove the following key result for our main theorem.

Lemma 3.2. *Let* Ω *be a bounded open set such that* $\overline{\Omega} \subseteq X$, *and let* $\mathcal{O} := \Omega \times]0, T[$, $T > 0$. *Let* $z_0 = (x_0, t_0) \in \partial\Omega \times]0, T[$ *be a* $\mathcal{L}$-*regular boundary point.*
Then there exists a function $b \in \mathcal{K}(\mathcal{O})$ *such that*

 (*i*) *b is an* $\mathcal{L}$-*barrier for* $\mathcal{O}$ *at* z_0;
(*ii*) $t \longmapsto b(x,t)$ *is monotone decreasing for every fixed* $x \in \Omega$.

Proof. Let Y be a bounded open set such that $\overline{\Omega} \subseteq Y \subseteq \overline{Y} \subseteq X$ and let $x_0 \in \Omega$. By Lemma 2.1 there exists a function $h \in C^\infty(Y, \mathbb{R})$ such that

(*a*) $h(x_0) = 0$ and $h(x) > 0 \ \forall x \neq x_0$.
(*b*) $\mathcal{L}_0 h > 0$ in Ω.

For a fixed $\delta \in]0, T_0[$ let us define

$$\widehat{h} : \overline{\Omega} \times [0, T] \longrightarrow \mathbb{R}, \quad \widehat{h}(x,t) = \begin{cases} h(x) & \text{if } \delta < t \leq T, \\ M & \text{if } 0 \leq t \leq \delta, \end{cases}$$

where $M = \sup_{\overline{\Omega}} h$.
   This function is $\mathcal{L}$-subharmonic in $\mathcal{O}_1 := \Omega \times]0, \delta[$ and in $\mathcal{O}_2 := \Omega \times]\delta, T[$ since

$$\mathcal{L}\widehat{h} = 0 \text{ in } \mathcal{O}_1 \quad \text{and} \quad \mathcal{L}\widehat{h} = \mathcal{L}_0 h > 0 \text{ in } \mathcal{O}_2.$$

On the other hand,

$$\limsup_{\substack{(x,t)\longrightarrow(\xi,\delta)\\ t<\delta}} \widehat{h}(x,t) = M = \limsup_{(x,t)\longrightarrow(\xi,\delta)} \widehat{h}(x,t).$$

Then, by Proposition 2.3,

$$\widehat{h} \in \underline{\mathcal{K}}(\Omega \times]0, T[).$$

Moreover,

$$t \longmapsto \widehat{h}(x,t) \text{ is monotone decreasing,}$$

for every fixed $x \in \overline{\Omega}$.

Let us now put

$$b := K^{\mathcal{O}}_{\widehat{h}|_{\partial \mathcal{O}}},$$

which is well defined and $\mathcal{K}$-harmonic in $\mathcal{O}$, since $\widehat{h}|_{\partial \mathcal{O}}$ is bounded and upper semicontinuous.

Moreover, by Lemma 3.1, $t \longmapsto b(x,t)$ is monotone decreasing for every fixed $x \in \Omega$.

It remains to show that b is an $\mathcal{L}$-barrier for $\mathcal{O}$ at z_0. To this end we first remark that

$$\widehat{h} \in \underline{\mathcal{U}}^{\mathcal{O}}_{\widehat{h}|_{\partial \mathcal{O}}},$$

so that

$$\widehat{h} \leq b \text{ in } \mathcal{O}.$$

This implies $b > 0$ in $\mathcal{O}$ since $\widehat{h}$ is strictly positive.

On the other hand, since $\widehat{h}|_{\partial \mathcal{O}}$ is continuous in a neighborhood of z_0, and z_0 is $\mathcal{L}$-regular for $\mathcal{O}$,

$$\lim_{z \longrightarrow z_0} b(z) = \lim_{z \longrightarrow z_0} K^{\mathcal{O}}_{\widehat{h}|_{\partial \mathcal{O}}}(z) = \widehat{h}(z_0) = \phi(x_0) = 0.$$

This completes the proof. $\quad \square$

4. Proof of Theorem 1.1

Let us keep the notation of Theorem 1.1 and split the proof in two steps.

(1) *If $x_0 \in \partial \Omega$ is $\mathcal{L}_0$-regular for Ω, then $z = (x_0, t_0)$ is $\mathcal{L}$-regular for $\mathcal{O}$.*

Indeed, the $\mathcal{L}_0$-regularity of x_0 implies the existence of a $\mathcal{L}_0$-harmonic barrier for Ω at x_0, i.e. a function $b_0 \in \mathcal{K}(\Omega)$ such that

$$b_0 > 0 \text{ in } \Omega \quad \text{and} \quad b_0 \longrightarrow 0 \text{ as } x \longrightarrow x_0.$$

It follows that

$$\widehat{b}(x,t) = b_0(x), \quad (x,t) \in \mathcal{O},$$

is $\mathcal{L}$-harmonic in $\mathcal{O}$ ($\mathcal{L}\widehat{b} = \mathcal{L}_0 b_0 = 0$). Moreover,

$$\widehat{b} > 0 \text{ in } \mathcal{O} \quad \text{and} \quad \widehat{b}(x,t) = b_0(x) \longrightarrow 0 \text{ as } (x,t) \longrightarrow (x_0, t_0).$$

Hence, $\widehat{b}$ is an $\mathcal{L}$-barrier function for $\mathcal{O}$ at z_0 and, as a consequence, z_0 is $\mathcal{L}$-regular for $\mathcal{O}$.

(2) *If $z = (x_0, t_0)$, $x_0 \in \Omega, 0 < t_0 < T$, is $\mathscr{L}$-regular for $\mathcal{O}$, then x_0 is $\mathscr{L}_0$-regular for Ω.*

Indeed, by Lemma 3.2, there exists a function $b \in \mathcal{K}(\mathcal{O})$ such that $b > 0$, $b(z) \longrightarrow 0$ as $z \longrightarrow z_0$ and

$$t \longmapsto b(x, t) \text{ is monotone decreasing } \quad \forall x \in \Omega.$$

It follows that, letting $b_0(x) = b(x, t_0)$,

$$\mathscr{L}_0 b_0 = \mathscr{L} b + \partial_t b = \partial_t b \le 0 \text{ in } \Omega.$$

Hence, b_0 is $\mathscr{L}_0$-superharmonic in Ω. Moreover, $b_0 > 0$ in Ω and

$$b_0(x) = b(x, t_0) \longrightarrow 0 \text{ as } x \longrightarrow x_0.$$

Therefore, b_0 is an $\mathscr{L}$-barrier for Ω at x_0, and x_0 is $\mathscr{L}_0$-regular.

5. An application to degenerate Ornstein–Uhlenbeck operators

In $\mathbb{R}^N$ let us consider the partial differential operator

$$L_0 = \mathrm{div}\,(A\nabla) + \langle Bx, \nabla \rangle, \tag{7}$$

where $A = (a_{ij})_{i, j=1,...,N}$ and $B = (b_{ij})_{i, j=1,...,N}$ are $N \times N$ real constant matrices, $x = (x_1, \ldots, x_N)$ is the point of $\mathbb{R}^N$, div, ∇ and $\langle \cdot, \cdot \rangle$ denote the divergence, the Euclidean gradient and the inner product in $\mathbb{R}^N$, respectively.

We suppose that the matrix A is symmetric, positive semidefinite and that it assumes the following block form

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix},$$

A_0 being a $p_0 \times p_0$ strictly positive definite matrix with $1 \le p_0 \le N$. Moreover, we assume the matrix B to be of the following type

$$B = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ B_1 & 0 & \ldots & 0 & 0 \\ 0 & B_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & B_r & 0 \end{bmatrix}, \tag{8}$$

where B_j is a $p_{j-1} \times p_j$ block with rank p_j $(j = 1, 2, ..., r)$, $p_0 \ge p_1 \ge ... \ge p_r \ge 1$ and $p_0 + p_1 + ... + p_r = N$.

Finally, letting

$$E(s) := \exp(-sB), \quad s \in \mathbb{R},$$

we assume that the following condition is satisfied:

$$C(t) = \int_0^t E(s) A E^T(s) \, ds \quad \text{is strictly positive definite for every } t > 0.$$

As it is quite well known this condition implies the hypoellipticity of L, see [15]. In that paper it is proved that the evolution counterpart of L_0, i.e. the operator

$$L = L_0 - \partial_t \text{ in } \mathbb{R}^{N+1},$$

is left translation invariant and homogeneous of degree two on the homogeneous group

$$\mathbb{K} = (\mathbb{R}^{N+1}, \circ, \delta_\lambda)$$

with composition law $\circ$ defined as follows

$$(x, t) \circ (x', t') = (x' + E(t')x, t + t')$$

and dilation δ_λ, $\lambda > 0$, defined by

$$\delta_\lambda : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{N+1}, \quad \delta_\lambda(x, t) = \delta_\lambda(x^{(p_0)}, x^{(p_1)}, \ldots, x^{(p_r)}, t)$$
$$:= (\lambda x^{(p_0)}, \lambda^3 x^{(p_1)}, \ldots, \lambda^{2r+1} x^{(p_r)}, \lambda^2 t),$$

where $x^{(p_i)} \in \mathbb{R}^{p_i}$, $i = 0, \ldots, r$.

The natural number $q := Q + 2$, with

$$Q := p_0 + 3p_1 + \ldots + (2r + 1)p_r, \tag{9}$$

is the homogeneous dimension of $\mathbb{K}$. In what follows we will write

$$\delta_\lambda(z) = \delta_\lambda(x, t) = (D_\lambda(x), \lambda^2 t),$$

where,

$$D_\lambda(x) = (\lambda x^{(p_0)}, \lambda^3 x^{(p_1)}, \ldots, \lambda^{2r+1} x^{(p_r)}, \lambda^2 t).$$

Obviously, $(D_\lambda)_{\lambda > 0}$ is a group of dilations in $\mathbb{R}^N$. The natural number Q in (9) is the homogeneous dimension of $\mathbb{R}^N$ w.r.t. the group $(D_\lambda)_{\lambda > 0}$.

The operator L has a fundamental solution Γ given by

$$\Gamma(z_0, z) := \gamma(z^{-1} \circ z_0), \quad z, z_0 \in \mathbb{R}^{N+1},$$

where $\circ$ is the composition law in $\mathbb{K}$, z^{-1} denotes the opposite of z in $\mathbb{K}$ and, for a suitable $C_Q > 0$,

$$\gamma(x,t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \frac{C_Q}{t^Q} \exp\left(-\frac{1}{4}\left|D_{\frac{1}{\sqrt{t}}}(x)\right|_C^2\right) & \text{if } t > 0, \end{cases}$$

where,

$$|y|_C^2 = \langle C^{-1}(1)y, y \rangle,$$

see again [15].

It is quite easy to recognize that our Tikhonov-type theorem applies to the operators L_0 and L. Hence, if Ω is a bounded open subset of $\mathbb{R}^N$, $x_0 \in \partial\Omega$ and $t_0 \in]-T, T[$, $T > 0$, we have:

$$x_0 \text{ is } L_0\text{-regular for } \Omega$$

$$\text{if and only if}$$

$$z_0 = (x_0, 0) \text{ is } L\text{-regular for } \mathcal{O}_T := \Omega \times]-T, T[.$$

On the other hand, in [11, Corollary 1.3] it is proved that

$$z_0 \text{ is } L\text{-regular for } \mathcal{O}_T$$

if, for a $\mu \in]0, 1[$, the following condition holds:

$$\sum_{k=1}^{\infty} \frac{|\mathcal{O}_{T,k}^c(z_0)|}{\mu^{\alpha(k)\frac{Q+2}{Q}}} = \infty, \tag{10}$$

where $\alpha(k) = k \log k$, $|\cdot|$ denotes the Lebesque measure in $\mathbb{R}^{N+1}$ and

$$\mathcal{O}_{T,k}^c(z_0) = \left\{ z \neq \mathcal{O}_T \; : \; \left(\frac{1}{\mu}\right)^{\alpha(k)} \leq \Gamma(z_0, z) \leq \left(\frac{1}{\mu}\right)^{\alpha(k+1)} \right\}.$$

We express now this condition in a more explicit form. To this end we let

$$A_k^c(x_0) = \left\{ (x,t) \in \mathbb{R}^{N+1} \mid x \notin \Omega, \gamma(z^{-1} \circ (x,0)) \geq \left(\frac{1}{\mu}\right)^{\alpha(k)} \right\}. \tag{11}$$

Then,

$$\mathcal{O}_{T,k}^c((x_0,0)) = (A_k(x_0) \smallsetminus A_{k+1}(x_0)) \cup \left\{ \gamma = \left(\frac{1}{\mu}\right)^{\alpha(k+1)} \right\}$$

$$\supseteq A_k(x_0) \smallsetminus A_{k+1}(z_0).$$

Hence, denoting for the sake of brevity,

$$d_k = |A_k(z_0)| \quad \text{and} \quad v = \mu^{\frac{(Q+2)}{Q}},$$

condition (10) is satisfied if

$$\sum_{k=1}^{\infty} \frac{d_k - d_{k+1}}{\nu^{\alpha(k)}} = \infty. \tag{12}$$

On the other hand, for every $p \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} \frac{d_k - d_{k+1}}{\nu^{\alpha(k)}}$$

$$= \frac{d_1}{\nu^{\alpha(1)}} + d_2 \left(\frac{1}{\nu^{\alpha(2)}} - \frac{2}{\nu^{\alpha(1)}} \right) + \cdots + d_p \left(\frac{1}{\nu^{\alpha(p)}} - \frac{2}{\nu^{\alpha(p-1)}} \right) - \frac{d_{p+1}}{\nu^{\alpha(p)}}$$

$$\leq (1 - \nu^{\log 2}) \sum_{k=1}^{p} \frac{d_k}{\nu^{\alpha(k)}} - \frac{d_{p+1}}{\nu^{\alpha(p)}}.$$

Then, since $\dfrac{d_{p+1}}{\nu^{\alpha(p)}} \longrightarrow 0$ as $p \to \infty$ (as we will see later) condition (12) is satisfied if

$$\sum_{k=1}^{\infty} \frac{d_k}{\mu^{\alpha(k)}} = \infty. \tag{13}$$

Keeping in mind the very definition of Γ, we have that $A_k(x_0)$ is equal to the following set

$$\left\{ (x, t) \in \mathbb{R}^{N+1} \mid x \in \Omega^c, t < 0, \left| D_{\frac{1}{\sqrt{|t|}}} (x_0 - E(|t|x)) \right|_C^2 < 2Q \log \frac{(C_Q \mu^{\alpha(k)})^{\frac{2}{Q}}}{t} \right\},$$

whereby, with the change of variables $y := x_0 - E(|t|)x$, $\tau = -t$, we get

$$d_k = \left| \left\{ (y, \tau) \mid \tau > 0, \ y \in x_0 - E(\tau)(\Omega^c), \left| D_{\frac{1}{\sqrt{|\tau|}}} \right|_C^2 < 2Q \log \frac{R_k}{\tau} \right\} \right|. \tag{14}$$

Here $R_k = (C_Q \mu^{\alpha(k)})^{\frac{2}{Q}}$ and $\Omega^c := \mathbb{R}^{N+1} \setminus \Omega$.

Therefore,

$$d_k \leq \left| \left\{ (y, \tau) \mid \tau > 0, \left| D_{\frac{1}{\sqrt{|\tau|}}} \right|_C^2 < 2Q \log \frac{R_k}{\tau} \right\} \right|$$

(using the change of variables $y = D_{\sqrt{R_k}}(\xi)$, $\tau = R_k s$)

$$= R_k^{\frac{Q+2}{Q}} \left| \left\{ (\xi, s) \mid s > 0, \left| D_{\sqrt{\frac{1}{s}}}(\xi) \right| \leq 2Q \log \frac{1}{s} \right\} \right|.$$

Hence, for a suitable dimensional constant $C_Q^* > 0$,

$$d_k \leq C_Q^* \mu^{\alpha(k)\frac{Q+2}{Q}} = C_Q^* v^{\alpha(k)}.$$

Then,

$$0 \leq \frac{d_{p+1}}{v^{\alpha(p)}} \leq C_Q^* \mu^{\alpha(p+1)-\alpha(p)} \longrightarrow 0 \text{ as } p \longrightarrow \infty,$$

since $0 < \mu < 1$ and $\alpha(p+1) - \alpha(p) = p \log \frac{p+1}{p} + \log (p+1) \longrightarrow \infty$.

We have completed the proof of the following criterion:

Let L be the Ornstein–Uhlenbeck-type operator in (7) *and let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set. Then, a point $x_0 \in \partial\Omega$ is L-regular for Ω if*

$$\sum_{k=1}^{\infty} \frac{d_k(\Omega, x_0)}{\mu^{\alpha(k)\frac{Q+2}{2}}} = \infty, \tag{15}$$

where $d_k(\Omega, x_0) := d_k$ is defined in (14).

We note that condition (15) holds if Ω satisfies the exterior cone-type condition introduced in [13]. Geometric boundary regularity criteria for wide classes of hypoelliptic operators are also established in [2], [12], [16], [17], [18], [21], [23], [24] and [25]. Thanks to our Theorem 1.1, several regularity results for evolution operators contained in the previous papers can be used to obtain boundary regularity criteria for operators of the type (1).

Acknowledgment

References

[1] W. Arendt, Resolvent positive operators and inhomogeneous boundary conditions, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) 29 (3) (2000) 639–670.

[2] A. Björn, J. Björn, U. Gianazza, M. Parviainen, Boundary regularity for degenerate and singular parabolic equations, Calc. Var. Partial Differ. Equ. 52 (3–4) (2015) 797–827.

[3] A. Banerjee, N. Garofalo, On the Dirichlet boundary value problem for the normalized p-Laplacian evolution, Commun. Pure Appl. Anal. 14 (1) (2015) 1–21.

[4] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, Stratified Lie Groups and Potential Theory for Their Sub-Laplacians, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[5] I. Babuška, R. Výborný, Reguläre und stabile Randpunkte für das Problem der Wärmeleitungsgleichung, Ann. Pol. Math. 12 (1962) 91–104.

[6] C. Constantinescu, A. Cornea, Potential Theory on Harmonic Spaces, Die Grundlehren der mathematischen Wissenschaften, Band 158, Springer-Verlag, New York-Heidelberg, 1972, with a preface by H. Bauer.

[7] C.Y. Chan, E.C. Young, Regular regions for parabolic and elliptic equations, Port. Math. 36 (1) (1977) 7–12.

[8] W. Fulks, A note on the steady state solutions of the heat equation, Proc. Am. Math. Soc. 7 (1956) 766–770.

[9] W. Fulks, Regular regions for the heat equation, Pac. J. Math. 7 (1957) 867–877.

[10] T. Kilpeläinen, P. Lindqvist, On the Dirichlet boundary value problem for a degenerate parabolic equation, SIAM J. Math. Anal. 27 (3) (1996) 661–683.

[11] A.E. Kogoj, E. Lanconelli, G. Tralli, Wiener-Landis criterion for Kolmogorov-type operators, Discrete Contin. Dyn. Syst., Ser. A 38 (5) (2018) 2467–2485.

[12] A.E. Kogoj, On the Dirichlet problem for hypoelliptic evolution equations: Perron–Wiener solution and a cone-type criterion, J. Differ. Equ. 262 (3) (2017) 1524–1539.

[13] A.E. Kogoj, A Zaremba type criterion for hypoelliptic degenerate Ornstein–Uhlenbeck operators, Discrete Contin. Dyn. Syst., Ser. S (2019), in press.

[14] A.E. Kogoj, S. Polidoro, Harnack inequality for hypoelliptic second order partial differential operators, Potential Anal. 45 (3) (2016) 545–555.

[15] E. Lanconelli, S. Polidoro, On a class of hypoelliptic evolution operators, in: Partial Differential Equations, II, Turin, 1993, Rend. Sem. Mat. Univ. Politec. Torino 52 (1) (1994) 29–63.

[16] E. Lanconelli, G. Tralli, F. Uguzzoni, Wiener-type tests from a two-sided gaussian bound, Ann. Mat. Pura Appl. 196 (1) (2017) 217–244.

[17] E. Lanconelli, F. Uguzzoni, Potential analysis for a class of diffusion equations: a Gaussian bounds approach, J. Differ. Equ. 248 (9) (2010) 2329–2367.

[18] M. Manfredini, The Dirichlet problem for a class of ultraparabolic equations, Adv. Differ. Equ. 2 (5) (1997) 831–866.

[19] P. Negrini, Punti regolari per aperti cilindrici in uno spazio β-armonico, Boll. Unione Mat. Ital., B (6) 2 (2) (1983) 537–547.

[20] O.A. Oleǐnik, E.V. Radkevič, Second Order Equations with Nonnegative Characteristic Form, Plenum Press, New York-London, 1973, translated from the Russian by Paul C. Fife.

[21] I.I. Skrypnik, On the necessity of the Wiener condition for singular parabolic equations with non-standard growth, Adv. Nonlinear Stud. 11 (3) (2011) 701–731.

[22] A.N. Tikhonov, The heat equation for several variables, Byull. Mosk. Gos. Univ. Mat. Mekh. 1 (9) (1938) 1–49.

[23] G. Tralli, F. Uguzzoni, Wiener criterion for X-elliptic operators, J. Differ. Equ. 259 (11) (2015) 6510–6527.

[24] G. Tralli, F. Uguzzoni, On a non-smooth potential analysis for Hörmander-type operators, Calc. Var. Partial Differ. Equ. 57 (2) (2018) 37.

[25] G. Tralli, F. Uguzzoni, A Wiener test à la Landis for evolutive Hörmander operators, arXiv:1903.04438, 2019, preprint.