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X-elliptic operators and X-control distances

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ABSTRACT. – Let L be a linear second order divergence form operator with non negative characteristic form. Let $X = (X_1, \dots, X_m)$ be a family of locally Lipschitz continuous vector fields on $\mathbb{R}^n$. Assuming that L is X-elliptic accordingly to definition (2) of the subsequent Introduction, we provide a condition on X for the weak solution, to $Lu = 0$ satisfies a "scale invariant" Harnack inequality.

1. Introduction

In this note we are concerned with a class of degenerate elliptic equations in divergence form:

$$Lu := \sum_{i,j=1}^{N} \partial_{x_j}(a_{ij}\partial_{x_i}). \tag{1}$$

We assume $a_{ji} = a_{ij} \in L^{\infty}_{\text{loc}}(\mathbb{R}^N)$ and the existence of a family $X = (X_1, \dots, X_m)$ of locally Lipschitz continuous vector fields on $\mathbb{R}^N$ such that, for a suitable constant $c > 0$,

$$\frac{1}{c}\sum_{j=1}^{m}\langle X_j(x), \xi_j\rangle^2 \le \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \le c\sum_{j=1}^{m}\langle X_j(x), \xi_j\rangle^2 \tag{2}$$

for every $x, \xi \in \mathbb{R}^N$. We call *X-elliptic* any operator whose characteristic form satisfies the inequality (2).

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The principal aim of this paper is to show a general condition on X assuring a *scale invariant* Harnack inequality for the positive weak solutions to $Lu = 0$.

A first basic hypothesis, that we shall assume without further comment throughout the paper, is the *X-connectivity* of $\mathbb{R}^N$, i.e. the existence of a X-subunit path connecting any pair of points. This hypothesis enables to define on $\mathbb{R}^N$ the X-control distance d_X, also called the Carnot-Carathéodory distance.

Our main result, Theorem 4.1, provides a condition on X for which the following property holds: every d_X-ball, with radius small enough, satisfies the *doubling condition* with respect to the Lebesgue measure and supports *a Poincaré-type inequality* with respect to the X-gradient (we refer to Section 3 for formal statements and definitions). As we will point out in Section 3, this property allows to extend to the X-elliptic operators the iteration technique introduced by Moser for proving the Harnack inequality for classical uniformly elliptic equations.

The idea to adapt the Moser procedure to a non-euclidean control distance first appeared in [17] in 1982, and was inspired by a 1968 paper by Kuptsov [33]. In [18] a particular control metric was studied, and was subsequently used in [19] in order to prove Hölder continuity — and Harnack inequality — for the weak — positive — solutions to a class of degenerate non-uniformly elliptic equations.

Since then, a lot of papers have been devoted to the study of Poincaré-type inequalities for vector fields, and to their deep connection with Sobolev-type inequalities. In Section 3 a brief survey of such developments is given. In this same section we also sketch how Moser's technique can be extended to the X-elliptic operators.

Section 2 is devoted to a short historical introduction to some real analysis methods in the euclidean setting and to their extension to the metric space of *homogeneous type*. This methods play a crucial role in Moser's procedure.

In Section 4, by using the notion of X-controllable almost exponential map recently introduced in [34], we prove our main result.

2. Some classical results

Many functional spaces arising in the theory of partial differential equations have been modeled on different metrics in $\mathbb{R}^N$ and inherit some of their most important properties. A distinctive feature of Sobolev spaces,

the natural functional setting for studying elliptic equations, deeply reflects the geometry of the euclidean metric: the embedding inequality

$$\|u\|_{L^p(\mathbb{R}^N)} \leq C\|\nabla u\|_{L^2(\mathbb{R}^N)}, \qquad p = \frac{2N}{N-2}$$

is a consequence of the isoperimetric property of the euclidean balls (see Fleming and Rishel [16] and Talenti [47]).

Other functional spaces modeled on the euclidean metric are those introduced by Morrey for studying regularity problems in the calculus of variations.

If $1 \leq p < \infty$, $\lambda \geq 0$ and Ω is an open bounded subset of $\mathbb{R}^N$, a function $u \in L^p(\Omega)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ if

$$\sup_B \frac{1}{|B|^\lambda} \int_{B\cap\Omega} |u|^p dx < \infty, \tag{3}$$

where the supremum is taken over the family of euclidean balls ($|B|$ denotes the Lebesgue measure of B). A remarkable variant of Morrey spaces was proposed and used by Campanato, for proving Hölder regularity results for solutions to elliptic equations and systems. A function $u \in L^p(\Omega)$ belongs to the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ if

$$\sup_B \frac{1}{|B|^\lambda} \int_{B\cap\Omega} |u - u_B|^p dx < \infty, \tag{4}$$

where u_B denotes the integral average of u on $B \cap \Omega$.

If $0 \leq \lambda < 1$, then $\mathcal{L}^{p,\lambda} = L^{p,\lambda}$. Moreover, if $\lambda > 1$ and Ω satisfies a weak regularity condition, then up to a modification on a set of zero measure, a function $u \in \mathcal{L}^{p,\lambda}$ is Hölder continuous of exponent $\alpha = N\frac{\lambda-1}{p}$.

The former of these results is due to Campanato [5], while the latter is due to Campanato [6] and G.N. Meyers [40]. When $\lambda = 1$, the Campanato space becomes the BMO space of functions with *bounded mean oscillation*. The space BMO was first introduced and studied by John and Nirenberg and it subsequently played an important role in the regularity theory of weak solutions to linear and non-linear elliptic equation, and in harmonic analysis.

John and Nirenberg proved the following result, since then called John and Nirenberg's Lemma [31].

LEMMA 2.1. – *Let B_0 be a fixed euclidean ball. If $u \in BMO(B_0)$ then*

$$|\{x \in B_0 \ / \ |u(x) - u_B| \geq t\}| \leq ce^{-bt}|B|$$

for every ball $B \subset B_0$. As a consequence,

$$\int_B e^{a|u-u_B|}dx \le c\,|B|$$

for every $B \subset B_0$. The positive constant a, b and c are independent of B.

This lemma was promptly used by Moser [37] for a crucial step of his proof of the Harnack inequality for positive weak solutions to elliptic equations in divergence form with bounded measurable coefficients:

$$Lu := \sum_i \partial_{x_i}(a_{ij}\partial_{x_j}u) = 0\,. \tag{5}$$

We briefly recall Moser's result: if u is a positive weak solution to equation (5), and B is an euclidean ball such that $4B \subset \Omega$, then

$$\sup_B u \le c \inf_B u \tag{6}$$

where c is independent of u and B. We call (6) *scale invariant Harnack inequality* for u.

Moser devised a new iterative method that allows to prove that every positive weak solution $u > 0$ to (5) is bounded and satisfies the following inequalities

$$\sup_B u \le c_p \left(\frac{1}{|2B|}\int_{2B} u^p\right)^{\frac{1}{p}} \qquad \inf_B u \ge c_p \left(\frac{1}{|2B|}\int_{2B} u^{-p}\right)^{-\frac{1}{p}} \tag{7}$$

for every $p > 0$. Consequently, if there exists a positive exponent p such that

$$\int_{2B} u^p dx \int_{2B} u^{-p}dx \le c\,|2B|^2$$

the inequality (6) follows from (7). Now this last inequality holds if $w = \log u \in BMO$. Indeed, if this is the case, John-Nirenberg's Lemma yields

$$\begin{aligned}\int_{2B} u^p dx \int_{2B} u^{-p}dx &= \int_{2B} e^{pw}dx \int_{2B} e^{-pw}dx \\ &\le \int_{2B} e^{p(w-w_{2B})}dx \int_{2B} e^{-p(w-w_{2B})}dx \\ &\le \left(\int_{2B} e^{p|w-w_{2B}|}dx\right)^2 \le c\,|2B|^2\,.\end{aligned}$$

In order to prove that $w = \log u$ is a bounded mean oscillation function, Moser first showed that

$$\int_B |\nabla w|^2 \le \frac{c}{r^2}|B|, \qquad r = \text{radius of } B\,,$$

for any euclidean ball B such that $2B \subset \Omega$. From this result, and applying the classical Poincaré inequality

$$\int_B |w - w_B|^2 \le cr^2 \int_B |\nabla w|^2\,,$$

Moser immediately deduced

$$\int_B |w - w_B| \le r\left(\int_B |\nabla w|^2\right)^{\frac{1}{2}} |B|^{\frac{1}{2}} \le c\,|B|$$

from which $w \in BMO$.

From inequality (6), by a simple real analysis argument, it easy to obtain the Hölder continuity of the weak solution to the equation (5), then providing a new proof of the celebrated De Giorgi's Theorem [12].

The euclidean distance properties also play an important role in the Calderon-Zygmund theory of singular integrals. If we wish to solve the equation $-\Delta u = f$, we are naturally led to the study of the newtonian potential

$$\Gamma * f(x) = \int_{\mathbb{R}^N} \Gamma(x-y)f(y)dy$$

where Γ denotes the fundamental solution of the laplacian. When the function f is Hölder continuous (and compactly supported) the potential $\Gamma * f$ has Hölder continuous second derivatives and satisfies the Laplace equation $-\Delta u = f$ in a classical sense. On the other hand, if the function f is merely continuous then, in general, $\Gamma * f \notin C^2$, and the equation $-\Delta u = f$ is not solvable in a classical sense, i.e. there do not exist functions $u \in C^2$ such that $-\Delta u = f$. However, if $f \in L^p(\mathbb{R}^N)$ and $1 < p < \infty$, then $\Gamma * f \in W^{2,p}_{\text{loc}}(\mathbb{R}^N)$ and

$$\Delta(\Gamma * f)(x) = -f(x), \qquad \text{a.e.}$$

This result easily follows from the Calderon-Zygmund Theorem:

for $i, j = 1, \dots, N$ let us denote $\omega_{i,j}(x) = \partial^2_{x_i,x_j}\Gamma(x)$. Then, for every $f \in L^p(\mathbb{R}^N)$, $1 < p < \infty$,

$$\lim_{\varepsilon\to 0}\int_{|x-y|\ge\varepsilon} \omega_{i,j}(x-y)f(y)dy \tag{8}$$

exists in $L^p(\mathbb{R}^N)$ (see Stein [45], II, Theorem 3).

This result is a consequence of a deep decomposition lemma of $\mathbb{R}^N$ into a family of almost disjoint euclidean balls on each of which either f or its mean oscillation is small.

It is important to mention that the John-Nirenberg Lemma is based on the previous Calderon-Zygmund decomposition.

We also stress that the existence of the limit in (8) depends on the symmetry properties of the *recision domain* $\{|x-y| \geq \varepsilon\}$, the same of the kernels $\omega_{i,j}$. These functions, indeed, are homogeneous with respect to the dilations

$$\delta(\lambda) : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \qquad \delta(\lambda) = \lambda x .$$

and

$$\int_{|y|=\varepsilon} \omega_{ij}(y) d\sigma(y) = 0, \qquad \forall \varepsilon > 0 .$$

All results cited above have been naturally generalized for studying parabolic equations, whose most important prototype is the heat equation in $\mathbb{R}^{N+1}$:

$$H = \Delta - \partial_t .$$

The fundamental solution Γ of H,

$$\Gamma(x,t) = (4\pi t)^{\frac{-N}{2}} e^{\frac{-|x|^2}{4t}}, \quad \text{for} \quad t > 0, \qquad \Gamma(x,t) = 0, \quad \text{for} \quad t \leq 0$$

is homogeneous of degree $-N$ with respect to the dilations

$$(9) \qquad \delta(\lambda)(x,t) = (\lambda x, \lambda^2 t) .$$

Therefore, if we wish to study the operator H in L^p spaces, we are led to consider singular integrals having kernels homogeneous with respect to the dilations $\delta(\lambda)$ in (9) and use, instead of the euclidean metric, a distance with the same kind of homogeneity.

This generalization was accomplished by B.F. Jones in 1963 [32]. This author extended the theory of Calderon-Zygmund to the integral kernels with the following homogeneity:

$$K(\lambda x, \lambda^m t) = \lambda^{-n} K(x,t) .$$

Some years later, in 1966, Fabes and Rivière [14] studied a more general class of kernels, homogeneous in the following sense:

$$(10) \qquad K(\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_N} x_N) = \lambda^{-\sum \alpha_j} K(x) ,$$

where $\alpha_1, \dots, \alpha_N$ are arbitrary real numbers greater than 1.

In [14], singular integrals related to the kernels (10) were studied in perfect analogy with the "isotropic" Calderon-Zygmund case, replacing the euclidean metric with a distance of the following type:

$$(11) \qquad d(x,y) = \sum_{j=1}^{N} |x_j - y_j|^{\frac{1}{\alpha_j}}$$

which is homogeneous of degree 1 with respect to the dilations

$$\delta(\lambda)(x) = (\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_N} x_N) .$$

An analogous metric had been already introduced by Barozzi for studying Hölder regularity properties of solutions to semi-elliptic equations [2]. Barozzi proposed a generalization of Morrey spaces modeled on the distance (11), and afterwards, Da Prato [13] introduced a similar generalization for Campanato spaces. To obtain Barozzi and Da Prato spaces, it is enough to replace in definitions (3) and (4) the euclidean balls with the balls of the distance (11).

Twenty years later, in 1979, Macias e Segovia [41] studied a wide generalization of Campanato spaces, which is modeled on metrics with weak homogeneity properties. Following a definition introduced by Coifman and Weiss [11], a triple (M, d, μ) is called a *homogeneous metric space* if d and μ are, respectively, a distance and a regular Borel measure on X, such that

$$A := \sup_{x \in M, r > 0} \frac{\mu(B_d(x,2r))}{\mu(B_d(x,r))} < \infty$$

where $B_d(x,r)$ denotes the d-balls with center x and radius r. Obviously $A \geq 1$. The real number

$$Q = \log_2 A .$$

is called the *homogeneous dimension* of (M,d,μ). It can be easily shown that

$$\mu(B_d(x,tr)) \leq A t^Q \mu(B_d(x,r))$$

for every $x \in M$, $r > 0$ e $t \geq 1$.

Let m denote the Lebesgue measure, d_e the euclidean distance and d_α the distance (11). Then

$$(\mathbb{R}^N, d_e, m), \qquad (\mathbb{R}^N, d_\alpha, m)$$

are homogeneous spaces of dimension, respectively, N and $\sum_j \alpha_j$.

Coifman and Weiss proved that in every homogeneous space a Calderon-Zygmund-type decomposition lemma holds. As a natural consequence, they developed a general theory of singular integrals which generalizes the classical elliptic and parabolic ones.

With the notion of homogeneous metric space, we are now in position to state the Macias and Segovia Theorems which generalize those of Campanato-Meyers and Da Prato. Let (M, d, μ) be a homogeneous space. If $1 \le p < \infty$ and $0 < \beta < \infty$, Macias and Segovia define the space $\mathrm{Lip}(\beta, p)$ as the class of the functions $u \in L^p_{\mathrm{loc}}(M, \mu)$ such that

$$\int_{B_d} |u - u_{B_d}|^p dx \le c(\mu(B_d))^{p\beta} .$$

They proved that every function $u \in \mathrm{Lip}(\beta, p)$ is almost everywhere equal to a function v which is Hölder continuous in the following sense:

$$|v(x) - v(y)| \le c(\mu(B_d))^\beta$$

for every ball B_d such that $x, y \in B_d$.

We end this section by quoting a paper of Lu [36], where the author studied a class of spaces containing that of Morrey-Sobolev and their generalization proposed by Barozzi.

A more particular extension of Morrey spaces was introduced by Citti and Di Fazio [7] in studying a class of Schrödinger equations related to second order sub-elliptic operators.

3. X-elliptic equations and X-control distance

The results of the previous section suggest the following general problem: given a second order partial differential operator in divergence form

$$Lu := \sum_{i,j=1}^{N} \partial_{x_j}(a_{i,j}\partial_{x_i}) \tag{12}$$

with nonnegative characteristic form

$$\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j \ge 0, \qquad \forall x, \xi \in \mathbb{R}^N \tag{13}$$

does there exist a distance d that plays for L the same role played by the euclidean distance with respect to the Laplace operator? In particular: is it possible to deduce the regularity properties of the "weak" solutions of $Lu = 0$ from the "geometric" properties of the d balls? A first rough answer to this problem is contained in a 1968 paper by Kuptsov [33]. If, roughly speaking, there exists an m family $X_1, \dots, X_m$ of C^∞ vector fields such that

(*i*) $\sum_{i,j=1}^N a_{i,j}(x)\xi_i\xi_j = \sum_{j=1}^m \langle X_j(x), \xi_j\rangle^2$

(*ii*) the Lie algebra generated by $X_1, \dots, X_m$ is free, stratified and with maximum rank at every point of $\mathbb{R}^N$,

then Kuptsov defined a metric modeled on the X_j's. Without providing a complete proof, Kuptsov stated that the Hölder continuity of the weak solutions to $Lu = 0$ can be proved by using the Moser iteration technique.

Unfortunately, one year earlier and under the only hypothesis of "maximum rank":

$$\mathrm{rank}\ \mathrm{Lie}(X_1, \dots, X_m) = N \qquad \forall x \in \mathbb{R}^N , \tag{14}$$

Hörmander [29] had proved that the weak solution to the equation

$$\sum_{j=1}^m X_j^2 u = f$$

is C^∞ if f is C^∞. Probably for this reason Kuptsov's work was almost completely ignored. Nevertheless, it contains a good idea which has been used later, independently and in different settings, by many authors.

To introduce the subject we first recall the definition of *control distance* related to a m family $X = (X_1, \dots, X_m)$ of locally Lipschitz continuous vector fields on $\mathbb{R}^N$. We will say that a piecewice regular curve $\gamma : [0,1] \to \mathbb{R}^N$ is an *X-trajectory* if there exist m functions $a_1, \dots, a_m : [0,1] \to \mathbb{R}$ such that

$$\dot\gamma(t) = \sum_{j=1}^m a_j(t) X_j(\gamma(t)), \quad \text{a.e. on} \quad [0,1].$$

We set

$$\|\gamma\| = \|\gamma\|_X := \sup_{t\in[0,1]} \left(\sum_{j=1}^m a_j^2(t)\right)^{\frac{1}{2}} .$$

Denoting by $\Gamma(X)$ the set of the X-trajectories, for every pair of points $x, y \in \mathbb{R}^N$ we define

$$d_X(x,y) = d(x,y) := \inf\{\|\gamma\| / \gamma \in \Gamma(X), \gamma(0) = x, \gamma(1) = y\} . \tag{15}$$

If $\mathbb{R}^N$ is X-connected, i.e. if the set to the right-hand side of (15) is non-empty for every $x, y \in \mathbb{R}^N$, then d is a metric called the *control* distance (or the *Carnot-Carathèodory* distance) related to $X_1, \dots, X_m$.

When the vector fields are smooth enough, a sufficient condition for the X-connectivity of $\mathbb{R}^N$ is that the linear space spanned by the commutators of the vector fields has dimension N at every point of $\mathbb{R}^N$ (Carathéodory [10], Razevskii [44], Chow [9], Hermann [27]). The Kuptsov metric previously recalled is (equivalent to) a control distance. Metrics which reflect commutation properties of C^∞ vector fields have been studied by Folland e Hung [15], Folland e Stein [22], Fefferman e Phong [21], Nagel, Stein e Wainger [42]. It should be noticed that the techniques employed in these papers always require the smoothness of the vector fields $X_1, \dots, X_m$.

In [18], for the first time, a control metric d_X related to a family of non-smooth vector fields $X_1, \dots, X_N$ was studied. It was obtained a full characterization of the geometry of the d_X-balls and a sharp estimate of their Lebesgue measure. The purpose of the paper was to prove the Hölder continuity for the weak solutions of an equation as in (12) where the operator is *"elliptic" with respect to the vector fields* $X_1, \dots, X_N$. The proof uses the Moser iteration technique by substituting the euclidean metric with the new distance d_X. In order to make this statement more precise, we need to introduce some additional notation and clarify the crucial steps of the Moser technique.

If $X_1, \dots, X_m$ is a family of vectors fields on $\mathbb{R}^N$, we say that the operator (12) is X-elliptic if there exists a constant $c > 0$ such that

$$\frac{1}{c}\sum_{j=1}^{m}\langle X_j(x), \xi_j\rangle^2 \le \sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j \le c\sum_{j=1}^{m}\langle X_j(x), \xi_j\rangle^2 .$$

We remark that every second order operator with non-negative characteristic form and sufficiently smooth coefficients is X-elliptic with respect to a suitable family X. Indeed, if the matrix (a_{ij}) is non-negative defined and $a_{ij} \in C^2$, then there exists a non-negative defined matrix (α_{ij}) with locally Lipschitz entries such that

$$\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j = \sum_{h=1}^{N}\left(\sum_{j=1}^{N}\alpha_{hj}\xi_j\right)^2, \qquad \forall \xi \in \mathbb{R}^N .$$

(Phillips and Sarason [43]). This remark tells us that X-elliptic operators fail to have good properties if the family X has no "coercitivity" properties.

The following condition seems to be a natural requirement:

$\mathbb{R}^N$ *is* X*-connected and* $(\mathbb{R}^N, d_X, m)$ *is a homogeneous metric space.*

In [18] it was proved that such condition is satisfied if

$$X = (\lambda_1\partial_{x_1}, \dots, \lambda_N\partial_{x_N})$$

and the λ_j's are non-negative functions possibly degenerating on the coordinate axis with a polynomial behavior. A particular case contained in [18] is the following:

$$X_j = \partial_{x_j}, \quad j = 1, \dots, p, \qquad Y_k = |x|^\alpha\partial_{y_k}, \quad k = 1, \dots, q,$$

where $\mathbb{R}^N = \mathbb{R}^p_x \times \mathbb{R}^q_y$ and α is any real positive number. The operator

$$L = \Delta_x + |x|^{2\alpha}\Delta_y \tag{16}$$

is $(X_1, \dots, X_p, Y_1, \dots, Y_q)$ elliptic. When α is a positive integer the operator L is contained in the class studied in [29] by Hörmander.(1)

Our definition of weak solution for a X-elliptic equation is the following. Given an open subset $M \subseteq \mathbb{R}^N$, we define

$$\langle u, v\rangle = \int_M uv + \int_M \sum_{j=1}^{m} X_j u X_j v := \int_M uv + \int_M \langle Xu, Xv\rangle$$

and denote by $W_X(M)$ and $W^0_X(M)$, respectively, the closure of $C^1(\bar M)$ and $C^1_0(M)$ with respect to the norm associated to the previous inner product. Xu stands for the X-*gradient* of u, i.e. $Xu = (X_1u, \dots, X_mu)$ where $X_ju = \langle X_j, Du\rangle$. We also denote by $W^{loc}_X(M)$ the space of functions $u \in L^2_{loc}(M)$ such that $\phi u \in W^0_X(M)$, for every $\phi \in C^1_0(M)$. The bilinear form

$$a(u, v) = \int_M \langle Xu, Xv\rangle, \qquad u \in C^1(\bar M), \quad v \in C^1_0(M)$$

can be continued to $W^{loc}_X \times W^0_X$.

(1) In a probably improper way, such operators are now commonly called of Grushin or Baouendi-Grushin type. In fact, in 1967 Baouendi [1] studied a boundary value problem for (16) on an open set contained in the half plane $x > 0$. He used a technique based on classical weighted Sobolev spaces. In 1970, Grushin studied a problem which is closest in spirit to our setting. He proved that when α is a positive integer, the hypoellipticity of the operators (16) can fail by adding lower order terms with complex coefficients. Grushin also provided a complete characterization of the hypoellipticity for such operators.

We say that a function $u \in W_X^{loc}(M)$ is a weak solution of $Lu = 0$ if $a(u,v) = 0$, for every $v \in W_X^0(M)$.

The Moser iteration technique can be straightforwardly adapted to X-*elliptic* operators if the following conditions are satisfied.

(I) $(\mathbb{R}^N, d_X, m)$ is a homogeneous metric space and the d_X-topology is equivalent to the euclidean one.

(II) For every $r > t$ there exists an X-Lipschitz continuous function ϕ supported in the ball $B_d(x,r)$, and equal to 1 on $B_d(x,t)$ such that

$$|X\phi| \leq \frac{c}{r-t},$$

c independent of r, t and x.

(III) There exists $p > 2$ such that the following Sobolev-type inequality holds:

$$\left(\frac{1}{|B|}\int_B |u|^p\right)^{\frac{1}{p}} \leq cr\left(\frac{1}{|B|}\int_B |Xu|^2\right)^2,$$

for every $u \in C_0^1(B)$and for any d_X-ball $B = B_d(x,r) \subset M$. c is independent of B and u.

(IV) For any d_X-ball $B = B_d(x,r)$ such that $\lambda B \subseteq M$ the following Poincaré-type inequality holds

$$\int_B |u - u_B| \leq cr \int_{\lambda B} |Xu|,$$

where c and λ are independent of u and B, and u_B denotes the integral average of u on B; λB stands for the ball with the same center of B and radius $r(\lambda B) = \lambda r(B)$.

We first show a consequence of properties (I)-(III).

PROPOSITION 3.1. – *Let us suppose that* (I)-(III) *are satisfied and let $u \in W_X^{loc}(M)$ be a positive weak solution to $Lu = 0$ in M. Then, there exists $\lambda > 1$ independent of u such that, if $\lambda B \subseteq M$,*

$$\sup_{tB} u \leq c_p \frac{1}{(r-t)^\nu}\left(\frac{1}{|rB|}\int_{rB} |u|^p\right)^{\frac{1}{p}}$$

and

$$\inf_{tB} u \geq c_p (r-t)^\nu \left(\frac{1}{|rB|}\int_{rB} |u|^{-p}\right)^{-\frac{1}{p}}$$

for any $p > 0$ and $\frac{1}{\lambda} < t < r < \lambda$. Moreover, by setting $w = \log u$,

$$\int_B |Xw| \leq \frac{c}{r}|B|, \qquad r = \text{radius of } B, \tag{17}$$

The constants c_p, c and ν are independent of B, u, r, and t.

Proof. – It is enough to follow the Moser procedure, as presented e.g. in [25], Section 8, by replacing in it the euclidean distance with d_X and the classical gradient with the X-gradient. □

Properties (I) and (IV), together with inequality (17) imply the following result.

PROPOSITION 3.2. – *Let us suppose that* (I) *and* (III) *are satisfied and let $u \in W_X^{loc}(M)$ be a positive weak solution to $Lu = 0$ in M. Then, there exist $c > 0$ and $\lambda > 1$ such that, if $w = \log u$*

$$\left|\left\{x \in B / |w - w_B| > \frac{1}{s}\right\}\right| \leq \frac{c}{s}|B|$$

for any d_X-ball B such that $\lambda B \subseteq M$. c and λ are independent of u and B.

Proof. – We have

$$\left|\left\{x \in B / |w - w_B| > \frac{1}{s}\right\}\right| \leq \frac{1}{s}\int_B |w - w_B| \leq \text{ (by (IV))}$$
$$\leq \frac{c}{s} r \int_{\lambda B} |Xw| \leq \text{(by (17))} \frac{c}{s}|\lambda B| \leq \text{(for the doubling condition) } \frac{c}{s}|B|. \ \square$$

Propositions 3.1 and 3.2 enable to use a Bombieri-Moser's Lemma (see [39], Lemma 2) in order to get the following proposition.

PROPOSITION 3.3. – *Let us suppose that* (I)-(IV) *are satisfied. Then, there exists $c > 0$ and $\lambda > 1$ such that*

$$\sup_B u \leq \inf_B u$$

for every $u \in W_X^{loc}(M)$ positive weak solution to $Lu = 0$ in M, and for any d_X-ball B such that $\lambda B \in M$.

Properties (I)-(IV) are not independent. The existence of cut-off functions was proved in some particular cases in [19], [8], [35]. Franchi, Serapioni and Serra Cassano [23], and Garofalo and Nhieu [24] have very recently proved that property (II) is *always a consequence* of (I). Moreover, properties (I), together with the following 1-1 Poincaré inequality

$$\int_B |u - u_B| \leq cr \int_{\lambda B} |Xu| \qquad \forall u \in C^1(\lambda \bar{B}) \tag{18}$$

imply properties (III).

A first result in this direction was proved by Saloff-Coste [46] for vector fields satisfying sub-elliptic estimates. In a general framework it has been proved by Biroli and Mosco [3]. Very interesting papers on this subject are [20] and [24]. In [20] Franchi, Lu and Wheeden proved that (I) implies the existence of representation formulas for functions as fractional integral transform of their X-gradient. From this representation formula the Poincaré-Sobolev inequality easily follows. In a very general setting, Garofalo and Nhieu [24] showed that Poincaré-Sobolev inequalities are merely a consequence of (I) and of the following weak Poincaré inequality

$$|\{x \in B \,/\, |u(x) - u_B| > t\} \leq \frac{c}{t} \int_{\lambda B} |Xu(y)| dy .$$

We also wish to mention the relevant works by Hayslasz and Hayslasz and Koskela who proved some deep connections between "distances", "Sobolev" and "Poincaré", in a general abstract setting: the metric spaces "without derivatives" ([26], [28]).

By using all these results, from Proposition 3.3 we obtain the following theorem.

THEOREM 3.4. – *Let us suppose that* (I) *and* (IV) *are satisfied. Then, there exists* $c > 0$ *and* $\lambda > 1$ *such that*

$$\sup_B u \leq \inf_B u$$

for every $u \in W_X^{loc}(M)$ *positive weak solution to* $Lu = 0$ *in* M, *and for any* d_X*-ball* B *such that* $\lambda B \subseteq M$.

In the next section, by using the notion of *almost exponential map* introduced in [34], we show a general condition on the family X which assures that conditions (I) and (IV) are satisfied.

4. X-controllable almost exponential map. Harnack inequality. Hölder continuity

Let $X = (X_1, \dots , X_m)$ be a family of locally Lipschitz continuous vector fields on $\mathbb{R}^N$. As always, we assume the X-connectivity of $\mathbb{R}^N$ and simply denote by d the X-control distance d_X. $B(x,r)$ will denote the d-ball of center x and radius r. We call *almost exponential* map on M, an open subset of $\mathbb{R}^N$, any C^1-function

$$E : M \times Q_E \longrightarrow \mathbb{R}^N ,$$

where Q_E is a neighborhood of the origin, such that $E(x,0) = x$ for every $x \in M$. We call Q_E the *maximal box* of E and assume the existence of a family of neighborhoods of the origin $(Q_E(x,r))_{x\in M, r>0}$ satisfying the following conditions:

(E1) $\bigcup_{x\in M, r>0} Q_E(x,r) \subseteq Q_E$ and $Q_E(x,r) \subseteq Q_E(x,r')$ if $r \leq r'$.

(E2) There exists $a > 0$ such that $Q_E(x,r) \subseteq Q_E(y,ar)$ if $d(x,y) \leq r$.

The almost exponential map E will be said X-controllable if there exists a measurable function

$$\gamma : M \times Q \times]0, +\infty[\dashrightarrow \mathbb{R}^N$$

with the following properties:

(C1) For any $x \in M$ and $h \in Q_E(x,r)$, $t \longmapsto \gamma(x,h,t)$ is a X-subunit path connecting x and $E(x,h)$ with a hitting time proportional to r, i.e. $\gamma(x,h,0) = x$ and $\gamma(x,h,T(x,h)) = E(x,h)$ for a suitable $T(x,h) \leq ar$. (the constant a is independent of x, h and r).

(C2) For any $(h,t) \in Q_E \times [0, +\infty[$, $x \longmapsto \gamma(x,h,t)$ is a one-to-one map having continuous first derivatives and jacobian determinant uniformly bounded away from zero, i.e.

$$b := \inf_{\Omega \times Q \times [0,T]} \left| \frac{\partial \gamma}{\partial x} \right| > 0 .$$

We call any function γ satisfying (C1) and (C2) a *control function* of E. A family $\mathcal{E}$ of X-controllable almost exponential maps will be said *complete* if for any r sufficiently small and for any $x \in M$ we can find a map $E \in \mathcal{E}$ such that:

($\mathcal{E}1$) $B(x,r) \subseteq E(x, Q_E(x,r))$;

($\mathcal{E}2$) $E(x,\cdot)$ is one-to-one on $Q_E(x,ar)$ and

$$\frac{1}{c}D(x,0) \le D_E(x,h) \le cD(x,0) \qquad \forall\, h \in Q_E(x,ar).$$

Here a is the positive constant in (E2), $c > 0$ is independent of x and r and

$$D(x,h) := \left|\det \frac{\partial}{\partial h} E(x,h)\right| .$$

From ($\mathcal{E}1$) and (C1) we easily obtain the following proposition.

PROPOSITION 4.1. – *Let $\mathcal{E}$ be a finite complete family of X-controllable almost exponential maps on M. Then there exists $r_0 > 0$ such that*

$$|B(x,r)| \le \sum_{E\in\mathcal{E}} |E(x,Q_E(x,r)| \le \frac{1}{q}|B(x,ar)| \tag{19}$$

for any $x \in M$ and $0 < r \le r_0$. Here $q = \#\mathcal{E}$ and a is a positive constant independent of x and r.

Proof. – The first inequality in (19) straightforwardly follows from ($\mathcal{E}1$). In order to show the second one it is enough to prove the inclusion $E(x,Q_E(x,r)) \subseteq B(x,ar)$. Indeed, if γ is a control function for E, for any $h \in Q(x,r)$ we have $\gamma(x,h,0) = x$, $\gamma(x,h,T(x,h)) = E(x,h)$, $T(x,h) \le a_E r$, where $a_E > 0$ only depends on E (see (C1)). Then, since $\gamma(x,h,\cdot)$ is X-subunit,

$$d(x,E(x,h)) \le T(x,h) \le a_E r .$$

Therefore, defining $a = \max\{a_E |\ E \in \mathcal{E}\}$, we have

$$Q(x,E(x,r)) \subseteq B(x,ar),$$

and the proposition is proved. □

If $\mathcal{E}$ is a finite family of X-controllable almost exponential maps, we set

$$\Lambda_{\mathcal{E}}(x,r) := \sum_{E\in\mathcal{E}} |E(x,Q_E(x,r)| .$$

Finally, we can state our main results.

THEOREM 4.2 (Harnack inequality). – *Let L be an X-elliptic operator and M an open subset of $\mathbb{R}^N$. Assume*

(H1) *There exists a finite complete family $\mathcal{E}$ of X-controllable almost exponential maps on M.*

(H2) *There exist $A > 1$ and $r_0 > 0$ such that*

$$\Lambda_{\mathcal{E}}(x,2r) \le A\ \Lambda_{\mathcal{E}}(x,r)$$

for any $x \in M$ and $0 < r < r_0$.

Then, any positive weak solution to

$$Lu = 0 \qquad in \qquad M$$

satisfies the Harnack inequality

$$\sup_B u \le c \inf_B u$$

for any d-ball B such that $\lambda B \subseteq M$. The constants c and λ are independent of u and B.

THEOREM 4.3 (Hölder continuity). – *Suppose the hypotheses of Theorem 4.2 are satisfied. Let $u \in W^{1,2}_{\mathrm{loc}}(M,X)$ be a weak solution to*

$$Lu = 0 \qquad in\ M .$$

Then, for any $x,y \in M$ such that $d(x,y) < \frac{r}{2}$ and $B(x,\lambda r) \subseteq \Omega$, we have

$$|u(x) - u(y)| \le c\left(\frac{d(x,y)}{r}\right)^{\alpha} \sup_{B(x,\lambda r)} u$$

c, λ and α are independent of u and r.

Proof. – Theorem 4.3 follows from Theorem 4.2 by a quite standard real analysis argument (see, e.g. [25] Theorem 8.22). By Theorem 3.4, in order to obtain Theorem 4.2 we have just to verify conditions (I) and (IV).

The doubling condition (I) follows from Proposition 4.1 and the hypothesis (H2). The Poincaré inequality (IV) holds thanks to Theorem 2.1 in [34]. Indeed our hypotheses together with the doubling condition for the d-balls, ensure that all the hypotheses of Theorem 2.1 in [34] are satisfied. □

We close the section by giving some applications of our main results.

EXAMPLE 4.4. – Let us consider in $\mathbb{R}^N$ a N-tuple of real functions $\lambda_1, \dots, \lambda_N$ satisfying the conditions of [18]. If we set $X_j = \lambda_j \partial_{x_j}$, $j = 1, \dots, N$, the family $X = (X_1, \dots, X_N)$ verifies the hypotheses of our Theorems 4.2 and 4.3 (see [34], Section 3). Then every weak solution to any X-elliptic equation is locally Hölder continuous with respect to d_X and, if it is positive, satisfies a *scale invariant* Harnack inequality on the d_X-balls. (We stress that the d-Hölder continuity implies the usual one since

$$d_X(x,y) \le c|x-y|^\alpha$$

for any x and y in a fixed compact $K \subseteq \mathbb{R}^N$; c and α only depending on K). These results were first proved in [19].

We explicitly remark that, in the particular case in which $\lambda_1 = \dots = \lambda_N = 1$, we obtain the classical De Giorgi's and Moser's Theorems for the classical uniformly elliptic equations.

EXAMPLE 4.5. – Let $X = (X_1, \dots, X_m)$ be a family of smooth vector fields satisfying the Hörmander condition

$$\operatorname{rank} \mathcal{L}(X_1, \dots, X_m)(x) = N \qquad \forall\, x \in \mathbb{R}^N .$$

By slightly improving a well known rapresentation theorem of the d_X-balls due to Nagel-Stein-Weinger, we can check that the hypotheses of our Theorem 4.2 are satisfied on every bounded open set $M \subseteq \mathbb{R}^N$. We directly refer to [34] for more details. Then, for the weak solutions to any X-elliptic equation, Hölder continuity and Harnack inequality hold.

These results are contained in a paper by Lu [35].

EXAMPLE 4.6. – In $\mathbb{R}^3$ we consider the pair of vector fields $X = (X_1, X_2)$ where $X_j = \partial_{x_1} + a_j \partial_{x_3}$, $j = 1, 2$. We assume $a_j \in C^1(\mathbb{R}^N, \mathbb{R})$ and

$$p : X_1 a_2 - X_2 a_1 > 0$$

at any point of $\mathbb{R}^3$ (note: $[X_1, X_2] = p\partial_{x_3}$). Due to the results of Section 5 in [34] the pair $X = (X_1, X_2)$ satisfies hypotheses (H1) and (H2) of Theorem 4.2 on every bounded open set $M \subseteq \mathbb{R}^3$.

Let us consider the operator

$$L = \operatorname{div}\,(AD)$$

where $A = [X_1, X_2][X_1, X_2]^T$, i.e.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ a & b & a^2+b^2 \end{pmatrix}$$

Since

$$< A\xi, \xi > = < X_1, \xi >^2 + < X_2, \xi >^2$$

L is an X-elliptic operator. Then, by Theorems 4.2 and 4.3 every weak solution to $Lu = 0$ is d-Hölder continuous and, if it is positive, it also satisfies the Harnack inequality. Moreover, since $d(x,y)$ is locally bounded from above by $c|x-y|^{\frac{1}{2}}$ (see [34], Section 5), d-Hölder continuity implies the classical one.

We would like to stress that L takes the following form

$$L = \partial_{x_1x_1} + \partial_{x_2x_2} + (a^2+b^2)\partial_{x_3x_3} + \partial_{x_1}(a\partial_{x_3}) + \\ + \partial_{x_2}(b\partial_{x_3}) + \partial_{x_3}(a\partial_{x_1} + b\partial_{x_3}) + \partial_{x_3}(a^2+b^2)$$

This kind of operators arises in studying the Levi-curvature equation in $\mathbb{C}^2$(see [4]).

REFERENCES

[1] M. S. BAOUENDI: "Sur une class d'operateurs elliptiques dégénérant au bord", *C.R.A.S.* Ser. A-B (1966).

[2] G. C. BAROZZI: "Su una generalizzazione degli spazi $L^{q,\tau}$ di Morrey", *Ann. Sc. Norm. Sup. Pisa* **19**, (1965).

[3] M. BIROLI, U. MOSCO: "Sobolev inequality on Homogeneous spaces", *Potential Anal.* **4**, (1995).

[4] G. CITTI: "C^∞regularity of solutions of the Levi equation", *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **15**, (1998).

[5] S. CAMPANATO: "Proprietà di inclusione per spazi di Morrey", *Ricerche di Matematica* **XII**, (1963).

[6] S. CAMPANATO: "Proprietà di hölderianità di alcune classi di funzioni", *Ann. Sc. Norm. Sup. Pisa* **XVIII**, (1964).

[7] G. CITTI, G. D. FAZIO: "Hölder continuity of the solutions for operators which are a sum of squares of vector fields plus a potential", *Proceedings of the A.M.S.* **122**, (1994).

[8] G. CITTI, N. GAROFALO, E. LANCONELLI: "Harnack' s inequality for sum of squares of vector fields plus a potential", *American J. of Math.* **115**, (1993).

[9] W. L. CHOW: "Über systeme von linearen partiellen differentialgleichungen erster ordming", *Math. Ann.* **117**, (1940).

[10] C. Carathéodory: "Untersuchungen über grundlagen der thermodynamik", *Math. Ann.* **67**, (1909).

[11] R. R. Coifman, G. Weiss: "Analyse harmonique non-commutative sur certain espaces homogénes", in "Lecture Notes in Math.", vol. 242, *Springer-Verlag*, 1971.

[12] E. De Giorgi: "Sulla differenziabilità e analiticità delle estremali degli integrali multipli", *Mem. Torino Cl. Sci. Fis. Mat. Natur.* **3**, (1957).

[13] G. D. Prato: "Spazi $L^{p,\theta}(\Omega, S)$ e loro proprietà", *Ann. Sc. Norm. Sup. Pisa* **69**, (1965).

[14] E. B. Fabes, N. Riviére: "Singular integrals with mixed homogeneity", *Studia Math.* **27**, (1966).

[15] G. Folland, H. T. Hung: "Non-isotropic Lipschitz spaces", in "Harmonic Analysis in Euclidean spaces", *Amer. Math. Soc.* Part 2, Providence, 1979.

[16] W. H. Fleming, R. Rishel: "An integral for total gradient variation", *Arch. Math.* **11**, (1960).

[17] B. Franchi, E. Lanconelli: "De Giorgi's Theorem for a class of strongly degenerate elliptic equations", *Atti Accad. Naz. Lincei* **72**, (1982).

[18] B. Franchi, E. Lanconelli: "Une métrique associeé á une classe d'operateurs elliptiques dégenéres" in Proceedings of the meeting *Linear Partial and Pseudo Differential operators, Rend. Sem. Mat. Univ. e Politec. Torino* (1982).

[19] B. Franchi, E. Lanconelli: "Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients", *Ann. Sc. Norm. Sup. Pisa* **10**, (1983).

[20] B. Franchi, G. Lu, R. L. Wheeden: "A relationship between Poincaré-Type inequalities and representation in spaces of homogeneous type", *Intl. Math. Research Notices* **1**, (1996).

[21] C. Fefferman, D. H. Phong: "Subelliptic eigenvalue problems", *Conference on Harmonic Analysis in honor of Antoni Zygmund*, vol. 2, Wadsworth, 1983.

[22] G. B. Folland, E. M. Stein: "Estimates for the $\bar{\partial}_b$ complex and analysis on the heisenberg group", *Comm. Pure Appl. Math.* **27**, (1974).

[23] B. Franchi, R. Serapioni, F. S. Cassano: "Approximation and imbedding theorems for weighted Sobolev spaces associated with Lipschitz continuous vector fields", *Bollettino dell'Unione Matematica Italiana*, (1996).

[24] N. Garofalo, D.M. Nhieu: "Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces", *Comm. Pure Appl. Math.* **49**, (1996).

[25] D. Gilbarg, N. Trudinger: "Elliptic Partial Differential Equations of Second Order", *Springer*, Berlin-Hedelberg-New York, 1983.

[26] P. Hajlasz: "Geometric approach to Sobolev space and badly degenerated elliptic equations", *GAKUTO Int. Ser., Math. Sci. Appl.* **7**, (1995).

[27] R. Hermann: "Differential geometry and the calculus of variations", *Academic Press*, New York London, 1968.

[28] P. Hajlasz, P.Koskela: "Sobolev met Poincaré", *Mem. Amer. Math. Soc.*, to appear.

[29] L.Hörmander: "Hypoelliptic second-order differential equations", *Acta Math.* **119**, (1967).

[30] D. Jerison: "The Poincaré inequality for vector fields satisfying Hörmander conditions", *Duke Math. J.* **53**, (1986).

[31] F. John, L. Nirenberg: "On functions of bounded mean oscillation", *Comm. Pure and Appl. Math.* **14**, (1961).

[32] B. F. Jones: "Singular integrals and parabolic equations", *Bull. A.M.S.* **69**, (1963).

[33] L. P. Kupcov: "Disuguaglianza di Harnack per soluzioni generalizzate di equazioni ellittiche degeneri del secondo ordine,(in russo)", *Differential'nie Uravnienija* **4**, (1968).

[34] E. Lanconelli, D. Morbidelli: "On the Poincaré inequality for vector fields", *Arkiv för Matematik*, to appear.

[35] G. Lu: "Embedding theorems on Campanato-Morrey spaces for vector fields and applications", *Approximation Theory Appl.* **14**, (1998).

[36] G. Lu: "Existence and size estimates for the Green' s functions of differential operators construncted from degenerate vector fields", *Commun. in PDE* **17**, (1992).

[37] J. Moser: "A new Proof of De Giorgi's Theorem Concerning the Regularity Problem for Elliptic Differential Equations", *Comm. Pure and Appl. Math.* **13**, (1960).

[38] J. Moser: "On Harnack's Theorem for elliptic differential equations", *Comm. Pure and Appl. Math.* **14**, (1961).

[39] J. Moser: "On a pointwise estimate for parabolic differential equations", *Comm. Pure and Appl. Math.* **24**, (1971).

[40] G. N. Meyers: "Mean oscillation over cubes and hölder continuity", *Proc. Amer. Math. Soc.* **15**, (1964).

[41] R. A. Macias, C. Segovia: "Lipschitz functions on spaces of homogeneous type", *Advances in Math.* **33**, (1979).

[42] A. Nagel, E. M. Stein, S. Wainger: "Balls and metric defined by vector fields I: basic properties", *Acta Math.* **155**, (1985).

[43] R. Phillips, L. Sarason: "Elliptic-parabolic equations of the second order", *J. Math. Mech.* **17**, (1967/68).

[44] P. K. Razevskii: "Sulla connettibilità di due punti qualunque di uno spazio non olonomo mediante una curva ammissibile", *Uc. Zap. Moskov. Gus. Ped Inst. inn, Libknehta Ser. FiZ. Mat.* **2**, (1938).

[45] E. M. Stein: "Harmonic Analysis", *Princeton Univ. Press*, 1993.

[46] L. Saloff-Coste: "A note on Poincaré, Sobolev and Harnack inequalities", *Internat. Math. Res. Notices* **2**, (1992).

[47] G. Talenti: "Best constant in Sobolev inequality", *Ann. Mat. Pura ed Applicata* **4**, (1976).

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