



A control condition for a weak Harnack inequality

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ABSTRACT

We introduce a new condition allowing to get a *weak Harnack* inequality for non-negative solutions to linear second order degenerate elliptic equations of X-elliptic type. Roughly speaking, our condition requires that the Euclidean balls of small radius are representable by means of X-controllable almost exponential maps.

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1. Introduction and main theorems

The Harnack inequality plays a crucial role in the theory of linear and nonlinear second order PDEs of elliptic type. In the last decades, however, degenerate elliptic equations with underlying Carnot–Carathéodory metric structures have appeared in literature. Mainly thanks to the works by Franchi and Lanconelli [1], Saloff-Coste [2], Garofalo and Nhieu [3], Franchi et al. [4], it is today very well known that scale invariant Harnack inequalities hold for such a new class of operators if the involved metric structures are of doubling type and the metric balls support a kind of Poincaré inequality (see below for more details). We would like to stress that these two conditions are usually quite difficult to be verified.

In this paper we introduce a new sufficient condition to have a *weak Harnack* inequality for non-negative solutions to linear second order degenerate elliptic equations of X-elliptic type.

The operators we are dealing with have the following divergence form:

$$\mathcal{L} := \sum_{i,j=1}^N \partial_{x_i} (a_{ij}(x) \partial_{x_j}) + \sum_{i=1}^N b_i(x) \partial_{x_i}, \quad (1)$$

where a_{ij} and b_i are measurable real functions in an open set $\Omega \subseteq \mathbb{R}^N$. We set $A = (a_{ij})_{i,j=1,\dots,N}$ and $b = (b_1, \dots, b_N)$.

We assume \mathcal{L} is X-elliptic in the sense first introduced in [5] and subsequently used in [6,7]. Precisely let $X = (X_1, \dots, X_m)$ be a family of vector fields in \mathbb{R}^N , $X_j = (c_{j1}, \dots, c_{jN})$, $j = 1, \dots, m$ where the c_{jk} 's are locally Lipschitz-continuous functions in \mathbb{R}^N . We say that \mathcal{L} in (1) is uniformly X-elliptic in Ω if

(E1) there exists a constant $\lambda > 0$ such that

$$\frac{1}{\lambda} \sum_{j=1}^m \langle X_j(x), \xi \rangle^2 \leq q_{\mathcal{L}}(x, \xi) \leq \lambda \sum_{j=1}^m \langle X_j(x), \xi \rangle^2 \quad \forall x, \xi \in \mathbb{R}^N, \quad (2)$$

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where $q_{\mathcal{L}}(x, \xi)$ is the characteristic form of \mathcal{L} given by

$$q_{\mathcal{L}}(x, \xi) := \langle A(x)\xi, \xi \rangle = \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j;$$

(E2) there exists a constant $\mathbf{b} \geq 0$ such that

$$\langle b(x), \xi \rangle^2 \leq \mathbf{b} \sum_{j=1}^N \langle X_j(x), \xi \rangle^2 \quad \forall x, \xi \in \mathbb{R}^N. \quad (3)$$

In (E1) and (E2) $\langle \cdot, \cdot \rangle$ denotes the usual inner product in \mathbb{R}^N .

If the family $X = (X_1, \dots, X_m)$ generates a control distance $d = d_X$ whose balls have the so called *doubling property* and support a Poincaré inequality with respect to the X -gradient, then every non-negative solution to $\mathcal{L}u = 0$ in Ω satisfies the inequality

$$\sup_{B_d} u \leq C \inf_{B_d} u \quad (4)$$

for every d -ball B_d such that the homotetic ball $2B_d$ is contained in Ω and the constant C in (4) is independent both of u and of the ball B_d (see e.g. [5,6] and the reference therein, see also [8]). This deep result, however, requires the two previous conditions, *doubling* plus *Poincaré*, which are not trivial at all to be verified. It is known that they are basically satisfied only in two particular cases:

(i) the vector fields X_j 's are smooth and satisfy the Hörmander rank condition:

$$\text{rank Lie}\{X_1, \dots, X_m\}(x) = N \quad \text{for every } x \in \Omega$$

(see [9,10]);

(ii) the vector fields X_j 's are of diagonal type i.e.

$$\begin{aligned} X_1 &= \partial_{x_1} \\ X_j &= \lambda_j(x_1, \dots, x_{j-1}) \partial_{x_j} \quad j = 2, \dots, N \end{aligned}$$

where the λ_j 's are locally Lipschitz functions satisfying some particular condition (see [1,11]).

We would also like to recall the paper [12] where a general condition for the Poincaré inequality was given. The condition in [12], however, basically requires to know the structure of the d -balls and their doubling property.

The aim of the present paper is to show that a *non scale invariant Harnack inequality* holds for \mathcal{L} if, roughly speaking, the *Euclidean balls* of small radius are representable by means of X -controllable almost exponential maps. Our approach is inspired by some ideas and results first appeared in the papers [13,12,5].

To precisely state our result we need to fix some definitions.

A path $\gamma : [0, T] \rightarrow \mathbb{R}^N$ is called X -subunit if it is absolutely continuous and

$$\langle \gamma'(t), \xi \rangle^2 \leq \sum_{j=1}^m \langle X_j(\gamma(t)), \xi \rangle^2, \quad \forall \xi \in \mathbb{R}^N, \text{ for a.e. } t \in [0, T].$$

Let $B = B(x_0, r) \subseteq \mathbb{R}^N$ be the Euclidean ball of center x_0 and radius r . We say that $E : B(x_0, r) \times B(0, R) \rightarrow \mathbb{R}^N$ is an *almost exponential map* in B of type (a, R) if the following conditions are satisfied:

(i) $E(x, \cdot)$ is C^1 and one to one on $B(0, R)$ and the Jacobian determinant $D(x, h) := |\det \frac{\partial E}{\partial h}(x, h)|$ satisfies the estimates

$$\frac{1}{a} \leq D(x, h) \leq a \quad \text{for every } x \in B \text{ and } h \in B(0, R).$$

(ii) For every $x \in B$ and $|h| \leq R$

$$\frac{1}{a}|h| \leq |E(x, h) - x| \leq a|h|.$$

(iii) For every $x \in B(x_0, r)$

$$E(x, B(0, R)) \supseteq B(x_0, r).$$

We say that the map E is X -controllable in Ω if there exists a function

$$\gamma : B(x_0, r) \times B(0, R) \times [0, T] \rightarrow \Omega$$

satisfying:

(C1) For any $(x, h) \in B(x_0, r) \times B(0, R)$, $t \mapsto \gamma(x, h, t)$ is a X -subunit path connecting x and $E(x, h)$, i.e. $\gamma(x, h, 0) = x$ and $\gamma(x, h, T(x, h)) = E(x, h)$ for a suitable $T(x, h) \leq T$.

(C2) For any $(h, t) \in B(0, R) \times [0, T]$, $x \mapsto \gamma(x, h, t)$ is a one to one map with continuous first derivatives and Jacobian bounded away from zero, i.e.

$$\inf_{\Gamma} \left| \det \frac{\partial \gamma}{\partial x} \right| \geq \frac{1}{a}, \quad \Gamma := B(x_0, r) \times B(0, R) \times [0, T].$$

(C3) For every $x \in B(x_0, r)$ and $h \in B(0, R)$,

$$|T(x, h)| \leq a|h|^\theta, \quad \theta = \frac{1}{a}.$$

Finally, we say that X *locally controls* the Euclidean balls in Ω if for every $x_0 \in \Omega$ there exists $r > 0$ and an almost exponential map E on $B(x_0, r)$ which is X -controllable in Ω .

For future references it is convenient to give another definition. We say that the operator \mathcal{L} in (1) has the *weak Harnack property* if for every compact set $K \subseteq \Omega$ there exists a positive constant $C = C(K, \Omega)$ such that

$$\sup_K u \leq C \inf_K u$$

for every non-negative weak solution to $\mathcal{L}u = 0$ in Ω . We directly refer to the next section for the definition of weak solution.

With the previous definitions we can state our main result.

Theorem 1.1. *Let \mathcal{L} be an X -elliptic operator in a bounded connected open set $\Omega \subseteq \mathbb{R}^N$. Then \mathcal{L} has the weak Harnack property if X locally controls the Euclidean balls in Ω .*

In order to show some applications of this theorem, the following propositions will be useful.

Proposition 1.2. *Let $Y = \{Y_1, \dots, Y_p\}$ be a family of smooth vector fields in an open set $\Omega \subseteq \mathbb{R}^N$. Assume the Hörmander rank condition*

$$\text{rank Lie}\{Y_1, \dots, Y_p\}(x) = N \quad \forall x \in \Omega$$

is satisfied. Then Y locally controls the Euclidean balls in Ω .

This proposition comes from the results contained in the paper [12], Section 4.

Proposition 1.2 can be also used in non-smooth cases thank to the following comparison result whose proof is quite obvious.

Proposition 1.3. *Let $X = \{X_1, \dots, X_m\}$ and $Y = \{Y_1, \dots, Y_p\}$ two systems of vector fields in $\Omega \subseteq \mathbb{R}^N$. Assume X is stronger than Y , that is: there exists a positive constant α such that*

$$\sum_{j=1}^p \langle Y_j(x), \xi \rangle^2 \leq \alpha \sum_{j=1}^m \langle X_j(x), \xi \rangle^2 \quad \forall \xi \in \mathbb{R}^N, \quad \forall x \in \Omega.$$

Then X locally controls the Euclidean balls in Ω if Y locally controls the Euclidean balls in Ω .

Combining **Propositions 1.3** and **1.2** together with **Theorem 1.1**, we obtain the following theorem as an application of our main result.

Theorem 1.4. *Let \mathcal{L} be an operator which is X -elliptic with respect to diagonal family of vector fields*

$$X = \{\lambda_1 \partial_{x_1}, \dots, \lambda_N \partial_{x_N}\}$$

where the λ_j 's are non-negative functions in a bounded open set $\Omega \subseteq \mathbb{R}^N$ such that

$$\lambda_1 \geq 1 \quad \text{and} \quad \lambda_j(x) \geq c_0 |x_1|^{\alpha_1} \cdots |x_{j-1}|^{\alpha_{j-1}} \quad \text{for every } x \in \Omega$$

for suitable positive constants c_0 and $\alpha_1, \dots, \alpha_{j-1}$. Then \mathcal{L} has the weak Harnack property.

Indeed, if m is a positive even integer such that $\alpha_j \leq m_j$ for every $j = 1, \dots, N-1$, the system

$$Y = \{\mu_1 \partial_{x_1}, \dots, \mu_N \partial_{x_N}\}$$

with $\mu_1 \equiv 1$ and $\mu_j(x) = (x_1 \cdots x_{j-1})^m$ is weaker than the system X . Moreover, an easy computation shows that

$$\text{rank Lie } Y = \{\mu_1 \partial_{x_1}, \dots, \mu_N \partial_{x_N}\}(x) = N \quad \forall x \in \mathbb{R}^N.$$

Then **Theorem 1.4** comes from **Propositions 1.3** and **1.2**. We would like to stress that the operators studied in [1,11,14] satisfy the assumptions of **Theorem 1.4**.

From **Theorem 1.1** a Liouville-type Theorem for homogeneous X -elliptic operators in principal form easily follow.

Theorem 1.5. Let \mathcal{L} be an X -elliptic operator in \mathbb{R}^N , in principal form, i.e. the coefficients b_i 's in (1) are identically zero. Assume that X is homogeneous of degree one with respect to a family of dilations in \mathbb{R}^N and that X locally controls the Euclidean ball near the origin. Then, every non-negative solution to

$$\mathcal{L}u = 0 \quad \text{in } \mathbb{R}^N$$

is constant.

A family of dilations in \mathbb{R}^N is one-parameter family $(\delta_\lambda)_{\lambda>0}$ of diagonal linear functions of the kind

$$\delta_\lambda : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad \delta_\lambda(x_1, \dots, x_N) = (\lambda^{\sigma_1}x_1, \dots, \lambda^{\sigma_N}x_N).$$

We say that $X = (X_1, \dots, X_m)$ is δ_λ -homogeneous of degree one if

$$X_j(u(\delta_\lambda(x))) = \lambda(X_j u)(\delta_\lambda(x)) \quad j = 1, \dots, m$$

for every $x \in \mathbb{R}^N$ and $\lambda > 0$, and for every smooth function u .

This Theorem will be proved in Section 5.

We mention that Liouville-type Theorems for X -elliptic operators are contained in the papers [6,7] (see also the references therein). We would like to stress that in all these papers the doubling property and the Poincaré inequality for the X -Carathéodory balls are required.

The remaining part of the paper is organized as follows. In Section 2 we fix the notion of weak solution for an X -elliptic operator. Section 3 is devoted to a preliminary result for the proof of Theorem 1.1, which will be given in Section 4. In Section 5 we prove the Liouville-type Theorem 1.5.

2. Weak solutions

Let us denote by $W(X, \Omega)$ the closure of the linear space

$$\{u \in C^1(\Omega) : \|u; W(X, \Omega)\| < \infty\}$$

with respect to the norm

$$\|u; W(X, \Omega)\| := \left(\sum_{j=1}^m \|X_j u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

$W_0(X, \Omega)$ will denote the closure of $C_0^1(\Omega)$ with respect the same norm. Obviously, the bilinear form

$$(u, v) \longmapsto a(u, v) = \int_{\Omega} \langle ADu, Dv \rangle dx, \quad D = (\partial_{x_1}, \dots, \partial_{x_N})$$

can be continued from $(C^1(\Omega) \cap W(X, \Omega)) \times C_0^1(\Omega)$ to $W(X, \Omega) \times W_0(X, \Omega)$. Moreover, the map

$$(u, v) \longmapsto \langle ADu, Dv \rangle$$

can be continued to $W(X, \Omega) \times W(X, \Omega)$. Therefore

$$\langle ADu, Dv \rangle$$

makes sense for every $u \in W(X, \Omega)$ and $v \in W(X, \Omega)$. Analogously, we can define $\sum_{j=1}^n b_j \partial_j u$ as a function of $L^2(\Omega)$ for every $u \in W(X, \Omega)$. This comes from hypothesis (3) that also gives

$$\left\| \sum_{j=1}^n b_j \partial_j u \right\|_{L^2(\Omega)}^2 \leq \mathbf{b} a(u, u).$$

We say that $u \in W^{loc}(X, \Omega)$ if $\varphi u \in W(X, \Omega)$ for every $\varphi \in C_0^1(\Omega)$. Finally, $u \in W^{loc}(X, \Omega)$ will be said a weak solution to $Lu = 0$ in Ω if

$$\mathcal{L}(u, \varphi) := a(u, \varphi) - \int_{\Omega} \left(\sum_{j=1}^n b_j \partial_j u \right) \varphi \, dx = 0$$

for every $\varphi \in C_0^1(\Omega)$.

If $\mathcal{L}(u, \varphi) \geq 0$ (≤ 0) for every $\varphi \in C_0^1(\Omega)$ we say that $\mathcal{L}u \leq 0$ (≥ 0) and that u is a weak supersolution (subsolution) to $Lu = 0$.

3. A preliminary result

In the paper [13] Franchi and Lanconelli proved that the X -elliptic operator \mathcal{L} has the weak Harnack property if the following two conditions hold:

(S) [Sobolev-type inequality] For every $x_0 \in \Omega$ there exists $p \in]2, \infty[$ and $r > 0$ such that

$$\|u\|_{L^p(B(x_0, r))} \leq c (\|Xu\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \quad (5)$$

for every $u \in C_0^1(\Omega)$, where $C = C(x_0, r, \Omega) > 0$ is independent of u .

(P) [Poincaré-type inequality] For every $x_0 \in \Omega$ there exist $r, \rho > 0$, $\rho \geq r$, such that

$$\int_{B(x_0, r) \times B(x_0, r)} |u(x) - u(y)|^2 dx dy \leq C \int_{B(x_0, \rho)} |Xu|^2 dz \quad (6)$$

for every $u \in C^1(\Omega)$, where $C = C(r, \rho) > 0$ is independent of u .

In this section we recognize that (S) and (P) follow from a *fractional embedding* inequality. Precisely we prove the following result.

Proposition 3.1. Let $B_r = B(x_0, r)$ and $B_\rho = B(x_0, \rho)$ be Euclidean balls such that $B_r \subseteq B_\rho \subseteq \overline{B_\rho}$. Assume there exists $\varepsilon \in]0, 1[$ and a positive constant C such that

$$\int_{B_r \times B_r} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\varepsilon}} dx dy \leq C \int_{B_\rho} |Xu|^2 dz \quad (7)$$

for every $u \in C^1(\Omega)$. Then (6) holds true and (5) holds for every $p \in]2, \frac{2N}{N-2\varepsilon}[$.

Proof. Inequality (6) trivially follows from (7) since

$$\int_{B_r \times B_r} |u(x) - u(y)|^2 dx dy \leq c_r \int_{B_r \times B_r} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\varepsilon}} dx dy$$

with $c_r := \sup_{x, y \in B_r} |x - y|^{N+2\varepsilon}$.

On the other hand, inequality (5) comes from a classical embedding Theorem for fractional Sobolev spaces. Indeed, if $2 < p < \frac{2N}{N-2\varepsilon}$, for every $u \in C^1(\Omega)$ one has

$$\|u\|_{L^p(B_r)} \leq C \left(\|u\|_{L^2(B_r)} + \left(\int_{B_r \times B_r} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\varepsilon}} dx dy \right)^{\frac{1}{2}} \right) \quad (8)$$

where C is independent of u (see e.g. [15], Theorem 7.57). Then (5) follows from (8) and (7). \square

4. Proof of Theorem 1.1

Since X locally controls the Euclidean balls in Ω , for every fixed $x_0 \in \Omega$ there exist $r, R > 0$ and an almost exponential map

$$E : B(x_0, r) \times B(0, R) \longrightarrow \mathbb{R}^N$$

which is X -controllable in Ω . This means that one can find a map $\gamma : B(x_0, r) \times B(0, R) \times [0, T] \longrightarrow \Omega$ satisfying conditions (C1), (C2) and (C3) in the introduction. We show that this implies inequality (7), from which, by Proposition 3.1, Theorem 1.1 will follow. First of all, thank to the properties of E , we have

$$\begin{aligned} \int_{B_r \times B_r} \frac{|u(x) - u(E(x, h))|^2}{|h|^{N+2\varepsilon}} dx dh &\geq \left(\frac{1}{a}\right)^{N+2\varepsilon} \int_{B_r} \left(\int_{B_R} \frac{|u(x) - u(E(x, h))|^2}{|x - E(x, h)|^{N+2\varepsilon}} dh \right) dx \\ &\geq \left(\text{using the change of variable } E(x, h) = y \text{ and} \right. \\ &\quad \left. \text{keeping in mind that } \left| \det \frac{\partial E}{\partial h} \right| \leq \frac{1}{a} \right) \\ &\quad \left(\frac{1}{a}\right)^{N+2\varepsilon+1} \int_{E(x, B_R)} \left(\int_{B_r} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\varepsilon}} dy \right) dx \\ &\geq \left(\frac{1}{a}\right)^{N+2\varepsilon+1} \int_{B_r \times B_r} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\varepsilon}} dx dy. \end{aligned}$$

Let us now estimates from above the left hand side of the previous sequence of inequalities. We have

$$\begin{aligned}
 \int_{|h| \leq R} \left(\int_{B_r} \frac{|u(x) - u(E(x, h))|^2}{|h|^{N+2\varepsilon}} dx \right) dh &= \int_{|h| \leq R} |h|^{-(N+2\varepsilon)} \left(\int_{B_r} \left| \int_0^{T(x, h)} \left(\frac{d}{dt} u(\gamma(x, h, t)) \right) dt \right|^2 dx \right) dh \\
 &= \int_{|h| \leq R} |h|^{-(N+2\varepsilon)} \left(\int_{B_r} \left| \int_0^{T(x, h)} \langle \nabla u(\gamma), \dot{\gamma} \rangle dt \right|^2 dx \right) dh \\
 &\leq (\text{keeping in mind that } \gamma \text{ is } X\text{-subunit and that } T(x, h) \leq a|h|^\theta) \\
 &\quad a \int_{|h| \leq R} |h|^{-(N+2\varepsilon)+\theta} \left(\int_{B_r} \left(\int_0^{a|h|^\theta} |Xu(\gamma)|^2 dt \right) dx \right) dh \\
 &= a \int_{|h| \leq R} |h|^{-(N+2\varepsilon)+\theta} \left(\int_{B_r} \left(\int_0^{a|h|^\theta} |Xu(\gamma(x, h, t))|^2 dt \right) dx \right) dh \\
 &\leq \left(\text{using the change of variable } \gamma(x, h, t) = z \text{ and} \right. \\
 &\quad \left. \text{keeping in mind that } \left| \det \frac{\partial \gamma}{\partial x} \right| \geq \frac{1}{a} \right) \\
 &\quad a^2 \int_{|h| \leq R} |h|^{-(N+2\varepsilon)+\theta} \left(\int_0^{a|h|^\theta} \left(\int_{\gamma(B_r, h, t)} |Xu(z)|^2 dz \right) dt \right) dh.
 \end{aligned}$$

On the other hand, since $|\gamma(x, h, t) - x_0| \leq \rho$ for every $(x, h, t) \in B_r \times B_r \times [0, T]$, we have

$$\gamma(B_r, h, t) \subseteq B(x_0, \rho) := B_\rho.$$

Therefore

$$\begin{aligned}
 \int_{|h| \leq R} \left(\int_{B_r} \frac{|u(x) - u(E(x, h))|^2}{|h|^{N+2\varepsilon}} dx \right) dh &\leq a^2 \int_{|h| \leq R} |h|^{-(N+2\varepsilon)+\theta} \left(\int_0^{a|h|^\theta} \left(\int_{B_\rho} |X(u)|^2 dz \right) dt \right) dh \\
 &= a^3 \int_{B_\rho} |Xu|^2 dz \int_{|h| \leq R} |h|^{-(N+2\varepsilon)+2\theta} dh \\
 &= (\text{choosing } 0 < \varepsilon < \theta) C_\varepsilon \int_{B_\rho} |Xu|^2 dz.
 \end{aligned}$$

This completes the proof of the Theorem.

5. Proof of Theorem 1.5

Let $u \geq 0$ be a weak solution to $\mathcal{L}u = 0$ in \mathbb{R}^N . Since $X = \{X_1, \dots, X_m\}$ is δ_R -homogeneous of degree one, letting

$$X_j := \sum_{k=1}^N c_{jk} \partial_{x_k}$$

we have

$$c_{jk}(\delta_R(x)) = R^{\sigma_k - 1} b_{jk}(x), \quad \text{for every } R > 0. \quad (9)$$

From the definition of weak solution we get

$$\begin{aligned}
 0 &= \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(x) \partial_{x_j} u(x) \partial_{x_i} \varphi(x) dx \\
 &= (\text{making the change of variable } x = \delta_R(y) \text{ and letting } Q = \sigma_1 + \dots + \sigma_N) \\
 &\quad R^Q \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(\delta_R(y)) (\partial_{x_j} u)(\delta_R(y)) \partial_{x_i} (\varphi(\delta_R(y))) dy.
 \end{aligned}$$

Let us now denote $u_R(y) := u(\delta_R(y))$. Then

$$\int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}(\delta_R(y)) R^{-\sigma_i - \sigma_j} \partial_{x_j} u_R(y) \partial_{x_i} (\varphi(\delta_R(y))) dy = 0.$$

If we multiply this identity by R^2 we obtain that u_R is a weak solution to

$$\mathcal{L}^{(R)} v := \sum_{i,j=1}^N \partial_{x_i} (a_{ij}^{(R)} \partial_{x_j} v) = 0$$

with $a_{ij}^{(R)}(y) = R^{2-\sigma_i-\sigma_j} a_{ij}(\delta_R(y))$. Using (9) it is easy to check that $\mathcal{L}^{(R)}$ is an X -elliptic operator with X -ellipticity constants independent of R . Then, since X locally controls the Euclidean ball in a neighborhood of the origin, there exists $r_0 > 0$ such that

$$\sup_{B(0,r_0)} u_R \leq C \inf_{B(0,r_0)} u_R, \quad C > 0 \text{ independent of } R.$$

This inequality implies

$$\sup_{B(0,Rr_0)} u \leq C \inf_{B(0,Rr_0)} u. \quad (10)$$

Since u is any non-negative solution to $\mathcal{L}u = 0$, inequality (10) also holds for $v := u - \inf_{\mathbb{R}^N} u$. Therefore

$$\sup_{B(0,Rr_0)} v \leq C \inf_{B(0,Rr_0)} v \quad \text{for every } R > 0.$$

Letting R go to infinity we obtain

$$0 \leq \sup_{\mathbb{R}^N} v \leq C \inf_{\mathbb{R}^N} v = 0.$$

Hence $v \equiv 0$ and $u \equiv \inf_{\mathbb{R}^N} u$. This completes the proof.

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