A RIGIDITY THEOREM FOR KOLMOGOROV-TYPE OPERATORS

ALESSIA E. KOGOJ AND ERMANNO LANCONELLI

To Enzo Mitidieri, on the occasion of his 70th birthday, with profound friendship and great admiration.

Abstract. Let $D \subseteq \mathbb{R}^n$, $n \geq 3$, be a bounded open set and let $x_0 \in D$. Assume that the Newtonian potential of D is proportional outside D to the Newtonian potential of a mass concentrated at $\{x_0\}$. Then D is a Euclidean ball centered at x_0. This Theorem, proved by Aharonov, Schiffer and Zalcman in 1981, was extended to the caloric setting by Suzuki and Watson in 2001. In this note, we show that Suzuki–Watson Theorem is a particular case of a more general rigidity result related to a class of Kolmogorov-type PDEs.

1. Introduction

1.1. Rigidity results in harmonic and caloric settings.

Let K be the fundamental solution with pole at the origin of the Laplacian Δ in $\mathbb{R}^n$, $n \geq 3$. We denote by $B(x_0, r)$ the Euclidean ball of $\mathbb{R}^n$ centered at x_0 with radius $r > 0$. For every $y \notin B(x_0, r)$ the function $x \longmapsto K(x - y)$ is harmonic in $B(x_0, r)$ so that, by Gauss Mean Value Theorem,

$$\int_{B(x_0,r)} K(x-y)\, dx = c\, K(x_0 - y), \tag{1.1}$$

where c is the volume of $B(x_0, r)$.

In 1981, Aharonov, Schiffer and Zalcman proved that identity (1.1) is a rigidity property of the Euclidean ball. More precisely, they proved (see [1], see also [5]) that a bounded open set D such that, for a point $x_0 \in D$ and a suitable positive constant c,

$$\int_D K(x-y)\, dx = c\, K(x_0 - y) \quad \forall\, y \notin D,$$

has to be the Euclidean ball $B(x_0, r)$ with Lebesgue measure equal to c.

Suzuki and Watson, in 2001, extended the previous theorem to the *heat balls* in $\mathbb{R}^{n+1}$, $n \geq 1$ (see [11]). To be more precise, we need some notation.

Let us denote by G the fundamental solution of the heat operator $\mathcal{H} := \Delta - \partial_t$ in $\mathbb{R}^{n+1} = \mathbb{R}^n_x \times \mathbb{R}_t$. We call *heat ball* with center $z_0 \in \mathbb{R}^{n+1}$ and $r > 0$ the following

2020 *Mathematics Subject Classification.* Primary 35H10, 35K65, 35K70, 53C24; Secondary 35K05, 35B99.

Key words and phrases. Degenerate parabolic equations, Kolmogorov-type operators, Hypoelliptic operators, Rigidity properties, Inverse Problems.

set

$$\Omega_r(z_0) := \left\{ z \in \mathbb{R}^{n+1} : G(z_0 - z) > \frac{1}{r} \right\}.$$

The caloric functions, i.e., the solutions to the heat equation

$$\mathcal{H}u = 0$$

can be characterized in terms of the following caloric mean value formula

$$u(z_0) = M_r(u)(z_0) := \frac{1}{r} \int_{\Omega_r(z_0)} u(\zeta) W(\zeta - z_0)\, d\zeta, \tag{1.2}$$

where W is the Pini–Watson kernel

$$W(\eta) = W(\xi, \tau) := \frac{1}{4} \left(\frac{|\xi|}{\tau} \right)^2, (\xi, \tau) \in \mathbb{R}^{n+1}, \tau \neq 0. \tag{1.3}$$

Indeed, a continuous function $u : O \longrightarrow \mathbb{R}$, O open subset of $\mathbb{R}^{n+1}$, is smooth and solves $\mathcal{H}u = 0$ in O if and only if

$$u(z_0) = M_r(u)(z_0)$$

for every $z_0 \in O$ and $r > 0$ such that $\overline{\Omega_r(z_0)} \subseteq O$ (see [12]).

As a consequence, for every heat ball $\Omega_r(z_0)$ and for every $z \notin \Omega_r(z_0)$, one has

$$\frac{1}{r} \int_{\Omega_r(z_0)} G(\zeta - z) W(\zeta - z_0)\, d\zeta = G(z_0 - z). \tag{1.4}$$

Indeed, if $0 < \rho < r$ and and $z \notin \Omega_r(z_0)$, $z \neq z_0$, then

$$\zeta \longmapsto G(\zeta - z)$$

is caloric in $\mathbb{R}^{n+1} \setminus \overline{\Omega_\rho(z_0)}$. Hence, by the caloric mean value formula (1.2),

$$\frac{1}{\rho} \int_{\Omega_\rho(z_0)} G(\zeta - z) W(\zeta - z_0)\, d\zeta = G(z_o - z).$$

From this identity, as $\rho \nearrow r$, one gets (1.4) in the case $z \neq z_0$. On the other hand, if $z = z_0$ identity (1.4) is trivial. Suzuki and Watson, extending Aharonov, Schiffer and Zalcman's Theorem to the caloric setting, proved that (1.4) is a rigidity property of the heat balls.

Their result reads as follows: let $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and let D be a bounded open set of $\mathbb{R}^{n+1}$. Assume that for a suitable constant $c > 0$,

$$\int_D G(\zeta - z) W(\zeta - z_0)\, d\zeta = c\, G(z_o - z) \quad \forall\, z \notin D.$$

Then, if

$$\zeta \longmapsto (\mathbb{1}_D - \mathbb{1}_{\Omega_c(z_0)})(\zeta) W(z_0 - \zeta) \in L^p \text{ for some } p > \frac{n}{2} + 1, \tag{1.5}$$

then

$$D = \Omega_c(z_0).$$

Here, and in what follows, $\mathbb{1}_E$ denotes the characteristic function: $\mathbb{1}_E(x) = 1$ if $x \in E$, $\mathbb{1}_E(x) = 0$ otherwise.

We want to stress that condition (1.5) replaces the condition $x_0 \in D$ of the harmonic case; its meaning is that D and $\Omega_c(z_0)$ are indistinguishable in the vicinity of z_0. The present authors, together with G. Tralli, in [7] partially extended this last rigidity theorem to a class of second order hypoelliptic operators containing in particular the prototype of the so called Kolmogorov operators.

In this note, we provide a full extension of Suzuki–Watson's rigidity Theorem to such a class of partial differential operators. Our technique is inspired by the one used in the paper [5] where harmonic characterizations of the Euclidean balls are proved.

1.2. **Our operators.**

We will deal with Partial Differential Operators of the type:

$$\mathcal{L} := \operatorname{div}(A\nabla) + \langle Bx, \nabla\rangle - \partial_t, \tag{1.6}$$

where $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, ∇ and $\langle\cdot,\cdot\rangle$ denote the gradient and the inner product in $\mathbb{R}^n$. $A = (a_{i,j})_{i,j=1,\dots,n}$ and $B = (b_{i,j})_{i,j=1,\dots,n}$ are $n \times n$ real constant matrices taking the following block form:

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ B_1 & 0 & \dots & 0 & 0 \\ 0 & B_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & B_r & 0 \end{bmatrix},$$

where A_0 is a $p_0 \times p_0$ $(1 \le p_0 \le n)$ symmetric and positive definite constant matrix and B_j is a $p_j \times p_{j-1}$ $(j = 1, 2, ..., r)$ block with rank equal to p_j. Moreover $p_0 \ge p_1 \ge ... \ge p_r \ge 1$ and $p_0 + p_1 + ... + p_r = n$.

We explicitly remark that the operator (1.6) becomes the heat operator if $A = \mathbb{I}_n$ - the identity matrix - and $B = 0$. In this case, with the previous notations, $p_0 = n$ and $p_1, \dots, p_r$ disappear.

It is quite well known that, under these block form assumptions on A and B, the operator $\mathcal{L}$ in (1.6) is hypoelliptic, i.e., every distributional solution u to $\mathcal{L}u = f$ is actually smooth whenever f is smooth (see [10], see also [2, Chapter 4, Section 4.3.4]). It is also well known that $\mathcal{L}$ is left translation invariant and homogeneous of degree two on the homogeneous Lie group

$$\mathbb{K} := (\mathbb{R}^{n+1}, \circ, \delta_\lambda)$$

whose composition law is the following one

$$(x, t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau),$$

with $E(\tau) = \exp(-\tau B)$; moreover the dilation $\delta_\lambda, \lambda > 0$, is the linear map from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$ whose Jacobian matrix is given by

$$D(\lambda) := \mathrm{diag}(\lambda \mathbb{I}_{p_0}, \lambda^3 \mathbb{I}_{p_1}, \ldots, \lambda^{2r+1} \mathbb{I}_{p_r}, \lambda^2),$$

being $\mathbb{I}_{p_j}$ the $p_j \times p_j$ identity matrix. We remark that

$$\det D(\lambda) = \lambda^Q,$$

with $Q := p_0 + p_1 + \ldots + (2k+1)p_r + 2$. This natural number is the *homogeneous dimension of* $\mathbb{K}$. Since $\mathbb{K}$ is a homogeneous Lie group in $\mathbb{R}^{n+1}$, the Lebesgue measure in $\mathbb{R}^{n+1}$ is left and right translation invariant on $\mathbb{K}$.

An explicit fundamental solution for (1.6) is given by

$$\Gamma(z, \zeta) := \gamma\left(\zeta^{-1} \circ z\right) \text{ for } z, \zeta \in \mathbb{R}^{n+1}, \tag{1.7}$$

where $\zeta^{-1} = (\xi, \tau)^{-1} = (-E(-\tau)\xi, -\tau)$ denotes the opposite of ζ with respect to the composition law in K and

$$\gamma(z) = \gamma(x,t) := \begin{cases} 0 & \text{for } t \le 0 \\ \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\left\langle C^{-1}(t)\, x, x\right\rangle\right) & \text{for } t > 0 \end{cases}. \tag{1.8}$$

Here $C(t)$ stands for the matrix

$$C(t) := \int_0^t E(s) A E^T(s)\, ds.$$

This matrix is strictly positive definite for every $t > 0$ and strictly negative definite for every $t < 0$ (see [10]). In the case of the heat operator, $C(t)$ is simply given by $t\mathbb{I}_n$.

The function γ, the fundamental solution of $\mathcal{L}$ with pole at the origin - the neutral element of $\mathbb{K}$ - is δ_λ-homogeneous of degree $2 - Q$, i.e.,

$$\gamma(\delta_\lambda(z)) = \lambda^{2-Q}\gamma(z) \quad \forall\, z \in \mathbb{R}^{n+1},\ \forall\, \lambda > 0.$$

1.3. **Mean Value formula.**

Let $\mathcal{L}$ be the operator (1.6). For every $z_0 \in \mathbb{R}^{n+1}$ and $r > 0$ we call $\mathcal{L}$*−ball with center* z_0 *and radius* $r > 0$ the following open set

$$\Omega_r(z_0) = \left\{ z \in \mathbb{R}^{n+1} : \Gamma(z_0, z) > \frac{1}{r} \right\}. \tag{1.9}$$

From (1.7) and (1.8), one easily verifies that $\Omega_r(z_0)$ is a non-empty bounded open set; moreover,

$$\bigcap_{r>0} \Omega_r(z_0) = \{z_0\}.$$

A continuous function $u : O \longrightarrow \mathbb{R}$, $O \subseteq \mathbb{R}^{n+1}$ open, actually is smooth in O and solves

$$\mathcal{L}u = 0 \text{ in } O$$

if and only if

$$u(z_0) = \frac{1}{r} \int_{\Omega_r(z_0)} u(\zeta) W(z_0^{-1} \circ \zeta)\ d\zeta, \tag{1.10}$$

for every $\mathcal{L}$-ball $\Omega_r(z_0)$ such that $\overline{\Omega_r(z_0)} \subseteq O$ (see [6], see also [7, Theorem 1.1]). In (1.10) the kernel W is defined as follows

$$W(z) = W(x,t) := \frac{\langle AC^{-1}(t)x, C^{-1}(t)x\rangle}{4}. \tag{1.11}$$

W is a well-defined and strictly positive almost everywhere in $\mathbb{R}^{n+1}$ smooth function. Indeed, since $A \geq 0$, $W \geq 0$ in $\mathbb{R}^{n+1} \setminus (\mathbb{R}^n \times \{0\})$. Moreover, as

$$C^{-1}(t) > 0 \quad \forall\, t > 0 \quad \text{ and } \quad C^{-1}(t) < 0 \quad \forall\, t < 0,$$

one has $W(x,t) = 0$ if and only if $x \in F_t := C(t)(\ker(A))$. Being $\operatorname{rank}(A) = p_0 > 0$, $C(t)(\ker(A))$ has dimension $n - p_0$, hence strictly less then n. It follows that F_t has n-measure equal to zero for every $t \neq 0$ and this implies that

$$F := \{(x,t) \in \mathbb{R}^{n+1} \mid W(x,t) = 0\}$$

has $n+1$-measure equal to zero.

We remark that when $\mathcal{L} = \mathcal{H}$, the kernel (1.11) becomes the Pini–Watson kernel in (1.3).

From the Mean Value formula (1.10), just proceeding as in the caloric case, one gets

$$\int_{\Omega_r(z_0)} \Gamma(\zeta, z) W(z_0^{-1} \circ \zeta)\ d\zeta = r\Gamma(z_0, z) \quad \forall\, z \notin \Omega_r(z_0). \tag{1.12}$$

1.4. **Main Theorem.**

The aim of this paper is to prove that identity (1.12) is a rigidity property of the $\mathcal{L}$-balls; equivalently, we want to extend the Suzuki–Watson Theorem to the $\mathcal{L}$-setting. Here is our main theorem in which Γ denotes the fundamental solution of the operator $\mathcal{L}$ in (1.6), $\Omega_r(z_0)$ is the $\mathcal{L}$-ball in (1.9) and Q is the homogeneous dimension of $\mathbb{K}$.

Theorem 1.1. *Let $z_0 \in \mathbb{R}^{n+1}$ and let D be a bounded open subset of $\mathbb{R}^{n+1}$ such that, for a suitable $r > 0$,*

$$\int_D \Gamma(\zeta, z) W(z_0^{-1} \circ \zeta)\ d\zeta = r\Gamma(z_0, z) \quad \forall\, z \in \mathbb{R}^{n+1} \setminus D. \tag{1.13}$$

If, moreover,

$$(\mathbb{1}_D - \mathbb{1}_{\Omega_r(z_0)}) W(z_0^{-1} \circ \cdot) \in L^p \text{ for some } p > \frac{Q}{2}, \tag{1.14}$$

then $D = \Omega_r(z_0)$.

We explicitly remark that identity (1.13) implies the inclusion

$$D \subseteq \mathbb{R}^n \times]-\infty, t_0[,$$

being $t_0 \in \mathbb{R}$ the time-component of z_0, i.e., $z_0 = (x_0, t_0)$ for a suitable $x_0 \in \mathbb{R}^n$. Indeed, the right hand side of (1.13) is equal to zero for every $z = (x,t)$ with $t \geq t_0$ while $\Gamma(\zeta, z) > 0$ if $\zeta = (\xi, \tau) \in D$ and $\tau > t$.

Due to the invariance of the Lebesgue measure with respect to the right (and left) translation on $\mathbb{K}$, identity (1.13) is equivalent to the following one

$$\int_{z_0^{-1}\circ D} \Gamma(\zeta, z) W(\zeta)\, d\zeta = r\Gamma(0, z) \quad \forall\, z \notin z_0^{-1} \circ D. \tag{1.15}$$

Then, it is enough to prove Theorem 1.1 in the case $z_0 = 0$.

Corollary 1.2. *Let $z_0 \in \mathbb{R}^{n+1}$ and let D be a bounded open subset of $\mathbb{R}^{n+1}$ such that, for a suitable $r > 0$,*

$$u(z_0) = \frac{1}{r}\int_D u(\zeta) W(z_0^{-1} \circ \zeta)\, d\zeta \tag{1.16}$$

for every non negative function u $\mathcal{L}$-harmonic in an open set containing $D \cup \{z_0\}$. Then

$$D = \Omega_r(z_0)$$

if condition (1.14) *is satisfied.*

Remark 1.3. If we replace condition (1.14) with the following stronger ones:

- (i) there exists a neighborhood V of z_0 s. t. $\Omega_r(z_0) \cap V = D \cap V$;
- (ii) $\overline{D} \setminus \{z_0\} \subset \mathbb{R}^n \times]-\infty, t_0[$, being $t_0 \in \mathbb{R}$ the time-component of z_0, i.e., $z_0 = (x_0, t_0)$ for a suitable $x_0 \in \mathbb{R}^n$;

then Theorem 1.1 becomes a particular case of Theorem 1.4 in [7].

The paper is organized as follows. In Section 2 we recall some notion and several results from Potential Analysis for the operator $\mathcal{L}$ and its adjoint $\mathcal{L}^*$. These will be the main ingredients and tools of our proof of Theorem 1.1. Section 3 will be devoted entirely to the proof of Theorem 1.1 and Section 4 to the proof of its Corollary 1.2.

2. Basic notions and results from Potential Analysis for $\mathcal{L}$ and $\mathcal{L}^*$

2.1. Harmonic and subharmonic functions for $\mathcal{L}$ and $\mathcal{L}^*$.

Let $\Omega \subseteq \mathbb{R}^{n+1}$ be open. A function $u : \Omega \to \mathbb{R}$ is called $\mathcal{L}^*$*-harmonic* -in short notation $u \in \mathcal{L}^*(\Omega)$ - if $u \in C^\infty(\Omega, \mathbb{R})$ and $\mathcal{L}^* u = 0$ in Ω. We explicitly observe that

$$\mathcal{L}^* := \operatorname{div}(A\nabla) - \langle Bx, \nabla\rangle + \partial_t,$$

and therefore, as $\mathcal{L}$, it is hypoelliptic.
Analogous meanings as above for $\mathcal{L}$-harmonic functions and for $\mathcal{L}(\Omega)$.

A bounded open set $V \subseteq \mathbb{R}^{n+1}$ is called $\mathcal{L}^*$*-regular* if for every function $\varphi \in C(\partial V, \mathbb{R})$ there exists a unique $\mathcal{L}^*$-harmonic function in V, denoted by H_φ^V, such that

$$\lim_{z\to\zeta} H_\varphi^V(z) = \varphi(\zeta) \quad \forall\, \zeta \in \partial V.$$

Analogous meaning for $\mathcal{L}$-regular set.

Let $\Omega \subseteq \mathbb{R}^{n+1}$ be open and let $u : \Omega \longrightarrow [-\infty, +\infty[$ be an upper semicontinuous function. We say that u is $\mathcal{L}^*$-subharmonic in Ω - in short notation $u \in \underline{\mathcal{L}}^*(\Omega)$ - if it satisfies the following conditions:

(i) $u > -\infty$ in a dense subset of Ω;
(ii) for every $\mathcal{L}^*$-regular open set V with $\overline{V} \subseteq \Omega$ and for every $\varphi \in C(\partial V, \mathbb{R})$ such that $u|_{\partial V} \leq \varphi$ one has $u \leq H_\varphi^V$ in V.

We shall denote by $\overline{\mathcal{L}}^*(\Omega)$ the family of the $\mathcal{L}^*$-superharmonic functions, i.e., the family of the functions v such that $-v \in \underline{\mathcal{L}}^*(\Omega)$.

2.2. **Maximum principle for $\mathcal{L}^*$-subharmonic functions.**
Let $\Omega \subseteq \mathbb{R}^{n+1}$ be a bounded open set and let $u \in \underline{\mathcal{L}}^*(\Omega)$ be such that

$$\limsup_{z \to \zeta} u(z) \leq 0 \quad \forall\, \zeta \in \partial\Omega.$$

Then

$$u \leq 0 \text{ in } \Omega$$

(see e.g. [3, Proposition 2.3]).

2.3. **Propagation of maxima along drift-trajectories.**
We call *drift-trajectory* of $\mathcal{L}^*$ any path of the type

$$s \longmapsto \gamma(s) := \alpha + s e_j \quad \text{ or } \quad s \longmapsto \gamma(s) := \alpha - s e_j,$$

where $0 \leq s \leq S, \quad \alpha \in \mathbb{R}^{n+1}, \quad e_j = (0, \dots, 0, \underset{j}{1}, 0, \dots, 0), \quad 0 \leq j \leq p_0.$

Then, if $u \in \overline{\mathcal{L}}^*(\Omega), \Omega$ open subset of $\mathbb{R}^{n+1}$, and $z_0 \in \Omega$ is such that

$$u(z_0) = \max_\Omega u,$$

then $u(\gamma(s)) = u(z_0)$ for every drift-trajectory $\gamma : [0, S] \longrightarrow \Omega$ such that $\gamma(0) = z_0$ (see e.g. [4], [8]).

2.4. **Γ-potentials.** Let Γ be the fundamental solution of $\mathcal{L}$ defined in (1.7). and μ be a non-negative Radon measure with compact support. We let $\Gamma_\mu : \mathbb{R}^{n+1} \longrightarrow [0, \infty]$,

$$\Gamma_\mu(z) := \int_{\mathbb{R}^{n+1}} \Gamma(z, \zeta)\, d\mu(\zeta).$$

Then, see e.g. [3, Proposition 4.1],

$$\Gamma_\mu \in \overline{\mathcal{L}}^*(\mathbb{R}^{n+1})$$

and

$$\mathcal{L}^* \Gamma_\mu = -\mu \quad \text{in the weak sense of the distributions.}$$

In particular Γ_μ is $\mathcal{L}^*$-harmonic in $\mathbb{R}^{n+1} \setminus \operatorname{supp}\mu$. We want to explicitly remark that $\mathbb{R}^{n+1} \setminus \operatorname{supp}\mu$ is the open set union of the family of the open sets O such that $\mu(O) = 0$.

2.5. **An inequality for the Γ-potentials of the $\mathcal{L}$-balls.**
For fixed $z_0 \in \mathbb{R}^{n+1}$ and $r > 0$, denote by μ the Radon measure in $\mathbb{R}^{n+1}$ such that

$$d\mu(\zeta) = \frac{1}{r} \mathbb{1}_{\Omega_r(z_0)}(\zeta) W(z_0^{-1} \circ \zeta)\ d\zeta, \tag{2.1}$$

where $\Omega_r(z_0)$ is the $\mathcal{L}$-ball centered at z_0 with radius r and W is the kernel (1.11). We have already noticed - see (1.12) - that

$$\Gamma_\mu(z) = \Gamma(z_0, z) \quad \forall\, z \notin \Omega_r(z_0).$$

From Corollary 3.2 in [9], we also get the following inequality

$$\Gamma_\mu(z) < \Gamma(z_0, z) \quad \forall\, z \in \Omega_r(z_0). \tag{2.2}$$

This inequality will play a crucial rôle in the proof of the Theorem 1.1. Here, we stress that, together with (1.11), it implies

$$\Gamma_\mu(z) \leq \Gamma(z_0, z) \quad \forall\, z \in \mathbb{R}^{n+1}. \tag{2.3}$$

2.6. **A convolution continuity result.**
Since γ is δ_λ-homogenous of degree $2 - Q$ we have

$$\gamma \in L^q_{\text{loc}}(\mathbb{R}^{n+1}) \quad \text{if } q(Q-2) < Q$$

or, equivalently, if

$$0 < \frac{1}{p} := 1 - \frac{1}{q} < \frac{2}{Q}.$$

Then, if $f \in L^p(\mathbb{R}^{n+1})$ for some $p > \dfrac{Q}{2}$ and, moreover, the support of f is compact, the function

$$z \longmapsto \Gamma(z) := \int_{\mathbb{R}^{n+1}} \Gamma(\zeta, z) f(\zeta)\ d\zeta$$

is a well-defined real continuous function in $\mathbb{R}^n$. The proof of this statement is completely standard if one remarks that, being the Lebesgue measure translation invariant in $\mathbb{K}$,

$$\int_{\mathbb{R}^{n+1}} \Gamma(\zeta, z) f(\zeta)\ d\zeta = \int_{\mathbb{R}^{n+1}} \gamma(z^{-1} \circ \zeta) f(\zeta)\ d\zeta = \int_{\mathbb{R}^{n+1}} \gamma(\eta) f(z \circ \eta)\ d\eta.$$

3. Proof of Theorem 1.1

As we already noticed, it is not restrictive to assume $z_0 = 0$. Let μ be the compactly supported Radon measure defined in (2.1), and let ν be the measure such that

$$d\nu(\zeta) = \frac{1}{r} \mathbb{1}_D(\zeta) W(z_0^{-1} \circ \zeta)\ d\zeta. \tag{3.1}$$

Since D is bounded, ν is also a compactly supported Radon measure. For our aims, it is crucial to remark that $\text{supp}\nu = \overline{D}$, hence

$$\partial D \subseteq \text{supp}\nu. \tag{3.2}$$

It is also crucial for us to remark that

$$\mu|_{\Omega_r(z_0) \cap D} = \nu|_{\Omega_r(z_0) \cap D}. \tag{3.3}$$

Let Γ_μ and Γ_ν be the Γ-potentials of μ and ν respectively, i.e.,

$$\Gamma_\mu(z) = \int_{\mathbb{R}^{n+1}} \Gamma(\zeta, z)\ d\mu(\zeta) = \frac{1}{r} \int_{\Omega_r(z_0)} \Gamma(\zeta, z) W(z_0^{-1} \circ \zeta)\ d\zeta, \tag{3.4}$$

and

$$\Gamma_\nu(z) = \int_{\mathbb{R}^{n+1}} \Gamma(\zeta, z)\ d\nu(\zeta) = \frac{1}{r} \int_{D} \Gamma(\zeta, z) W(z_0^{-1} \circ \zeta)\ d\zeta. \tag{3.5}$$

From (1.12) and assumption (1.13), we have

$$\Gamma_\mu(z) = \Gamma(z_0, z) \quad \forall\, z \notin \Omega_r(z_0) \tag{3.6}$$

and

$$\Gamma_\nu(z) = \Gamma(z_0, z) \quad \forall\, z \notin D. \tag{3.7}$$

Therefore,

$$\Gamma_\mu(z) = \Gamma_\nu(z) \quad \forall\, z \in \mathbb{R}^{n+1} \setminus (\Omega_r(z_0) \cup D). \tag{3.8}$$

Let us remark that

$$(\Gamma_\mu - \Gamma_\nu)(z) = \int_{\mathbb{R}^{n+1}} \Gamma(z, \zeta) f(\zeta)\ d\zeta,$$

where

$$f(\zeta) := \frac{1}{r} (\mathbb{1}_{\Omega_r(z_0)} - \mathbb{1}_D)(\zeta) W(z_0^{-1} \circ \zeta).$$

From assumption (1.14), the function $f \in L^p(\mathbb{R}^{n+1})$ for a suitable $p > \frac{Q}{2}$. Furthermore, as $\Omega_r(z_0)$ and D are bounded, f has compact support. Then,

$$\Gamma_\mu - \Gamma_\nu \in C(\mathbb{R}^{n+1}, \mathbb{R}) \tag{3.9}$$

(see Subsection 2.6). Then, since $\Gamma\mu$ is finite at any point (see (2.3)), we have

$$\Gamma_\nu(z) < \infty \quad \forall\, z \in \mathbb{R}^{n+1}.$$

With all these results at hand, we can prove Theorem 1.1 with a procedure inspired by the one used in [5]. We split our procedure into several steps in which we simply denote by Ω the $\mathcal{L}$-ball $\Omega_r(z_0)$.

Step I. The aim of this step is to prove the inequality

$$\Gamma_\mu \leq \Gamma_\nu \text{ in } \mathbb{R}^{n+1} \setminus \Omega. \tag{3.10}$$

To this end, keeping in mind (3.8) and remarking that

$$\mathbb{R}^{n+1} \setminus \Omega \subseteq (\mathbb{R}^{n+1} \setminus (\Omega \cup D)) \cup D,$$

it is enough to prove that

$$\Gamma_\mu(z) \leq \Gamma_\nu(z) \quad \forall\, z \in D. \tag{3.11}$$

Define

$$\mu_1 := \mu|_{\Omega\setminus D} \quad \text{and} \quad \nu_1 := \nu|_{D\setminus\Omega}. \tag{3.12}$$

Then μ_1 and ν_1 are compactly supported Radon measures whose Γ-potentials are finite at any point of $\mathbb{R}^{n+1}$. If $z \in D$ we have

$$\begin{aligned}(\Gamma_\mu - \Gamma_\nu)(z) &= \int_{\Omega\cap D} \Gamma(z,\zeta)\ d\mu(\zeta) + \int_{\Omega\setminus D} \Gamma(z,\zeta)\ d\mu(\zeta) \\ &\quad - \left(\int_{D\cap\Omega} \Gamma(z,\zeta)\ d\nu(\zeta) + \int_{D\setminus\Omega} \Gamma(z,\zeta)\ d\nu(\zeta) \right),\end{aligned}$$

so that, keeping in mind that

$$\mu|_{\Omega\cap D} = \nu|_{\Omega\cap D}$$

(see (3.3)), we get

$$(\Gamma_\mu - \Gamma_\nu)(z) = (\Gamma_{\mu_1} - \Gamma_{\nu_1})(z) \quad \forall\, z \in D. \tag{3.13}$$

We let

$$\hat{u} := \Gamma_{\mu_1} - \Gamma_{\nu_1}. \tag{3.14}$$

Since D is open, one has the following inclusions:

$$D \cap \operatorname{supp} \mu_1 \subseteq D \cap (\overline{\Omega \setminus D}) \subseteq D \cap (\mathbb{R}^n \setminus D) = \emptyset.$$

As a consequence,

$$\Gamma_{\mu_1} \text{ is } \mathcal{L}^*\text{-harmonic in } D$$

(see Subsection 2.4). It follows that

$$\hat{u} \in \underline{\mathcal{L}}^*(D).$$

We claim that

$$\limsup_{D\ni z\to\zeta} \hat{u}(z) \le 0 \quad \forall\, \zeta \in \partial D. \tag{3.15}$$

Taking this claim for granted, for a moment, from the Maximum Principle for $\mathcal{L}^*$-subharmonic functions (see Subsection 2.2), we get

$$\hat{u} \le 0 \text{ in } D.$$

Then, by (3.13) and (3.14),

$$\Gamma_\mu - \Gamma_\nu \le 0 \text{ in } D.$$

This proves (3.11), hence (3.10).

We are left with the proof of claim (3.15). Keeping again in mind (3.14) and (3.13), and using the continuity of the function $\Gamma_\mu - \Gamma_\nu$ - see (3.9) - for every $\zeta \in \partial D$ we have

$$(3.16)\quad \limsup_{D\ni z\to\zeta} \hat{u}(z) = \limsup_{D\ni z\to\zeta}(\Gamma_{\mu_1}(z)-\Gamma_{\nu_1}(z)) = \limsup_{D\ni z\to\zeta}(\Gamma_{\mu}(z)-\Gamma_{\nu}(z)) = \lim_{z\to\zeta}(\Gamma_{\mu}(z)-\Gamma_{\nu}(z)) = \Gamma_{\mu}(\zeta)-\Gamma_{\nu}(\zeta).$$

On the other hand, by (2.3), $\Gamma_\mu(\zeta) \le \Gamma(z_0,\zeta)$, while, since $\zeta \in \partial D$ so that $\zeta \notin D$, from (3.7) it follows $\Gamma_\nu(\zeta) = \Gamma(z_0,\zeta)$. Using this information in (3.16) we finally get

$$\limsup_{D\ni z\to\zeta} \hat{u}(z) \le 0.$$

This proves the claim (3.15) and completes the proof of (3.10).

Step II. The aim of this step is the proof of the inclusion

$$(3.17)\qquad D \subseteq \Omega.$$

Since supp $\mu = \overline{\Omega}$, the potential function Γ_μ is $\mathcal{L}^*$-harmonic in $\mathbb{R}^{n+1}\setminus\overline{\Omega}$. Therefore

$$(3.18)\qquad v := \Gamma_\mu - \Gamma_\nu \in \underline{\mathcal{L}}^*(\mathbb{R}^{n+1}\setminus\overline{\Omega}).$$

We know, by *Step I*, that

$$(3.19)\qquad v \le 0 \text{ in } \mathbb{R}^{n+1}\setminus\overline{\Omega}.$$

On the other hand, by (3.8),

$$(3.20)\qquad v \equiv 0 \text{ in } \mathbb{R}^{n+1}\setminus(\Omega\cup D).$$

Now, let $z = (x,t)$ be an arbitrary point of $\mathbb{R}^{n+1}\setminus\overline{\Omega}$. From the very definition of Ω ($=\Omega_r(z_0)$ and $z_0 = 0$) the bounded subset of $\mathbb{R}^n$

$$\Omega_t := \left\{\xi\in\mathbb{R}^n \mid (\xi,t)\in\overline{\Omega}\right\}$$

is empty or convex. As a consequence, for every fixed $j\in\{1,\dots,p_0\}$,

$$z + se_j \in \mathbb{R}^{n+1}\setminus\overline{\Omega} \quad \forall\, s\ge 0$$

or

$$z - se_j \in \mathbb{R}^{n+1}\setminus\overline{\Omega} \quad \forall\, s\ge 0.$$

To fix ideas, let us suppose that the first case occurs. Then, since $\Omega\cup D$ is bounded, there exists $S>0$ such that

$$z^* := z + Se_j \in \mathbb{R}^{n+1}\setminus(\Omega\cup D)$$

and

$$z + se_j \in \mathbb{R}^{n+1}\setminus\overline{\Omega} \quad \forall\, s\in[0,S].$$

Then, by (3.20) and (3.19),

$$v(z^*) = 0 = \max_{\mathbb{R}^{n+1}\setminus\overline{\Omega}} v.$$

The propagation of maxima for $\mathcal{L}^*$-subharmonic functions (see Subsection 2.3) implies that

$$v(z + se_j) = v(z^*) = 0 \quad \forall\, s \in [0, S].$$

In particular, for $s = 0$, we get $v(z) = 0$. Since z is an arbitrary point of $\mathbb{R}^{n+1} \setminus \overline{\Omega}$, we have so proved that

$$v \equiv 0 \text{ in } \mathbb{R}^{n+1} \setminus \overline{\Omega}.$$

This means that

$$\Gamma_\mu = \Gamma_\nu \text{ in } \mathbb{R}^{n+1} \setminus \overline{\Omega}.$$

On the other hand, as we have already observed, Γ_μ is $\mathcal{L}^*$-harmonic in $\mathbb{R}^{n+1} \setminus \overline{\Omega}$. As a consequence,

$$\nu = -\mathcal{L}^*(\Gamma_\nu) = -\mathcal{L}^*(\Gamma_\mu) = 0 \text{ in } \mathbb{R}^{n+1} \setminus \overline{\Omega},$$

i.e.,

$$\nu(\mathbb{R}^{n+1} \setminus \overline{\Omega}) = 0,$$

or, equivalently,

$$\text{supp } \nu \subseteq \overline{\Omega}.$$

Then $D \subseteq \overline{\Omega}$ since supp $\nu = \overline{D}$. As a consequence, $D \subseteq \text{int}(\overline{\Omega})$. On the other hand, keeping in mind that γ is δ_λ-homogeneous of degree $2 - Q$, it is easy to show that

$$\text{int}(\overline{\Omega}) = \Omega.$$

Hence, from the last inclusion, we get

$$D \subseteq \Omega.$$

Step III. In this final step we prove that

$$D = \Omega. \tag{3.21}$$

We argue by contradiction and assume $D \neq \Omega$. In this case, since $D \subseteq \Omega$ by *Step II*, there exists $z \in \Omega$ such that $z \notin D$. Then, by inequality (2.2)

$$\Gamma(z_0, z) > \Gamma_\mu(z). \tag{3.22}$$

On the other hand, since $D \subseteq \Omega$, from (3.3) we get

$$\mu|_D = \nu. \tag{3.23}$$

Moreover, since $z \notin D$,

$$\Gamma_\nu(z) = \Gamma(z_o, z). \tag{3.24}$$

Putting (3.22), (3.23) and (3.24) together we have

$$\begin{aligned} \Gamma(z_0, z) &> \Gamma_\mu(z) = \int_\Omega \Gamma(\zeta, z)\, d\mu(\zeta) \\ &\geq \int_D \Gamma(\zeta, z)\, d\mu(\zeta) = \int_D \Gamma(\zeta, z)\, d\nu(\zeta) \\ &= \Gamma_\nu(z) = \Gamma(z_o, z), \end{aligned}$$

that is $\Gamma(z_0, z) > \Gamma(z_0, z)$. This contradiction proves (3.21) and completes the proof of Theorem 1.1.

4. Proof of Corollary 1.2

We first remark that Corollary's assumptions imply

$$D \subseteq \mathbb{R}^n \times]-\infty, t_0[, \tag{4.1}$$

if t_0 is the time component of z_0 (i.e. $z_0 = (x_0, t_0)$ for a suitable $x_0 \in \mathbb{R}^n$). Indeed, assume by contradiction that

$$D \cap (\mathbb{R}^n \times [t_0, \infty[) \neq \emptyset.$$

Then, since D is bounded, there exists $z^* = (x^*, t^*) \notin D$ such that $, t^* > t_0$ and

$$D^* := D \cap (\mathbb{R}^n \times]t^*, \infty[) \neq \emptyset.$$

Let us now consider the function

$$u^*(\zeta) := \Gamma(\zeta, z^*), \ \zeta \in \mathbb{R}^{n+1}.$$

Since $z^* \notin D, u^*$ is $\mathcal{L}$-harmonic and nonnegative in D. Moreover

$$u^*(z_0) = \Gamma(z_0, z^*) = 0 \quad \text{and} \quad u^* > 0 \text{ in } D^*.$$

Then, by assumption (1.16)

$$u^*(z_0) = \frac{1}{r} \int_D u^*(\zeta) W(z_0^{-1} \circ \zeta) \, d\zeta.$$

This is a contradiction since $u^*(z_0) = 0$ and

$$\int_D u^*(\zeta) W(z_0^{-1} \circ \zeta) \, d\zeta \geq \int_{D^*} u^*(\zeta) W(z_0^{-1} \circ \zeta) \, d\zeta > 0.$$

This contradiction proves the inclusion (4.1).

Let us now observe that, for every $z \notin D, z \neq z_0$, the function

$$\zeta \longmapsto \Gamma(\zeta, z)$$

is non-negative and $\mathcal{L}$-harmonic in an open set, precisely $\mathbb{R}^{n+1} \setminus \{z\}$, containing $D \cup \{z_0\}$. Then by assumption (1.16)

$$\int_D \Gamma(\zeta, z) W(z_0^{-1} \circ \zeta) \, d\zeta = r \Gamma(z_0, z) \tag{4.2}$$

for every $z \notin D, \ z \neq z_0$. On the other hand, identity (4.2) is trivial if $z = z_0$ since $\Gamma(z_0, z_0) = 0$ and, being $D \subseteq \mathbb{R}^n \times]-\infty, t_0[$, $\Gamma(\zeta, z_0) = 0$ for every $\zeta \in D$ so that

$$\int_D \Gamma(\zeta, z) W(z_0^{-1} \circ \zeta) \, d\zeta = 0.$$

We have so proved that D satisfies the assumption of Theorem 1.1, keeping in mind that we are assuming condition (1.14). Then, by Theorem 1.1,

$$D = \Omega_r(z_0).$$

Acknowledgment. The first author has been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] D. Aharonov, M. M. Schiffer, and L. Zalcman. Potato kugel. *Israel J. Math.*, 40(3-4):331–339, 1981.

[2] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians.* Springer Monographs in Mathematics. Springer, Berlin, 2007.

[3] C. Cinti and E. Lanconelli. Riesz and Poisson-Jensen representation formulas for a class of ultraparabolic operators on Lie groups. *Potential Anal.*, 30(2):179–200, 2009.

[4] C. Cinti, K. Nyström, and S. Polidoro. A note on Harnack inequalities and propagation sets for a class of hypoelliptic operators. *Potential Anal.*, 33(4):341–354, 2010.

[5] G. Cupini and E. Lanconelli. On the harmonic characterization of domains via mean formulas. *Matematiche (Catania)*, 75(1):331–352, 2020.

[6] G. Cupini and E. Lanconelli. On mean value formulas for solutions to second order linear PDEs. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 22(2):777–809, 2021.

[7] A. E. Kogoj, E. Lanconelli, and G. Tralli. An inverse mean value property for evolution equations. *Adv. Differential Equations*, 19(7-8):783–804, 2014.

[8] A. E. Kogoj and S. Polidoro. Harnack inequality for hypoelliptic second order partial differential operators. *Potential Anal.*, 45(3):545–555, 2016.

[9] A. E. Kogoj and G. Tralli. Blaschke, Privaloff, Reade and Saks theorems for diffusion equations on Lie groups. *Potential Anal.*, 38(4):1103–1122, 2013.

[10] E. Lanconelli and S. Polidoro. On a class of hypoelliptic evolution operators. *Rend. Sem. Mat. Univ. Politec. Torino*, 52(1):29–63, 1994. Partial differential equations, II (Turin, 1993).

[11] N. Suzuki and N. A. Watson. A characterization of heat balls by a mean value property for temperatures. *Proc. Amer. Math. Soc.*, 129(9):2709–2713, 2001.

[12] N. A. Watson. *Introduction to Heat Potential Theory.* Mathematical Surveys and Monographs vol.182, Amer. Math. Soc., Providence RI, 2012.

Dipartimento di Scienze Pure e Applicate (DiSPeA), Università degli Studi di Urbino Carlo Bo, Piazza della Repubblica 13, 61029 Urbino (PU), Italy.
Email address: alessia.kogoj@uniurb.it

Dipartimento di Matematica, Alma Mater Studiorum Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy.
Email address: ermanno.lanconelli@unibo.it