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Blaschke, Privaloff, Reade and Saks Theorems for Diffusion Equations on Lie Groups

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Abstract We prove some asymptotic characterizations for the subsolutions to a class of diffusion equations on homogeneous Lie groups. These results are the diffusion counterpart of the classical Blaschke, Privaloff, Reade and Saks Theorems for harmonic functions.

Keywords Diffusion equations · Ultraparabolic equations · Subsolutions · Mean-value operators · Homogeneous Lie groups

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1 Introduction

In this paper we are interested in characterizing the subsolutions of certain hypoelliptic ultraparabolic operators with underlying homogeneous Lie group structures. This class of operators has been introduced by the first author and Lanconelli in [10] and it contains, as particular cases, the heat operators on Carnot groups and the Kolmogorov type operators first studied in [14]. It also contains the operators constructed with the *link procedure* introduced in [11]. A characterization of such subsolutions in terms of suitable mean-value operators has been already singled out by Cinti in [5], who extended to our setting several results obtained by Watson for the classical heat operator [25, 26, 28, 29]. We would like to pursue this approach further by using some *asymptotic average operators*.

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For the Laplace operator this kind of asymptotic characterization has a long history, starting with the papers by Blaschke [2] and Privaloff [19]. Beckenbach and Radó in [1] characterized the continuous subsolutions of the Laplacian by comparing the solid and the surface average (see also the recent paper [7]). In [23], Saks proved a relation between the operators of Blaschke and Privaloff and the symmetric derivative of the *mass distribution* associated to a subharmonic function (see also [20] and [18]): this result is a kind of second order differentiability almost everywhere for subharmonic functions. Finally, in [21] Reade introduced his asymptotic version of the Beckenbach-Radó condition. Recently, these classical results have been extended to a wide class of sub-elliptic operators by Bonfiglioli and Lanconelli in [3]. In the parabolic setting, some of these conditions were first investigated by Pini, who proved for the heat operator Blaschke and Privaloff type conditions in [16] and the analogous of the Saks theorem in [17]. We would like to quote also the papers [26] and [29] where some asymptotic behaviors of heat averages have been established.

The present paper is organized as follows. In Section 2 we recall the structure of the operators and we state the main results. In Section 3 we show some properties of the mean-value operators and a Nevanlinna-type theorem is proved. In Section 4 we prove the asymptotic characterizations. We exploit a technique which finds its origins in a paper by Kozakiewicz [12] and it has been recently used in [3]. In Section 5 we prove our Saks-type theorem. To do this, the homogeneity properties of our setting play a key role. Finally, in the Appendix we give an explicit proof of some characterizations of the subsolutions already present in the literature.

2 Definitions and Statement of the Main Results

We consider a linear second order operator of the following type

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0 - \partial_t \qquad \text{in } \mathbb{R}^{N+1}.$$

In our notations, a generic point of $\mathbb{R}^{N+1}$ is denoted by $z = (x, t)$ ($x \in \mathbb{R}^N$, $t \in \mathbb{R}$) and the X_j's are smooth vector fields depending just on the x-variables, i.e.

$$X_j = \sum_{k=1}^{N} \alpha_{jk}(x)\partial_{x_k}, \qquad j = 0, \dots, m,$$

where $\alpha_{jk} \in C^\infty(\mathbb{R}^N, \mathbb{R})$. We also define the *drift* Y as the vector field in $\mathbb{R}^{N+1}$

$$Y := X_0 - \partial_t.$$

As in the paper [10], let us make the following two assumptions on $\mathcal{L}$:

(A1) there exists a homogeneous Lie group in $\mathbb{R}^{N+1}$,

$$\mathbb{L} = (\mathbb{R}^{N+1}, \circ, d_\lambda)$$

such that

$X_1, \dots, X_m, Y$ are left translation invariant on $\mathbb{L}$;
$X_1, \dots, X_m$ are d_λ-homogeneous of degree one, whereas the vector field Y is d_λ-homogeneous of degree two.

(A2) For every $(x, t), (y, \tau) \in \mathbb{R}^{N+1}$, $t > \tau$, there exists an *$\mathcal{L}$-admissible* path $\eta : [0, T] \longrightarrow \mathbb{R}^{N+1}$ such that $\eta(0) = (x, t)$, $\eta(T) = (y, \tau)$. The curve η is called $\mathcal{L}$-admissible if it is absolutely continuous and satisfies

$$\eta'(s) = \sum_{j=1}^{m} \lambda_j(s) X_j(\eta(s)) + \mu(s) Y(\eta(s)) \qquad \text{a.e. in } [0, T]$$

for suitable piecewise constant real functions $\lambda_1, \dots, \lambda_m$, and μ, $\mu \geq 0$.

We recall that the hypothesis (A2) implies the Hörmander condition

$$\text{rank Lie}\{X_1, \dots, X_m, Y\}(z) = N + 1 \text{ for every } z \in \mathbb{R}^{N+1}$$

and therefore the hypoellipticity of $\mathcal{L}$. The previous assumptions yield also that the composition law $\circ$ is euclidean in the t-variable, i.e.

$$(x, t) \circ (y, \tau) = (S(x, t, y, \tau), t + \tau)$$

for a suitable smooth function S with values in $\mathbb{R}^N$, and the dilation d_λ takes the form

$$d_\lambda(x, t) = (D_\lambda(x), \lambda^2 t) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_N} x_N, \lambda^2 t)$$

for some positive integers $\sigma_1, \dots, \sigma_N$. The natural number

$$Q = \sum_{k=1}^{N} \sigma_k + 2$$

is the homogeneous dimension of $\mathbb{L}$ with respect to d_λ.

If Ω is an open subset of $\mathbb{R}^{N+1}$, we say that a smooth function $u : \Omega \longrightarrow \mathbb{R}$ satisfying $\mathcal{L}u = 0$ is *$\mathcal{L}$-harmonic* in Ω. Moreover, we say that a bounded open set $V \subset \mathbb{R}^{N+1}$ is *$\mathcal{L}$-regular* if, for any $\varphi \in C(\partial V)$, there exists a unique $\mathcal{L}$-harmonic function H_φ^V such that

$$\lim_{z \to z_0} H_\varphi^V(z) = \varphi(z_0) \qquad \text{for every} \quad z_0 \in \partial V.$$

Finally, we say that an upper semi-continuous (u.s.c) function $u : \Omega \longrightarrow [-\infty, +\infty[$ is *$\mathcal{L}$-subharmonic* in Ω if u is finite in a dense subset of Ω and if, for every $\mathcal{L}$-regular set $V \subset \overline{V} \subset \Omega$, we have

$$u \leq H_\varphi^V \quad in \quad V$$

for every $\varphi \in C(\partial V)$ such that $\varphi \geq u$ on ∂V. We denote by $\underline{\mathcal{S}}(\Omega)$ the family (actually, the cone) of the $\mathcal{L}$-subharmonic functions in Ω. Concerning the functions u of class C^2 in Ω, we recall that $u \in \underline{\mathcal{S}}(\Omega)$ if and only if $\mathcal{L}u \geq 0$ in Ω by the maximum principle.

For any fixed $z_0 \in \mathbb{R}^{N+1}$, there exists a global fundamental solution $\Gamma(\cdot, z_0)$ for $\mathcal{L}$ with pole at z_0, which is very well-behaved. We refer the reader to [10] for a complete list of the properties of Γ; for our purposes we would like to remark the following

$\Gamma(z, z_0) = \Gamma(z_0^{-1} \circ z, 0) =: \Gamma(z_0^{-1} \circ z)$ for every $z, z_0 \in \mathbb{R}^{N+1}$, $z \neq z_0$;
$\Gamma(d_\lambda(z)) = \lambda^{-Q+2}\Gamma(z)$, for every $z \in \mathbb{R}^{N+1} \setminus \{0\}$, $\lambda > 0$.

Given $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$, we define the *$\mathcal{L}$-ball of center z_0 and radius r* as

$$\Omega_r(z_0) = \left\{ z \in \mathbb{R}^{N+1} \ : \ \Gamma(z_0, z) = \Gamma(z^{-1} \circ z_0) > \frac{1}{r^{Q-2}} \right\}.$$

We explicitly stress that $\Gamma(z_0, z)$ is the fundamental solution of $\mathcal{L}^* = \sum_{j=1}^m X_j^2 - Y$. By the recalled properties of Γ, it turns out that

$$\Omega_r(z_0) = z_0 \circ \Omega_r(0) = z_0 \circ \delta_r(\Omega_1(0)) \qquad \text{and} \qquad |\Omega_r(z_0)| = r^Q |\Omega_1|.$$

Throughout this paper, we always denote with $|E|$ the Lebesgue measure of a subset E of $\mathbb{R}^{N+1}$.

It has been pointed out in [5] that, on the $\mathcal{L}$-balls, some representation formulas for smooth functions hold true. To be precise, if $\Omega \subseteq \mathbb{R}^{N+1}$ is an open set containing z_0, for every $u \in C^2(\Omega)$ we have

$$\begin{aligned} u(z_0) &= \int_{\partial\Omega_r(0)} k(z) u(z_0 \circ z)\, d\sigma(z) - \int_{\Omega_r(z_0)} \left(\Gamma(z^{-1} \circ z_0) - \frac{1}{r^{Q-2}} \right) \mathcal{L}u(z)\, dz \\ &=: m_r(u)(z_0) - n_r(\mathcal{L}u)(z_0) \qquad \text{for every } \overline{\Omega}_r(z_0) \subseteq \Omega \end{aligned} \tag{2.1}$$

where

$$k(z) = \frac{|\nabla_X \Gamma(0, z)|^2}{|\nabla_z \Gamma(0, z)|}, \quad \nabla_z = (\partial_{x_1}, \ldots, \partial_{x_n}, \partial_t) \quad \text{and} \quad \nabla_X = (X_1, \ldots, X_m).$$

Via Federer's co-area formula, we also have

$$\begin{aligned} u(z_0) &= \frac{1}{r^{Q-2}} \int_{\Omega_r(z_0)} K(z^{-1} \circ z_0) u(z)\, dz - \frac{Q-2}{r^{Q-2}} \int_0^r l^{Q-3} n_l(\mathcal{L}u)(z_0)\, dl \\ &=: M_r(u)(z_0) - N_r(\mathcal{L}u)(z_0) \qquad \text{for every } \overline{\Omega}_r(z_0) \subseteq \Omega \end{aligned} \tag{2.2}$$

where

$$K(z^{-1} \circ z_0) = \frac{|\nabla_X \Gamma(z_0, z)|^2}{\Gamma^2(z_0, z)}.$$

We explicitly note that the kernel K is left translation invariant, unlike k. If $\mathcal{L} = \Delta - \partial_t$ is the classical heat operator, the kernel K becomes the one appearing in the mean-value Theorem for caloric functions of Watson [25]. As in [27], we could also obtain different *solid representation formulas* by integrating Eq. 2.1 against other functions.

If $u : \Omega \longrightarrow [-\infty, \infty[$ is just an u.s.c function, for any $\overline{\Omega}_r(z_0) \subseteq \Omega$ we can consistently define

$$m_r(u)(z_0) = \int_{\partial\Omega_r(0)} k(z) u(z_0 \circ z)\, d\sigma(z).$$

This number is allowed to be $-\infty$, but it makes sense and it is bounded above since the kernel is non negative and u is bounded above on compact sets. Hence, for any $\overline{\Omega}_r(z_0) \subseteq \Omega$, we can also define

$$M_r(u)(z_0) = \frac{Q-2}{r^{Q-2}} \int_0^r l^{Q-3} m_l(u)(z_0)\, dl, \tag{2.3}$$

which is coherent to our previous definition. By using these mean-value operators, it is possible to characterize the solutions and the subsolutions of $\mathcal{L}$.

In what follows, we denote by $D(u)$ the set where a real-extended valued function u takes finite values. Moreover, if $z_0 \in \Omega$, we define

$$R_{z_0} := \sup\{r > 0 \ : \ \overline{\Omega_r(z_0)} \subset \Omega\}.$$

In the literature, it is already known the following characterization for the $\mathcal{L}$-subharmonic functions (see [5, 6]).

Theorem A *Let $u : \Omega \longrightarrow [-\infty, \infty[$ be an u.s.c. function. Let us suppose that $D(u)$ is dense in Ω. Then, the following statements are equivalent:*

(i) $u \in \underline{\mathcal{S}}(\Omega)$.
(ii) $u(z_0) \leq M_r(u)(z_0)$ *for every* $z_0 \in \Omega$ *and* $0 < r < R_{z_0}$.
(iii) $u(z_0) \leq m_r(u)(z_0)$ *for every* $z_0 \in \Omega$ *and* $0 < r < R_{z_0}$.
(iv) *For every* $z_0 \in \Omega$, *the function* $r \longmapsto M_r(u)(z_0)$ *is monotone increasing on* $(0, R_{z_0})$ *and* $\lim_{r\to 0^+} M_r(u)(z_0) = u(z_0)$.
(v) *For every* $z_0 \in \Omega$, *the function* $r \longmapsto m_r(u)(z_0)$ *is monotone increasing on* $(0, R_{z_0})$ *and* $\lim_{r\to 0^+} m_r(u)(z_0) = u(z_0)$.
(vi) $u \in L^1_{\mathrm{loc}}(\Omega)$, $\mathcal{L}u \geq 0$ *in the distribution sense and* $\lim_{r\to 0^+} M_r(u)(z_0) = u(z_0)$ *for every* $z_0 \in \Omega$.

The aim of this paper is to prove some characterizations and properties of the $\mathcal{L}$-subharmonic functions in terms of suitable asymptotic average operators. We also take the opportunity to complete the proof of the equivalence of the conditions (v), (vi) and the $\mathcal{L}$-subharmonicity, since we did not find an explicit proof. We postpone it to the Appendix, although it is assumed throughout the paper.

Let us give the statement of our result which extends to this setting the theorems by Blaschke, Privaloff and Reade.

Theorem 2.1 *Let $u : \Omega \longrightarrow [-\infty, \infty[$ be an u.s.c. function. Let us suppose that $D(u)$ is dense in Ω. Then, the following statements are equivalent:*

(i) $u \in \underline{\mathcal{S}}(\Omega)$.
(vii) *(Blaschke) For every* $z_0 \in D(u)$,

$$\limsup_{r\to 0^+} \frac{m_r(u)(z_0) - u(z_0)}{r^2} \geq 0.$$

(viii) *(Privaloff) For every* $z_0 \in D(u)$,

$$\limsup_{r\to 0^+} \frac{M_r(u)(z_0) - u(z_0)}{r^2} \geq 0.$$

(ix) *(Beckenbach-Radó) For every* $z_0 \in \Omega$, $M_r(u)(z_0) \leq m_r(u)(z_0)$ *for any* $r \in (0, R_{z_0})$ *and* $\lim_{r\to 0^+} m_r(u)(z_0) = u(z_0)$.
(x) *(Reade) For every* $z_0 \in D(u)$,

$$\liminf_{r\to 0^+} \frac{m_r(u)(z_0) - M_r(u)(z_0)}{r^2} \geq 0$$

and $\lim_{r\to 0^+} m_r(u)(z_0) = u(z_0)$.

By condition (vi) and Riesz-Schwartz's Representation Theorem, if u is $\mathcal{L}$-subharmonic in Ω there exists a positive Radon measure μ_u in Ω such that $\mathcal{L}u = \mu_u$ in $\mathcal{D}'(\Omega)$. We shall call μ_u the $\mathcal{L}$-Riesz measure related to u. Given $z_0 \in \Omega$, we define the $\mathcal{L}$-symmetric derivative of μ_u at z_0 as the limit, when it exists,

$$D_s\mu_u(z_0) = \lim_{r\to 0^+} \frac{\mu_u(\Omega_r(z_0))}{|\Omega_r(z_0)|}.$$

The following Saks-type theorem gives sharper versions of Blaschke, Privaloff and Reade conditions.

Theorem 2.2 *Let $u \in \underline{S}(\Omega)$ and let μ_u be the $\mathcal{L}$-Riesz measure related to u. Then, for almost every point $z_0 \in \Omega$, we have*

$$\lim_{r\to 0^+} \frac{m_r(u)(z_0) - u(z_0)}{r^2} = \left(\frac{Q-2}{2}\,|\Omega_1|\right) D_s\mu_u(z_0).$$

$$\lim_{r\to 0^+} \frac{M_r(u)(z_0) - u(z_0)}{r^2} = \left(\frac{(Q-2)^2}{2Q}\,|\Omega_1|\right) D_s\mu_u(z_0).$$

$$\lim_{r\to 0^+} \frac{m_r(u)(z_0) - M_r(u)(z_0)}{r^2} = \left(\frac{Q-2}{Q}\,|\Omega_1|\right) D_s\mu_u(z_0).$$

In particular, this happens when the $\mathcal{L}$-symmetric derivative exists and it is finite at z_0.

The proofs of Theorems 2.1 and 2.2 are given respectively in Sections 4 and 5. In the following Section, we deal with the properties of the average operators $m_r(u)(z)$ and $M_r(u)(z)$ for $u \in \underline{S}(\Omega)$: a study of their behavior is useful for a better understanding. Following the arguments present in [3] (Section 6), we discuss the finiteness of these operators, the continuity as functions of r and the $\mathcal{L}$-subharmonicity as functions of z. Furthermore, we study more in detail how these operators shrink to u as r goes to 0 and we finally prove a Nevanlinna-type theorem.

3 Properties of Average Operators

Let u be a subharmonic function in Ω and let μ_u be its $\mathcal{L}$-Riesz measure. The key point of our investigation is the Riesz Representation Theorem for $\underline{S}(\Omega)$. In [5] (Theorem 5.6), Cinti proved that, for any bounded open set $V \subseteq \overline{V} \subseteq \Omega$, there exists a function h which is $\mathcal{L}$-harmonic in V such that

$$u(z) = -\int_{\overline{V}} \Gamma(z,\zeta)\,d\mu_u(\zeta) + h(z) \qquad \text{for every } z \in V. \tag{3.1}$$

The fact that $-\int_{\overline{V}} \Gamma(z,\zeta)d\mu_u(\zeta)$ is $\mathcal{L}$-subharmonic can be found in [6] (Proposition 4.1). It is a consequence of the $\mathcal{L}$-subharmonicity of the functions $-\Gamma(\cdot,\zeta)$ for any $\zeta \in \mathbb{R}^{N+1}$ (we have to define $-\Gamma(\zeta,\zeta) = 0$ if we want to make $-\Gamma(\cdot,\zeta)$ u.s.c.).

The formula 3.1 allows us to deduce the properties of $m_r(u)$ and $M_r(u)$ from the ones of $m_r(-\Gamma(\cdot,\zeta))$ and $M_r(-\Gamma(\cdot,\zeta))$. That is why it is crucial to compute explicitly the average of the fundamental solution. This is exactly the aim of the following proposition. In the case of classical parabolic operators with variable coefficients, it was proved by Garofalo and Lanconelli in [8] (Lemma 5.2).

Proposition 3.1 *Let r be a positive number and $z_0, \zeta \in \mathbb{R}^{N+1}$. Then we have*

$$m_r(-\Gamma(\cdot,\zeta))(z_0) = \max\left\{-\Gamma(z_0,\zeta), -\frac{1}{r^{Q-2}}\right\}. \tag{3.2}$$

Proof Consider first the case where $z_0 = \zeta$ and $r > 0$. Since $\Gamma(z) = 0$ for any $z \in \partial\Omega_r(0)$, we have $m_r(-\Gamma(\cdot,\zeta))(\zeta) = 0 = -\Gamma(\zeta,\zeta)$.

Fix now $z_0 \neq \zeta$ and r such that $\Gamma(z_0,\zeta) < \frac{1}{r^{Q-2}}$. Then $\Gamma(\cdot,\zeta)$ is harmonic in a neighborhood of $\overline{\Omega_r(z_0)}$ and by Eq. 2.1 we get $m_r(-\Gamma(\cdot,\zeta))(z_0) = -\Gamma(z_0,\zeta)$.

Consider finally $z_0 \neq \zeta$ and r such that $\Gamma(z_0,\zeta) > \frac{1}{r^{Q-2}}$. We want to exploit the representation formula 2.1 by using some cut-off functions. Take an euclidean ball $B_e(\zeta, 2\rho)$ centered at ζ with radius $2\rho > 0$ which is compactly contained in $\Omega_r(z_0)$. For every $0 < \varepsilon < \frac{\rho}{2}$, let us define a C^∞-function ψ_ε such that $\psi_\varepsilon \equiv 0$ in $B_e(\zeta,\varepsilon)$ and $\psi_\varepsilon \equiv 1$ out of $B_e(\zeta, 2\varepsilon)$. Since $\psi_\varepsilon = 1$ on $\partial\Omega_r(z_0)$ and $\psi_\varepsilon\Gamma(\cdot,\zeta)$ is smooth, by Eq. 2.1 we get

$$\begin{aligned} m_r(\Gamma(\cdot,\zeta))(z_0) &= m_r(\psi_\varepsilon\Gamma(\cdot,\zeta))(z_0) \\ &= \Gamma(z_0,\zeta) + \int_{B_e(\zeta,\rho)} \left(\Gamma(z^{-1}\circ z_0) - \frac{1}{r^{Q-2}}\right)\mathcal{L}(\psi_\varepsilon\Gamma(\cdot,\zeta))(z)\,dz. \end{aligned}$$

We shall prove that the second term in the r.h.s tends to $\frac{1}{r^{Q-2}} - \Gamma(z_0,\zeta)$, as $\varepsilon \to 0^+$. In order to prove it, let us fix a function $\varphi \in C_0^\infty$ such that $\varphi \equiv 1$ in $B_e(\zeta,\rho)$ and it vanishes out of $B_e(\zeta, 2\rho)$. Hence, the term in question is equal to

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}} \left(\Gamma(z^{-1}\circ z_0) - \frac{1}{r^{Q-2}}\right)\varphi(z)\mathcal{L}(\psi_\varepsilon\Gamma(\cdot,\zeta))(z)\,dz \\ &\quad = \int_{\mathbb{R}^{N+1}} \mathcal{L}^*\left(\left(\Gamma(z_0,\cdot) - \frac{1}{r^{Q-2}}\right)\varphi\right)(\psi_\varepsilon(z) - 1)\Gamma(z,\zeta)\,dz \\ &\quad\quad + \int_{\mathbb{R}^{N+1}} \mathcal{L}^*\left(\left(\Gamma(z_0,\cdot) - \frac{1}{r^{Q-2}}\right)\varphi\right)\Gamma(z,\zeta)\,dz. \end{aligned}$$

Since $\left(\Gamma(z_0,\cdot) - \frac{1}{r^{Q-2}}\right)\varphi$ is a C_0^∞-function, the properties of the fundamental solution imply that the last integral is equal to $\frac{1}{r^{Q-2}} - \Gamma(z_0,\zeta)$. On the other hand, since $\Gamma(\cdot,\zeta) \in L^1_{\text{loc}}$, the former integral tends to 0 as $\varepsilon \to 0^+$. Therefore, $m_r(-\Gamma(\cdot,\zeta))(z_0) = -\frac{1}{r^{Q-2}}$.

Summing up, we have proved the proposition for every $\zeta, z_0 \in \mathbb{R}^{N+1}$ and every $r > 0$ such that $\Gamma(z_0,\zeta) \neq \frac{1}{r^{Q-2}}$. Since $m_r(-\Gamma(\cdot,\zeta))(z_0)$ is increasing as a function of r by condition (v) of Theorem A, the relation 3.2 holds true for every $r > 0$. □

Let us study the properties of the surface averages of the fundamental solution. We already know that, for every $z_0, \zeta \in \mathbb{R}^{N+1}$, the function $r \mapsto m_r(-\Gamma(\cdot,\zeta))(z_0)$ is increasing on the interval $(0, +\infty)$, but we can also note this is a continuous function of r. Moreover, for every $r > 0$ and $\zeta \in \mathbb{R}^{N+1}$, the function $z \mapsto m_r(-\Gamma(\cdot,\zeta))(z)$ is $\mathcal{L}$-subharmonic in $\mathbb{R}^{N+1}$ since it is the maximum between two $\mathcal{L}$-subharmonic functions. Finally we stress that, for every $r > 0$, we have $m_r(-\Gamma(\cdot,\zeta))(z_0) \geq -\frac{1}{r^{Q-2}}$ for any $z_0, \zeta \in \mathbb{R}^{N+1}$.

Let us now focus our attention on the solid averages. By using Eq. 2.3, a straightforward consequence of the last proposition is the following.

Corollary 3.2 *Let $z_0, \zeta \in \mathbb{R}^{N+1}$ and $r > 0$. Then we have*

$$M_r(-\Gamma(\cdot,\zeta))(z_0) = \begin{cases} -\frac{1}{r^{Q-2}}\left(1+\log(r^{Q-2}\Gamma(z_0,\zeta))\right) & \text{if } \zeta \in \Omega_r(z_0) \\ -\Gamma(z_0,\zeta) & \text{otherwise} \end{cases}$$

In order to study also the properties of this function, we need a lemma which is contained and proved in [3] (Lemma 6.2). Since we are going to use it again, we recall the statement for the sake of convenience.

Lemma 3.3 *Let $(\Lambda, d\lambda)$ be a measure space and suppose that $\{u_l\}_{l\in\Lambda}$ is a family of $\mathcal{L}$-subharmonic functions in an open set $O \subseteq \mathbb{R}^{N+1}$. Suppose furthermore that, for any $z \in O$, $u_l(z)$ is $d\lambda$-measurable in Λ and that one of the following conditions is satisfied:*

$u_l \le 0$ in O for every $l \in \Lambda$;
$\{u_l\}_{l\in\Lambda}$ is uniformly bounded from above and $\lambda(\Lambda) < +\infty$.

Then, if the function

$$U : O \longrightarrow [-\infty, +\infty), \qquad U(z) = \int_\Lambda u_l(z)d\lambda(l) \quad \text{for } z \in O$$

is finite in a dense subset of O, we have $U \in \underline{S}(O)$.

By recalling that $M_r(-\Gamma(\cdot,\zeta))(z) = \frac{Q-2}{r^{Q-2}}\int_0^r l^{Q-3} m_l(-\Gamma(\cdot,\zeta))(z)\,dl$, we can use the last lemma with $\Lambda = (0, r)$, $d\lambda(l) = \frac{Q-2}{r^{Q-2}} l^{Q-3} dl$. Then, we get that, for every $r > 0$ and $\zeta \in \mathbb{R}^{N+1}$, the function $z \mapsto M_r(-\Gamma(\cdot,\zeta))(z)$ is $\mathcal{L}$-subharmonic in $\mathbb{R}^{N+1}$. Moreover, for every $z_0, \zeta \in \mathbb{R}^{N+1}$, $M_r(-\Gamma(\cdot,\zeta))(z_0)$ is an increasing C^1-function of r, since it is the integral of a continuous function. Finally, we stress that, for a fixed positive r, $M_r(-\Gamma(\cdot,\zeta))(z_0)$ is not bounded by below because of the presence of the logarithm.

Let us transfer these properties from the function $-\Gamma(\cdot,\zeta) \in \underline{S}(\mathbb{R}^{N+1})$ to a general $\mathcal{L}$-subharmonic function u.

Proposition 3.4 *Let $u \in \underline{S}(\mathbb{R}^{N+1})$. Then, for any $r > 0$, the functions $z \mapsto m_r(u)(z)$, $M_r(u)(z)$ are $\mathcal{L}$-subharmonic in $\mathbb{R}^{N+1}$.*

Proof Fix $r > 0$. We want to take the surface average to both sides of the Riesz representation formula 3.1. Since the $\mathcal{L}$-subharmonicity is a local property, we consider a bounded neighborhood U of some $p \in \mathbb{R}^{N+1}$. Take V such that it contains $\Omega_r(U) := \{z \in \mathbb{R}^{N+1} : z \in \Omega_r(z_0) \text{ for some } z_0 \in U\}$. By Tonelli's theorem, we have

$$m_r\left(\int_V \Gamma(\cdot,\zeta)d\mu_u(\zeta)\right)(z) = \int_V m_r(\Gamma(\cdot,\zeta))(z)d\mu_u(\zeta).$$

Since the $\mathcal{L}$-harmonic functions are equal to their averages, we get

$$m_r(u)(z) = h(z) + \int_V m_r(-\Gamma(\cdot,\zeta))(z)d\mu_u(\zeta) \tag{3.3}$$

for any $z \in U$. We recall that $m_r(u)(z) \geq u(z) > -\infty$ at almost every z. Hence, the right-hand side is $\mathcal{L}$-subharmonic by Lemma 3.3. Therefore we have $m_r(u)(\cdot) \in \underline{\mathcal{S}}(\mathbb{R}^{N+1})$. We can use again Lemma 3.3 in order to get $M_r(u)(\cdot) \in \underline{\mathcal{S}}(\mathbb{R}^{N+1})$. □

If $u \in \underline{\mathcal{S}}(\Omega)$, the result of the last proposition is still true where everything is well defined. As a matter of fact, if we put $\Omega_\varepsilon := \{z \in \Omega \,:\, \overline{\Omega}_\varepsilon(z) \subseteq \Omega\}$ for a fixed $\varepsilon > 0$ such that this is not an empty set, then $m_r(u)(\cdot)$, $M_r(u)(\cdot)$ are $\mathcal{L}$-subharmonic in Ω_ε for any $0 < r < \varepsilon$.

Now we analyze the finiteness of the average operators and their behavior as functions of r.

Proposition 3.5 *Let $u \in \underline{\mathcal{S}}(\Omega)$. For every $z_0 \in \Omega$ and $0 < r < R_{z_0}$, $m_r(u)(z_0)$ is finite. Moreover, for any fixed $z_0 \in \Omega$, the function $r \mapsto m_r(u)(z_0)$ is continuous on the interval $(0, R_{z_0})$.*

Proof Fix $z_0 \in \Omega$ and $0 < r_0 < R_{z_0}$. Let V be an open neighborhood of $\Omega_{r_0}(z_0)$ such that it is compactly contained in Ω. By Eqs. 3.3 and 3.2, for any $0 < r < r_0$ we get

$$m_r(u)(z_0) = h(z_0) + \int_{\overline{V}} \max\left\{-\Gamma(z_0, \zeta), -\frac{1}{r^{Q-2}}\right\} d\mu_u(\zeta).$$

Of course we have $\max\{-\Gamma(z_0, \zeta), -\frac{1}{r^{Q-2}}) \geq -\frac{1}{r^{Q-2}}$. Since μ_u is a Radon measure, we have also $\mu_u(\overline{V}) < +\infty$. Hence we deduce that $m_r(u)(z_0) > -\infty$. Moreover, since the function h depends on r_0 but not on r, this shows also that $\rho \mapsto m_\rho(u)(z_0)$ is continuous at $\rho = r$ by the dominated convergence theorem. The arbitrariness of r_0 concludes the argument. □

Provided that $M_r(u)(z_0)$ is finite for some positive r (and then for every $r < R_{z_0}$), the last Proposition implies that $r \mapsto M_r(u)(z_0)$ is a C^1-function by Eq. 2.3. For example, this happens when $z_0 \in D(u)$, since $M_r(u)(z_0) \geq u(z_0)$. We would like to stress that, unlike the surface average, $M_r(u)(z_0)$ is not forced to be finite. In [27] (p. 255), Watson gives an explicit example in the parabolic case where this infiniteness appears.

We now study the behavior of the average operators as r goes to 0. By condition (v) and (iv) of Theorem A, we know that $\lim_{r\to 0^+} m_r(u)(z_0) = u(z_0)$ and $\lim_{r\to 0^+} M_r(u)(z_0) = u(z_0)$ if $u \in \underline{\mathcal{S}}(\Omega)$ and $z_0 \in \Omega$. We would like to say more. In the case of the Laplace operator, the second condition means that $\frac{1}{|B_e(z_0,r)|}\int_{B_e(z_0,r)} u(z)dz \to u(z_0)$ for every z_0 if u is Δ-subharmonic ($B_e(z_0, r)$ is the Euclidean ball). In [9] (p. 144) there is an improvement of this result: every z_0 such that $u(z_0) > -\infty$ is actually a Lebesgue point if u is Δ-subharmonic. The following proposition is the analogous result in our setting.

Proposition 3.6 *Let be $u \in \underline{\mathcal{S}}(\Omega)$ and $z_0 \in D(u)$. Then, we have*

$$\lim_{r\to 0^+} \int_{\partial\Omega_r(0)} k(z)\, |u(z_0 \circ z) - u(z_0)|\; d\sigma(z)$$
$$= \lim_{r\to 0^+} \frac{1}{r^{Q-2}} \int_{\Omega_r(0)} K(z^{-1})\, |u(z_0 \circ z) - u(z_0)|\; dz = 0.$$

Proof Fix $\varepsilon > 0$. Put $c = u(z_0) + \frac{\varepsilon}{2}$. Since the kernel is non-negative and its integral is equal to 1, we have

$$\int_{\partial\Omega_r(0)} k(z)\,|u(z_0 \circ z) - u(z_0)|\; d\sigma(z) \leq \int_{\partial\Omega_r(0)} k(z)\,|u(z_0 \circ z) - c|\; d\sigma(z) + \frac{\varepsilon}{2}.$$

The function u is u.s.c and therefore there exists $0 < r_0 < R_{z_0}$ such that $u(z_0 \circ z) \leq u(z_0) + \frac{\varepsilon}{2} = c$ if $z \in \overline{\Omega_r(0)}$ for every $r < r_0$. Hence, the r.h.s. of the last formula is equal to

$$c - \int_{\partial\Omega_r(0)} k(z) u(z_0 \circ z) d\sigma(z) + \frac{\varepsilon}{2} = u(z_0) - m_r(u)(z_0) + \varepsilon$$

for $0 < r < r_0$. Recalling that $u(z_0) \leq m_r(u)(z_0)$ by the $\mathcal{L}$-subharmonicity, we have just proved that

$$\int_{\partial\Omega_r(0)} k(z)\,|u(z_0 \circ z) - u(z_0)|\; d\sigma(z) \leq \varepsilon$$

for every $0 < r < r_0$. This is the first half of the claim. The second half is completely analogous. □

Finally, we want to prove a Nevanlinna-type formula. This gives a quantitative result about the monotonicity property of the surface averages. We follow an argument used by Watson in [29] (Theorem 1) for the heat operator.

Proposition 3.7 *Let $u \in \underline{\mathcal{S}}(\Omega)$ and let μ_u be the $\mathcal{L}$-Riesz measure related to u. For every $z_0 \in \Omega$ and $0 < \rho < r < R_{z_0}$, we have*

$$m_r(u)(z_0) - m_\rho(u)(z_0) = (Q-2) \int_\rho^r \frac{\mu_u(\Omega_t(z_0))}{t^{Q-1}}\, dt$$

and

$$\begin{aligned} u(z_0) &= m_r(u)(z_0) - (Q-2) \int_0^r \frac{\mu_u(\Omega_t(z_0))}{t^{Q-1}}\, dt \\ &= m_r(u)(z_0) - \int_{\Omega_r(z_0)} \left(\Gamma(z^{-1} \circ z_0) - \frac{1}{r^{Q-2}} \right) d\mu_u(z). \end{aligned} \tag{3.4}$$

Proof Let V be an open bounded neighborhood of $\Omega_r(z_0)$ such that $\overline{V} \subset \Omega$. The formula 3.3 holds true for $m_r(u)(z_0)$ and $m_\rho(u)(z_0)$. Reminding that every term is finite in such formulas, we get

$$\begin{aligned} & m_r(u)(z_0) - m_\rho(u)(z_0) \\ &\quad = \int_{\overline{V}} \left(\min\left\{ \Gamma(z_0, \zeta), \frac{1}{\rho^{Q-2}} \right\} - \min\left\{ \Gamma(z_0, \zeta), \frac{1}{r^{Q-2}} \right\} \right) d\mu_u(\zeta) \\ &\quad = \int_{\Omega_r(z_0)} \left(\min\left\{ \Gamma(z_0, \zeta), \frac{1}{\rho^{Q-2}} \right\} - \frac{1}{r^{Q-2}} \right) d\mu_u(\zeta) \\ &\quad = \int_{\Omega_r(z_0) \setminus \Omega_\rho(z_0)} \left(\Gamma(z_0, \zeta) - \frac{1}{r^{Q-2}} \right) d\mu_u(\zeta) + \left(\frac{1}{\rho^{Q-2}} - \frac{1}{r^{Q-2}} \right) \mu_u(\Omega_\rho(z_0)). \end{aligned}$$

By substituting $\Gamma(z_0,\zeta)-\frac{1}{r^{Q-2}}=\int_{\frac{1}{r^{Q-2}}}^{\Gamma(z_0,\zeta)} ds$ and $\frac{1}{\rho^{Q-2}}-\frac{1}{r^{Q-2}}=\int_{\frac{1}{r^{Q-2}}}^{\frac{1}{\rho^{Q-2}}} ds$, we can change the integrals and we get

$$\begin{aligned}
m_r(u)(z_0)-m_\rho(u)(z_0) &= \int_{\frac{1}{r^{Q-2}}}^{\frac{1}{\rho^{Q-2}}} \mu_u\left(\Omega_{\left(\frac{1}{s}\right)^{\frac{1}{Q-2}}}(z_0)\right) ds\\
&= (Q-2)\int_\rho^r \frac{\mu_u(\Omega_t(z_0))}{t^{Q-1}}\,dt.
\end{aligned}$$

This proves the first part of the Proposition. By letting $\rho\to 0^+$ and recalling that $\lim_{\rho\to 0^+} m_\rho(u)(z_0)=u(z_0)$, we deduce the formula 3.4. The equality

$$\int_{\Omega_r(z_0)}\left(\Gamma(z^{-1}\circ z_0)-\frac{1}{r^{Q-2}}\right) d\mu_u(z)=(Q-2)\int_0^r \frac{\mu_u(\Omega_t(z_0))}{t^{Q-1}}\,dt$$

is obtained in the same manner. □

The formula 3.4 can be seen also as a Poisson-Jensen type formula involving the surface averages. In [6] (Section 6), Cinti and Lanconelli proved necessary and sufficient conditions in order to obtain some Poisson-Jensen type formulas on general domains. In [5] (Theorem 5.7), it is proved a Poisson-Jensen type formula involving the solid averages. Actually, if $u\in\underline{\mathcal{S}}(\Omega)$, Cinti proved that

$$\begin{aligned}
u(z_0) &= M_r(u)(z_0)-\frac{Q-2}{r^{Q-2}}\int_0^r l^{Q-3}\int_{\Omega_l(z_0)}\left(\Gamma(z^{-1}\circ z_0)-\frac{1}{r^{Q-2}}\right) d\mu_u(z)\,dl\\
&= M_r(u)(z_0)-\frac{(Q-2)^2}{r^{Q-2}}\int_0^r l^{Q-3}\left(\int_0^l \frac{\mu_u(\Omega_t(z_0))}{t^{Q-1}}\,dt\right) dl \qquad (3.5)
\end{aligned}$$

for every $z_0\in\Omega$ and every $0<r<R_{z_0}$. At least for $z_0\in D(u)$, we could deduce this also by integrating Eq. 3.4.

Remark 3.8 Since $m_r(u)(z_0)$ is always finite for $u\in\underline{\mathcal{S}}(\Omega)$, formula 3.4 implies that

$$u \text{ is finite at } z_0 \quad\Leftrightarrow \int_0^r \frac{\mu_u(\Omega_t(z_0))}{t^{Q-1}}dt \text{ is finite for some positive } r.$$

In [5] (Theorem 6.2), Cinti studies also the integrability of $\frac{\mu(\Omega_t(z_0))}{t^{Q-1}}$ at $t=+\infty$. If we put together her result and the last remark, we have the following theorem. In the case of the sub-Laplacians, this is proved in [4] (Theorem 9.6.1). For the heat operator, we refer the reader to [29] (Theorem 7).

Theorem 3.9 *Let μ be a Radon measure in $\mathbb{R}^{N+1}$ and let $z_o\in\mathbb{R}^{N+1}$. Then, μ is the $\mathcal{L}$-Riesz measure of a bounded-above $\mathcal{L}$-subharmonic function in $\mathbb{R}^{N+1}$ with $u(z_0)>-\infty$ if and only if it is satisfied the following condition:*

$$\int_0^{+\infty} \frac{\mu_u(\Omega_t(z_0))}{t^{Q-1}}dt<+\infty.$$

4 Proof of Theorem 2.1

We have to prove several implications in order to get Theorem 2.1: we are going to exploit the equivalences of Theorem A.

Remark 4.1 We start noting that the implications (iii) $\Rightarrow$ (vii) and (ii) $\Rightarrow$ (viii) are trivial. We can easily get also (v) $\Rightarrow$ (ix) and (ix) $\Rightarrow$ (x). As a matter of fact, by the definition of $M_r(u)(z_0)$ and the monotonicity of $l \mapsto m_l(u)(z_0)$, we deduce

$$M_r(u)(z_0) = \frac{Q-2}{r^{Q-2}} \int_0^r l^{Q-3} m_l(u)(z_0)\, dl \le \frac{Q-2}{r^{Q-2}} \int_0^r l^{Q-3} m_r(u)(z_0)\, dl = m_r(u)(z_0)$$

for any $z_0 \in \Omega$ and $r \in (0, R_{z_0})$. This proves (v) $\Rightarrow$ (ix). About (ix) $\Rightarrow$ (x), we first note that the condition $\lim_{r\to 0^+} m_r(u)(z_0) = u(z_0)$ implies $\lim_{r\to 0^+} M_r(u)(z_0) = u(z_0)$. Hence, for $z_0 \in D(u)$, $m_r(u)(z_0)$ and $M_r(u)(z_0)$ are both finite if r is small enough: in this way (x) is an easy consequence of (ix).

We have just reduced the problem to proving that the Blaschke-type condition, the Privaloff-type condition and the Reade-type condition imply the definition of $\mathcal{L}$-subharmonicity.

We first prove that this is true for smooth functions by using the representation formulas 2.1 and 2.2. To this aim, for any $r > 0$ and $z_0 \in \mathbb{R}^{N+1}$, let us define the positive numbers

$$q_r(z_0) = \int_{\Omega_r(z_0)} \left(\Gamma(z^{-1} \circ z_0) - \frac{1}{r^{Q-2}} \right) dz,$$
$$Q_r(z_0) = \frac{Q-2}{r^{Q-2}} \int_0^r l^{Q-3} q_l(z_0)\, dl,$$
$$w_r(z_0) = q_r(z_0) - Q_r(z_0).$$

Since the Lebesgue measure is left translation invariant, we have

$$q_r(z_0) = \int_{\Omega_r(0)} \left(\Gamma(z^{-1}) - \frac{1}{r^{Q-2}} \right) dz = q_r(0)$$

for every $z_0 \in \mathbb{R}^{N+1}$. Because of the independence of the point z_0, in what follows we are going to omit it and to use the notations q_r, Q_r and w_r. By the recalled properties of homogeneity of Γ, an easy computations shows that

$$q_r = \int_{\Omega_r(0)} \int_{\frac{1}{r^{Q-2}}}^{\Gamma(z^{-1})} ds\, dz = \int_{\frac{1}{r^{Q-2}}}^{+\infty} \left| \Omega_{\left(\frac{1}{s}\right)^{\frac{1}{Q-2}}}(0) \right| ds$$
$$= (Q-2) \int_0^r \frac{|\Omega_t(0)|}{t^{Q-1}}\, dt = (Q-2)\, |\Omega_1| \int_0^r t\, dt = \frac{Q-2}{2}\, |\Omega_1|\, r^2.$$

Therefore, for every $r > 0$, we have

$$\begin{aligned} q_r &= \frac{Q-2}{2}\,|\Omega_1|\,r^2, \\ Q_r &= \frac{(Q-2)^2}{2Q}\,|\Omega_1|\,r^2, \\ w_r &= \frac{Q-2}{Q}\,|\Omega_1|\,r^2. \end{aligned} \tag{4.1}$$

Proposition 4.2 *For every $z_0 \in \mathbb{R}^{N+1}$ and for every $u \in C^2(\overline{\Omega_R(z_0)}, \mathbb{R})$, we have*

$$\begin{aligned} \mathcal{L}u(z_0) &= \lim_{r\to 0^+} \frac{m_r(u)(z_0) - u(z_0)}{q_r} \\ &= \lim_{r\to 0^+} \frac{M_r(u)(z_0) - u(z_0)}{Q_r} \\ &= \lim_{r\to 0^+} \frac{m_r(u)(z_0) - M_r(u)(z_0)}{w_r}. \end{aligned}$$

Proof Given $\varepsilon > 0$, the continuity of $\mathcal{L}u$ implies the existence of $0 < \rho < R$ such that $\sup_{z\in\Omega_\rho(z_0)} |\mathcal{L}u(z) - \mathcal{L}u(z_0)| < \varepsilon$. Then, for $0 < r < \rho$, we get

$$\begin{aligned} &|m_r(u)(z_0) - u(z_0) - \mathcal{L}u(z_0)q_r| \\ &= \left| \int_{\Omega_r(z_0)} \left(\Gamma(z^{-1}\circ z_0) - \frac{1}{r^{Q-2}} \right) (\mathcal{L}u(z) - \mathcal{L}u(z_0))dz \right| \\ &\le \int_{\Omega_r(z_0)} \left(\Gamma(z^{-1}\circ z_0) - \frac{1}{r^{Q-2}} \right) \sup_{\Omega_\rho(z_0)} |\mathcal{L}u(z) - \mathcal{L}u(z_0)|dz \le \varepsilon q_r. \end{aligned}$$

This means

$$m_r(u)(z_0) - u(z_0) - \mathcal{L}u(z_0)q_r = o(q_r), \quad as \;\; r \to 0$$

and so the first equality holds true. Analogously, we have

$$M_r(u)(z_0) - u(z_0) - \mathcal{L}u(z_0)Q_r = o(Q_r), \quad as \;\; r \to 0.$$

By comparing the last two equalities, we get

$$m_r(u)(z_0) - M_r(u)(z_0) - \mathcal{L}u(z_0)w_r = o(r^2) = o(w_r), \quad as \;\; r \to 0.$$

□

Since q_r, Q_r and w_r are equal to r^2 up to positive constants, the last proposition proves our main theorem in the case of smooth functions. In order to prove it in all its generality, we exploit a result that, for the Laplace operator, is due to Kozakiewicz [12]. The idea of transferring it in a more general setting and of using it for the asymptotic conditions is already in [3]. Let us fix some notations. We denote by $\mathcal{U}(\Omega)$ the set of functions $v : U \longrightarrow [-\infty, +\infty[$ for some open set $U \subset \Omega$ which are u.s.c and finite on a dense subset of U. Moreover, we denote by $\mathcal{F}(\Omega)$ the set of the real-extended valued functions defined on some subset of Ω.

Theorem 4.3 *Let $G : \mathcal{U}(\Omega) \longrightarrow \mathcal{F}(\Omega)$ be a map satisfying*

[k1] *for $v \in \mathcal{U}(\Omega)$, $G(v)$ is defined on $D(v)$;*
[k2] *$G(h) = \mathcal{L}(h)$ for every $h \in \mathcal{U}(\Omega)$ of class C^2.*

Let $u : \Omega \longrightarrow [-\infty, +\infty[$ be an u.s.c function, finite in a dense subset of Ω, such that $G(u) \geq 0$ in $D(u)$. Suppose finally that, for every open set $V \subset \Omega$ and every $\phi \in C^2(V, \mathbb{R})$, the following conditions are satisfied:

[k3] *$G(u + \phi) = G(u) + G(\phi)$;*
[k4] *for every local maximum point ζ of $u - \phi$, we have $G(u - \phi)(\zeta) \leq 0$.*

Then u is $\mathcal{L}$-subharmonic in Ω.

Proof Let V be an $\mathcal{L}$-regular set with $\overline{V} \subset \Omega$. We have to show that, for every $\varphi \in C(\partial V, \mathbb{R})$ with $u \leq \varphi$ on ∂V, we have $u \leq H_\varphi^V$ in V. For any $\varepsilon > 0$, let us consider th function $v_\varepsilon = u - H_\varphi^V - \varepsilon e^t$. This function is defined on V and finite on $D(u) \cap V$. By the compactness of $\overline{V}$, there exists a point $\zeta \in \overline{V}$ such that $\sup_{U \cap V} v_\varepsilon = \sup_V v_\varepsilon$ for every open neighborhood U of ζ. Arguing as in [3] (Theorem 5.3), we can state that ζ cannot belong to V. Therefore, we can deduce that $sup_V v_\varepsilon \leq 0$. Letting $\varepsilon \to 0^+$, we have $u - H_\varphi^V \leq 0$ in V. □

Let us note that the condition [k4] is closely related with the usual notion of viscosity subsolution.

We complete the proof of Theorem 2.1 by showing that the asymptotic operators in conditions (vii), (viii) and (x) satisfy the assumptions of the last theorem.

Proposition 4.4 *Under the notations and the hypotheses of Theorem* A*, if u satisfies the Blaschke-type condition* (vii)*, the Privaloff-type condition* (viii) *or the Reade-type condition* (x)*, then $u \in \underline{\mathcal{S}}(\Omega)$.*

Proof For any $v \in \mathcal{U}(\Omega)$, let us set

$$G(v)(z) := \limsup_{r \to 0^+} \frac{m_r(v)(z) - v(z)}{q_r}, \qquad \text{for } z \in D(v).$$

By our choice of G, [k1] is satisfied. The condition [k2] is ensured by Proposition 4.2. Actually, for C^2-functions the lim sup in the definition of G is a limit: that's why also [k3] holds true. Finally, if ζ is a local maximum point for $v \in \mathcal{U}(\Omega)$, we have $v(\zeta) > -\infty$ and $m_r(v)(\zeta) \leq v(\zeta)$. Hence, [$k$4] is fulfilled too. Since condition (vii) means exactly that $G(u) \geq 0$ in $D(u)$, by Theorem 4.3 we get $u \in \underline{\mathcal{S}}(\Omega)$.

The implication (viii) $\Rightarrow$ (i) is completely analogous by putting

$$G(v)(z) := \limsup_{r \to 0^+} \frac{M_r(v)(z) - v(z)}{Q_r} \qquad \text{for } v \in \mathcal{U}(\Omega), \quad z \in D(v).$$

The Reade-type condition is slightly more delicate. For any $v \in \mathcal{U}(\Omega)$, we set

$$G(v)(z) := \liminf_{r \to 0^+} \frac{m_r(v)(z) - M_r(v)(z)}{w_r}, \qquad \text{for } z \in D(v).$$

We agree to put $m_r(v)(z) - M_r(v)(z) = -\infty$ if $m_r(v)(z) = -\infty$. We can easily get assumptions [k1], [k2] and [k3] as above. In order to get [k4], we need the Lemma

5.10 in [3]. We can apply it since we know that $\lim_{r\to 0^+} m_r(u)(z_0) = u(z_0)$. By this Lemma, if ζ is a local maximum point for $u-\varphi$, there exists a sequence $\{r_k\}$ decreasing to 0 such that $m_\rho(u-\varphi)(\zeta) \geq m_{r_k}(u-\varphi)(\zeta) - \frac{1}{k} w_{r_k}$ for every $\rho \in (0, r_k]$ and every k. This implies $M_{r_k}(u-\varphi)(\zeta) \geq m_{r_k}(u-\varphi)(\zeta) - \frac{1}{k} w_{r_k}$ for all k. Therefore, we get

$$G(u-\varphi)(\zeta) \leq \lim_{k\to+\infty} \frac{1}{k} = 0,$$

which is condition [k4]. In this way, the proof of this proposition and of Theorem 2.1 are complete. □

5 Proof of Theorem 2.2

Keeping in mind Eq. 4.1, we can restate Theorem 2.2 as follows.

Theorem 2.2 *Let $u \in \underline{\mathcal{S}}(\Omega)$ and let μ_u be the $\mathcal{L}$-Riesz measure related to u. Then, for almost every point $z_0 \in \Omega$, we have*

$$\begin{aligned} D_s\mu_u(z_0) &= \lim_{r\to 0^+} \frac{m_r(u)(z_0) - u(z_0)}{q_r} \\ &= \lim_{r\to 0^+} \frac{M_r(u)(z_0) - u(z_0)}{Q_r} \\ &= \lim_{r\to 0^+} \frac{m_r(u)(z_0) - M_r(u)(z_0)}{w_r}. \end{aligned}$$

In particular, this happens when the $\mathcal{L}$-symmetric derivative exists and it is finite at z_0.

We would like to stress that this is the analogous of Proposition 4.2 for non-smooth subharmonic functions.

First, we are going to show that the $\mathcal{L}$-symmetric derivative of a given positive Radon measure μ in Ω exists almost everywhere. To this aim, let us define

$$n((x,t)) = \max\left\{|x_1|^{\frac{1}{\sigma_1}}, \ldots, |x_N|^{\frac{1}{\sigma_N}}, |t|^{\frac{1}{2}}\right\}$$

for $(x,t) \in \mathbb{R}^{N+1}$. The function n is d_λ-homogeneous of degree one. Given $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$, we can define the $\mathbb{L}$-ball of radius r centered at z_0 as

$$B_r(z_0) = \left\{z \in \mathbb{R}^{N+1} \ : \ n(z_0^{-1} \circ z) < r\right\}.$$

It is easy to see that $B_r(z_0) = z_0 \circ B_r(0) = z_0 \circ \delta_r(B_1(0))$ and therefore we have $|B_r(z_0)| = r^Q |B_1|$. By considering these balls and the Lebesgue measure, in [24] it is showed that some classical arguments concerning the maximal function still work. In particular, if φ is a locally integrable function, then

$$\lim_{r\to 0^+} \frac{\int_{B_r(z_0)} \varphi(z)\, dz}{|B_r(z_0)|} = \varphi(z_0) \qquad \text{for almost every } z_0.$$

We would like to plug in the previous relation the $\mathcal{L}$-balls instead of the $\mathbb{L}$-balls. Let us note that there exists a positive number C such that $\Omega_1(0) \subseteq B_C(0)$. Hence, for every $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$, by dilating and translating we get

$$\Omega_r(z_0) \subseteq B_{Cr}(z_0),$$
$$|\Omega_r(z_0)| = \alpha\, |B_{Cr}(z_0)| \qquad \text{with } \alpha = \frac{1}{C^Q}\frac{|\Omega_1|}{|B_1|}.$$

According to the definition in [22] (Chapter 7), we can say that the $\mathcal{L}$-balls $\Omega_r(z_0)$ *shrink to* z_0 *nicely* with respect to the $\mathbb{L}$-balls $B_r(z_0)$. Let us now recall that the Lebesgue decomposition of μ with respect to the Lebesgue measure is $d\mu(z) = \varphi\, dz + ds(z)$, where φ is a locally integrable non-negative function and ds is singular with respect to dz. By arguing as in [22] (Theorem 7.14) we have

$$\lim_{r\to 0^+} \frac{\int_{\Omega_r(z_0)} \varphi(z)\, dz}{|\Omega_r(z_0)|} = \varphi(z_0) \quad \text{and} \quad \lim_{r\to 0^+} \frac{s(\Omega_r(z_0))}{|\Omega_r(z_0)|} = 0$$

for almost every $z_0 \in \Omega$. Hence, this implies

$$\lim_{r\to 0^+} \frac{\mu(\Omega_r(z_0))}{|\Omega_r(z_0)|} = \varphi(z_0)$$

for almost every $z_0 \in \Omega$. Therefore the $\mathcal{L}$-symmetric derivative of μ exists almost everywhere. Moreover, since a $L^1_{\mathrm{loc}}(\Omega)$-function is finite almost everywhere in Ω, we have that at almost every point $z_0 \in \Omega$, $D_s\mu(z_0)$ exists and it is finite.

Remark 5.1 Let us consider the particular case of the $\mathcal{L}$-Riesz measure μ_u related to a $\mathcal{L}$-subharmonic function u. Take a point $z_0 \in \Omega$ such that $D_s\mu_u(z_0)$ exists and it is finite. By the definition, we get that $\frac{\mu_u(\Omega_t(z_0))}{t^{Q-1}}$ behaves as $tD_s\mu_u(z_0)$ as t goes to 0^+. Then, the Remark 3.8 implies that $u(z_0)$ is finite.

We can now complete the proof of our Saks-type theorem by exploiting the Poisson-Jensen representation formulas stated in Section 3.

Proof of Theorem 2.2 Fix a point $z_0 \in \Omega$ such that $D_s\mu_u(z_0)$ exists and it is finite. Given $\varepsilon > 0$, there exists $0 < \rho < R_{z_0}$ such that $\left|\frac{\mu_u(\Omega_r(z_0))}{|\Omega_r(z_0)|} - D_s\mu_u(z_0)\right| \le \varepsilon$ for $0 < r < \rho$. Hence, by Eq. 3.4, we get for $0 < r < \rho$ that

$$\begin{aligned} |m_r(u)(z_0) - u(z_0) - q_r D_s\mu_u(z_0)| &= \left|(Q-2)\int_0^r \frac{\mu_u(\Omega_t(z_0))}{t^{Q-1}}\, dt - q_r D_s\mu_u(z_0)\right| \\ &= (Q-2)\,|\Omega_1| \left|\int_0^r t\left(\frac{\mu_u(\Omega_t(z_0))}{|\Omega_r(z_0)|} - D_s\mu_u(z_0)\right) dt\right| \\ &\le \varepsilon q_r. \end{aligned}$$

This means

$$m_r(u)(z_0) - u(z_0) - q_r D_s\mu_u(z_0) = o(q_r), \quad \textit{as} \;\; r \to 0$$

and so the first equality holds true. Analogously, by exploiting Eq. 3.5, we have

$$M_r(u)(z_0) - u(z_0) - Q_r D_s\mu_u(z_0) = o(Q_r), \quad as \quad r \to 0.$$

Since every term is finite, we can compare the last two equalities and we get

$$m_r(u)(z_0) - M_r(u)(z_0) - w_r D_s\mu_u(z_0) = o(r^2) = o(w_r), \quad as \quad r \to 0.$$

Even if $D_s\mu_u(z_0) = +\infty$, we can prove that $D_s\mu_u(z_0) = \lim_{r\to 0^+} \frac{m_r(u)(z_0)-u(z_0)}{q_r}$. As a matter of fact, for any $M > 0$, there exists $0 < \rho < R_{z_0}$ such that

$$\frac{\mu_u(\Omega_t(z_0))}{t^{Q-1}} \geq |\Omega_1|\, Mt \qquad \text{for } 0 < r < \rho.$$

The formula 3.4 implies that $m_r(u)(z_0) - u(z_0) \geq Mq_r$ for $0 < r < \rho$ and therefore $\lim_{r\to 0^+} \frac{m_r(u)(z_0)-u(z_0)}{q_r} = +\infty$.

Analogously, we get $\lim_{r\to 0^+} \frac{M_r(u)(z_0)-u(z_0)}{Q_r} = +\infty$. □

Appendix

For the sake of completeness, we start proving that the conditions (v) and (vi) in Theorem A are equivalent to the $\mathcal{L}$-subharmonicity. These facts are already stated in [6], but we have not found an explicit proof. Since it does not seem so trivial to us, we show it explicitly.

Remark A.1 Firstly, we would like to remark that, if we drop the condition $\lim_{r\to 0^+} M_r(u)(z_0) = u(z_0)$, (vi) is not equivalent to (i). As a matter of fact, the function $u = \chi_{\{0\}}$ is clearly u.s.c, locally integrable and $\mathcal{L}u = 0$ in the sense of distributions. But u is not $\mathcal{L}$-subharmonic, since cannot be sub-mean at 0.

The implication (i) $\Rightarrow$ (vi) is already proved. Indeed, by arguing as in [15] (Theorem 1), it is proved in [5] (Proposition 2.1) that (i) implies that $u \in L^1_{\mathrm{loc}}$ and $\mathcal{L}u \geq 0$ in the distribution sense. Moreover, the condition $\lim_{r\to 0^+} M_r(u)(z_0) = u(z_0)$ is a part of the implication (i) $\Rightarrow$ (iv).

Proposition A.2 *Let u be as in the Theorem* A. *Let us suppose $u \in L^1_{\mathrm{loc}}$ and $\mathcal{L}u \geq 0$ in the sense of distributions. Then, the function $r \longmapsto M_r(u)(z_0)$ is monotone increasing on* $(0, R_{z_0})$ *for every $z_0 \in \Omega$.*

Proof Since $u \in L^1_{\mathrm{loc}}$ and $\mathcal{L}u \geq 0$ in the sense of distributions, there exists the $\mathcal{L}$-Riesz measure related to u. Hence (see [5], Theorem 5.1), for every bounded set $V \subset \overline{V} \subset \Omega$, there is a function h which is $\mathcal{L}$-harmonic in V, such that

$$u(z) = -\int_{\overline{V}} \Gamma(\zeta^{-1} \circ z)\, d\mu_u(\zeta) \, + \, h(z) \qquad \text{for almost every } z \in \ V.$$

We can now argue as in [13] (Theorem 1.3) and state that, for every bounded set $V \subset \overline{V} \subset \Omega$, there exists a decreasing sequence u_n of smooth functions, which converge to u for almost every $z \in V$, such that $\mathcal{L}u_n \geq 0$ in the classical sense.

Now we can fix a point $z_0 \in \Omega$ and $0 < r_1 < r_2 < R_{z_0}$. Take a bounded set V such that $\overline{\Omega}_{r_2}(z_0) \subset V \subset \overline{V} \subset \Omega$. Let us consider a sequence u_n as above. For smooth subharmonic functions, by differentiating the representation formula 2.2 with respect to r, we know that $r \mapsto M_r(u_n)(z_0)$ is monotone increasing. Hence, we have $M_{r_1}(u_n)(z_0) \leq M_{r_2}(u_n)(z_0)$ for every n. Letting $n \to +\infty$, by the monotone convergence theorem, we get $M_{r_1}(u)(z_0) \leq M_{r_2}(u)(z_0)$. □

By the last proposition, we easily get (vi) $\Rightarrow$ (iv)($\Leftrightarrow$ (i)). Let us now consider the equivalence between (v) and the property of subharmonicity.

Proposition A.3 *With the notations and the hypotheses of Theorem* A*, we have*

$$\text{(i)} \Leftrightarrow \text{(v)}.$$

Proof The implication (v) $\Rightarrow$ (iv) is straightforward. As we have already noted in Remark 4.1, the condition

$$u(z_0) = \lim_{r \to 0^+} m_r(u)(z_0) \qquad \text{implies} \qquad u(z_0) = \lim_{r \to 0^+} M_r(u)(z_0).$$

Moreover, fixing $z_0 \in \Omega$ and $0 < r_1 < r_2 < R_{z_0}$, we would like $M_{r_1}(u)(z_0) \leq M_{r_2}(u)(z_0)$. We can assume $M_{r_1}(u)(z_0) > -\infty$, so that $m_l(u)(z_0)$ is finite for almost every $l \in (0, r_1)$. By the monotonicity assumption, it has to be true for every $l \in (0, r_2]$ and we get

$$\begin{aligned}
&\frac{M_{r_2}(u)(z_0)}{Q-2} - \frac{M_{r_1}(u)(z_0)}{Q-2} \\
&\quad = \left(\frac{1}{r_2^{Q-2}} - \frac{1}{r_1^{Q-2}}\right) \int_0^{r_1} l^{Q-3} m_l(u)(z_0)\, dl + \frac{1}{r_2^{Q-2}} \int_{r_1}^{r_2} l^{Q-3} m_l(u)(z_0)\, dl \\
&\quad \geq m_{r_1}(u)(z_0) \left(\left(\frac{1}{r_2^{Q-2}} - \frac{1}{r_1^{Q-2}}\right) \int_0^{r_1} l^{Q-3}\, dl + \frac{1}{r_2^{Q-2}} \int_{r_1}^{r_2} l^{Q-3}\, dl \right) = 0.
\end{aligned}$$

Viceversa, suppose $u \in \underline{S}(\Omega)$. In order to avoid any possible circular reasoning, we are not going to exploit any result of Section 3. Fix a point $z_0 \in \Omega$. We want to prove first that $\lim_{r \to 0^+} m_r(u)(z_0) = u(z_0)$. If $u(z_0) > -\infty$, given $\varepsilon > 0$ there exists a neighborhood of z_0 where $u(z) \leq u(z_0) + \varepsilon$ since u is u.s.c. in Ω. Then, for every r sufficiently small, we get

$$u(z_0) \leq m_r(u)(z_0) \leq u(z_0) + \varepsilon,$$

where the first inequality comes from the implication (i) $\Rightarrow$ (iii). If $u(z_0) = -\infty$, the upper semi-continuity implies $\lim_{z \to z_0} u(z) = u(z_0)$ and therefore we have $\lim_{r \to 0^+} m_r(u)(z_0) = -\infty$. It remains to prove the monotonicity of the surface averages. Fix $0 < r_1 < r_2 < R_{z_0}$ and consider a bounded set V such that $\overline{\Omega}_{r_2}(z_0) \subset V \subset \overline{V} \subset \Omega$. Since $u \in \underline{S}(\Omega)$, formula 3.1 holds true. Like in the last Proposition, we can build up a decreasing sequence u_n of smooth subharmonic functions converging to u everywhere in V (because Eq. 3.1 is everywhere satisfied and not just almost everywhere). For smooth functions, it follows directly from the representation formula 2.1 that

$$m_{r_1}(u_n)(z_0) - m_{r_2}(u_n)(z_0) = -\int_{\Omega_{r_2} \setminus \Omega_{r_1}} \left(\Gamma(z^{-1} \circ z_0) - \frac{1}{r_2^{Q-2}} \right) \mathcal{L}u_n(z)\, dz$$
$$+ \left(\frac{1}{r_2^{Q-2}} - \frac{1}{r_1^{Q-2}} \right) \int_{\Omega_{r_1}(z_0)} \mathcal{L}u_n(z)\, dz \leq 0.$$

Therefore, $m_{r_1}(u_n)(z_0) \leq m_{r_2}(u_n)(z_0)$ for every n. Letting $n \to +\infty$, by the monotone convergence theorem, we get $m_{r_1}(u)(z_0) \leq m_{r_2}(u)(z_0)$. □

By using these approximation methods, we can actually prove all the conditions implicated by (i) in Theorem A.

References

1. Beckenbach, E.F., Radó, T.: Subharmonic functions and surfaces of negative curvature. Trans. Am. Math. Soc. **35**, 662–674 (1933)
2. Blaschke, W.: Ein Mittelwersatz und eine kennzeichnende Eigenschaft des logarithmischen Potentials. Leipz. Ber. **68**, 3–8 (1916)
3. Bonfiglioli, A., Lanconelli, E.: Subharmonic functions in sub-Riemannian settings. J. Eur. Math. Soc. (2012, to appear)
4. Bonfiglioli, A., Lanconelli, E., Uguzzoni, F.: Stratified Lie Groups and Potential Theory for Their Sub-Laplacians. Springer, Berlin (2007)
5. Cinti, C.: Sub-solutions and mean value operators for ultraparabolic equations on Lie groups. Math. Scand. **101**, 83–103 (2007)
6. Cinti, C., Lanconelli, E.: Riesz and Poisson–Jensen representation formulas for a class of ultraparabolic operators on Lie groups. Potential Anal. **30**, 179–200 (2009)
7. Freitas, P., Matos, J.P.: On the characterization of harmonic and subharmonic functions via mean-value properties. Potential Anal. **32**, 189–200 (2010)
8. Garofalo, N., Lanconelli, E.: Asymptotic behavior of fundamental solutions and potential theory of parabolic operators with variable coefficients. Math. Ann. **283**, 211–239 (1989)
9. Hörmander, L.: Notions of Convexity. Birkhäuser, Boston (2007)
10. Kogoj, A.E., Lanconelli, E.: An invariant Harnack inequality for a class of hypoelliptic ultraparabolic equations. Mediterr. J. Math. **1**, 51–80 (2004)
11. Kogoj, A.E., Lanconelli, E.: Link of groups and homogeneous Hörmander operators. Proc. Am. Math. Soc. **135**, 2019–2030 (2007)
12. Kozakiewicz, W.: Un théorème sur les opérateurs et son application à la théorie des Laplaciens généralisés. C. R. Soc. sc. Varsovie **26**, 18–24 (1933)
13. Lanconelli, E., Pascucci, A.: Superparabolic functions related to second order hypoellitic operators. Potential Anal. **11**, 303–323 (1999)
14. Lanconelli, E., Polidoro, S.: On a class of hypoelliptic evolution operators. Rend. Sem. Mat. Univ. Pol. Torino **52**(1), 29–63 (Partial Differ. Equ.) (1994)
15. Negrini, P., Scornazzani, V.: Superharmonic functions and regularity of boundary points for a class of elliptic-parabolic partial differential operators. Bollettino UMI An. Funz. Appl. Serie VI, Vol. III-C **1**, 85–106 (1984)
16. Pini, B.: Maggioranti e minoranti delle soluzioni delle equazioni paraboliche. Ann. Math. Pures Appl. **37**, 249–264 (1954)
17. Pini, B.: Su un integrale analogo al potenziale logaritmico. Boll. Unione Mat. Ital. **9**(3), 244–250 (1954)
18. Potts, D.H.: A note on the operators of Blaschke and Privaloff. Bull. Am. Math. Soc. **54**, 782–787 (1948)

19. Privaloff, I.: Sur les fonctions harmoniques. Rec. Math. Moscou (Mat. Sbornik) **32**, 464–471 (1925)
20. Privaloff, I.: On a theorem of S. Saks. Rec. Math. Moscou (Mat. Sbornik) **9**, 457–460 (1941)
21. Reade, M.: Some remarks on subharmonic functions. Duke Math. J. **10**, 531–536 (1943)
22. Rudin, W.: Real and Complex Analysis. McGraw-Hill, New York (1987)
23. Saks, S.: On the operators of Blaschke and Privaloff for subharmonic functions. Rec. Math. Moscou (Mat. Sbornik) **9**, 451–456 (1941)
24. Stein, E.M.: Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton (1995)
25. Watson, N.A.: A theory of subtemperatures in several variables. Proc. Lond. Math. Soc. **26**, 385–417 (1973)
26. Watson, N.A.: Nevanlinna's first fundamental theorem for supertemperatures. Math. Scand. **73**, 49–64 (1993)
27. Watson, N.A.: Volume mean values of subtemperatures. Colloq. Math. **86**, 253–258 (2000)
28. Watson, N.A.: Elementary proofs of some basic subtemperatures theorems. Colloq. Math. **94**, 111–140 (2002)
29. Watson, N.A.: A generalized Nevanlinna theorem for supertemperatures. Ann. Acad. Sci. Fenn. Math. **28**, 35–54 (2003)