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One-Side Liouville Theorems for a Class of Hypoelliptic Ultraparabolic Equations

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1. Introduction

The aim of this paper is to show a one-side Liouville theorem for a class of hypoelliptic ultraparabolic equations and for their "stationary" counterpart.

The operators we shall deal with are of the following type:

$$\mathcal{L} = \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x)\partial_{x_j}) + \sum_{i=1}^{N} b_i(x)\partial_{x_i} - \partial_t \qquad \text{in } \mathbb{R}^{N+1}, \tag{1.1}$$

where the coefficients a_{ij} and b_i are smooth functions defined in $\mathbb{R}^N$. The matrix $A = (a_{ij})$, $i,j = 1,\dots,N$, is supposed to be symmetric and nonnegative definite at any point of $\mathbb{R}^N$.

Throughout the paper we shall denote by $z = (x,t)$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, the point of $\mathbb{R}^{N+1}$ and by Y the vector field in $\mathbb{R}^{N+1}$

$$Y := \sum_{i=1}^{N} b_i(x)\partial_{x_i} - \partial_t. \tag{1.2}$$

Moreover, we shall denote by $\mathcal{L}_0$ the *stationary* part of $\mathcal{L}$, i. e.

$$\mathcal{L}_0 = \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x)\partial_{x_j}) + \sum_{i=1}^{N} b_i(x)\partial_{x_i}. \tag{1.3}$$

We assume the following hypotheses.

(H1) $\mathcal{L}$ is hypoelliptic in $\mathbb{R}^{N+1}$ and homogeneous of degree two with respect to the group of dilations $(d_\lambda)_{\lambda>0}$ given by

$$\begin{aligned} d_\lambda(x,t) &= (D_\lambda(x), \lambda^2 t) \\ D_\lambda(x) &= D_\lambda(x_1,\dots,x_N) = (\lambda^{\sigma_1}x_1,\dots,\lambda^{\sigma_N}x_N), \end{aligned} \tag{1.4}$$

where $\sigma = (\sigma_1,\dots,\sigma_N)$ is an N-tuple of natural numbers satisfying $1 = \sigma_1 \le \sigma_2 \le \dots \le \sigma_N$. $\mathcal{L}$ is d_λ-homogeneous of degree two if

$$\mathcal{L}(u(d_\lambda(x,t))) = \lambda^2(\mathcal{L}u)(d_\lambda(x,t)) \qquad \forall u \in C^\infty(\mathbb{R}^{N+1}).$$

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(H2) For every $(x,t),(y,\tau) \in \mathbb{R}^{N+1}$, $t > \tau$, there exists an $\mathcal{L}$- admissible path $\eta : [0,T] \longrightarrow \mathbb{R}^{N+1}$ such that $\eta(0) = (x,t)$, $\eta(T) = (y,\tau)$.

An *$\mathcal{L}$-admissible* path is any continuous path η which is the sum of a finite number of diffusion and drift trajectories.
A *diffusion trajectory* is a curve η satisfying, at any points of its domain, the inequality

$$(\langle \eta'(s), \xi \rangle)^2 \le \langle \hat{A}(\eta(s)\xi, \xi \rangle \qquad \forall \xi \in \mathbb{R}^N.$$

Here $\langle , \rangle$ denotes the inner product in $\mathbb{R}^{N+1}$ and $\hat{A}(z) = \hat{A}(x,t) = \hat{A}(x)$ stands for the $(N+1) \times (N+1)$ matrix

$$\hat{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

A *drift trajectory* is a positively oriented integral curve of Y.

Throughout the paper we shall denote by Q the homogeneous dimension of $\mathbb{R}^{N+1}$ with respect to the dilations (1.4), i.e.

$$Q = \sigma_1 + \ldots + \sigma_N + 2$$

and we assume

$$Q \ge 5.$$

Then, the D_λ-homogeneous dimension of $\mathbb{R}^N$ is $Q - 2 \ge 3$.
We explicitly remark that the smoothness of the coefficients of $\mathcal{L}$ and the homogeneity assumption in (H1) imply that the a_{ij}' s and the b_i' s are polynomial functions (see [L], Lemma 2).

For any $z = (x,t) \in \mathbb{R}^{N+1}$ we define the d_λ-homogeneous norm $|\cdot|$ by

$$|z| = |(x,t)| := (|x|^4 + t^2)^{\frac{1}{4}}$$

where

$$|x| = |(x_1, \ldots, x_N)| = \left(\sum_{j=1}^N (x_j^2)^{\frac{\sigma}{\sigma_j}} \right)^{\frac{1}{2\sigma}}, \quad \sigma = \prod_{j=1}^N \sigma_j.$$

The class of the operators just introduced contains the one recently considered in [**KL**]. In particular, it contains the heat operators on Carnot groups, the prototype of Kolmogorov operators and the operators obtained by *linking* the previous ones (see [**KL**], Example 9.3 and 9.7). An example of operators satisfying our hypotheses (H1) and (H2), and not contained in [**KL**] is given by $\mathcal{L} = \partial_{x_1}^2 + x_1^3 \partial_{x_2} - \partial_t$ in $\mathbb{R}^3$.

The main result of this paper is the following Liouville-type theorem.

THEOREM 1.1. *Let $u : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ be a (smooth) solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{N+1}$. Suppose $u \ge 0$ and*

$$u(0,t) = O(t^m) \qquad \text{as} \quad t \longrightarrow \infty \tag{1.5}$$

for some $m \ge 0$. Then

$$u = \text{const.} \qquad \text{in} \quad \mathbb{R}^{N+1}. \tag{1.6}$$

Before proceeding we want to note that condition (1.5) cannot be removed in order to get (1.6). Indeed, for example, the function

$$u(x,t) = \exp(x_1 + x_2 + \ldots + x_N + Nt), \qquad x \in \mathbb{R}^N, \ t \in \mathbb{R},$$

is nonnegative, non-constant and satisfies the heat equation

$$\Delta u - \partial_t = 0 \qquad \text{in } \mathbb{R}^{N+1}, \qquad \Delta = \sum_{j=1}^N \partial_{x_j}^2.$$

We stress that u does not satisfy condition (1.5) since $u(0,t) = \exp(Nt)$.

From Theorem 1.1 a Liouville type theorem for $\mathcal{L}_0$ follows.

COROLLARY 1.2. *Let $v : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a (smooth) solution*[1] *to $\mathcal{L}_0 v = 0$ in $\mathbb{R}^N$. Then, if $v \ge 0$,*

$$v = \text{const.} \qquad \text{in } \mathbb{R}^N.$$

PROOF. The function

$$u : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}, \qquad u(x,t) = v(x)$$

satisfies $\mathcal{L}u = 0$ in $\mathbb{R}^{N+1}$. Moreover, $u \ge 0$ and

$$u(0,t) = v(0) \qquad \forall t \in \mathbb{R}.$$

Then, by Theorem 1.1, $u = \text{const.}$ in $\mathbb{R}^{N+1}$ so that $v = \text{const.}$ in $\mathbb{R}^N$. □

This Corollary extends to the present class of operators the Liouville Theorem 7.1 in [**KL**]. A Liouville type theorem for a very wide class of partial differential operators, homogeneous with respect to a group of dilations, was proved by Luo Xuebo in [**L**]. Luo Xuebo' s Theorem, which extends previous results by Geller [**G**] and Rothschild [**R**], also applies to our operators and, in this context, reads as follows.

THEOREM. *Let u be a tempered distribution satisfying, in the weak sense of distributions, the equation*

$$\mathcal{L}u = 0 \qquad \text{in } \mathbb{R}^{N+1}.$$

Then u is a polynomial function.

This result reduces the proof of Theorem 1.1 to the proof of the following

MAIN LEMMA. *Let $u : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ be a nonnegative smooth solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{N+1}$ satisfying condition (1.5). Then,*

$$u(z) = O(|z|^n) \qquad \text{as } |z| \longrightarrow \infty$$

for a suitable $n > 0$.

This Lemma, together with Luo Xuebo' s Theorem, immediately gives the

[1]Obviously, $\mathcal{L}_0$ is hypoelliptic in $\mathbb{R}^N$ since $\mathcal{L}$ is hypoelliptic in $\mathbb{R}^{N+1}$. Then, every distributional solution to $\mathcal{L}_0 v = 0$ is smooth.

PROOF OF THEOREM 1.1. Let u be a solution to $\mathcal{L}u = 0$ satisfying the hypotheses of Theorem 1.1. By the Main Lemma, u is a tempered distribution so that, by Luo Xuebo's Theorem, u is a polynomial function. Then, $u = u_0 + \ldots + u_m$, where u_k $(k = 0, 1, \ldots, m)$ is a polynomial function d_λ-homogeneous of degree k and $u_m \geq 0$, since $u \geq 0$. On the other hand, being $\mathcal{L}u = 0$ and $\mathcal{L}u_k$ d_λ-homogeneous of degree $k-2$, if $k \geq 2$, we have $\mathcal{L}u_k = 0$ for every $k = 0, 1, \ldots, m$. In particular $\mathcal{L}u_m = 0$. Since u_m is nonnegative and d_λ-homogeneous of degree $m \geq 0$, there exists $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$ such that

$$u_m(z_0) = \inf_{\mathbb{R}^{N+1}} u_m.$$

By the strong maximum principle (see next section, Proposition 2.2) we then have

$$u_m(x,t) = u_m(x_0, t_0) \qquad \forall\ (x,t) \in \mathbb{R}^N \times]-\infty, t_0[.$$

Since u_m is a polynomial function, this obviously implies

$$u_m(x,t) = u_m(x_0, t_0) \qquad \forall\ (x,t) \in \mathbb{R}^{N+1}.$$

Then $m = 0$ and $u \equiv u_0$ i.e. u is a constant function. □

2. A Harnack Inequality

In this section we shall prove the following Harnack inequality for nonnegative solutions to $\mathcal{L}u = 0$.

THEOREM 2.1. *Let $u : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ be a nonnegative solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{N+1}$. Then, there exist two positive constants $C = C(\mathcal{L})$ and $\theta = \theta(\mathcal{L})$ such that*

$$\sup_{C_{\theta r}} u \leq C u(0, r^2) \qquad \forall\ r > 0, \tag{2.1}$$

where, for $\rho > 0$, C_ρ denotes the d_λ-symmetric ball

$$C_\rho := \{z \in \mathbb{R}^{N+1} \mid |z| < \rho\}.$$

In order to prove this result, our main tool is a Mean-Value Theorem for the $\mathcal{L}$-harmonic functions, i.e. for the solutions to $\mathcal{L}u = 0$.

From hypotheses (H1) and (H2), by easily adapting the procedure already used in [**LP1**], [**BLU**] and [**KL**], we can prove the existence of a fundamental solution $\Gamma(z,\zeta)$ of $\mathcal{L}$ with the following properties.

(i) Γ is smooth in $\{(z,\zeta) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \mid z \neq \zeta\}$,
(ii) $\Gamma(\cdot,\zeta) \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ and $\mathcal{L}\Gamma(\cdot,\zeta) = -\delta_\zeta$ for every $\zeta \in \mathbb{R}^{N+1}$,
(iii) $\Gamma(z,\cdot) \in L^1_{\text{loc}}(\mathbb{R}^{N+1})$ and $\mathcal{L}^*\Gamma(z,\cdot) = -\delta_z$ for every $z \in \mathbb{R}^{N+1}$,
(iv) $\limsup_{\zeta \to z} \Gamma(z,\zeta) = \infty$ for every $z \in \mathbb{R}^{N+1}$,
(v) $\Gamma(0,\zeta) \longrightarrow 0$ as $\zeta \longrightarrow \infty$, $\Gamma(0, d_\lambda(\zeta)) = \lambda^{-Q+2}\Gamma(0,\zeta)$,
(vi) $\Gamma((x,t),(\xi,\tau)) \geq 0, > 0$ iff $t > \tau$,
(vii) $\Gamma((x,t),(\xi,\tau)) = \Gamma((x,0),(\xi,\tau - t))$.

In (iii) $\mathcal{L}^*$ denotes the formal adjoint of $\mathcal{L}$. We would like to stress that property (vi) follows from the invariance of $\mathcal{L}$ with respect to the translations parallel to the t-axis. The second part of property (vi) can be proved as in [**KL**], Section 2, by using the following strong maximum principle.

PROPOSITION 2.2. *Let u be a nonnegative solution to the equation $\mathcal{L}u = 0$ in the halfspace*

$$S := \mathbb{R}^N \times]-\infty, t_0[, \quad t_0 \in \mathbb{R}.$$

Suppose there exists a point $z_1 = (x_1, t_1) \in S$ such that

$$u(x_1, t_1) = 0.$$

Then $u = 0$ in $\mathbb{R}^N \times]-\infty, t_1[$.

PROOF. Let us denote by $P_{z_1}(S)$ the propagation set of z_1 in S, i.e. the set

$$P_{z_1}(S) = \{z \in S \ : \ \text{there exists an } \mathcal{L}\text{-admissible path } \eta : [0,T] \longrightarrow S \text{ s. t. } \eta(0) = z_1,\ \eta(T) = z\}.$$

The hypothesis (H2) implies $P_{z_1}(S) = \mathbb{R}^N \times]-\infty, t_1[$. On the other hand since z_1 is a minimum point of u and the minimum spreads all over P_{z_1} (see [**A**]), we get

$$u(z) = u(z_1) \qquad \forall\ z \in \mathbb{R}^N \times]-\infty, t_1[.$$

Then, the assertion follows since $u(z_1) = 0$. □

For every $(0,T) \in \mathbb{R}^{N+1}$ and $r > 0$ we define the *$\mathcal{L}$-ball* centered at $(0,T)$ and with radius r, as follows

$$\Omega_r(0,T) := \left\{\zeta \in \mathbb{R}^{N+1} \ : \ \Gamma((0,T),\zeta) > \left(\frac{1}{r}\right)^{Q-2}\right\}.$$

Then, if $\mathcal{L}u = 0$ in $\mathbb{R}^{N+1}$, the following Mean Value formula holds

$$u(0,T) = \left(\frac{1}{r}\right)^{Q-2} \int_{\Omega_r(0,T)} K(T,\zeta)\, u(\zeta)\, d\zeta, \tag{2.2}$$

where

$$K(T,\zeta) = \frac{< A(\xi)\nabla_\xi\Gamma, \nabla_\xi\Gamma >}{\Gamma^2}, \qquad \zeta = (\xi,\tau),$$

and Γ stands for $\Gamma((0,T),(\xi,\tau))$. Moreover, $<,>$ denotes the inner product in $\mathbb{R}^N$ and ∇_ξ is the gradient operator $(\partial_{\xi_1}, \ldots, \partial_{\xi_N})$.

Formula (2.2) is one of the numerous extensions of the classical Gauss Mean Value Theorem for harmonic functions. For a proof of it we directly refer to [**LP2**], Theorem 1.5.

The following lemmas will be crucial for our purposes.

LEMMA 2.3. *Let U be an open connected subset of $\mathbb{R}^{N+1}$. Let $u : U \longrightarrow \mathbb{R}$ be a smooth function such that*

$$A(x)\nabla_x u(x,t) = 0, \quad Yu(x,t) = 0 \qquad \forall\ (x,t) \in U. \tag{2.3}$$

Then u is constant in U.

PROOF. Let us denote by X_k the vector field

$$X_k := \sum_{j=1}^N a_{kj}\partial_{x_j}.$$

Since $\mathcal{L}$ is hypoelliptic and its coefficients are polynomial functions, the following rank condition holds (see [**D**])

$$\text{rank Lie}(X_1, \ldots, X_N, Y)(x,t) = N+1 \qquad \forall\ (x,t) \in \mathbb{R}^{N+1}. \tag{2.4}$$

On the other hand, by hypothesis (2.3),

$$Zu = 0 \quad \text{in } U \qquad \forall\ Z \in \text{Lie}(X_1, \ldots, X_N, Y).$$

Then, by the rank condition (2.4), $\nabla_z u(z) = 0$ at any point $z \in U$, and u is constant. □

LEMMA 2.4. *The closed set*

$$U := \{\zeta = (\xi, \tau) \ : \ K(T, \zeta) = 0, \ \tau < T\}$$

does not contain interior points.

PROOF. We argue by contradiction and assume $K(T, \zeta) = 0$ for every ζ in a non empty connected open set $U \subseteq \mathbb{R}^N \times]-\infty, T[$. Then, letting $h(\zeta) := \Gamma((0,T), \zeta)$, we have

$$A(\xi)\nabla_\xi h(\xi, \tau) = 0 \qquad \forall (\xi, \tau) \in U,$$

hence $\operatorname{div}(A\nabla h) \equiv 0$ in U. The $\mathcal{L}^*$-harmonicity of h now gives $Yh \equiv 0$ in U. Thus, by Lemma 2.3, $h = $ const. in U. This is absurd because $h(\zeta) = h(\xi, \tau) = \Gamma((0,0), (\xi, \tau - T))$ and $z \longmapsto \Gamma(0, z)$ is d_λ-homogeneous of degree $2 - Q \neq 0$. □

LEMMA 2.5. *There exists a positive constant $\theta = \theta(\mathcal{L})$ such that*

$$C_\theta \subseteq \Omega_{r_0}(0, 1).$$

PROOF. By the property (vi) of Γ, it is $\Gamma((0,1),(0,0)) > 0$. Then, for a suitable positive constant r_0 and θ_0, we have

$$\Gamma((0,1), \zeta) > \left(\frac{1}{r_0}\right)^{Q-2} \qquad \forall\, \zeta \in C_\theta.$$

This means that

$$C_{\theta_0} \subseteq \Omega_{r_0}(0, 1)$$

and the assertion is proved. □

We are now in the position to give the proof of Theorem 2.1.
Next Lemma easily follows from Theorem 7.1 in [**B**].

LEMMA 2.6. *Let (u_n) be a sequence of $\mathcal{L}$-harmonic function in an open set $\Omega \subseteq \mathbb{R}^{N+1}$:*

$$\mathcal{L}u_n = 0 \quad \text{in } \Omega \qquad \forall n \in \mathbb{N}.$$

Suppose (u_n) is monotone increasing and convergent in a dense subset of Ω. Then (u_n) converges at any point of Ω to a smooth function u such that $\mathcal{L}u = 0$ in Ω.

PROOF OF THEOREM 2.1. Since $\mathcal{L}$ is d_λ-homogeneous of degree two, it is enough to prove inequality (2.1) for $r = 1$. We argue by contradiction and assume that (2.1), with $r = 1$, is false. Then, there exists a sequence (u_n) of nonnegative $\mathcal{L}$-harmonic functions such that

$$\sup_{C_\theta} u_n \geq 4^n u_n(0, 1). \tag{2.5}$$

By the Mean Value formula (2.2),

$$u_n(0,1) = \left(\frac{1}{r_0}\right)^{Q-2} \int_{\Omega_{r_0}(0,1)} K(1, \zeta)\, u_n(\zeta)\, d\zeta, \quad n \in \mathbb{N}, \tag{2.6}$$

so that, since $\Omega_{r_0}(0,1) \supseteq C_\theta$, see Lemma 2.5,

$$u_n(0,1) \geq \left(\frac{1}{r_0}\right)^{Q-2} \int_{C_\theta} K(1, \zeta)\, u_n(\zeta)\, d\zeta.$$

On the other hand, by inequality (2.5) and Lemma 2.4, u_n and $K(1, \cdot)$ are strictly positive in a non-empty open subset of C_θ. It follows that $u_n(0,1) > 0$ for every $n \in \mathbb{N}$. Let us now put

$$v_n = \frac{u_n}{u_n(0,1)} \qquad \text{and} \qquad v = \sum_{n=1}^{\infty} \frac{v_n}{2^n}.$$

From the Mean Value formulas (2.6) we obtain

$$1 = v(0) = \left(\frac{1}{r_0}\right)^{Q-2} \int_{\Omega_{r_0}(0,1)} K(1, \zeta)\, v(\zeta)\, d\zeta,$$

so that, $v < \infty$ at any point of

$$T := \{\zeta \in \Omega_{r_0}(0,1) \ : \ K(1, \zeta) > 0\}.$$

By Proposition 2.2 the closure of T contains $\Omega_{r_0}(0,1)$. Then, by Lemma 2.6, v is finite and smooth in $\Omega_{r_0}(0,1)$. In particular v is continuous in C_θ. Then,

$$\sup_{C_\theta} v < \infty. \tag{2.7}$$

On the other hand, by inequality (2.5),

$$\sup_{C_\theta} \theta \geq \sup_{C_\theta} \frac{v_n}{2^n} = \frac{1}{2^n} \sup \frac{u_n}{u_n(0)} \geq 2^n.$$

Hence $\sup_{C_\theta} v \geq 2^n$ for every $n \in \mathbb{N}$. This contradicts (2.7) and proves the Theorem. □

With Theorem 2.1 at hand, the Main Lemma stated in the Introduction easily follows.

PROOF OF MAIN LEMMA. Let u be a nonnegative $\mathcal{L}$-harmonic function in $\mathbb{R}^{N+1}$ satisfying the growth condition (2.2). Then, by Theorem 2.1,

$$\sup_{|z| \leq \theta r} u(z) \leq Cu(0, r^2) \leq C_1(1 + r^{2n}).$$

This obviously implies

$$u(z) \leq C_2(1 + |z|^{2n}) \qquad \forall z \in \mathbb{R}^N.$$

□

References

[A] K. Amano, *Maximum principle for degenerate elliptic-parabolic operators*, Indiana Univ. Math. J. **29** (1979), 545–557.

[B] J.M. Bony, *Principe de maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier, Grenoble **19** (1969), 277–304.

[BLU] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, *Uniform gaussian estimates of the fundamental solutions for heat operators on Carnot groups*, Advances Diff. Equat. **7** (2002), 1153–1192.

[D] M. Derridj, *Un problème aux limites pour une classe d' operatéurs du second ordre hypoelliptiques*, Ann. Inst. Fourier (Grenoble) **21** (1971), 147–171.

[G] D. Geller, *Liouville' s Theorem for homogeneous groups*, Comm. in Partial Diff. Eq. **8** (1983), 1665–1677.

[KL] A.E. Kogoj and E. Lanconelli, *An invariant Harnack inequality for a class of hypoelliptic ultraparabolic equations*, Mediterr. J. Math., to appear.

[L] Luo Xuebo, *Liouville' s Theorem for homogeneous differential operators*, Comm. in Partial Diff. Eq. **22** (1997), 1813–1848.

[LP1] E. Lanconelli and A. Pascucci, *On the fundamental solution for hypoelliptic second order partial differential equations with non-negative characteristic form*, Ricerche di matematica **43** (1999), 81–106.

[LP2] ———, *Superparabolic Functions Related to Second Order Hypoelliptic Operators*, Potential Analysis **11** (1999), 303–323.

[R] L.P. Rothschild, *A remark on hypoellipticity of homogeneous invariant differential operators on nilpotent Lie groups*, Comm. P.D.E. **8** (1983), 1679–1682.

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