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Liouville theorems in halfspaces for parabolic hypoelliptic equations

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Abstract We prove some one-side Liouville-type theorems in halfspaces for a class of evolution hypoelliptic equations. The operators we deal with are left translation invariant, and homogeneous of degree two, on homogeneous Lie groups on $\mathbb{R}^{N+1}$.

Keywords Parabolic operators · Liouville Theorems · Liouville Theorems in halfspaces

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1 Introduction and main results

This paper deals with *one-side* Liouville theorems in *halfspaces* for the class of linear second order hypoelliptic operators first studied in [7].

Our main results apply, in particular, to the classical heat operator $H := \Delta - \partial_t$ in $\mathbb{R}^{N+1}$. In this case they read as follows.

Theorem A *Let u be a classical solution to the heat equation*

$$Hu := \Delta u - \partial_t u \quad \textit{in} \quad \mathbb{R}^N \times]-\infty, 0[.$$

(1) If $u \geq 0$, then

$$\lim_{t \longrightarrow -\infty} u(x,t)$$

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exists for every $x \in \mathbb{R}^N$ and is independent of x. Precisely

$$\lim_{t \longrightarrow -\infty} u(x,t) = \inf u$$

for every $x \in \mathbb{R}^N$.

(2) *If u is continuous up to $t = 0$, $u \geq 0$ and*

$$u(x,0) = O(|x|^n) \quad \textit{as } x \longrightarrow \infty$$

for a suitable $n \in \mathbb{R}$, then

$$u \equiv \textit{constant}.$$

It is well known that one side Liouville Theorems on all $\mathbb{R}^{N+1}$ does not hold for the heat operator.

Indeed, if $u \geq 0$ solves the heat equation in $\mathbb{R}^{N+1}$, then we cannot conclude that $u \equiv$ constant. For example, the function $u(x,t) = e^{x_1+\ldots+x_N+Nt}$ is a non-negative and non constant solution to $Hu = 0$ in $\mathbb{R}^{N+1}$.

Liouville properties in halfspaces for solutions to classical parabolic equations have been investigated by many authors. We only quote the papers by Hirschman [6], Glagoleva [3], [4], Tavkhelidze [9], Bear [1], which contain results closer to the ones of Theorem A. We directly refer to the references in these papers for related theorems on these subjects.

The aim of the present paper is to extend Theorem A to the class of operators of the following type.

$$\mathscr{L} = \sum_{j=1}^{m} X_j^2 + X_0 - \partial_t \qquad \text{in } \mathbb{R}^{N+1}, \tag{1}$$

where the X_j' s are first order differential operators in $\mathbb{R}^N$ with smooth coefficients, i.e.

$$X_j = \sum_{j=1}^{N} a_j^{(k)} \partial_{x_k}, \qquad j = 0, \ldots, m,$$

and $a_j^{(k)} \in C^\infty(\mathbb{R}^N, \mathbb{R})$ for every j and k.

We shall denote by $z = (x,t)$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, the point of $\mathbb{R}^{N+1}$, and by Y the first order operator

$$Y := X_0 - \partial_t \qquad \text{in } \mathbb{R}^{N+1}.$$

We shall also denote by

$$\mathscr{L}_0 := \sum_{j=1}^{m} X_j^2 + X_0$$

the *stationary* part of $\mathscr{L}$. Then, with these notations

$$\mathscr{L} = \mathscr{L}_0 - \partial_t.$$

As in the paper [7] we assume the following hypotheses on $\mathscr{L}$.

(H1) There exists a homogeneous Lie group in $\mathbb{R}^{N+1}$,

$$\mathbb{L} = (\mathbb{R}^{N+1}, \circ, d_\lambda)$$

such that
(i) $X_1, \ldots, X_m, Y$ are left translation invariant on $\mathbb{L}$.
(ii) $X_1, \ldots, X_m$ are d_λ-homogeneous of degree one and Y is d_λ-homogeneous of degree two.

(H2) For every $(x,t),(y,\tau) \in \mathbb{R}^{N+1}$, $t > \tau$, there exists an *$\mathscr{L}$-admissible* path $\eta : [0,T] \longrightarrow \mathbb{R}^{N+1}$ such that $\eta(0) = (x,t)$, $\eta(T) = (y,\tau)$. The curve η is called $\mathscr{L}$-admissible if it is absolutely continuous and satisfies

$$\eta'(s) = \sum_{j=1}^{m} \lambda_j(s) X_j(\eta(s)) + \mu(s) Y(\eta(s)) \qquad \text{a.e. in } [0,T]$$

for suitable piecewise constant real functions $\lambda_1, \ldots, \lambda_m$, and μ, $\mu \geq 0$.

We have already remarked in [7] that the hypothesis (H2) implies the hypoellipticity of $\mathscr{L}$ and $\mathscr{L}_0$. Moreover, from (H1) one obtain the following form of $\circ$ and d_λ:

$$(x,t) \circ (y,\tau) = (S(x,t,y,\tau), t+\tau),$$

$$d_\lambda(x,t) = (D(\lambda)x, \lambda^2 t),$$

where S is a smooth function with values in $\mathbb{R}^N$ and $D(\lambda) = \operatorname{diag}(\lambda^{\sigma_1}, \ldots, \lambda^{\sigma_N})$ is a dilation in $\mathbb{R}^N$, $1 \leq \sigma_1 \leq \ldots \leq \sigma_N$.

We shall denote by $Q = \sigma_1 + \ldots + \sigma_N + 2$ the homogeneous dimension of $\mathbb{L}$ and assume, as in [7], $Q \geq 5$.

Throughout the paper we shall denote

$$|z| = |(x,t)| := \left(|x|^4 + t^2\right)^{\frac{1}{4}},$$

where, if $x = (x_1, \ldots, x_N)$,

$$|x| := \left(\sum_{j=1}^{N} (x_j^2)^{\frac{\sigma}{\sigma_j}} \right)^{\frac{1}{2\sigma}}, \qquad \sigma = \prod_{j=1}^{N} \sigma_j.$$

We also put

$$d(z,\zeta) = |\zeta^{-1} \circ z|.$$

Then, $(z,\zeta) \longmapsto d(z,\zeta)$ is a pseudo-metric in $\mathbb{R}^{N+1}$. Precisely, for a suitable $C > 0$,

(i) $d(z,\zeta) \leq C d(\zeta,z)$;
(ii) $d(z,\zeta) \leq C(d(z,z_1) + d(z_1,\zeta))$ for every $z, z_1, \zeta \in \mathbb{R}^{N+1}$.

Throughout the paper we shall write $d(z)$ instead of $d(0,z) = |z^{-1}|$. Obviously, from (i), we have

$$\frac{1}{C}|z| \le d(z) \le C|z|.$$

In order to state our main theorem we need some more notation.

Let $\hat{\gamma} : [0,\infty[\longrightarrow \mathbb{R}^N$ be a continuous curve such that

$$\limsup_{s\to\infty} \frac{|\hat{\gamma}(s)|^2}{s} < \infty.$$

Then, the path

$$s \longmapsto \gamma(s) = (\hat{\gamma}(s), T-s), \qquad T \in \mathbb{R},$$

will be called a *$\mathscr{L}$-parabolic trajectory*.

Obviously the curve

$$s \longmapsto \gamma(s) = (\alpha, T-s), \qquad \alpha \in \mathbb{R}^N,\ T \in \mathbb{R},$$

is a $\mathscr{L}$-parabolic trajectory. We shall prove in Section 5 that every integral curve of the vector field Y also is a $\mathscr{L}$-parabolic trajectory (see Lemma 3). We would like to explicitly remark that the $\mathscr{L}$-parabolic trajectory $s \longmapsto (\alpha, T-s)$ need not be an integral curve of Y, see Example 2 in Section 5.

With the notion of $\mathscr{L}$-parabolic trajectory at hand we can state our first main result.

Theorem 1 *Let u be a bounded below solution to the equation*

$$\mathscr{L}u = 0$$

in the halfspace

$$S_T = \mathbb{R}^N \times]-\infty, T[, \qquad T \in \mathbb{R}.$$

Then

$$\lim_{s\to-\infty} u(\gamma(s)) = \inf_{S_T} u$$

for every $\mathscr{L}$-parabolic trajectory γ.

From the proof of this theorem we will see that it can be sharpened as follows

Corollary 1 *Let u be a nonnegative solution to*

$$\mathscr{L}u = 0$$

in the halfspace $S_T = \mathbb{R}^N \times]-\infty, T[,\ T \in \mathbb{R}$. Let $K \subseteq S$ be compact and $\gamma : [0,\infty[\to S$ be a $\mathscr{L}$-parabolic curve such that

$$\gamma(0) \in K.$$

Then

$$\lim_{s\to\infty} u(\gamma(s)) = \inf_{S_T} u$$

uniformly w.r. to $\gamma(0) \in K$.

For classical parabolic equations Theorem 1 and Corollary 1 were first proved by Glagoleva [3]. A somehow stronger version for the heat equation have been proved by Bear [1].

To clarify the meaning of our Theorem 1 let us give an example

Example 1 The Kolmogorov operator

$$\mathscr{L} = \partial_{x_1}^2 + x_1\partial_{x_2} - \partial_t \qquad \text{in } \mathbb{R}^3$$

satisfies hypotheses (H1) and (H2). The relevant homogeneous group of $\mathscr{L}$ is

$$\mathbb{L} = (\mathbb{R}^3, \circ, d_\lambda)$$

with composition law $\circ$ and dilation d_λ given by, respectively,

$$(x_1, x_2, t_1) \circ (y_1, y_2, t_2) = (x_1 + y_1, x_2 + y_2 + t_2 x_1, t_1 + t_2)$$

and

$$d_\lambda(x_1, x_2, t_1) = (\lambda x_1, \lambda^3 x_2, \lambda^2 t)$$

(see [7], Remark 9.5).
In the present case we have

$$Y = x_1\partial_{x_2} - \partial_t$$

whose integral curves are

$$s \longmapsto (\alpha, \beta + \alpha s, T - s), \qquad s \geq 0.$$

Then, if u is a non negative solution to $\mathscr{L}u = 0$ in the halfspace $\mathbb{R}^2 \times]-\infty, 0[$, we have

$$\inf_{\mathbb{R}^2 \times]-\infty, 0[} u = \lim_{s \to \infty} u(a, b + as, T - s)$$

for every $a, b, T \in \mathbb{R}, T \leq 0$.

From Theorem 1 one easily obtain the following Liouville property for $\mathscr{L}_0$, first proved in [7] Corollary 8.3 (see also [8] Corollary 1.2).

Corollary 2 *[Liouville Theorem for $\mathscr{L}_0$.] Let u be a nonnegative entire solution to*

$$\mathscr{L}_0 u = 0 \qquad \text{in } \mathbb{R}^N.$$

Then u is constant.

We now state our second main theorem.

Theorem 2 *Let $u \in C^\infty(\mathbb{R}^N \times]-\infty,0[) \cap C(\mathbb{R}^N \times]-\infty,0])$ be a nonnegative solution to*

$$\mathscr{L}u = 0 \quad \text{in } \mathbb{R}^N \times]-\infty,0[.$$

If there exists $n \in \mathbb{R}$ such that

$$u(x,0) = O(|x|^n) \quad \text{as } |x| \to \infty,$$

then u is constant.

In the case of the classical heat operator a stronger version of this corollary was proved by Hirschman and Bear, respectively in [6] and [1].

The paper is organized as follows.

In Section 2 we show a new version of the *Harnack inequality* for positive solution to $\mathscr{L}u = 0$ proved in [7] (Theorem 3).

In Section 3 we provide some gaussian estimates of Γ, the fundamental solution of $\mathscr{L}$, and of their derivatives (Theorem 4). These estimates are new and have some independent interest.

In Section 4 we prove that every solution to $\mathscr{L}u = 0$ in the halfspace $\mathbb{R}^N \times]-\infty,0[$ can be extended to a solution of the same equation in all $\mathbb{R}^N$ provided that u is continuous up to $t = 0$ and $u(x,0) = O(|x|^n)$ as $x \longrightarrow \infty$ (Theorem 5).

Finally, in Section 5, we prove Theorem 1, Corollary 2 and Theorem 2.

2 Harnack inequality

In this Section we prove a Harnack inequality which is a restatement of the one proved in [7], Theorem 7.1.

For every $M > 0$ and $z_0 \in \mathbb{R}^{N+1}$ let us denote

$$P(M) := \{(x,t) \in \mathbb{R}^N \times]-\infty,0[: \ |x|^2 \leq -Mt\}$$

and

$$P_{z_0}(M) := z_0 \circ P.$$

Then, we have the following theorem

Theorem 3 *Let u be a non-negative solution to*

$$\mathscr{L}u = 0 \qquad \text{in } \mathbb{R}^N \times]-\infty,0[.$$

Then, for every $z_0 \in \mathbb{R}^N \times]-\infty,0[$ and $M > 0$ there exists a constant $C = C(M)$, independent of z_0 and u, such that

$$\sup_{P_{z_0}} u \leq C u(z_0).$$

Proof We split the proof into three steps.

(I) Let u be a nonnegative solution to $\mathscr{L}u = 0$ in a open set $\Omega \supseteq \mathbb{R}^N \times]-\infty, 0]$. Then, for every $M > 0$ there exists a positive constant $C = C(M)$. independent of u, such that

$$\sup_{|x| \le M} u(x,-1) \le Cu(0,0).$$

This can be proved by proceeding exactly as in [7], pag.67.

(II) Let u be a nonnegative solution to $\mathscr{L}u = 0$ in an open set $\Omega \supseteq \mathbb{R}^N \times]-\infty, 0]$. Then, for every $M > 0$ there exists a constant $C = C(M)$ independent of u, such that

$$\sup_{P(M)} u \le Cu(0,0).$$

Proof Let $(\bar{x},\bar{t}) \in P(M)$ and put $\lambda := \sqrt{|\bar{t}|}$, $w_\lambda(z) = u(\delta_\lambda(z))$, $z \in \delta_{\frac{1}{\lambda}}(\Omega)$. Since $\delta_{\frac{1}{\lambda}}(\Omega)$ is an open set containing $\mathbb{R}^N \times]-\infty, o]$, by step (I) we have

$$\sup_{|x| \le M} w_\lambda(x,-1) \le Cw_\lambda(0,0) = Cu(0,0).$$

On the other hand

$$\sup_{|x| \le M} w_\lambda(x,-1) = \sup\{u(y,\bar{t}) \ : \ |y| \le M\sqrt{|\bar{t}|}\} \ge u(\bar{x},\bar{t}).$$

Then

$$u(\bar{x},\bar{t}) \le Cu(0,0) \qquad \forall (\bar{x},\bar{t}) \in P(M).$$

(III) Let us now assume that u satisfies the hypotheses of the Theorem and define

$$v(z) = u(z_0 \circ z), \quad z \in \Omega,$$

where $\Omega := z_0^{-1} \circ (\mathbb{R}^N \times]-\infty, 0[)$ is an open set containing $\mathbb{R}^N \times]-\infty, 0]$. We can then apply the conclusion of Step (II) to the function v obtaining

$$\sup_{P(M)} v \le Cv(0) \Longleftrightarrow \sup_{z_0 \circ P(M)} u \le Cu(z_0).$$

The Theorem is proved. □

3 Estimates for the derivatives of Γ

The aim of this Section is to prove gaussian estimates for the derivatives of Γ, the fundamental solution of $\mathscr{L}$ with pole at the origin.

If $\beta = (\beta_1, \ldots, \beta_m, \beta_{m+1})$ is a multi-index, we shall denote by X^β the following differential operator

$$X^\beta := X_1^{\beta_1} \circ \ldots \circ X_m^{\beta_m} \circ Y^{\beta_{m+1}}$$

and put

$$|\beta| = \beta_1 + \ldots + \beta_m + 2\beta_{m+1}.$$

Then X^β is a left translation invariant partial differential operator on $\mathbb{L}$ which is δ_λ-homogeneous of degree $|\beta|$.

Theorem 4 *For every multi-index β we have*

$$|X^\beta \Gamma(z)| \le C_\beta t^{-\frac{Q-2}{2}-\frac{|\beta|}{2}} \exp\left(-\frac{d^2(z)}{Ct}\right), \tag{2}$$

for every $z=(x,t)\in\mathbb{R}^{N+1}, t>0$, where C and C_β are positive constant, C_β depending on β.

For the proof of this Theorem we need the following lemma.

Lemma 1 *Let u be a nonnegative solution to*

$$\mathscr{L}u = 0 \text{ in } \Omega,$$

where Ω is an open set containing the strip $\mathbb{R}^N\times]-r^2,0]$. Then

$$\left|X^\beta u(0,-\frac{r^2}{2})\right| \le C_\beta r^{-|\beta|} u(0,0), \tag{3}$$

where $C_\beta>0$ is independent of u and r.

Proof Due to the homogeneity of $\mathscr{L}$ and X^β, it is enough to prove (3) for $r=1$. By Theorem 7.1 in [2], there exists $z_1,\dots,z_p\in\mathbb{R}^N\times]-1,0[$ such that

$$|X^\beta u(0,-\frac{1}{2})| \le C_\beta \sum_{j=1}^p u(z_j). \tag{4}$$

Let $M>0$ be such that

$$\{z_1,\dots,z_p\}\subseteq P(M).$$

By the Theorem 3.1, we have

$$u(z_j)\le Cu(0,0) \qquad \forall\ j=1,\dots,p,$$

where C only depends on M.

Using this inequality in (4), we get (3) for $r=1$ and the Lemma is proved. □

Corollary 3 *Let u be a nonnegative solution to*

$$\mathscr{L}u = 0 \qquad in\ \mathbb{R}^N\times]0,\infty[.$$

Then,

$$|X^\beta u(z)| \le C_\beta |t|^{\frac{-\beta}{2}} u\left(z\circ\left(0,\frac{t}{2}\right)^{-1}\right), \tag{5}$$

for every $z=(x,t)\in\mathbb{R}^N\times]0,\infty[$ for every multi-index $\beta=(\beta_1,\dots,\beta_m,\beta_{m+1})$.

Proof Let $z \in \mathbb{R}^N \times]0,\infty[$ be fixed and define

$$v(\zeta) = u\left(z \circ \left(0, -\frac{t}{2}\right)^{-1} \circ \zeta\right).$$

Since the time component of the composition law is the euclidean one, the function v is defined for every $\zeta = (\xi, \tau)$ with $\tau > -\frac{3}{2}t$. Then we can apply the previous lemma to the function v by choosing $r^2 = t$. We obtain

$$|X^\beta v\left(0, -\frac{t}{2}\right)| \le C_\beta t^{-\frac{\beta}{2}} v(0,0). \tag{6}$$

On the other hand, since X^β is left translation invariant,

$$X^\beta v\left(0, -\frac{t}{2}\right) = (X^\beta u)\left(z \circ \left(0, -\frac{t}{2}\right)^{-1} \circ \left(0, -\frac{t}{2}\right)\right) = X^\beta u(z).$$

Moreover $v(0,0) = u\left(z \circ \left(0, -\frac{t}{2}\right)^{-1}\right)$. By replacing these identities in (6), we obtain (5). □

We are now ready to give the

Proof (of Theorem 4) By the previous Corollary applied to the function $u := \Gamma$, we have

$$|X^\beta \Gamma(z)| \le C_\beta t^{-\frac{\beta}{2}} \Gamma\left(z \circ \left(0, -\frac{t}{2}\right)^{-1}\right) \tag{7}$$

On the other hand, from the gaussian estimates of Γ proved in [7], Section 5,

$$\Gamma\left(z \circ \left(0, \frac{t}{2}\right)^{-1}\right) \le C\left(\frac{t}{2}\right)^{-\frac{Q-2}{2}} \exp\left(-\frac{2d^2\left(z \circ \left(0, \frac{t}{2}\right)^{-1}\right)}{Ct}\right). \tag{8}$$

We now use the pseudo triangular inequality and the pseudo-symmetry of d to estimate

$$\begin{aligned} d\left(z \circ \left(0, \frac{t}{2}\right)^{-1}\right) &\ge Cd(z) - \frac{1}{C} d\left(0, \frac{t}{2}\right) \\ &= Cd(z) - \frac{1}{C} d(\delta_{\sqrt{\frac{t}{2}}}(0,1)) \\ &= Cd(z) - \frac{1}{C}\sqrt{\frac{t}{2}}\, d(0,1) \\ &= Cd(z) - C_1\sqrt{t} \end{aligned}$$

for suitable positive constants C and C_1. □

By using this estimates together with (8) and (7), we get (2).

4 An extension result

The aim of this section is to prove the following theorem.

Theorem 5 *Let us denote by S the halfspace*

$$S = \mathbb{R}^N \times]-\infty, 0[.$$

Let $u \in C^\infty(S) \cap C^0(\overline{S})$ *be a nonnegative solution to*

$$\mathscr{L} u = 0 \qquad \text{in } S.$$

Assume that

$$u(x,0) = O(|x|^n) \qquad \text{as } |x| \longrightarrow \infty,$$

for a suitable $n \in \mathbb{R}$.

Then, if we define

$$v : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}, \quad v(x,t) = \begin{cases} u(x,t) & \text{if } t \leq 0, \\ \int_{\mathbb{R}^N} \Gamma(x,t;\xi,0) u(\xi,0)\, d\xi & \text{if } t > 0 \end{cases}$$

we have

(i) $v \in C^\infty(\mathbb{R}^{N+1})$ *and* $\mathscr{L} v = 0$ *in* $\mathbb{R}^{N+1}$;
(ii) $v \geq 0$ *and* $v(0,t) = O(t^{\frac{n}{2}})$ *as* $t \longrightarrow \infty$;
(iii) $v|_S = u$.

To prove this theorem we shall use, together with the estimates of Γ and of its derivatives, a Gauss-Koebe type characterization of the solution to $\mathscr{L} u = 0$. Let us start by recalling the average operators already used in [7], Section 7. For $r > 0$ and $z_0 \in \mathbb{R}^{N+1}$, we define the $\mathscr{L}$-ball of radius r and "center" z as follows

$$\Omega_r(z) = \left\{ \zeta \in \mathbb{R}^{N+1} \ : \ \Gamma(\zeta^{-1} \circ z) > \left(\frac{1}{r}\right)^{Q-2} \right\}. \tag{9}$$

We explicitly remark that, if we denote $z = (x,t)$

$$\Omega_r(z) \subseteq \mathbb{R}^N \times]-\infty, t[. \tag{10}$$

Indeed, the "time" component of $\zeta^{-1} \circ z$ is $t - \tau$, where τ is the time component of ζ. Then, the assertion follows, keeping in mind that $\Gamma = 0$ in the halfspace $\mathbb{R}^N \times]-\infty, 0]$.

Let us now define the $\mathscr{L}$-average operator:

$$M_r u(z) := \left(\frac{1}{r}\right)^{Q-2} \int_{\Omega_r(z)} u(\zeta) K(\zeta^{-1} \circ z)\, d\zeta,$$

where

$$K(z) = \frac{\nabla_{\mathscr{L}} \Gamma(z)}{\Gamma(z)^2}, \qquad \nabla_{\mathscr{L}} = (X_1, \ldots, X_m),$$

see (7.5) in [7]. In that paper is proved that

$$u(z) = M_r u(z)$$

for every smooth solution to $\mathscr{L}u = 0$ in O and for every $\mathscr{L}$-ball $\Omega_r(z)$ such that $\overline{\Omega_r(z)} \subseteq O$.

We also have the reverse part of this Mean-Value Theorem. More precisely, directly from Corollary 3.4 and Example 2 in [5] we obtain

Proposition 1 *Let $u : O \longrightarrow \mathbb{R}$, $O \subseteq \mathbb{R}^{N+1}$ open, be a continuous function with the following Mean Value property: for every $z \in O$ there exists $r_z > 0$ such that*

$$u(z) = M_r u(z) \qquad \text{for every } r \in]0, r_z[.$$

Then $u \in C^\infty(O)$ and $\mathscr{L}u = 0$.

Corollary 4 *Let $u : O \longrightarrow \mathbb{R}$, $O \subseteq \mathbb{R}^{N+1}$ open, be a continuous function satisfying the following condition: there exists $t_0 \in \mathbb{R}$ such that*

$$u \in C^\infty(O_{t_0}), \quad \mathscr{L}u = 0 \text{ in } O_{t_0},$$

where O_{t_0} denotes the open set

$$O_{t_0} = \{(x,t) \in O \mid t \neq t_0\}.$$

Then $u \in C^\infty(O)$ and $\mathscr{L}u = 0$ in O.

Proof First of all, keeping in mind the inclusion (10), for every $z \in O_{t_0}$ there exists $r = r_z > 0$ such that

$$\overline{\Omega_r(z)} \subseteq O_{t_0} \qquad \text{for every } r \in]0, r_z[.$$

Then, since u is a smooth solution to $\mathscr{L}u = 0$ in O_{t_0},

$$u(z) = M_r(u)(z) \qquad \text{for every } z \in O_{t_0} \text{ and } 0 < r < r_z.$$

Let us now consider a point $z_0 = (x_0, t_0) \in O$ and a real number $r_{z_0} > 0$ such that $\overline{\Omega_r(z_0)} \subseteq O$ for every $r \in]0, r_{z_0}[$. For every fixed $r \in]0, r_{z_0}[$, we choose a sequence (z_j) in O_{t_0} such that $\overline{\Omega_r(z_j)} \subseteq O_{t_0}$. Then, by the continuity of u and the mean value property in O_{t_0}, we have

$$\begin{aligned} u(z_0) &= \lim_{j\to\infty} u(z_j) = \lim_{j\to\infty} M_r u(z_j) \\ &= \lim_{j\to\infty} \int_{\Omega_r(0,0)} u_j(z_j \circ \zeta^{-1}) K(\zeta)\, d\zeta \\ &= \int_{\Omega_r(0,0)} u(z_0 \circ \zeta^{-1}) K(\zeta)\, d\zeta \ = M_r u(z_0). \end{aligned}$$

This proves that u satisfies the hypotheses of Proposition 1. Thus $u \in C^\infty(O)$ and $\mathscr{L}u = 0$ in O. The Corollary is proved. □

Let us now show a result having an independent interest.

Proposition 2 *Let $\varphi : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a continuous function such that $\varphi(\xi) = O(|\xi|^n)$ as $|\xi| \longrightarrow \infty$. Define*

$$U(z) = U(x,t) := \int_{\mathbb{R}^N} \Gamma((\xi,0)^{-1} \circ z)\varphi(\xi)\, d\xi, \quad x \in \mathbb{R}^N,\ t > 0.$$

Then U is a solution to the Cauchy problem

$$\mathscr{L}U = 0 \quad \text{in } \mathbb{R}^N \times]0,\infty[,$$
$$\lim_{z \to (x_0,0)} U(z) = \varphi(x_0) \qquad \forall x_0 \in \mathbb{R}^N.$$

Moreover

$$U(0,t) = O(t^{\frac{n}{2}}) \quad \text{as } t \to \infty. \tag{11}$$

Proof From the estimates of Γ and of their derivatives (Theorem 3.1) we immediately obtain that U is a well defined and smooth function in $\mathbb{R}^N \times]0,\infty[$.

Moreover, by deriving under the integral

$$\mathscr{L}U(z) = \int_{\mathbb{R}^N} (\mathscr{L}\Gamma)((\xi,0)^{-1} \circ z)\varphi(\xi)\, d\xi = 0$$

for every $z \in \mathbb{R}^N \times]0,\infty[$.

Since

$$\int_{\mathbb{R}^N} \Gamma((\xi,o)^{-1} \circ z)\, d\xi = 1,$$

(see [7], Section 1), we can write

$$U(z) - \varphi(x_0) = \int_{\mathbb{R}^N} \Gamma((\xi,0)^{-1} \circ z)(\varphi(\xi) - \varphi(x_0))\, d\xi.$$

From this identity, by using very standard devices we get

$$\lim_{(x,t) \to (x_0,0), t>0} U(x,t) = \varphi(x_0), \quad \forall x_0 \in \mathbb{R}^N.$$

Finally we prove the asymptotic behavior (11).

From the estimates of Γ and the assumption on φ, for a suitable constant $C > 0$ we have

$$\begin{aligned}
|U(0,t)| &\le Ct^{-\frac{Q-2}{2}} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{Ct} d^2\left((\xi,0)^{-1} \circ (0,t)\right)\right)(1 + |\xi|^n)\, d\xi \\
&= \text{(using the change of variable } \xi = D_{\sqrt{t}}\eta \text{)} \\
&\quad C\int_{\mathbb{R}^N} \exp\left(-\frac{1}{C} d^2((\eta,0)^{-1} \circ (0,1))\right)(1 + t^{\frac{n}{2}}|\eta|^n)\, d\eta \\
&\le \text{(if } t > 1 \text{)}\ Ct^{\frac{n}{2}} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{C} d^2((\eta,0)^{-1} \circ (0,1))\right)(1 + |\eta|^n)\, d\eta \\
&= C^* t^{\frac{n}{2}}.
\end{aligned}$$

This completes the proof. □

We close this section by giving the

Proof (of Theorem 5) From Proposition 2 it follows that $v \in C^\infty(\mathbb{R}^{N+1} \setminus (\mathbb{R}^N \times \{0\})) \cap C(\mathbb{R}^{N+1})$. Moreover

$$\mathscr{L} v = 0 \quad \text{in } \mathbb{R}^{N+1} \setminus (\mathbb{R}^N \times \{0\}).$$

Now Corollary 4 shows that v is a smooth solution to $\mathscr{L} v = 0$ in $\mathbb{R}^{N+1}$. Finally, again from Proposition 2,

$$v(0,t) = O(t^{\frac{n}{2}}) \quad \text{as } t \to \infty.$$

□

5 Proof of the main theorems

We start with the proof of the following lemma

Lemma 2 *Let $\gamma : [0,\infty[\longrightarrow \mathbb{R}^{N+1}$, $\gamma(s) = (\hat{\gamma}(s), T - s)$ be a $\mathscr{L}$-parabolic trajectory. Then, there exists $M = M(\gamma) \in \mathbb{R}$ with the following property: for every $z \in \mathbb{R}^{N+1}$ there exists $t^* = t^*(z, M)$ such that*

$$\gamma(s) \in P_z(M) \qquad \forall s \geq t^*.$$

Proof Since γ is a $\mathscr{L}$-parabolic trajectory, there exists $M_0 > 0$ such that

$$|(\hat{\gamma}(s))|^2 \leq M_0 s \qquad \forall s \geq 1.$$

As a consequence, for every $z \in \mathbb{R}^{N+1}$ and $s \geq t^* = \max\{1, d^2(z)\}$, we have

$$\begin{aligned} d^2(z \circ \gamma(s)) &\leq C(d(z) + d(\gamma(s)))^2 \\ &\leq C_1(\sqrt{s} + \sqrt{M_0 s})^2 = C_1(1 + \sqrt{M_0})^2 s := Ms. \end{aligned}$$

□

We now prove the following lemma

Lemma 3 *Every integral curve of Y is a $\mathscr{L}$-parabolic trajectory.*

Proof To begin with we remark that the statement is equivalent to the following one:

$$\limsup_{s \to \infty} \frac{|\gamma_j(s)|^{\frac{1}{\sigma_j}}}{\sqrt{s}} < \infty \qquad \forall j = 1, \ldots, N, \tag{12}$$

where γ_j is the j-th component of γ:

$$\gamma(s) = (\gamma_1(s), \ldots, \gamma_N(s), \gamma_{N+1}(0) - s), \qquad s \geq 0.$$

Since Y is d_λ-homogeneous of degree two, we have

$$Y = \sum_{j=1; \sigma_j \geq 2}^{N} = a_j(x_1, \ldots, x_{j-1}) \partial_{x_j} - \partial_t$$

where a_j is a polynomial function d_λ-homogeneous of degree $\sigma_j - 2$. Then,

$$j_0 := \max\{j \in \{1,\dots,N\} : a_j \equiv 0\} > 1.$$

It follows that

$$\gamma_j' = 0 \qquad \text{for } 1 \le j \le j_0;$$
$$\gamma_j' = a_j(\gamma_1,\dots,\gamma_{j-1}) \qquad \text{for } j_0 < j \le N.$$

Then

$$\gamma_j(s) = \gamma_j(0) \qquad \forall\, s \ge 0,\ 1 \le j \le j_0,$$

so that (12) holds for $1 \le j \le j_0$. Arguing by induction we assume (12) holds for every $i \le j$ and prove that it holds for $i = j+1$. Then

$$\gamma_i(s) = \omega_i(s) s^{\frac{\sigma_i}{2}}, \qquad \omega_i \text{ bounded },\ 1 \le i \le j,$$

and

$$\begin{aligned}
\gamma_{i+1}' &= a_{i+1}(\gamma_1,\dots,\gamma_i)\\
&= a_{i+1}(\omega_1 s^{\frac{\sigma_1}{2}},\dots,\omega_i s^{\frac{\sigma_i}{2}})\\
&= (\text{since } a_{i+1} \text{ is } d_\lambda\text{-homogeneous of degree } \sigma_{i+1}-2)\\
&= s^{\frac{(\sigma_{j+1}-2)}{2}} a_{i+1}(\omega_1,\dots,\omega_i)\\
&= \omega_{i+1}^*(s) s^{\frac{(\sigma_{j+1}-2)}{2}}, \qquad \omega_{i+1}^* \text{ bounded.}
\end{aligned}$$

This completes the proof. □

Remark 1 We would explicitly remark that there are $\mathscr{L}$-parabolic trajectory that are not integral curve of Y. It is enough to observe that

$$\gamma(s) = (0,\dots,0,T-s), \qquad s \ge 0,\ T \in \mathbb{R},$$

is a $\mathscr{L}$-parabolic trajectory. Obviously this γ will be a Y integral curve if and only if $Y(0) = -\partial_t$. This condition is not satisfied, in general, by the operator in our class, as the following example shows.

Example 2 Let

$$\mathscr{L} = \partial_{x_1}^2 + (x_1\partial_{x_2} - \partial_{x_3})^2 + \partial_{x_2} - \partial_t.$$

This operator satisfies our hypoteses with respect to the homogeneous Lie group

$$\mathbb{L} = (\mathbb{R}^4, \circ, \delta_\lambda)$$

with composition law and dilations given by

$$(x_1,x_2,x_3,t)\circ(y_1,y_2,y_3,\tau) = (x_1+y_1, x_2+y_2+y_3x_1, x_3+y_3, t+\tau),$$

$$\delta_\lambda(x_1,x_2,x_3,t) = (\lambda x_1, \lambda^2 x_2, \lambda x_3, \lambda^2 t).$$

We explicitly remark that $Y = \partial_{x_2} - \partial_t$, hence

$$Y(0) \ne -\partial_t.$$

To prove that $\mathscr{L}$ satisfies hypotheses (H1) and (H2) we let

$$X_1 = \partial_{x_1} \qquad \text{and} \qquad X_2 = x_1 \partial_{x_2} - \partial_{x_3}$$

and we remark that

$$\mathscr{L} = X_1^2 + X_2^2 + [X_1, X_2] - \partial_t.$$

The operator

$$\mathscr{K} = X_1^2 + X_2$$

is the Kolmogorov-type operator in $\mathbb{R}^2 \times \mathbb{R}$ related to the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

(see [7], Example 9.3). Then, X_1 and X_2 are left translation invariant with respect to the composition law

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2 + y_3 x_1, x_3 + y_3).$$

It follows that $\partial_{x_2} = [X_1, X_2]$ is invariant too with respect the same law. Therefore the operator $\mathscr{L}$ is left translation invariant on $\mathbb{L}$. A direct computation shows that δ_λ is an automorphism of $(\mathbb{R}^4, \circ)$. Then $\mathscr{L}$ satisfies (H1).
Let us now show that $\mathscr{L}$ also satisfies (H2). Suppose we are given $(x,t), (y,\tau) \in \mathbb{R}^4$ with $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\tau < t$. Since rank $\mathrm{Lie}\{X_1, X_2\} = 3$ at any point, by Caratheodory-Chows Theorem there exists a $\mathscr{L}$-admissible path connecting (x,t) and $(y_1, y_2 - \tau + t, y_3, t) =: (\overline{y}, t)$. At this point, we can connect $(\overline{y}, t)$ with (y, τ) with a positively oriented integral curve of Y. This complete the proof that $\mathscr{L}$ satisfies (H1) and (H2).

Proof (of Theorem 1) We may suppose $\inf u = 0$. Let $\varepsilon > 0$ be fixed and choose a point $z_\varepsilon \in S$ such that

$$u(z_\varepsilon) < \varepsilon.$$

By the Harnack inequality of Theorem 2.1, there exists $C = C(M) > 0$, independent of z_ε, such that

$$\sup_{z_\varepsilon \circ P(M)} u \leq C u(z_\varepsilon).$$

Since γ is a parabolic trajectory, we have

$$\gamma(s) \in z_\varepsilon \circ P(M) \qquad \forall s > T$$

where $T = T(z_\varepsilon, M)$. Then

$$u(\gamma(s)) \leq C\varepsilon \qquad \forall s > T.$$

This proves the assertion because C is independent of ε. □

Proof (of Theorem 2) Let v be the function of Theorem 5 extending u to all $\mathbb{R}^{N+1}$. Then

$$v \geq 0, \quad \mathscr{L}v = 0 \quad \text{in } \mathbb{R}^{N+1}, \quad v(0,t) = O(t^{\frac{n}{2}}) \quad \text{as } t \longrightarrow \infty.$$

By Corollary 8.3 in [7], $v \equiv 0$ in $\mathbb{R}^{N+1}$. Hence $u \equiv 0$ in its domain. □

Proof (of Corollary 2) If we let $U(x,t) = u(x)$, then U is a nonnegative solution to $\mathscr{L}U = 0$ in $\mathbb{R}^{N+1}$. For every fixed $x \in \mathbb{R}^N$, let us put

$$\gamma_x(s) = (x, -s), \qquad s \geq 0.$$

Obviously, γ_x is a $\mathscr{L}$-parabolic trajectory. Then

$$\begin{aligned}\inf_{\mathbb{R}^N} u &= \inf_{\mathbb{R}^{N+1}} U \\ &= \text{(by Theorem 1) } \lim_{s\to\infty} U(\gamma_x(s)) \\ &= \lim_{s\to\infty} U(x,-s) = u(x)\end{aligned}$$

for every $x \in \mathbb{R}^N$, so that u is constant. □

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