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Liouville theorem for X-elliptic operators

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ABSTRACT

We prove a Liouville-type theorem for a class of degenerate elliptic operators of the form

$$\mathcal{L}u := \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}\partial_{x_j}u) + \sum_{i=1}^{N} b_i\partial_{x_i}u.$$

$\mathcal{L}$ is supposed to be X-elliptic, with respect to a family $X = (X_1, \dots, X_m)$ of locally Lipschitz continuous vector fields, in the sense introduced in [E. Lanconelli, A.E. Kogoj, X-elliptic operators and X-control distances, Contributions in Honor of the Memory of Ennio De Giorgi, Ricerche di Matematica 49 (Suppl.) (2000) 223–243].

1. Introduction

In this paper, we prove a Liouville Theorem for a class of linear second order partial differential operators in $\mathbb{R}^N$ of the form

$$\mathcal{L}u := \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}\partial_{x_j}u) + \sum_{i=1}^{N} b_i\partial_{x_i}u, \tag{1}$$

where $a_{ij} = a_{ji}$ and b_i are measurable functions in $\mathbb{R}^N$. We set $A = (a_{ij})_{i,j=1,\dots,N}$ and $b = (b_1, \dots, b_N)$.

Our general assumption is that $\mathcal{L}$ is uniformly X-elliptic, in the following sense. Let $X := \{X_1, \dots, X_m\}$ be a family of vector fields in $\mathbb{R}^N$, $X_j = (c_{j1}, \dots, c_{jN})$, $j = 1, \dots, m$, where the c_{jk}' s are locally Lipschitz continuous functions in $\mathbb{R}^N$. As usual, we identify the vector valued function X_j with the linear first order partial differential operator

$$X_j = \sum_{k=1}^{N} c_{jk}\partial_{x_k}.$$

Then, we say that the operator $\mathcal{L}$ in (1) is uniformly X-elliptic in $\mathbb{R}^N$ if

(E1) there exists a constant $\lambda > 0$ such that

$$\frac{1}{\lambda}\sum_{j=1}^{m}\langle X_j(x), \xi\rangle^2 \le q_{\mathcal{L}}(x,\xi) \le \lambda\sum_{j=1}^{m}\langle X_j(x), \xi\rangle^2 \quad \forall x, \xi \in \mathbb{R}^N, \tag{2}$$

where $q_{\mathcal{L}}(x,\xi)$ is the characteristic form of $\mathcal{L}$ given by

$$q_{\mathcal{L}}(x,\xi) := \langle A(x)\xi, \xi\rangle = \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j;$$

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(E2) there exists a function $\mathbf{b} \geq 0$ such that

$$\langle b(x), \xi \rangle^2 \leq \mathbf{b}^2(x) \sum_{j=1}^{N} \langle X_j(x), \xi \rangle^2 \quad \forall x, \xi \in \mathbb{R}^N. \tag{3}$$

In (E1) and (E2) $\langle , \rangle$ denotes the usual inner product in $\mathbb{R}^N$.

The notion of X-elliptic operator was explicitly introduced in 2000 in the paper [1]. However several families of operators falling into the X-elliptic class were already present in literature, see e.g. [2–6]. More recently, X-elliptic operator have been widely studied in [7], where a Maximum Principle, a non homogeneous Harnack inequality and a Liouville Theorem are proved.

1.1. Basic assumptions

To introduce our basic assumptions on the geometric structure underlying the operator $\mathcal{L}$ we need to recall the definition of Carnot-Carathéodory distance (or control distance) $d = d_X$ related to the family X. A piecewise regular path $\gamma : [0, 1] \longrightarrow \mathbb{R}^N$ is an X-trajectory if

$$\dot{\gamma}(t) = \sum_{j=1}^{m} a_j(t) X_j(\gamma(t)) \quad \text{a.e. in } [0, 1].$$

We set

$$|\gamma| = \sup_{t \in [0,1]} \left(\sum_{j=1}^{m} a_j^2(t) \right)^{\frac{1}{2}}$$

and denote by $\Gamma(x, y)$ the set of the X-trajectories connecting the points $x, y \in \mathbb{R}^N$. One defines

$$d(x, y) = d_X(x, y) := \inf\{|\gamma| \ : \ \gamma \in \Gamma(x, y)\}.$$

If $\Gamma(x, y) \neq \emptyset$ for every $x, y \in \mathbb{R}^N$, then d is a metric called the control distance related to X.
Then, we assume the following hypotheses are satisfied:

(H1) The control distance $d = d_X$ is well defined, the metric space $(\mathbb{R}^N, d)$ is complete and the d-topology is the Euclidean one. Moreover there exists $A > 1$ such that the following *doubling condition* holds

$$0 < |B_{2r}| \leq A|B_r|, \tag{4}$$

for every d-ball B_r of radius r, $0 < r < \infty$, hereafter $|E|$ will denote the Lebsgue measure of the set $E \subseteq \mathbb{R}^N$.

(H2) There exist positive constants C, ν such that the following *Poincaré inequality* holds

$$\fint_{B_r} |u - u_r| \mathrm{d}x \leq C\, r \fint_{B_{\nu r}} |Xu| \mathrm{d}x, \quad \forall u \in C^1(\overline{B_{\nu r}}) \tag{5}$$

for any d-ball B_r, with $u_r = \fint_{B_r} u\mathrm{d}x := \frac{1}{|B_r|} \int_{B_r} u\mathrm{d}x$.

Xu denote the *X-gradient* of u, i.e.

$$Xu = (X_1 u, \ldots, X_m u).$$

Finally, about the lower order terms of $\mathcal{L}$ we shall assume the following condition

(LT)

$$\mathbf{b}(x) \leq \frac{C}{1 + d(x)} \quad \text{for every } x \in \mathbb{R}^N, \tag{6}$$

where C is a suitable positive constant, $\mathbf{b}$ is the function in Eq. (3) and $d(x) = d(x, 0)$.

Then, our main result reads as follows. For the notion of *weak solution* we directly refer to the next subsection.

Theorem 1 (*Main Theorem*). *Let $\mathcal{L}$ be an X-elliptic operator and let u a be a nonnegative weak solution to the equation*

$$\mathcal{L}u = 0 \quad \text{in } \mathbb{R}^N.$$

Assume, together with (H1) *and* (H2) *, one of the following hypotheses is satisfied.*

(1.1) *The lower order terms b_j's are identically zero.*
(1.2) *Condition (LT) is verified and there exists a sequence $r_j \nearrow \infty$ such that $\partial B_{r_j}(0)$ is connected.*

Then, u is constant in $\mathbb{R}^N$.

Remark 2. In (1.2) the assumption on the connectedness of $\partial B_r(0)$ cannot be removed, as the following example shows (see [7], Remark 5.5).
The equation

$$u'' + \frac{2x}{1+x^2}u' = 0$$

has a nonnegative bounded solution which is non constant: $u(x) = \pi/2 + \arctan x$. Condition (LT) is satisfied but $\partial B_r(0) = \partial]-r, r[= \{-r, r\}$ is not connected.

1.2. *Weak solutions to* $\mathcal{L}u = 0$

To give the definition of *weak solution* it is convenient to first show some consequences of hypotheses (H1) and (H2).

Remark 3. The doubling condition (4) implies that $(\mathbb{R}^N, d)$ is a homogeneous space, in the following sense: for every d-ball of radius r one can find at most M disjoint balls of radius $r/2$, with M only depending on the doubling constant A (see [8], page 67). It follows that every d-bounded set $F \subseteq \mathbb{R}^N$ is d-totally bounded so that, since $(\mathbb{R}^N, d)$ is complete, $\overline{F}^d$ is d-compact, hence Euclidean compact, hence Euclidean bounded. Vice-versa, if $F \subseteq \mathbb{R}^N$ is Euclidean bounded, then its Euclidean closure $\overline{F}$ is Euclidean compact, hence d-compact, hence d-bounded.

Remark 4. $(\mathbb{R}^N, d)$ is a *lenght* space, i.e. the distance between any pair of points equals the infimum of the length of rectifiable paths joining them. This easily follows from the fact that if $\gamma : [0, T] \longrightarrow \mathbb{R}^N$ is a X-subunit curve, i.e. $|\gamma| \le 1$, and $0 \le t_1 \le t_2 \le \cdots \le t_p = T$, then $d(\gamma(t_{k-1}), \gamma(t_k)) \le t_k - t_{k-1}$ for $k = 1 \ldots, p$ and hence $T \ge$ metric length of γ.

It is well known that conditions (4) and (5) and the topological properties of d imply the following

Remark 5. For every $x, y \in \mathbb{R}^N$ there exists a *d-segment* connecting x and y, i.e. there exists a continuous curve $\gamma : [0, 1] \longrightarrow \mathbb{R}^N$ such that $\gamma(0) = x,\ \gamma(1) = y$ and

$$d(x, y) = d(x, \gamma(t)) + d(\gamma(t), y) \quad \forall t \in [0, 1]$$

(see [9], Lemma 3; see also [2,10]).

Remark 6. For every ball $B(x, r)$ and every $\lambda \in]0, 1[$ the ring

$$B(x, r) \setminus B(x, \lambda r)$$

is non empty. Indeed since $\mathbb{R}^N$ is Euclidean unbounded, hence d-unbounded, there exists $y \in \mathbb{R}^N$ such that $d(x, y) > r$. Let $\gamma : [0, 1] \longrightarrow \mathbb{R}^N$ be a d-segment connecting x and y. Then

$$g : [0, 1] \longrightarrow \mathbb{R}^N, \quad g(t) = d(x, \gamma(t))$$

is a continuous function such that $g(0) = 0$ and $g(1) = d(x, y) > r$. Then there exists $t \in]0, 1[$ such that $g(t) \in]\lambda r, r[$. Thus

$$\lambda r < d(x, \gamma(t)) < r, \quad \text{i.e. } \gamma(t) \in B(x, r) \setminus B(x, \lambda r).$$

Remark 7 (*Reverse Doubling*). There exists $\theta \in]0, 1[$, only depending on the doubling constant A such that

$$|B(x_0, r)| \le \theta |B(x, 2r)| \quad \forall x \in \mathbb{R}, \forall r > 0.$$

This follows from the previous remark, just proceeding exactly as in [10], Section 2.4.

Remark 8 (*Poincaré-Sobolev Inequality*). Let $Q = \log_2 A$, where A is the constant in the doubling condition (4), $1 < q < Q$ and $p = qQ/(Q - q)$. Then

$$\left(\fint_{B_r} |u - u_r|^p \mathrm{d}x \right)^{\frac{1}{p}} \le S \left(r \fint_{B_{\theta r}} |Xu|^q \mathrm{d}x \right)^{\frac{1}{q}}, \tag{7}$$

for any d-ball B_r, for any $u \in C^1(\overline{B_{\theta r}})$.
The constant S and θ only depend on p, the doubling constant A and the constant C and ν in the Poincaré inequality (5). (See [11], Theorem 5.1; see also [9,12].) Since it is not restrictive to assume the doubling constant $A > 4$, from now on we suppose $Q > 2$.

Remark 9 (*Sobolev Inequality*). For every bounded open set Ω there exists a positive constant C such that

$$\|u\|_{L^p(\Omega)} \le C \|Xu\|_{L^2(\Omega)} \quad \forall u \in C_0^1(\Omega),\ p = \frac{2Q}{Q-2}. \tag{8}$$

Obviously, it is enough to prove this statement when Ω is a d-ball $B_r = B(x, r)$. Let $u \in C_0^1(B_r)$. Then

$$\|u\|_{L^p(B_r)} = \|u\|_{L^p(B_{2r})} \le \|u - u_{B_{2r}}\|_{L^p(B_{2r})} + |u_{B_{2r}}|\, |B_{2r}|^{\frac{1}{p}}.$$

On the other hand

$$|u_{B_{2r}}| \le \frac{1}{|B_{2r}|}\int_{B_r} |u| \mathrm{d}y \le \frac{|B_r|^{1-\frac{1}{p}}}{|B_{2r}|}\|u\|_{L^p(B_r)}.$$

By using this inequality in the previous one, we get

$$\begin{aligned}\|u\|_{L^p(B_r)} &\le \|u - u_{B_{2r}}\|_{L^p(B_{2r})} + \left(\frac{|B_r|}{|B_{2r}|}\right)^{1-\frac{1}{p}} \|u\|_{L^p(B_r)} \\ &\le \text{(by the Reverse doubling of Remark 7)} \|u - u_{B_{2r}}\|_{L^p(B_{2r})} + \theta^{1-\frac{1}{p}}\|u\|_{L^p(B_r)}.\end{aligned}$$

Hence

$$\begin{aligned}\|u\|_{L^p(B_r)} &\le \frac{1}{1-\theta^{1-\frac{1}{p}}}\|u - u_{B_{2r}}\|_{L^p(B_{2r})} \\ &\le \text{(by the Poincar'e inequality (5), and keeping in mind that } u \text{ is supported in } B_r)\frac{C}{1-\theta^{1-\frac{1}{p}}}\|Xu\|_{L^2(B_r)}.\end{aligned}$$

Now, we are able to give our notion of *weak solution* to $\mathcal{L}u = 0$.

Let Ω be a bounded open subset of $\mathbb{R}^N$. Due to the Sobolev inequality (8) the function $u \mapsto \|Xu\|_{L^2(\Omega)}$ is a norm in $C_0^1(\Omega)$ and, as usual, we define the space $W_0^1(\Omega, X)$ as the closure of $C_0^1(\Omega)$ with respect to this norm.

If $u \in W_0^1(\Omega, X)$ then $X_j u$ exists in the sense of distributions and $X_j u \in L^2(\Omega)$ for $j = 1, \ldots, m$. Hence the X-gradient is well defined for any $u \in W_0^1(\Omega, X)$. We denote by $W^1(\Omega, X)$ the space

$$\{u \in L^2(\Omega) : Xu \in L^2(\Omega)\}.$$

We have the following inclusions

$$W_0^1(\Omega, X) \subset W^1(\Omega, X) \subset W_{\mathrm{loc}}^1(\Omega, X),$$

with an obvious meaning for $v \in W_{\mathrm{loc}}^1(\Omega, X)$.

To define the notion of weak solution to the equation $\mathcal{L}u = 0$ in Ω, we introduce the bilinear form

$$L(u, v) = \int_\Omega \{\langle ADu, Dv\rangle + \langle b, Du\rangle v\}\mathrm{d}x,$$

for $u \in C^1(\Omega)$ and $v \in C_0^1(\Omega)$. Since $A \ge 0$, we have

$$|L(u, v)| \le \int_\Omega \{\langle ADu, Du\rangle^{\frac{1}{2}}\langle ADv, Dv\rangle^{\frac{1}{2}} + |\langle b, Du\rangle||v|\}\mathrm{d}x.$$

Therefore, since $\mathcal{L}$ is X-elliptic,

$$|L(u, v)| \le \lambda \int_\Omega |Xu||Xv|\mathrm{d}x + \int_\Omega \mathbf{b}|Xu||v|\mathrm{d}x. \tag{9}$$

Since Ω is bounded (hence d-bounded) and $\mathbf{b} \in L^\infty(\mathbb{R}^N)$ the bilinear form $(u, v) \mapsto L(u, v)$ can be extended continuously to $W^1(\Omega, X) \times W_0^1(\Omega, X)$.

A function $u \in W_{\mathrm{loc}}^1(\mathbb{R}^N, X)$ is a *weak solution* to $\mathcal{L}u = 0$, if

$$L(u, v) = 0 \quad \forall v \in C_0^1(\mathbb{R}^N).$$

1.3. Comparison with previous results

Liouville-type theorems for several classes of X-elliptic operators or, equivalently, for degenerate elliptic operators with underlying Carnot-Carathéodory structures, are present in literature. Korany and Stanton proved a Liouville theorem for the Heisenberg Laplacian in [13]. This theorem was extended by Varopoulos [5] to the Laplacians on general Lie groups with polynomial growth. Liouville-type Theorems for sub-Laplacians on stratified Lie groups are also contained in [14]. Lancia and Marchi [6] proved a Liouville Theorem for operators which are X-elliptic with respect to the vector fields generating the Lie algebra of the Heisenberg group in $\mathbb{R}^{2n+1}$. General X-elliptic operators satisfying our Assumption (E1) and (E2), (H1) and (H2) have been studied in [7]. In that paper, a Liouville Theorem is proved, assuming a weaker form of (6) and the extra condition that the vector fields of the family X are homogeneous of degree one with respect to a family of dilations in $\mathbb{R}^N$. Here we strongly relax this condition, just assuming the connectedness of $\partial B_r(0)$ for a divergent sequence of radius.

2. Global cut-off function

Our proof of Theorem 1 is based on a result concerning "global"cut-off function. Precisely:

Theorem 10. *Let B_{r_1} and B_{r_2} be two concentric d-balls with $0 < r_1 < r_2 < \infty$. Then there exists $\eta \in W_0^1(B_{r_2})$ such that $\eta = 1$ a.e. in B_{r_1} and*

$$|X\eta| \le \frac{2}{r_2 - r_1} \quad \text{a.e. in } B_{r_2}.$$

To prove this Theorem we need the following Lemma, a consequence of Proposition 2.9 and inequality (2.2) in [15].

Lemma 11. *Let $x_0 \in \mathbb{R}^N$ be fixed and define*

$$\rho : \mathbb{R}^N \longrightarrow \mathbb{R}, \quad \rho(x) = d(x_0, x).$$

Then $\rho \in W^1_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$|X\rho| \le 1 \quad \text{a.e. in } \mathbb{R}^N. \tag{10}$$

Proof. By Proposition 2.9 in [15] we know that $\rho \in W^1_{\mathrm{loc}}(\mathbb{R}^N)$ with $|X\rho| \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$. We want to show that

$$|X_j\rho| \le 1 \quad \text{a.e. in } \mathbb{R}^N, j = 1, \dots, m.$$

Obviously, from these inequalities, (10) follows. For every $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$ sufficiently small, define

$$\exp(tX_j)(x) := \gamma(x, t),$$

where $\gamma(x, \cdot)$ is the solution to the Cauchy problem $\dot\gamma = X_j(\gamma)$, $\gamma(0) = x$. We explicitly remark that for every fixed compact set K there exists $T = T(K) > 0$ such that $\exp(tX_j)(x)$ is well defined for every $x \in K$ and $t \in]0, T[$. Then, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$ the function

$$\phi(t) := \int_{\mathbb{R}^N} \Delta\rho(x, t)\varphi(x)\mathrm{d}x, \tag{11}$$

with

$$\Delta\rho(x, t) := \frac{\rho\left(\exp(tX_j)(x)\right) - \rho(x)}{t}$$

is well defined for $0 < t < T = T(\operatorname{supp}\varphi)$.

From the definition of control distance, and keeping in mind that $t \mapsto \exp(tX_j)(x)$ is a X-sub-unit curve, we get

$$\begin{aligned}\rho(\exp(tX_j)(x)) - \rho(x) &= d(x_0, \exp(tX_j)(x)) - d(x_0, x)\\ &\le d(x_0, x) + d(x, \exp(tX_j)(x)) - d(x_0, x) \le t.\end{aligned}$$

Then $|\Delta\rho(x, t)| \le 1$. As a consequence

$$|\phi(t)| \le \int_{\mathbb{R}^N} |\varphi(t)|\mathrm{d}x.$$

On the other hand

$$\begin{aligned}\phi(t) &= \frac{1}{t}\left(\int_{\mathbb{R}^N} \rho(\exp(tX_j)(x))\varphi(x)\mathrm{d}x - \int_{\mathbb{R}^N} \rho(x)\varphi(x)\mathrm{d}x\right)\\ &= (\text{ by using the change of variable } y = \exp(tX_j)(x) \iff x = \exp(-tX_j)(y))\\ &= \frac{1}{t}\left(\int_{\mathbb{R}^N} \rho(y)\varphi(\exp(-tX_j)(y))\mathcal{J}(y, t)\mathrm{d}y - \int_{\mathbb{R}^N} \rho(x)\varphi(x)\mathrm{d}x\right), \quad \text{where } \mathcal{J}(y, t) = \left|\det\left(\frac{\partial x}{\partial y}\right)\right|.\end{aligned}$$

Therefore, letting

$$\Delta^*\varphi(y, t) := \frac{\varphi(\exp(-tX_j)(y))\mathcal{J}(y, t) - \varphi(y)}{t},$$

we have

$$\phi(t) = \int_{\mathbb{R}^N} \rho(y)\Delta^*\varphi(y, t)\mathrm{d}y.$$

Now we claim that

$$\lim_{t\to 0}\phi(t)=\int_{\mathbb{R}^N} X\rho(x)\varphi(x)\mathrm{d}x. \tag{12}$$

Taking this claim for granted, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^N}(X\rho)(x)\varphi(x)\mathrm{d}x\right| &= \left|\lim_{t\to 0}\phi(t)\right| \\ &\le \limsup_{t\to 0}\int_{\mathbb{R}^N}|\Delta\rho(x,t)||\varphi(x)|\mathrm{d}x \\ &\le \int_{\mathbb{R}^N}|\varphi(x)\mathrm{d}x.\end{aligned}$$

Then

$$\left|\int_{\mathbb{R}^N}(X\rho)(x)\varphi(x)\mathrm{d}x\right| \le \int_{\mathbb{R}^N}|\varphi(x)|\mathrm{d}x$$

for every $\varphi\in C_0^\infty(\mathbb{R}^N)$. This implies

$$\sup\operatorname{ess}|X\rho|\le 1.$$

We are thus left with the proof of the claim. We may suppose there exists a compact set K containing the support of $\Delta^*\varphi(\cdot,t)$ for every $t\in]0,T[$. Choosing a function $\psi\in C_0^\infty(\mathbb{R}^N)$ such that $\psi=1$ on K, we set $\rho_0:=\rho\psi$. Then $\rho_0\in W_0^1(\Omega,X)$ being Ω a bounded open set containing K. Then, there exists a sequence $(\rho_k)_{k\ge 1}$ in $C_0^1(\Omega)$ such that

$$\rho_k\longrightarrow\rho_0 \quad \text{as } k\longrightarrow\infty,\ \text{in } W_0^1(\Omega,X).$$

Therefore

$$\begin{aligned}\phi(t)-\int_{\mathbb{R}^N}(X\rho)\varphi\mathrm{d}x &= \int_{\mathbb{R}^N}\rho_0\Delta^*\varphi(\cdot,t)\mathrm{d}x-\int_{\mathbb{R}^N}(X\rho_0)\varphi\mathrm{d}x \\ &= \int_{\mathbb{R}^N}(\rho_0-\rho_k)\Delta^*\varphi(\cdot,t)\mathrm{d}x+\int_{\mathbb{R}^N}\rho_k\Delta^*\varphi(\cdot,t)\mathrm{d}x-\int_{\mathbb{R}^N}(X\rho_0)\varphi\mathrm{d}x \\ &= \int_{\mathbb{R}^N}(\rho_0-\rho_k)\Delta^*\varphi(\cdot,t)\mathrm{d}x+\int_{\mathbb{R}^N}\Delta\rho_k(\cdot,t)\varphi\mathrm{d}x-\int_{\mathbb{R}^N}(X_j\rho_0)\varphi\mathrm{d}x \\ &= \int_{\mathbb{R}^N}(\rho_0-\rho_k)\Delta^*\varphi(\cdot,t)\mathrm{d}x+\int_{\mathbb{R}^N}(\Delta\rho_k(\cdot,t)-X_j\rho_k)\varphi\mathrm{d}x+\int_{\mathbb{R}^N}(X_j\rho_k-X_j\rho_0)\varphi\mathrm{d}x.\end{aligned} \tag{13}$$

From inequality (2.2) in [15] we get

$$\begin{aligned}|\Delta^*\varphi(x,t)| &\le \frac{1}{t}|\varphi(\exp(tX_j))(x)-\varphi(x)|+\frac{1}{t}|\varphi(\exp(tX_j)(x))||\mathcal{J}(x,t)-1| \\ &\le \frac{1}{t}|\varphi(\exp(tX_j))(x)-\varphi(x)|+C_1\sup_K|\varphi|\end{aligned}$$

for a suitable constant $C_1>0$. Thus

$$\begin{aligned}\limsup_{t\longrightarrow 0}|\Delta^*\varphi(\cdot,t)| &\le |X_j\varphi|+C_1\sup|\varphi| \\ &\le C_2 \quad \text{on } K.\end{aligned}$$

Using this estimate in (13), we obtain

$$\begin{aligned}\limsup_{t\longrightarrow 0}\left|\phi(t)-\int_{\mathbb{R}^N}(X\rho)\varphi\mathrm{d}x\right| &\le C_2|\Omega|^{\frac{1}{2}}\|\rho_0-\rho_k\|_{L^2(\Omega)} \\ &\quad +\limsup_{t\longrightarrow 0}\int_{\mathbb{R}^N}|\Delta\rho_k(\cdot,t)-X_j\rho_k||\varphi|\mathrm{d}x+\|X_j\rho_k-X_j\rho_0\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}.\end{aligned}$$

On the other hand, since ρ_k is C^1, $\Delta_{\rho_k}(x,t)\longrightarrow X_j\rho_k(x)$ as $t\longrightarrow 0$, for every $x\in\Omega$. Moreover, the mean value theorem gives $\Delta\rho_k(x,t)=X_j\varphi(\exp(\tau X_j)(u))$ for a suitable $\tau\in]0,t[$. Therefore

$$\sup_K|\Delta\rho_k(\cdot,t)-X_j\rho_k|\le 2\sup_K|X_j\varphi|$$

and the Lebesgue Dominated Convergence Theorem implies

$$\lim_{t\longrightarrow 0}\int_{\mathbb{R}^N}|\Delta_{\rho_k}(\cdot,t)-X_j\rho_k|\,|\varphi|\mathrm{d}x=0.$$

Hence

$$\limsup_{t\longrightarrow 0}\left|\phi(t)-\int_{\mathbb{R}^N}(X_j\rho)\varphi\mathrm{d}x\right|\le C_2|\Omega|^{\frac{1}{2}}\|\rho_k-\rho\|_{L^2(\Omega)}+\|X_j\rho_k-X_j\rho_0\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}\quad\text{for every } k\in\mathbb{N}.$$

Letting k go to infinity at the right hand side we get

$$\limsup_{t\longrightarrow 0}\left|\phi(t)-\int_{\mathbb{R}^N}(X_j\rho)\varphi\mathrm{d}x\right|\le 0.$$

This completes the proof. $\square$

Proof of Theorem 10. Let f be a real smooth function defined on $\mathbb{R}$ such that

$$f=1\quad\text{on } [-r_1,r_1],\qquad f=0\quad\text{on } \mathbb{R}\setminus[-r,r]\quad\text{for a suitable } r\in]r_1,r_2[,\quad |f'|\le\frac{2}{r_2-r_1}.$$

Define

$$\eta:\mathbb{R}\longrightarrow\mathbb{R},\quad \eta(x)=f(d(x_0,x)).$$

Then

$$\eta\in W_0^1B(x_0,r_2),\quad \eta=1\text{ on } B(x_0,r_1)$$

and, by the previous lemma

$$|X\eta|=|\eta'(d(x_0,\cdot))Xd(x_0,0)|\le\frac{2}{r_2-r_1}\quad\text{a.e.}$$

In the first identity we have used the Leibnitz-type rule for the X-derivative of the composite functions (see e.g. Proposition 2.1 in [7]). The proof is complete. $\square$

3. Proof of the main theorem

The proof of our main result requires some preliminary lemma.

Lemma 12 (*Main Lemma*). *Let $u\in W^1_{\mathrm{loc}}(\mathbb{R}^N,X)$ be a nonnegative solution to*

$$\mathcal{L}u=0\quad \text{in } \mathbb{R}^N.$$

Then, for every d-ball $B_r(x_0)$, $x_0\in\mathbb{R}^N$, $r>0$, we have

$$\sup_{B_r(x_0)}u\le C\inf_{B_r(x_0)}u$$

where C depends on the constants λ, A, C_P, ν and on

$$\mathbf{b}_r^*(x_0):=\sup_{B_\rho(z)\subseteq B_{4r}(x_0)}\rho\left(\fint_{B_\rho(z)}\mathbf{b}^{2p}\right)^{\frac{1}{2p}}.$$

Proof. Let u be a nonnegative weak solution to the equation $\mathcal{L}u=0$. Then

$$L(u,v)=0\tag{14}$$

for every compactly supported $v\in W^1(\mathbb{R}^N)$. By replacing u with $u+\varepsilon$, if necessary, we can assume u bounded away from zero.

Let B_{4r} be a d-ball of radius $4r$ centered at a point $x_0\in\mathbb{R}^N$. By plugging into (14) test functions v of the kind $v=\eta^2u^\beta$ with $\beta\neq-1$ and $\eta\in W^1(\mathbb{R}^N)$, supp $\eta\subset B_{4r}$, we get (see [7], pagg. 1849-50):

$$\|\eta w\|_{L^q_*(B_{4r})}\le C(1+|\beta+1|)^{1+\mu}\left(\|\eta w\|_{L^2_*(B_{4r})}\|wX\eta\|_{L^2_*(B_{4r})}\right),\tag{15}$$

where $w=u^{\beta+1}$ and $\mu=\frac{Q}{2p-Q}$. Here we use the notation

$$\|u\|_{L^s_*(B_r)}=\left(\fint_{B_r}|u|^s\mathrm{d}x\right)^{\frac{1}{s}}.$$

The constant C in the previous inequality only depends on λ, the X-ellipticity constant of $\mathcal{L}$ in (2), A in the doubling condition (4), C_P and ν in the Poincaré inequality (5) and $\mathbf{b}_r^*(x_0)$.

In particular we stress that C is independent on u, β and r.

At this point, we choose η as the cut off function in Theorem 10. Precisely, given r_1 and r_2 such that

$$1 < \frac{r_1}{r} < \frac{r_2}{r} < 2,$$

we choose $\eta \in W_0^1(B_{2r})$ satisfying

$$\eta \equiv 1 \quad \text{in } B_{r_1} \quad \text{and} \quad |X\eta| \leq \frac{2}{r_2 - r_1}$$

(B_{r_i} denotes the d-ball of radius r_i centered at x_0).

Using this η in (15) we obtain

$$|w|_{L_*^q(B_{r_1})} \leq 2C(1 + |\beta + 1|)^{1+\mu} \left(1 + \frac{r}{r_2 - r_1}\right) \|w\|_{L_*^2(B_{r_2})}. \tag{16}$$

By applying these estimates on the sequences (B_k) and (r_k) given by

$$\beta_k + 1 = \theta^k s \quad \text{with } s > 1,\ \theta = \frac{q}{2},\ r_k = r\left(1 + \frac{1}{2^k}\right)$$

and letting k tend to infinity, we obtain

$$\sup_{B_r} u \leq C \left(\fint_{B_{2r}} u^s \mathrm{d}x \right)^{\frac{1}{s}}.$$

(See the Moser-iteration procedure as presented in [16], pag. 197; see also [7], pag. 1850.) The constant C in the previous inequality only depends on s and the structural constants $\lambda, A, C_P, \theta, \mu$. Analogously, iterating (16), on a sequence of negative β's, we get

$$\inf_{B_r} u \geq C \left(\fint_{B_{3r}} u^{-p_0} \mathrm{d}x \right)^{-\frac{1}{p_0}}$$

for every p_0 small enough. Here $C > 0$ only depends on p_0 and the structural constants.

On the other hand, inequality (16), with the choice of a suitable $\beta \in]-1, 0[$, gives

$$\left(\fint_{B_{2r}} u^s \mathrm{d}x \right)^{\frac{1}{s}} \leq C \left(\fint_{B_{3r}} u^{p_0} \mathrm{d}x \right)^{\frac{1}{p_0}}.$$

Summing up, we have

$$\sup_{B_r} u \leq C \left(\fint_{B_{3r}} u^{p_0} \mathrm{d}x \right)^{\frac{1}{p_0}} \tag{17}$$

$$\inf_{B_r} u \geq C \left(\fint_{B_{3r}} u^{-p_0} \mathrm{d}x \right)^{-\frac{1}{p_0}} \tag{18}$$

for every p_0 sufficiently small, and $C > 0$ only depending on p_0 and the structural constants.

We now use a John-Nirenberg-type Lemma to prove that

$$\left(\fint_{B_{3r}} u^{p_0} \mathrm{d}x \right)^{\frac{1}{p_0}} \leq C \left(\fint_{B_{3r}} u^{-p_0} \mathrm{d}x \right)^{-\frac{1}{p_0}} \tag{19}$$

for every d-ball B_{3r} centered at a point x_0. This, together with (17) and (18), will complete the proof of the theorem.

Let B_ρ be a d-ball centered at a point $z \in B_{4r}(x_0)$, $B_\rho(z) \subseteq B_{4r}(x_0)$. Plugging into (14) the test function $v = \eta \log u$, with $\eta \in W^1(\mathbb{R}^N, X)$, supp $\eta \subseteq B_{2\rho}$, $\eta \equiv 1$ in B_ρ, and letting $w = \log u$ we obtain

$$\int_{\mathbb{R}^N} |\eta X w|^2 \mathrm{d}x \leq C \int_{\mathbb{R}^N} \left((\mathbf{b}\eta)^2 + |X\eta|^2 \right) \mathrm{d}x.$$

Then, choosing the cut off function η in such a way that $|X\eta| \le 2\rho$ (see Theorem 10), also using the doubling condition (4), we get

$$\fint_{B_\rho} |Xw|^2 \mathrm{d}x \le C \left(\fint_{B_{2\rho}} \mathbf{b}^2 \mathrm{d}x + \frac{1}{\rho^2} \right) \le \frac{C}{\rho^2} \left(\rho^2 \|\mathbf{b}\|^2_{L^{2p}_*(B_{2\rho})} + 1 \right) \le \frac{C}{\rho^2} (\mathbf{b}^* + 1).$$

As a consequence, by using the Poincaré inequality (5), we obtain

$$\fint_{B_\rho} |w - w_\rho| \mathrm{d}x \le C,$$

where C is a positive structural constant. Then, the following John-Nirenberg estimate holds

$$\int_{B_{3r}} \exp(p_0 |w - w_{3r}|) \mathrm{d}x \le C$$

for every ball B_{3r} centered at a point of A_r. Here p_0 and C are suitable positive structural constants, see [17], Theorem 0.3 and Theorem 0.4. Finally, this inequality implies

$$\begin{aligned} \fint_{B_{3r}} u^{-p_0} \mathrm{d}x \fint_{B_{3r}} u^{p_0} \mathrm{d}x &= \fint_{B_{3r}} \exp(-p_0 w) \mathrm{d}x \fint_{B_{3r}} \exp(p_0 w) \mathrm{d}x \\ &\le \left(\fint_{B_{3r}} \exp(p_0 |w - w_{3r}|) \mathrm{d}x \right)^2 \end{aligned}$$

which obviously implies (19). □

From the proof of the previous lemma we immediately obtain the following lemma

Lemma 13 (*Invariant Harnack Inequality*). *Let $u \in W^1_{\mathrm{loc}}(\mathbb{R}^N, X)$ be a nonnegative weak solution to*

$$\mathcal{L}u = 0 \quad in\ \mathbb{R}^N.$$

Assume the lower order terms b_j's of $\mathcal{L}$ are identically zero. Then

$$\sup_{B_r} u \le C \inf_{B_r} u \tag{20}$$

for every d-ball B_r, with $C > 0$ independent of u and r.

Lemma 14. *Let $B_r(x_0)$ be a d-ball such that*

$$\partial B_r(x_0) \quad \text{is connected.}$$

Let $0 < \theta < 1$. Then there exists $x_1, \ldots, x_p \in \partial B_r(x_0)$, with p independent of r, such that:

(i)

$$\partial B_r(x_0) \subseteq \bigcup_{j=1}^p B_{\theta r}(x_j);$$

(ii) *letting $K_j := B_{\theta r}(x_j) \cap \partial B_r(x_0)$,*

$$\left(\bigcup_{j=1}^m K_j \right) \cap K_{m+1} \ne \emptyset \quad \text{for } m = 1, \ldots, p-1.$$

Proof. (i) Let $\mathcal{B} = (B_{\frac{\theta}{4}r}(x_j))_{j \in \mathcal{J}}$ be a maximal family of disjoint balls of radius $\frac{\theta}{4}r$, centered at a point of $\partial B_r(x_0)$. Since the d-topology has a countable basis of open sets, the family $\mathcal{B}$ is countable. Due to the maximality of $\mathcal{B}$, for every $x \in \partial B_r(x_0)$ there exists $j \in \mathcal{J}$ such that $B_{\frac{\theta}{4}r}(x) \cap B_{\frac{\theta}{4}r}(x_j) \ne \emptyset$. Hence $B_{\theta r}(x_j) \supseteq B_{\frac{\theta}{4}r}(x) \ni x$. This shows that $(B_{\theta r}(x_j))_{j \in \mathcal{J}}$ covers $B_r(x_0)$. On the other hand, since $\mathcal{B}$ is disjoint, for every finite set $\{1, \ldots, p\} \subseteq \mathcal{J}$ we have

$$\sum_{j=1}^p |B_{\frac{\theta}{4}r}(x_j)| = \left| \bigcup_{j=1}^p B_{\frac{\theta}{4}r}(x_j) \right| \le |B_R(x_0)|, \qquad R = \left(1 + \frac{\theta}{4}\right) r. \tag{21}$$

Moreover, since $B_{2R}(x_j) \supseteq B_R(x_0)$, for every $j \in \{1, \ldots, p\}$ we have

$$\begin{aligned} |B_R(x_0)| \le |B_{2R}(x_j)| &\le \text{(by the doubling condition)} \\ & A \left(\frac{2(4+\theta)}{\theta} \right)^Q \left| B_{\frac{\theta}{4}r}(x_j) \right| = A_Q \left| B_{\frac{\theta}{4}r}(x_j) \right|, \end{aligned}$$

where $A_Q = A\left(\frac{2(4+\theta)}{\theta}\right)^Q$. Using this estimate in (21), we get

$$\frac{p}{A_Q}|B_R(x_0)| \le |B_R(x_0)|,$$

hence $p \le A_Q$.

(ii) By removing from the family $(B_{\frac{r}{2}}(x_j))_{j=1,\dots,p}$ the balls having empty intersection with $\partial B_r(x_0)$ we may assume $K_j \ne \emptyset$ for every $j \in \{1, \dots, p\}$. Now, there exists at least one K_j such that $K_i \cap K_j \ne \emptyset$. Indeed, otherwise,

$$\Omega_1 := B_{\frac{r}{2}}(x_1) \quad \text{and} \quad \Omega_2 := \bigcup_{j=2}^{p} B_{\frac{r}{2}}(x_j)$$

would be open sets satisfying

$$\Omega_1 \cap \Omega_2 \cap \partial B_r(x_0) = \emptyset, \quad \Omega_1 \cap \partial B_r(x_0) \ne \emptyset \ne \Omega_2 \cap \partial B_r(x_0) \quad \Omega_1 \cup \Omega_2 \supseteq \partial B_r(x_0),$$

in contradiction with the connectedness of $\partial B_r(x_0)$. Thus, we may suppose $K_1 \cap K_2 \ne \emptyset$. The same argument as above proves that $(K_1 \cup K_2) \cap K_j \ne \emptyset$ for a suitable $j > 2$. Then, we may assume $(K_1 \cup K_2) \cap K_3 \ne \emptyset$. The proof can be completed by iterating this procedure. □

Lemma 15. *Let $K_1, \dots, K_p$ be any family of non empty sets such that*

$$\left(\bigcup_{j=1}^{m} K_j\right) \cap K_{m+1} \ne \emptyset \quad \textit{for } m = 1, \dots, p-1.$$

Assume we are given a nonnegative function $u : K \longrightarrow \mathbb{R}$, $K = \bigcup_{j=1}^{m} K_j$, such that

$$\sup_{K_j} u \le C \inf_{K_j} u \quad \textit{for } j = 1, \dots, m.$$

Then

$$\sup_{K} u \le C^p \inf_{K} u. \tag{22}$$

Proof. Let $x, y \in K_1 \cup K_2$ and choose a point $z \in K_1 \cap K_2$. If $x, y \in K_1$ or $x, y \in K_2$ we have $u(x) \le Cu(y)$. If $x \in K_1$ and $y \in K_2$, or $x \in K_2$ and $y \in K_1$, we have $u(x) \le Cu(z)$ and $u(z) \le Cu(y)$, so that, since $u \ge 0$, $u(x) \le C^2u(y)$. Thus, in any case, being $C \ge 1$,

$$u(x) \le C^2 u(y) \quad \text{for every } x, y \in K_1 \cup K_2.$$

Let us now take $x, y \in \bigcup_{j=1}^{3} K_j$ and choose $z \in (K_1 \cup K_2) \cap K_3$. If $x, y \in K_1 \cup K_2$ or $x, y \in K_3$ we have $u(x) \le C^2u(y)$ or $u(x) \le Cu(y)$. If $x \in K_1 \cup K_2$ and $y \in K_3$ we have $u(x) \le C^2u(z)$ and $u(z) \le Cu(y)$ hence $u(x) \le C^3u(y)$. The same inequality obviously holds if $x \in K_3$ and $y \in K_1 \cup K_2$. Thus, in any case

$$u(x) \le C^3 u(y) \quad \text{for every } x, y \in \bigcup_{j=1}^{3} K_j.$$

Iterating this procedure we obtain

$$u(x) \le C^p u(y) \quad \text{for every } x, y \in \bigcup_{j=1}^{p} K_j.$$

This inequality obviously implies (22). □

Now we can give the proof of our main result.

Proof of Theorem 1. We first assume $b_j \equiv 0$ for every $j = 1, \dots, N$. Then, in this case, by Eq. (20), the following Harnack inequality holds

$$\sup_{B_r} v \le C \inf_{B_r} v$$

for every nonnegative weak solution v of $\mathcal{L}v = 0$ in $\mathbb{R}^N$ and for every d ball B_r, the constant C being independent of r and v. Then, if $u \ge 0$ and solves $\mathcal{L}u = 0$ in $\mathbb{R}^N$, letting $v = u - \inf_{\mathbb{R}^N} u$ we have $v \ge 0$ and $\mathcal{L}v = 0$. As a consequence

$$\sup_{B_r(0)} v \le C \inf_{B_r(0)} v$$

so that, letting r go to infinity, we get $0 \le \sup_{\mathbb{R}^N} v \le C \inf_{\mathbb{R}^N} v = 0$. Hence $v \equiv 0$ and $u \equiv \inf_{\mathbb{R}^N} u$.

Let us now assume hypothesis (1.2) is satisfied. From the Main Lemma we have

$$\sup_{B_{\frac{r_j}{8}}} v \le C \inf_{B_{\frac{r_j}{8}}} v$$

for every nonnegative weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^N$ and for every d-ball $B_{\frac{r}{2}}$ centered at a point of $\partial B_{r_j}(0)$, with $C > 0$ independent on u and on the ball $B_{\frac{r_j}{8}}$. Indeed the constant C in the previous Harnack inequality only depends on the structural constant and on

$$\sup_{y\in\partial B_{r_j}(0)} \mathbf{b}^*_{\frac{r_j}{8}}(y) := \sup_{y\in\partial B_{r_j}(0)} \sup_{B_\rho(z)\subseteq B_{\frac{r_j}{2}}(y)} \rho \left(\fint_{B_\rho(z)} \mathbf{b}^{2p} \right)^{\frac{1}{2p}}.$$

By using the assumption (LT), with an easy computation we recognize that the right hand side can be bounded by a structural constant $\mathbf{b}^*$ independent of j. Then, by applying Lemma 14 with $\theta = 1/8$ and Lemma 15, we obtain

$$\sup_{\partial B_{r_j}(0)} \le C \inf_{\partial B_{r_j}(0)} \quad \text{for every } j \ge 1, \tag{23}$$

where C is independent of v and j. Let us now define $v = u - \inf_{\mathbb{R}^N} u$, where u is any nonnegative solution to $\mathcal{L}u = 0$ in $\mathbb{R}^N$. Then, $v \ge 0$ and $\mathcal{L}v = 0$ in $\mathbb{R}^N$. By using (23) and the Maximum Principle in [7], Theorem 3.1, we obtain

$$\sup_{B_{r_j}(0)} v = \sup_{\partial B_{r_j}(0)} v \le C \inf_{\partial B_{r_j}(0)} v = C \inf_{B_{r_j}(0)} v.$$

Letting r_j go to infinity, we get

$$0 \le \sup_{\mathbb{R}^N} v \le C \inf_{\mathbb{R}^N} v = 0.$$

Hence $v \equiv 0$ and $u \equiv \inf_{\mathbb{R}^N} u$. □

4. Further comments and results

It is well known that Liouville Theorem follows from invariant Harnack inequality (see the proof of Theorem 1). From our Harnack inequality of Lemma 13, with another standard argument we get the following result.

- *Let hypotheses of* Lemma 13 be *satisfied. Then there exists* $\alpha > 0$ *such that every weak solution to* $\mathcal{L}u = 0$ *such that*

$$\lim_{r\longrightarrow\infty} \frac{1}{r^\alpha} \sup_{B_r(0)} |u| = 0$$

must be constant.

Indeed, the invariant Harnack inequality (20) implies

$$|u(x) - u(y)| \le C \left(\frac{d(x,y)}{r} \right)^\alpha \sup_{B_r(0)} |u|$$

for suitable $C > 0$ and $0 < \alpha \le 1$, independent of r. (See e.g. Theorem 5.3 [10], see also [16] pages 190–191.). Then, for every fixed $x, y \in \mathbb{R}^N$, letting r go to infinity we get $|u(x) - u(y)| = 0$, that is $u \equiv$ const. We would also like to remark that the noteworthy Colding–Minicozzi' s Theorem 0.7 in [18] can be extended to the X-elliptic operators in principal part. Indeed, the existence of a global cut-off function given by our Theorem 10, and the Caccioppoli-type estimate (16) allow us to, verbatim, repeat the proof in [18]. For reading convenience, we explicitly state here the result

- *Let* $\mathcal{L}$ *as in* (1) *with* $b_i \equiv 0$ *for* $i = 1, \dots, N$. *Assume* (E1) *and* (E2), (H1) *and* (H2) *are satisfied. Then, for every fixed* $\alpha > 0$ *the linear space of the weak solution to* $\mathcal{L}u = 0$ *in* $\mathbb{R}^N$ *satisfying*

$$\sup_{r\ge 1} \left(\frac{1}{r^\alpha} \sup_{B_r(0)} |u| \right) < \infty$$

is finite dimensional.

Suitable versions of this result for a class of linear second order operators with nonnegative characteristic form, smooth coefficients and which are homogeneous with respect to a group of dilations in $\mathbb{R}^N$, are contained in [19].

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