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Weighted L^p-Liouville theorems for hypoelliptic partial differential operators on Lie groups

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Abstract. We prove *weighted* L^p-Liouville theorems for a class of second-order hypoelliptic partial differential operators $\mathcal{L}$ on Lie groups $\mathbb{G}$ whose underlying manifold is n-dimensional space. We show that a natural weight is the right-invariant measure $\check{H}$ of $\mathbb{G}$. We also prove Liouville-type theorems for C^2 subsolutions in $L^p(\mathbb{G}, \check{H})$. We provide examples of operators to which our results apply, jointly with an application to the uniqueness for the Cauchy problem for the evolution operator $\mathcal{L} - \partial_t$.

1. Introduction and main results

The aim of this paper is to obtain L^p-Liouville properties for hypoelliptic linear second-order partial differential operators $\mathcal{L}$ (with nonnegative characteristic form), which are left-invariant on a Lie group $\mathbb{G}$ on n-dimensional space $\mathbb{R}^n$. We shall obtain *weighted* L^p-Liouville theorems; in that the right-invariant measure of the group $\mathbb{G}$ will play a crucial and natural rôle, as we will shortly explain.

Precisely, we assume that $\mathcal{L}$ has the following structure: $\mathcal{L}$ is a linear second-order PDO (with vanishing zero-order term) on n-dimensional space $\mathbb{R}^n$ whose quadratic form is positive semidefinite at every point of $\mathbb{R}^n$; more explicitly, to fix the notation, we require that $\mathcal{L}$ has the coordinate form:

$$\mathcal{L} = \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j}, \tag{1.1}$$

with functions $a_{i,j}, b_j \in C^\infty(\mathbb{R}^n, \mathbb{R})$, and the matrix $A(x) := (a_{i,j}(x))$ is symmetric and positive semidefinite for every $x \in \mathbb{R}^n$.

Our assumptions are the following three:

(ND) $\mathcal{L}$ is *non-totally degenerate* at every $x \in \mathbb{R}^n$, that is $A(x) \neq 0$ for every $x \in \mathbb{R}^n$.

(HY) $\mathcal{L}$ is *hypoelliptic* in every open subset of $\mathbb{R}^n$, that is, if $U \subseteq W \subseteq \mathbb{R}^N$ are open sets, any $u \in \mathcal{D}'(W)$ which is a weak solution to $\mathcal{L}u = h$ in $\mathcal{D}'(U)$, with $h \in C^\infty(U, \mathbb{R})$, is itself a smooth function on U.

(LI) There exists a Lie group $\mathbb{G} = (\mathbb{R}^n, \cdot)$ such that $\mathcal{L}$ is *left-invariant* on $\mathbb{G}$.

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REMARK 1.1. Actually, under hypothesis (LI), assumption (ND) is a very mild condition; indeed, it is easy to check that, if $\mathcal{L}$ is left-invariant, then (ND) holds true if and only if there exists $x_0 \in \mathbb{R}^n$ such that $A(x_0) \neq 0$ (which is equivalent to requiring that $\mathcal{L}$ is not a merely first-order operator).

We observe that a set of explicit necessary and sufficient conditions ensuring hypothesis (LI) has been recently given by Biagi and the first-named author in [1].

We now fix a notation: in what follows we shall denote by $\check{H}$ a fixed right-invariant measure on the Lie group $\mathbb{G}$ in assumption (LI). Since any two right-invariant measures differ by a positive scalar multiple, we fix once and for all $\check{H}$ in the following (explicit way): given $x \in \mathbb{R}^n$ we set

$$\rho_x : \mathbb{R}^n \to \mathbb{R}^n, \quad \rho_x(y) := y \cdot x$$

to denote the right translation by x; then it is easy to verify that the measure

$$E \mapsto \check{H}(E) := \int_E \frac{1}{\det\big(\mathcal{J}_{\rho_x}(e)\big)}\, \mathrm{d}x \tag{1.2}$$

(defined on the Lebesgue measurable sets $E \subseteq \mathbb{R}^n$) is a right-invariant measure on the Lie group $\mathbb{G}$. Here and in the sequel we agree to denote by dx the Lebesgue integration on $\mathbb{R}^n$. The notation $\check{H}$ comes from the usual duality existing between left-invariant measures μ and right-invariant measures $\check{\mu}$:

$$\mu \longmapsto \check{\mu} \quad \text{where } \check{\mu}(E) = \mu(\iota(E)),$$

where ι is the group inversion on $\mathbb{G}$. Even if we will not use any Haar measure H of $\mathbb{G}$, we prefer to use the symbol $\check{H}$ to avoid any confusion with left invariance and, at the same time, in order to emphasize the rôle of right invariance in our Liouville results.

Throughout, $L^p(\mathbb{R}^n, \check{H})$ (for any $p \in [1, \infty]$) will denote the associated L^p-space on $\mathbb{G} \equiv \mathbb{R}^n$ with respect to the measure $\check{H}$.

In the sequel, we say that a function $u \in C^2(\mathbb{R}^n, \mathbb{R})$ is

- $\mathcal{L}$-*harmonic* on $\mathbb{R}^n$ if $\mathcal{L}u = 0$ on $\mathbb{R}^n$;
- $\mathcal{L}$-*subharmonic* on $\mathbb{R}^n$ if it satisfies $\mathcal{L}u \geq 0$ on $\mathbb{R}^n$.

We are now ready to state the main results of this paper, the following weighted L^p-Liouville theorems.

THEOREM 1.2. *(Weighted L^p-Liouville theorem for the $\mathcal{L}$-harmonic functions) Suppose that $\mathcal{L}$ satisfies assumptions* (ND)*,* (HY)*,* (LI)*.*

Let $u \in C^\infty(\mathbb{R}^n, \mathbb{R})$ be an $\mathcal{L}$-harmonic function.

Then $u \equiv 0$ if one of the following conditions is satisfied:

(i) $u \in L^p(\mathbb{R}^n, \check{H})$ *for some* $p \in [1, \infty[$*;*

(ii) $u \geq 0$ *and* $u^p \in L^1(\mathbb{R}^n, \check{H})$ *for some* $p \in]0, 1[$.

$\check{H}$ denotes the right-invariant measure on $\mathbb{G}$ defined in (1.2).

The classical form of Liouville's theorem for $\mathcal{L}$-harmonic functions (i.e., under the assumption $\mathcal{L}u = 0$ and the "one-side" bound $u \geq 0$ on the whole space) cannot be expected under our general hypotheses where operators with first-order terms as in (1.1) are allowed: for example, the classical Heat operator

$$\mathcal{L} = \sum_{j=1}^{n} (\partial_{x_j})^2 - \partial_t \quad \text{in } \mathbb{R}^{n+1} = \mathbb{R}^n_x \times \mathbb{R}_t$$

satisfies all the assumptions (ND), (HY) and (LI) (the latter w.r.t. the usual structure $\mathbb{G} = (\mathbb{R}^{n+1}, +)$), but the function $\exp(x_1 + \cdots + x_n + n\,t)$ is $\mathcal{L}$-harmonic and nonnegative in space $\mathbb{R}^{n+1}$. "One-side" Liouville-type theorems for some classes of homogeneous operators are proved in [11–13].

Our second main result, for $\mathcal{L}$-subharmonic functions, is the following one:

THEOREM 1.3. *(Weighted L^p-Liouville theorem for the $\mathcal{L}$-subharmonic functions) Suppose that $\mathcal{L}$ satisfies assumptions* (ND), (HY), (LI). *Let $u \in C^2(\mathbb{R}^n, \mathbb{R})$ be an $\mathcal{L}$-subharmonic function on $\mathbb{R}^n$.*

If $u \in L^p(\mathbb{R}^n, \check{H})$ for some $p \in [1, \infty[$, then $u \leq 0$.

In particular, any nonnegative $\mathcal{L}$-subharmonic function is identically zero, provided that $u \in L^p(\mathbb{R}^n, \check{H})$ for some $p \in [1, \infty[$.

For the proofs of Theorems 1.2 and 1.3 we closely follow the techniques recently introduced by Lanconelli and the second-named author in [14, Th. 1.1, 1.2, 1.3], where unimodular Lie groups are considered (with $\check{H}$ equal to the Lebesgue measure): the ideas introduced in [14] can be adapted to our (more general) framework, since they rely on a very versatile technique based on the use of convex functions of the global solution to $\mathcal{L}u = 0$, together with a general representation formula (of Poisson–Jensen type; see also (2.8)). The novelty of our case is the use of the right-invariant measure $\check{H}$; this allows us to encompass new examples, of interest, as the following one.

EXAMPLE 1.4. Let us consider in $\mathbb{R}^{n+1} = \mathbb{R}^n_x \times \mathbb{R}_t$ the *Kolmogorov-type operators*

$$\mathcal{L} = \mathrm{div}(A\nabla) + \langle Bx, \nabla\rangle - \partial_t, \tag{1.3}$$

where A and B are constant $n \times n$ real matrices, and A is symmetric and positive semidefinite. Let us define the matrix

$$E(s) := \exp(-sB), \quad s \in \mathbb{R}.$$

Then the operator $\mathcal{L}$ in (1.3) satisfies assumption (LI) w.r.t. the Lie group $\mathbb{G} = (\mathbb{R}^{n+1}, \cdot)$ with composition law

$$(x, t) \cdot (x', t') = (x' + E(t')x, t + t').$$

Since $\det(E(t)) = \exp(-t\,\mathrm{trace}(B))$, according to formula (1.2) the associated right-invariant measure $\check{H}$ is equal to

$$\mathrm{d}\check{H}(x, t) = e^{t\,\mathrm{trace}(B)}\,\mathrm{d}x\mathrm{d}t. \tag{1.4}$$

Moreover, if we assume that the matrix

$$\int_0^t E(s)\, A\, (E(s))^T \,\mathrm{d}s \quad \text{is positive definite for all } t > 0, \tag{1.5}$$

then $\mathcal{L}$ is hypoelliptic (see e.g., [15]; see also [5, Sections 4.1.3, 4.3.4]) so that hypothesis (HY) is satisfied as well. Condition (1.5) also encloses condition (ND) (since (1.5) cannot hold if $A = 0$). Hence, under condition (1.5), the operator $\mathcal{L}$ satisfies all our assumptions and the weighted L^p-Liouville Theorems 1.2 and 1.3 hold true w.r.t. the explicit measure $\check{H}$ in (1.4).

For a class of operators (encompassing the above hypoelliptic operator $\mathcal{L}$), we also prove a uniqueness result for the Cauchy problem (see Sect. 4.3); for simplicity we here state this result for the above operator $\mathcal{L}$ (see Proposition 4.2 for the larger class of operators to which this uniqueness result applies):

COROLLARY 1.5. *Let us denote by Ω the half-space $\{(x, t) \in \mathbb{R}^{n+1} : t > 0\}$. If $\mathcal{L}$ is the operator (1.3) and if the hypoellipticity condition (1.5) is satisfied, any classical solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ to the Cauchy problem*

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \\ u(x,t) = 0 & \text{for } t = 0 \end{cases}$$

is identically zero on Ω if it holds that

$$\int_0^\infty \int_{\mathbb{R}^n} |u(x,t)|^p \, e^{t\,\mathrm{trace}(B)} \,\mathrm{d}x \,\mathrm{d}t < \infty,$$

for some $p \in [1, \infty)$.

Other examples, appearing in the literature, of operators satisfying conditions (ND), (HY) and (LI) are:

(i) the classical Kolmogorov–Fokker–Planck operator

$$\mathcal{K} = \sum_{j=1}^n (\partial_{x_j})^2 + \sum_{j=1}^n x_j\, \partial_{x_{n+j}} - \partial_t,$$

in $\mathbb{R}^{2n+1} = \mathbb{R}^{2n}_x \times \mathbb{R}_t$ (it is of the form (1.3) and it satisfies (1.5));

(ii) $\mathcal{L} = L - \partial_t$ in $\mathbb{R}^3 = \mathbb{R}^2_x \times \mathbb{R}_t$, where $L = \frac{1}{2}(\partial_{x_1})^2 - (x_1 + x_2)\,\partial_{x_1} + x_1\,\partial_{x_2}$ (L belongs to a class recently studied by Da Prato and Lunardi, [6]); the associated right-invariant measure is

$$e^{-t}\,\mathrm{d}t\,\mathrm{d}x_1\,\mathrm{d}x_2;$$

(iii) the operators $\mathcal{L}$ considered by Lanconelli and the first-named author in [4], together with their evolution counterparts $\mathcal{L} - \partial_t$; since this class of PDOs furnishes a wide gallery of new examples for weighted L^p-Liouville theorems, we shall describe them in detail in Sect. 4.

Before giving the plan of the paper, we mention some related references from the existing literature:

- When hypothesis (LI) holds in the stronger form requiring that $\mathbb{G}$ is a homogeneous group w.r.t. a family of dilations (see [5, Section 1.3] for the relevant definition) and $\mathcal{L}$ is a homogeneous operator, Theorem 1.2 follows from a general Liouville-type theorem by Geller [8, Theorem 2].
- Yet in presence of dilation homogeneity (but not necessarily under the left-invariance condition (LI)), Luo extended Geller's theorem to homogeneous hypoelliptic operators (see [16, Theorem 1]). The theorems of Geller and of Luo cannot be applied to subharmonic functions (as in Theorem 1.3 above).
- For special classes of Lie groups $\mathbb{G}$ (namely, for stratified Lie groups), L^1-Liouville theorems on half-spaces have been proved by Uguzzoni [19] and by the second-named author [10]. See also [3] (and [5, Chapter 5, Section 5.8]) for Harnack–Liouville and asymptotic Liouville theorems for stratified Lie groups.
- The L^∞-Liouville property does not hold, in general: see Priola and Zabczyk [18] (see also [14, Remark 8.1]).

The plan of the paper is as follows. Section 2 recalls the techniques in [14], while in Sect. 3 we prove Theorems 1.2 and 1.3. Finally, Sect. 4 provides examples of operators to which our results apply, together with an application to the uniqueness of the Cauchy problem for a class of evolution operators.

2. Background results and recalls

Here and throughout the rest of the paper, we assume that $\mathcal{L}$ is as in (1.1) and that the matrix $A(x) = (a_{i,j}(x))$ of the second-order part of $\mathcal{L}$ is symmetric and positive semidefinite for every $x \in \mathbb{R}^n$. This will be tacitly understood.

REMARK 2.1. (a) Suppose that $\mathcal{L}$ satisfies hypothesis (LI). Since (by the Campbell–Baker–Hausdorff Theorem; see e.g., [2]) it is non-restrictive to assume that any Lie group is endowed with an analytic structure, the coefficients of $\mathcal{L}$ can be supposed to be of class C^ω. We shall assume the latter fact throughout. Moreover, by also using the Poincaré–Birkhoff–Witt Theorem, one can prove that assumption (LI) (together with the facts that the quadratic form of $\mathcal{L}$ be positive semidefinite and be associated with a symmetric matrix) implies that $\mathcal{L}$ is a sum of squares of vector fields plus a drift.

(b) We pass from (1.1) to the quasi-divergence form

$$\mathcal{L} = \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^n a_{i,j}(x)\,\frac{\partial}{\partial x_j}\Big) + \sum_{j=1}^n \Big(b_j(x) - \sum_{i=1}^n \frac{\partial a_{i,j}(x)}{\partial x_i}\Big)\frac{\partial}{\partial x_j}, \tag{2.1}$$

and we set

$$X_i := \sum_{j=1}^{n} a_{i,j}(x)\,\frac{\partial}{\partial x_j} \qquad (i = 1, \dots, n),$$
$$X_0 := \sum_{j=1}^{n} \Big(b_j(x) - \sum_{i=1}^{n} \frac{\partial a_{i,j}(x)}{\partial x_i}\Big)\frac{\partial}{\partial x_j}. \tag{2.2}$$

With this notation, (2.1) becomes

$$\mathcal{L} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(X_i) + X_0. \tag{2.3}$$

If $\mathcal{L}$ satisfies the hypoellipticity condition (HY), due to the results in [17] (and the C^ω regularity in (a) above), then the vector fields $X_0, X_1, \dots, X_n$ fulfill Hörmander's maximal rank condition, [9].

Remarks (a) and (b) also motivate the fact that our examples of PDOs satisfying assumptions (HY) and (LI) (see Sect. 4) will fall into the hypoellipticity class of the Hörmander operators.

Then we fix a notation: if $A = (a_{i,j})$ is the second-order matrix of $\mathcal{L}$ as in (1.1), and if u is of class C^1 on some open set, we set

$$\Psi_A(u)(x) := \sum_{i,j=1}^{n} a_{i,j}(x)\,\partial_{x_i} u(x)\,\partial_{x_j} u(x). \tag{2.4}$$

Notice that, since A is positive semidefinite, one has $\Psi_A(u) = \langle A(x)\nabla u(x), \nabla u(x)\rangle \geq 0$.

In [14, Lemma 4.2] it is proved the following result.

LEMMA A. *Let $\mathcal{L}$ be as in (1.1) and let Ψ_A be as in (2.4), where A is the second-order matrix of $\mathcal{L}$. Suppose that the vector fields $X_0, X_1, \dots, X_n$ in (2.2) fulfill Hörmander's maximal rank condition.*

Let $\Omega \subseteq \mathbb{R}^n$ be a connected open set and suppose that $u \in C^1(\Omega, \mathbb{R})$.

Then the following facts are equivalent:

(1) *u is constant in Ω;*
(2) *$X_0 u, X_1 u, \dots, X_n u$ all vanish in Ω;*
(3) *$\Psi_A(u) \equiv 0$ and $X_0 u \equiv 0$ in Ω;*
(4) *u is $\mathcal{L}$-harmonic on Ω and $\Psi_A(u) \equiv 0$ in Ω.*

Due to its relevance in the sequel, we provide the proof of this lemma for the sake of completeness.

Proof. Since (by hypothesis) $X_0, X_1, \dots, X_n$ are bracket-generating vector fields, the equivalence of (1) and (2) follows from the well-known connectivity theorem of Carathéodory–Chow–Rashevsky (see e.g., [5, Chapter 19]).

Next we recall that, given a symmetric positive semidefinite matrix A, then

$$\langle A\xi, \xi\rangle = 0 \quad \text{if and only if} \quad A\xi = 0.$$

As a consequence, $\Psi_A(u)(x) = 0$ if and only if $\nabla u(x)$ is in the kernel of $A(x)$, but this latter condition (due to the very definition of $X_1, \dots, X_n$) is equivalent to the fact that $X_1 u(x) = \cdots = X_n u(x) = 0$. Summing up,

$$\Psi_A(u) \equiv 0 \quad \text{if and only if} \quad X_1 u, \dots, X_n u \equiv 0. \tag{2.5}$$

Hence (3) is equivalent to (2). Finally, the equivalence of (3) and (4) is a consequence of (2.3), taking into account (2.5). □

The rôle of $\Psi_A(u)$ is clear from the following formula: if $u \in C^2(\Omega, \mathbb{R})$ (for some open set $\Omega \subseteq \mathbb{R}^n$) and $F \in C^2(\mathbb{R}, \mathbb{R})$ one has

$$\mathcal{L}(F(u)) = F'(u)\, \mathcal{L}u + F''(u)\, \Psi_A(u). \tag{2.6}$$

This formula has been exploited in [14], together with the use of convex functions $F(u)$ of the global solution u to $\mathcal{L}u = 0$, along with a representation formula of Poisson–Jensen type.

The latter is recalled in the next result, which is crucial for our purposes (see [14, Theorem 2.3] for the proof):

THEOREM B. *Suppose that $\mathcal{L}$ satisfies assumptions* (ND) *and* (HY). *Then there exists a basis $\mathcal{B}$ for the Euclidean topology of $\mathbb{R}^N$, whose elements are bounded open sets, with the following property:*

for every $\Omega \in \mathcal{B}$, and for every $x \in \overline{\Omega}$, there exist two Radon measures ν_x^{Ω} on $\overline{\Omega}$ and μ_x^{Ω} on $\partial\Omega$ such that, for any $v \in C^2(\overline{\Omega}, \mathbb{R})$, one has the representation formula

$$v(x) = \int_{\partial\Omega} v(y)\, \mathrm{d}\mu_x^{\Omega}(y) - \int_{\overline{\Omega}} \mathcal{L}v(y)\, \mathrm{d}\nu_x^{\Omega}(y), \quad \forall\, x \in \overline{\Omega}. \tag{2.7}$$

Moreover, if assumption (LI) *holds true, fixing a bounded open neighborhood Ω of e (the neutral element of $\mathbb{G}$) as above, then we have*

$$u(x) = \int_{\partial\Omega} u(x \cdot y)\, \mathrm{d}\mu(y) - \int_{\overline{\Omega}} (\mathcal{L}u)(x \cdot y)\, \mathrm{d}\nu(y), \tag{2.8}$$

for every $x \in \mathbb{R}^n$ and every $u \in C^2(\mathbb{R}^n, \mathbb{R})$. Here we have set, for brevity,

$$\nu := \nu_e^{\Omega}, \quad \mu := \mu_e^{\Omega}. \tag{2.9}$$

In view of the central use of representation formula (2.8), we fix some notation.

DEFINITION 2.2. For any $u \in C(\mathbb{R}^n, \mathbb{R})$ and any $x \in \mathbb{R}^n$, we set

$$\begin{aligned} M(u)(x) &:= \int_{\partial\Omega} u(x \cdot y)\, \mathrm{d}\mu(y), \\ N(u)(x) &:= \int_{\overline{\Omega}} u(x \cdot y)\, \mathrm{d}\nu(y). \end{aligned} \tag{2.10}$$

Hence (2.8) can be written as follows

$$u(x) = M(u)(x) - N(\mathcal{L}u)(x) \qquad \forall\, x \in \mathbb{R}^n, \quad \forall\, u \in C^2(\mathbb{R}^n, \mathbb{R}). \tag{2.11}$$

Distinctive properties of the operators M, N are proved in [14, Lemma 3.2], which we here recall:

PROPOSITION C. *Let $u \in C(\mathbb{R}^n, \mathbb{R})$ and let M and N be the operators in (2.10).*

(i) *If $u \geq 0$, then $M(u), N(u) \geq 0$;*
(ii) *$M(u), N(u) \in C(\mathbb{R}^n, \mathbb{R})$.*
(iii) *If $N(u) \equiv 0$ (or $M(u) \equiv 0$) and $u \geq 0$, then $u \equiv 0$.*

3. Proof of the weighted L^p-Liouville theorems

For the rest of the paper, we assume that $\mathcal{L}$ satisfies assumption (ND), (HY) and (LI). A main tool in the proof of our L^p-Liouville theorems is the following Lemma 3.1. It shows the rôle of the right-invariant measure $\check{H}$ with respect to the operator M. Lemma 3.1 and Corollary 3.2 are the versions, respectively, of [14, Lemma 3.1] and of [14, Proposition 4.3], where we drop the assumptions that $\mathbb{G}$ be unimodular and that $\check{H}$ be the Lebesgue measure.

LEMMA 3.1. *Let $u \in C(\mathbb{R}^n, \mathbb{R})$ be such that $u \in L^1(\mathbb{R}^n, \check{H})$, where $\check{H}$ is the right-invariant measure on $\mathbb{G} = (\mathbb{R}^n, \cdot)$ introduced in (1.2).*

Then $M(u) \in L^1(\mathbb{R}^n, \check{H})$ and

$$\int_{\mathbb{R}^n} M(u)(x)\, \mathrm{d}\check{H}(x) = \int_{\mathbb{R}^n} u(x)\, \mathrm{d}\check{H}(x). \tag{3.1}$$

Proof. It is a consequence of Fubini Theorem. We skip the proof of the fact that $M(u) \in L^1(\mathbb{R}^n, \check{H})$, since it follows by a similar argument as the following one. We have:

$$\begin{aligned}
\int_{\mathbb{R}^n} M(u)(x)\, \mathrm{d}\check{H}(x) &= \int_{\mathbb{R}^n} \left(\int_{\partial\Omega} u(x \cdot y)\, \mathrm{d}\mu(y) \right) \mathrm{d}\check{H}(x) \\
&= \int_{\partial\Omega} \left(\int_{\mathbb{R}^n} u(x \cdot y)\, \mathrm{d}\check{H}(x) \right) \mathrm{d}\mu(y) \\
(\check{H} \text{ is right-invariant on } \mathbb{G}) \quad &= \int_{\partial\Omega} \left(\int_{\mathbb{R}^n} u(x)\, \mathrm{d}\check{H}(x) \right) \mathrm{d}\mu(y) \\
&= \left(\int_{\mathbb{R}^n} u(x)\, \mathrm{d}\check{H}(x) \right) \left(\int_{\partial\Omega} \mathrm{d}\mu(y) \right) \\
&= \int_{\mathbb{R}^n} u(x)\, \mathrm{d}\check{H}(x).
\end{aligned}$$

In the last equality we have used identity $\mu(\partial\Omega) = 1$, coming from (2.8) with $u \equiv 1$. □

COROLLARY 3.2. *Let $u \in C^2(\mathbb{R}^n, \mathbb{R})$ be an $\mathcal{L}$-subharmonic function. If $u \in L^1(\mathbb{R}^n, \check{H})$, then u is actually $\mathcal{L}$-harmonic on $\mathbb{R}^n$.*

Proof. From (2.11) we have $N(\mathcal{L}u) = M(u) - u$ on $\mathbb{R}^n$. By Lemma 3.1, $u \in L^1(\mathbb{R}^n, \check{H})$ implies $M(u) \in L^1(\mathbb{R}^n, \check{H})$, whence $N(\mathcal{L}u) \in L^1(\mathbb{R}^n, \check{H})$ too. From (3.1) we also get

$$\int_{\mathbb{R}^n} N(\mathcal{L}u)\,\mathrm{d}\check{H} = \int_{\mathbb{R}^n} M(u)\,\mathrm{d}\check{H} - \int_{\mathbb{R}^n} u\,\mathrm{d}\check{H} = 0.$$

On the other hand, since $\mathcal{L}u \geq 0$, we have $N(\mathcal{L}u) \geq 0$ in $\mathbb{R}^n$ (see Proposition C-(i)). Therefore $N(\mathcal{L}u) = 0$ $\check{H}$-almost everywhere in $\mathbb{R}^n$. Since $\check{H}$ is equal to a (smooth) positive density times the Lebesgue measure on $\mathbb{R}^n$ (see (1.2)), we infer that

$$N(\mathcal{L}u) = 0 \quad \text{Lebesgue almost everywhere in } \mathbb{R}^n. \tag{3.2}$$

From $u \in C^2$, we get $\mathcal{L}u \in C$ so that, by Proposition C-(ii), $N(\mathcal{L}u)$ is continuous. As a consequence of (3.2) it follows $N(\mathcal{L}u) \equiv 0$. Finally, the $\mathcal{L}$-subharmonicity of u and an application of Proposition C-(iii) shows that u is $\mathcal{L}$-harmonic in $\mathbb{R}^n$. □

Now, we are in the position to prove Theorems 1.2 and 1.3 proceeding along the lines of [14]. First we need a result from Lie group theory: This comes from the characterization of compact groups in terms of the finiteness of the Haar measure (see e.g., [7, Proposition 1.4.5]). We give the (very short) details for completeness.

LEMMA 3.3. *The only constant function belonging to $L^1(\mathbb{R}^n, \check{H})$ is the null function.*

Proof. We argue by contradiction: We assume the existence of a non-vanishing constant function in $L^1(\mathbb{R}^n, \check{H})$, which is equivalent to requiring that $\check{H}(\mathbb{R}^n) < \infty$. If this happens, we can find a compact neighborhood U of the neutral element e of $\mathbb{G}$, and at most a finite family of mutually disjoint sets

$$U \cdot x_1, \dots, U \cdot x_k, \quad \text{with } k \text{ maximal}.$$

Here we have used the right invariance of $\check{H}$, ensuring that $\check{H}(U \cdot x_i) = \check{H}(U) > 0$, for any $i = 1, \dots, k$. We set $K := \bigcup_{i=1}^k U \cdot x_i$, which is clearly a compact set in $\mathbb{R}^n$.

From the maximality of k, it is simple to recognize that, for any $x \in \mathbb{R}^n$, one has $K \cap (K \cdot x) \neq \emptyset$. This shows that $\mathbb{R}^n = K^{-1} \cdot K$, which is absurd since the latter is a compact set. Hence $\check{H}(\mathbb{R}^n) = \infty$. □

We are ready to give the proofs of our main results.

Proof. (of Theorem 1.2.) Let u be a (smooth) solution to $\mathcal{L}u = 0$ in $\mathbb{R}^n$.

(i) Assume $u \in L^p(\mathbb{R}^n, \check{H})$ (for some $1 \leq p < \infty$) and consider $v := F(u)$, where

$$F : \mathbb{R} \longrightarrow \mathbb{R}, \quad F(t) = (\sqrt{1 + t^2} - 1)^p.$$

It is easy to check that

- $F \in C^2(\mathbb{R}, \mathbb{R})$;
- $0 \le F(t) \le |t|^p$ for every $t \in \mathbb{R}$;
- $F''(t) > 0$ for every $t \neq 0$.

Then $v \in C^2(\mathbb{R}, \mathbb{R})$, $v \in L^1(\mathbb{R}^n, \check{H})$ (since $u \in L^p(\mathbb{R}^n, \check{H})$ and $|F(u)| \le |u|^p$) and

$$\mathcal{L}v \overset{(2.6)}{=} F'(u)\,\mathcal{L}u + F''(u)\,\Psi_A(u) = F''(u)\,\Psi_A(u) \ge 0.$$

Therefore, by Corollary 3.2, $\mathcal{L}v = 0$ so that, since $F''(u) > 0$ if $u \neq 0$,

$$\Psi_A(u) = 0 \ \text{ in } \Omega_0 := \{x \in \mathbb{R}^n \mid u(x) \neq 0\}. \tag{3.3}$$

If $\Omega_0 = \emptyset$ we are done, since we aim to prove that $u \equiv 0$. Assume, by contradiction, that $\Omega_0 \neq \emptyset$. Keeping in mind that $\mathcal{L}u = 0$ in $\mathbb{R}^n$ by hypothesis and that $\Psi_A(u) = 0$ on Ω_0 by construction, from Lemma A-(4) we get that u is constant on every non-empty connected component O of Ω_0.

If $\partial O \neq \emptyset$, since $\partial O \subseteq \partial \Omega_0$ (and clearly $u = 0$ on $\partial\Omega_0$), then $u \equiv 0$ in O, in contradiction with the very definition of Ω_0. Thus, $\partial O = \emptyset$, i.e., $\Omega_0 = \mathbb{R}^n$ and u is constant on $\mathbb{R}^n$. Now, the assumption $u \in L^p(\mathbb{R}^n, \check{H})$ jointly with Lemma 3.3, shows that $u \equiv 0$, in contradiction with $\Omega_0 \neq \emptyset$. This ends the proof of Theorem 1.2 under assumption (i).

(ii) Assume $u \ge 0$ and $u^p \in L^1(\mathbb{R}^n, \check{H})$ for some $p \in]0, 1[$. Define $v := F(u)$, with

$$F : [0, \infty[\longrightarrow \mathbb{R}, \quad F(t) = (1 + t)^p - 1.$$

F has the following properties:

- $F \in C^\infty([0, \infty), \mathbb{R})$;
- $0 \le F(t) \le t^p$ for every $t \ge 0$;
- $F''(t) < 0$ for every $t \ge 0$.

Therefore, $v \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $v \in L^1(\mathbb{R}^n, \check{H})$ and

$$\mathcal{L}v \overset{(2.6)}{=} F'(u)\,\mathcal{L}u + F''(u)\,\Psi_A(u) = F''(u)\,\Psi_A(u) \le 0.$$

Thus, by Corollary 3.2 applied to $-v$, we infer that $\mathcal{L}v = 0$ in $\mathbb{R}^n$ so that the above identity yields $0 = F''(u)\,\Psi_A(u)$. As a consequence, since $F''(u) < 0$ (recall that $u \ge 0$ by assumption), we get

$$\Psi_A(u) \equiv 0 \ \text{ in } \mathbb{R}^n.$$

Since $\mathcal{L}u = 0$ in $\mathbb{R}^n$ by hypothesis, a direct application of Lemma A-(4) proves that u is constant in $\mathbb{R}^n$. Since u^p belongs to $L^1(\mathbb{R}^n, \check{H})$, we are entitled to apply Lemma 3.3 and infer that $u \equiv 0$ in $\mathbb{R}^n$, and this ends the proof. □

We end the section with the proof of our weighted L^p-Liouville Theorem for the $\mathcal{L}$-subharmonic functions.

Proof. (of Theorem 1.3.) Let $u \in C^2(\mathbb{R}^n, \mathbb{R})$ be $\mathcal{L}$-subarmonic and let it belong to $L^p(\mathbb{R}^n, \check{H})$ (for some $p \in [1, \infty)$). We aim to prove that

$$\Omega_+ := \{x \in \mathbb{R}^n \mid u(x) > 0\} = \emptyset.$$

We argue by contradiction and assume that $\Omega_+ \neq \emptyset$. Let us consider the function

$$F : \mathbb{R} \longrightarrow \mathbb{R}, \quad F(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ \big(\sqrt[4]{1+t^4} - 1\big)^p & \text{if } t > 0. \end{cases}$$

It is easy to recognize that:

(i) $F \in C^2(\mathbb{R}, \mathbb{R})$, F is increasing and convex;
(ii) $F' > 0$ and $F'' > 0$ in $]0, \infty[$;
(iii) $0 \leq F(t) \leq t^p$ for every $t \geq 0$.

We define $v := F(u)$ on $\mathbb{R}^n$. Then $v \in C^2(\mathbb{R}, \mathbb{R})$ and, by property (iii) above, $v \in L^1(\mathbb{R}^n, \check{H})$. Moreover, by identity (2.6),

$$\mathcal{L}v = F'(u)\, \mathcal{L}u + F''(u)\, \Psi_A(u) \geq 0,$$

since $\mathcal{L}u \geq 0$ and $F', F'' \geq 0$ by (i). Summing up, v is $\mathcal{L}$-subharmonic in space and in $L^1(\mathbb{R}^n, \check{H})$: Corollary 3.2 then implies that $\mathcal{L}v \equiv 0$, whence

$$F'(u)\, \mathcal{L}u + F''(u)\, \Psi_A(u) = 0 \quad \text{in } \mathbb{R}^n.$$

Taking into account property (ii) of F, we obtain

$$\mathcal{L}u = 0 \quad \text{and} \quad \Psi_A(u) = 0 \quad \text{in } \Omega_+.$$

We are therefore entitled to apply Lemma A-(4) on every connected component O of Ω_+, and derive that u is constant on O.

If $\partial O \neq \emptyset$, since $\partial O \subseteq \partial \Omega_+$ (and clearly $u = 0$ on $\partial \Omega_+$), then $u \equiv 0$ in O, in contradiction with the definition of Ω_+. Thus, $\partial O = \emptyset$, so that $\Omega_+ = \mathbb{R}^n$ and u is constant on $\mathbb{R}^n$. As in the proof of Theorem 1.2, by invoking Lemma 3.3 we get that $u \equiv 0$, in contradiction with $\Omega_+ \neq \emptyset$. □

4. Examples and an application

We now give new examples of PDOs to which the L^p-Liouville theorems apply.

4.1. Matrix-exponential groups

We denote the points of $\mathbb{R}^{1+n}$ by (t, x), with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Let B be a real square matrix of order n; following [4, Section 2], we say that the *matrix-exponential group* $\mathbb{G}(B)$ related to B is $(\mathbb{R}^{1+n}, \cdot)$ endowed with the product

$$(t, x) \cdot (t', x') = \big(t + t', x + \exp(t\, B)\, x'\big), \qquad t, t' \in \mathbb{R}, \;\; x, x' \in \mathbb{R}^n.$$

A basis for the Lie algebra of $\mathbb{G}(B)$, say $\mathfrak{g}(B)$, is $\{\partial_t, X_1, \dots, X_n\}$, where

$$X_j := \sum_{k=1}^{n} a_{k,j}(t)\,\partial_{x_k} \quad (j = 1, \dots, n),$$

where $a_{k,j}(t)$ is the entry of position (k, j) of the matrix $\exp(t\,B)$. The neutral element of $\mathbb{G}(B)$ is $(0, 0)$. The right-invariant measure $\check{H}$ in (1.2) is equal to the Lebesgue measure in $\mathbb{R}^{1+n}$, since (in block form) we have

$$\mathcal{J}\rho_{(t',x')}((0,0)) = \begin{pmatrix} 1 & 0 \\ Bx' & \mathbb{I}_n \end{pmatrix} \qquad (\mathbb{I}_n \text{ is the identity matrix of order } n).$$

On the other hand, since any Haar measure on $\mathbb{G}(B)$ is a (positive) scalar multiple of

$$\frac{1}{\det \mathcal{J}\tau_{(t,x)}((0,0))}\,\mathrm{d}t\mathrm{d}x$$

(here $\tau_{(t,x)}$ denotes the left translation by (t, x)), and since

$$\mathcal{J}\tau_{(t,x)}((0,0)) = \begin{pmatrix} 1 & 0 \\ 0 & \exp(tB) \end{pmatrix}, \tag{4.1}$$

then $\mathbb{G}(B)$ is unimodular (with $\mathrm{d}t\mathrm{d}x$ as left/right-invariant measure) if and only if $\mathrm{trace}(B) = 0$. Thus, our results here are contained in [14] *only when* $\mathrm{trace}(B) = 0$.

More precisely, the L^p-Liouville Theorems 1.2 and 1.3 hold true *for any matrix B* (with L^p standing for the usual $L^p(\mathbb{R}^{1+n})$ space), and for any second-order operator $\mathcal{L}$ which is a polynomial of degree 2 in $\partial_t, X_1, \dots, X_n$ (hence it is left-invariant), provided that $\mathcal{L}$ also fulfills our structure hypotheses (ND) and (HY). For example, $\mathcal{L}$ may be of the form

$$X_1^2 + \cdots + X_n^2 + (\partial_t)^2$$

(a sum of square of Hörmander vector fields), or of the forms

$$X_1^2 + \cdots + X_n^2 - \partial_t, \qquad X_1^2 + \cdots + X_n^2 + \partial_t$$

(evolution Hörmander operators, with drift terms $\pm\partial_t$). These operators are non-degenerate (recall that the vector fields $X_1, \dots, X_n$ are associated with the columns of the non-null matrix $\exp(tB)$), and they are hypoelliptic, due to Hörmander hypoellipticity condition (since $\{\partial_t, X_1, \dots, X_n\}$ is a basis of $\mathfrak{g}(B)$).

More degenerate operators are allowed, as in the next example.

EXAMPLE 4.1. Following [4, Section 3], if B takes on the special "companion" form

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

(for some assigned real numbers $a_0, a_1, \ldots, a_{n-1}$), then

$$\exp(tB) = \begin{bmatrix} u_1(t) & u_1'(t) & \cdots & u_1^{(n-1)}(t) \\ \vdots & \vdots & \vdots & \vdots \\ u_n(t) & u_n'(t) & \cdots & u_n^{(n-1)}(t) \end{bmatrix},$$

where $\{u_1(t), \ldots, u_n(t)\}$ is the fundamental system of solutions of the n-th-order constant-coefficient ODE

$$u^{(n)}(t) + a_{n-1}\, u^{(n-1)}(t) + \cdots + a_1\, u'(t) + a_0\, u(t) = 0.$$

Following our previous notation for the basis $\{\partial_t, X_1, \ldots, X_n\}$ of $\mathfrak{g}(B)$, we have

$$X_j = u_1^{(j-1)}(t)\, \partial_{x_1} + \cdots + u_n^{(j-1)}(t)\, \partial_{x_n} \quad (j = 1, \ldots, n).$$

This shows that it is sufficient to consider the two vector fields

$$\partial_t \quad \text{and} \quad X_1 = u_1(t)\, \partial_{x_1} + \cdots + u_n(t)\, \partial_{x_n}$$

to Lie-generate the whole of $\mathfrak{g}(B)$. Therefore, the five operators

$$(\partial_t)^2 + (X_1)^2, \quad (\partial_t)^2 \pm X_1, \quad (X_1)^2 \pm \partial_t$$

are Hörmander operators (hence hypoelliptic) to which our L^p-Liouville results apply (with measure $\check{H}$ equal to the Lebesgue measure in $\mathbb{R}^{1+n}$). When $a_{n-1} \neq 0$, the associated matrix-exponential group $\mathbb{G}(B)$ is *not* unimodular, so that these operators are not comprised in [14].

4.2. The inverse group of $\mathbb{G}(B)$

We have the following examples:

(a) Let $\mathbb{G}(B)$ be the group constructed in Sect. 4.1. Following [4, Section 4],

we can interchange right and left multiplications of $\mathbb{G}(B)$, obtaining the group $\widehat{\mathbb{G}}(B) := (\mathbb{R}^{1+n}, \widehat{\cdot})$ (referred to as the *inverse group of* $(\mathbb{G}(B), \cdot)$), where

$$(t, x)\,\widehat{\cdot}\,(t', x') = \big(t + t', x' + \exp(t'\, B)\, x\big), \qquad t, t' \in \mathbb{R},\ x, x' \in \mathbb{R}^n. \tag{4.2}$$

Hence, a basis for the Lie algebra of $\widehat{\mathbb{G}}(B)$ is $\{T, \partial_{x_1}, \ldots, \partial_{x_n}\}$, where

$$T := \partial_t + \sum_{i,j=1}^{n} b_{i,j}\, x_j\, \partial_{x_i}.$$

The neutral element of $\widehat{\mathbb{G}}(B)$ is $(0, 0)$ and the associated right-invariant measure $\check{H}$ in (1.2) is equal to

$$\mathrm{d}\check{H}(t, x) = e^{-t\,\mathrm{trace}(B)}\, \mathrm{d}t\mathrm{d}x, \tag{4.3}$$

and it is easy to recognize that $\widehat{\mathbb{G}}(B)$ is unimodular (with $\mathrm{d}t\mathrm{d}x$ as left/right-invariant measure) if and only if $\mathrm{trace}(B) = 0$.

As a consequence, the L^p-Liouville Theorems 1.2 and 1.3 hold true for any matrix B, with L^p standing for $L^p(\mathbb{R}^{1+n}, \check{H})$ with $\check{H}$ as in (4.3), and for any second-order operator $\mathcal{L}$ which is a polynomial of degree 2 in $T, \partial_{x_1}, \dots, \partial_{x_n}$, provided that $\mathcal{L}$ also fulfills our structure hypotheses (ND) and (HY). For example, if we use the compact notations $\Delta_x := \sum_{j=1}^n (\partial_{x_j})^2$ and $T = \partial_t + \langle Bx, \nabla_x \rangle$, $\mathcal{L}$ may be of the form

$$\mathcal{L}_1 = \Delta_x + \Big(\partial_t + \langle Bx, \nabla_x \rangle\Big)^2$$

(a sum of square of Hörmander vector fields), or of the form (replacing B with $-B$)

$$\mathcal{L}_2 = \Delta_x + \langle Bx, \nabla_x \rangle - \partial_t$$

(an evolution Hörmander operator, with drift term $\langle Bx, \nabla_x \rangle - \partial_t$), which is a left-invariant evolution PDO of Kolmogorov–Fokker–Planck type.

(b) Many other examples inspired by the previous case are available of more degenerate operators to which our results apply: for example, when

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

we have $T = \partial_t + x_1\, \partial_{x_2}$ and it is sufficient to consider ∂_{x_1} to obtain the Hörmander system $\{T, \partial_{x_1}\}$ in $\mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}^2_x$. As a consequence, our Theorems 1.2 and 1.3 (with $\check{H}$ equal to the Lebesgue measure on $\mathbb{R}^3$) apply to the two Hörmander operators

$$\mathcal{L}_1 = (\partial_{x_1})^2 + (\partial_t + x_1\, \partial_{x_2})^2, \quad \mathcal{L}_2 = (\partial_{x_1})^2 - \partial_t - x_1\, \partial_{x_2}.$$

The associated group law (4.2) is

$$(t, x_1, x_2) \widehat{\cdot} (t', x_1', x_2') = \Big(t + t', x_1 + x_1', x_2 + x_2' + x_1 t'\Big),$$

which defines the so-called polarized Heisenberg group. We remark that $\mathcal{L}_2$ is a degenerate ultraparabolic operator of Kolmogorov–Fokker–Planck type.

More generally, one can consider a matrix of the form

$$B = \begin{pmatrix} 0 & 0 \\ \mathbb{I}_n & 0 \end{pmatrix} \quad (\mathbb{I}_n \text{ is the } n \times n \text{ identity matrix}),$$

and the associated $\widehat{\mathbb{G}}(B)$ group. In this case the measure $\check{H}$ is the Lebesgue measure on $\mathbb{R}^{2n+1}$ and a meaningful operator to which our results apply is

$$\mathcal{K} = \sum_{j=1}^n (\partial_{x_j})^2 + \sum_{j=1}^n x_j\, \partial_{x_{n+j}} - \partial_t,$$

the classical Kolmogorov–Fokker–Planck operator (see example (i) in the Introduction).

(c) Yet another example is given by the matrix

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

so that $T = \partial_t + (x_1 + x_2)\partial_{x_1} - x_1 \partial_{x_2}$ and the associated right-invariant measure is

$$\mathrm{d}\check{H}(t, x_1, x_2) = e^{-t}\, \mathrm{d}t\mathrm{d}x_1\mathrm{d}x_2.$$

An operator fulfilling our hypotheses (ND), (HY), (LI) is therefore

$$\mathcal{L} = \left(\tfrac{1}{\sqrt{2}}\, \partial_{x_1}\right)^2 - T = \tfrac{1}{2}(\partial_{x_1})^2 - (x_1 + x_2)\partial_{x_1} + x_1\partial_{x_2} - \partial_t,$$

considered in example (ii) in the Introduction.

(d) In general, if the operator $\mathcal{L}$ satisfies our structure conditions (ND), (HY), (LI) (the latter w.r.t. the group $\mathbb{G} = (\mathbb{R}^n, \cdot)$), we can add an extra variable $t \in \mathbb{R}$ thus obtaining a new evolution operator

$$\mathcal{H} := \mathcal{L} - \partial_t \quad \text{on } \mathbb{R}^{n+1} = \mathbb{R}^n_x \times \mathbb{R}_t,$$

to which Theorems 1.2 and 1.3 can be applied: it is suffice to consider the Lie group obtained as a direct product of $\mathbb{G}$ with the group $(\mathbb{R}_t, +)$, by taking into account the right-invariant product measure

$$\mathrm{d}\check{H}(x)\, \mathrm{d}t.$$

4.3. An application to the uniqueness of the Cauchy problem

Suppose that the operator $\mathcal{L}$ in $\mathbb{R}^{1+n}$ (whose points are denoted by (t, x), with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$) satisfies our structure assumptions (ND), (HY), (LI); assume furthermore that $\mathcal{L}$ has the following "Heat-type" form:

$$\mathcal{L} = L - \partial_t, \quad \text{where} \quad L = \sum_{i,j=1}^{n} a_{i,j}(x)\, \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j}. \tag{4.4}$$

Then we prove the following uniqueness result:

PROPOSITION 4.2. *Let us denote by Ω the half-space $\{(t, x) \in \mathbb{R}^{1+n} : t > 0\}$. Under the above assumptions and notation on $\mathcal{L}$, any classical solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ to the Cauchy problem*

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega \\ u(t, x) = 0 & \text{for } t = 0 \end{cases} \tag{4.5}$$

is identically zero on Ω if it holds that $u \in L^p(\Omega, \check{H})$ for some $p \in [1, \infty)$. As usual, $\check{H}$ denotes the right-invariant measure (1.2) on the group $\mathbb{G}$ for which hypothesis (LI) *holds.*

Proof. Let us denote by $\overline{u}$ the trivial prolongation of u on $\mathbb{R}^{1+n}$ obtained by setting $\overline{u}$ to be 0 when $t < 0$. Clearly, $u \in C(\mathbb{R}^{1+n}, \mathbb{R}) \cap L^p(\mathbb{R}^{1+n}, \check{H})$, as $u \in L^p(\Omega, \check{H})$. We claim

$$\overline{u} \in C^\infty(\mathbb{R}^{1+n}, \mathbb{R}) \quad \text{and} \quad \mathcal{L}\overline{u} = 0 \text{ on } \mathbb{R}^{1+n}. \tag{4.6}$$

Once we have proved this, an application of Theorem 1.2 to $\overline{u}$ will prove that $\overline{u} \equiv 0$ on $\mathbb{R}^{1+n}$, i.e., $u = 0$ on Ω.

We are then left to prove the claimed (4.6). Since $\mathcal{L}$ is hypoelliptic by assumption (HY), (4.6) will follow if we show that $\mathcal{L}\overline{u} = 0$ on $\mathbb{R}^{1+n}$ in the weak sense of distributions. To this aim, let $\varphi \in C_0^\infty(\mathbb{R}^{1+n})$. We have the following computation:

$$\begin{aligned}
\int_{\mathbb{R}^{1+n}} \overline{u}\, \mathcal{L}^*\varphi &= \int_{\mathbb{R}^{1+n}} \overline{u}\, (L^*\varphi + \partial_t\varphi) = \int_0^\infty \Big(\int_{\mathbb{R}^n} u\, (L^*\varphi + \partial_{x_n}\varphi)\, \mathrm{d}x\Big)\mathrm{d}t \\
&= \lim_{\epsilon\to 0^+} \int_\epsilon^\infty \Big(\int_{\mathbb{R}^n} u\, L^*\varphi\, \mathrm{d}x + \int_{\mathbb{R}^n} u\, \partial_t\varphi\, \mathrm{d}x\Big)\mathrm{d}t \\
&\quad \text{(by (4.4), } L \text{ operates only in the } x\text{-variable and integration by parts is allowed)} \\
&= \lim_{\epsilon\to 0^+} \int_\epsilon^\infty \Big(\int_{\mathbb{R}^n} Lu\, \varphi\, \mathrm{d}x + \int_{\mathbb{R}^n} u\, \partial_t\varphi\, \mathrm{d}x\Big)\mathrm{d}t \\
&\quad \text{(we use (4.5) and (4.4), ensuring that } Lu = \partial_t u \text{ on } \Omega\text{)} \\
&= \lim_{\epsilon\to 0^+} \int_\epsilon^\infty \Big(\int_{\mathbb{R}^n} \partial_t(u\,\varphi)\, \mathrm{d}x\Big)\mathrm{d}t = \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^n} \Big(\int_\epsilon^\infty \partial_t(u\,\varphi)\, \mathrm{d}t\Big)\mathrm{d}x \\
&= \lim_{\epsilon\to 0^+} -\int_{\mathbb{R}^n} u(\epsilon, x)\, \varphi(\epsilon, x)\, \mathrm{d}x = 0.
\end{aligned}$$

In the last identity we used the initial condition of (4.5) and a simple dominated-convergence argument (since $u \in C(\overline{\Omega})$). This completes the proof. □

We explicitly remark that, among our examples, the operators

- $\mathcal{L}_2$ in Sect. 4.2-(a),
- $\mathcal{L}_2$ and $\mathcal{K}$ in Sect. 4.2-(b),
- $\mathcal{L}$ in Sect. 4.2-(c),
- $\mathcal{H}$ in Sect. 4.2-(d)

all satisfy the structure assumptions in (4.4), so that Proposition 4.2 can be applied to them. Corollary 1.5 in the Introduction is a particular case of Proposition 4.2 (obtained by replacing the matrix B with $-B$).

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