WIENER-LANDIS CRITERION
FOR KOLMOGOROV-TYPE OPERATORS

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Abstract. We establish a necessary and sufficient condition for a boundary point to be regular for the Dirichlet problem related to a class of Kolmogorov-type equations. Our criterion is inspired by two classical criteria for the heat equation: the Evans–Gariepy's Wiener test, and a criterion by Landis expressed in terms of a series of caloric potentials.

1. **Introduction.** Aim of this paper is to establish a necessary and sufficient condition for the regularity of a boundary point for the Dirichlet problem related to a class of hypoelliptic evolution equations of Kolmogorov-type. Our criterion is inspired both to the Evans–Gariepy's Wiener test for the heat equation, and to a criterion by Landis, for the heat equation too, expressed in terms of a series of caloric potentials.

The partial differential operators we are dealing with are of the following type

$$\mathcal{L} = \operatorname{div}(A\nabla) + \langle Bx, \nabla \rangle - \partial_t, \tag{1}$$

where $A = (a_{i,j})_{i,j=1,\ldots,N}$ and $B = (b_{i,j})_{i,j=1,\ldots,N}$ are $N \times N$ real and constant matrices, $z = (x,t) = (x_1,\ldots,x_N,t)$ is the point of $\mathbb{R}^{N+1}$, $\nabla = (\partial_{x_1},\ldots,\partial_{x_N})$, div and $\langle \ , \ \rangle$ stand for the gradient, the divergence and the inner product in $\mathbb{R}^N$, respectively.

The matrix A is supposed to be symmetric and positive semidefinite. Moreover, letting

$$E(s) := \exp(-sB), \quad s \in \mathbb{R},$$

we assume that the following condition is satisfied: the matrix

$$C(t) = \int_0^t E(s)AE^T(s)\,ds \text{ is strictly positive definite for every } t > 0.$$

The matrix $C(t)$ appears as the covariance matrix of a stochastic process relevant to $\mathcal{L}$. As it is quite well known, the condition $C(t) > 0$ for $t > 0$ (which is equivalent to the so-called Kalman condition) is necessary and sufficient for the hypoellipticity of $\mathcal{L}$ in (1), i.e to the smoothness of u whenever $\mathcal{L}u$ is smooth (see, e.g., [8]; see also [14]). We also assume the operator $\mathcal{L}$ to be homogeneous of degree two with respect to a group of dilations in $\mathbb{R}^{N+1}$. As we will recall in Section 2, this is equivalent to assume A and B taking the blocks form (4) and (5).

Under the above assumptions, one can apply results and techniques from potential theory in abstract Harmonic Spaces, as presented, e.g, in [1]. As a consequence, for every bounded open set $\Omega \subseteq \mathbb{R}^{N+1}$ and for every function $f \in C(\partial\Omega, \mathbb{R})$, the Dirichlet problem

$$\mathcal{L}u = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = f, \tag{2}$$

has a *generalized solution* H_f^Ω in the sense of Perron–Wiener–Brelot–Bauer. The function H_f^Ω is smooth and solves the equation in (2) in the classical sense. However, it may occur that H_f^Ω does not assume the boundary datum. A point $z_0 \in \partial\Omega$ is called $\mathcal{L}$-*regular* for Ω if

$$\lim_{z \to z_0} H_f^\Omega(z) = f(z_0) \quad \forall\, f \in C(\partial\Omega, \mathbb{R}).$$

Aim of this paper is to obtain a characterization of the $\mathcal{L}$-regular boundary points in terms of a series involving $\mathcal{L}$-potentials of regions in Ω^c, the complement of Ω, within different level sets of Γ, the fundamental solution of $\mathcal{L}$. More precisely, if $z_0 \in \partial\Omega$ and $\lambda \in\,]0, 1[$ are fixed, we define for $k \in \mathbb{N}$

$$\Omega_k^c(z_0) = \left\{ z \in \Omega^c : \left(\frac{1}{\lambda}\right)^{k \log k} \leq \Gamma(z_0, z) \leq \left(\frac{1}{\lambda}\right)^{(k+1)\log(k+1)} \right\} \cup \{z_0\}.$$

Then, our main result is the following

Theorem 1.1. *Let Ω be a bounded open subset of $\mathbb{R}^{N+1}$ and let $z_0 \in \partial\Omega$. Then z_0 is $\mathcal{L}$-regular for $\partial\Omega$ if and only if*

$$\sum_{k=1}^\infty V_{\Omega_k^c(z_0)}(z_0) = +\infty. \tag{3}$$

Here and in what follows, if F is a compact subset of $\mathbb{R}^{N+1}$, V_F will denote the $\mathcal{L}$-*equilibrium potential* of F, and $\operatorname{cap}(F)$ will denote its $\mathcal{L}$-*capacity*. We refer to Section 3 for the precise definitions.

From Theorem 1.1, one easily obtains a corollary resembling the Wiener test for the classical Laplace and Heat operators.

Corollary 1.2. *Let Ω be a bounded open subset of $\mathbb{R}^{N+1}$ and $z_0 \in \partial\Omega$. The following statements hold:*

(i) *if*

$$\sum_{k=1}^\infty \frac{\operatorname{cap}(\Omega_k^c(z_0))}{\lambda^{k \log k}} = +\infty$$

then z_0 is $\mathcal{L}$-regular;

(ii) *if z_0 is $\mathcal{L}$-regular then*

$$\sum_{k=1}^{\infty} \frac{\mathrm{cap}\left(\Omega_k^c(z_0)\right)}{\lambda^{(k+1)\log(k+1)}} = +\infty.$$

We can make the sufficient condition for the $\mathcal{L}$-regularity more concrete and more geometrical with the following corollary.

Corollary 1.3. *Let Ω be a bounded open subset of $\mathbb{R}^{N+1}$ and $z_0 \in \partial\Omega$. If*

$$\sum_{k=1}^{\infty} \frac{|\Omega_k^c(z_0)|}{\lambda^{\frac{Q+2}{Q}k\log k}} = +\infty$$

then z_0 is $\mathcal{L}$-regular. In particular, the $\mathcal{L}$-regularity of z_0 is ensured if Ω has the exterior $\mathcal{L}$-cone property at z_0.

If E is a subset of either $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$, $|E|$ stands for the relative Lebesgue measure. Moreover, Q is the homogeneous dimension recalled in Section 2, and the $\mathcal{L}$-cone property will be defined precisely in Section 7. We just mention here that it is a natural adaptation of the parabolic cone condition to the homogeneities of the operator $\mathcal{L}$.

Before proceeding, we would like to comment on Theorem 1.1 and Corollary 1.2.

A boundary point regularity test for the heat equation involving infinite sum of (caloric) equilibrium potentials was showed by Landis in [17]. In Landis' proof the caloric potentials are built with respect to regions where the Gaussian kernel takes values in $[\rho_k, \rho_{k+1}]$, being ρ_k a suitable sequence of positive real numbers such that $\frac{\rho_{k+1}}{\rho_k} \nearrow +\infty$. A similar test for a Kolmogorov equation in $\mathbb{R}^3$ was obtained by Scornazzani in [19]. Our Theorem 1.1 contains, extends, and improves the criterion in [19]. The Wiener test for the heat equation was proved by Evans and Gariepy in [3]. It was extended to parabolic operators with variable coefficients in [5, 4] and to the heat equation on the Heisenberg group in [7].

The extension of such a criterion to the Kolmogorov operators (1) is an open, and seemingly difficult, problem. Our Corollary 1.2, which is a straightforward consequence of Theorem 1.1, is a Wiener-type test giving necessary and sufficient conditions which look "almost the same". As a matter of fact, in Theorem 1.1 we have considered the $\mathcal{L}$-potentials of the compact sets $\Omega_k^c(z_0)$ which are built by the difference of two consecutive super-level sets of $\Gamma(z_0, \cdot)$. These level sets correspond with the sequence of values $\lambda^{-k\log k}$. The exact analogue of the Evans-Gariepy criterion would have required the sequence with integer exponents λ^{-k}. The presence of the logarithmic term, which makes the growth of the exponents slightly superlinear, is crucial for our proof of Theorem 1.1. Moreover, such presence is also the responsible for the non-equivalence of the necessary and the sufficient condition in Corollary 1.2. To complete our historical comments, we mention that a potential analysis for Kolmogorov operators of the kind (1) first appeared in [19], in [6], and in [14]. We also mention that the cone criterion contained in Corollary 1.3 has been recently proved in [10], where such a boundary regularity test has been showed for classes of operators more general than (1). For further bibliographical notes concerning Wiener-type tests for both classical and *degenerate* operators, we refer the reader to [15].

The paper is organized as follows. In Section 2 we show some structural properties of $\mathcal{L}$ and fix some notations. Section 3 is devoted to the potential theory for $\mathcal{L}$, while in Section 4 a crucial estimate of the ratio between the fundamental

solution Γ at two different poles is proved. In Section 5 the *only if* part of Theorem 1.1 is proved. The *if* part, the core of our paper, is proved in Section 6, where the estimates of Section 4 play a crucial rôle. Section 7 is devoted to the proof of Corollary 1.2 and Corollary 1.3.

2. **Structural properties of $\mathcal{L}$.** In [14, Section 1] it is proved that the operator $\mathcal{L}$ is left-translation invariant with respect to the Lie group $\mathbb{K}$ whose underlying manifold is $\mathbb{R}^{N+1}$, endowed with the composition law

$$(x, t) \circ (\xi, s) = (\xi + E(s)x, t + s).$$

Furthermore, a fundamental solution for $\mathcal{L}$ (see, e.g., [11, 14]) is given by

$$\Gamma(z, \zeta) = \Gamma\left(\zeta^{-1} \circ z\right) \text{ for } z, \zeta \in \mathbb{R}^{N+1},$$

where,

$$\Gamma(z) = \Gamma(x, t) = \begin{cases} 0 & \text{for } t \leq 0, \\ \dfrac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp\left(-\tfrac{1}{4}\left\langle C^{-1}(t)x, x\right\rangle - t\,\mathrm{tr}B\right) & \text{for } t > 0. \end{cases}$$

We assume the operator $\mathcal{L}$ to be homogeneous of degree two with respect to a group of dilations. This last assumption, together with the hypoellipticity of $\mathcal{L}$, implies that the matrices A and B take the following form with respect to some basis of $\mathbb{R}^N$ (see again [14, Section 1]):

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} \tag{4}$$

for some $p_0 \times p_0$ symmetric and positive definite matrix A_0 ($p_0 \leq N$), and

$$B = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ B_1 & 0 & 0 & 0 & 0 \\ 0 & B_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & B_n & 0 \end{bmatrix}, \tag{5}$$

where B_j is a $p_{j-1} \times p_j$ block with rank p_j ($j = 1, 2, ..., n$), $p_0 \geq p_1 \geq ... \geq p_n \geq 1$ and $p_0 + p_1 + ... + p_n = N$. For such a choice we have $\mathrm{tr}B = 0$, and the family of automorphisms of $\mathbb{K}$ making $\mathcal{L}$ homogeneous of degree two can be taken as

$$\begin{aligned} \delta_r : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{N+1}, \quad \delta_r(x, t) &= \delta_r(x^{(p_0)}, x^{(p_1)}, \dots, x^{(p_n)}, t) \\ &:= \left(rx^{(p_0)}, r^3 x^{(p_1)}, \dots, r^{2n+1} x^{(p_n)}, r^2 t\right), \end{aligned}$$

$$x^{(p_i)} \in \mathbb{R}^{p_i}, \quad i = 0, \dots, n, \quad r > 0.$$

We denote by $Q+2$ $(= p_0 + 3p_1 + ... + (2n+1)p_n + 2)$ the *homogeneous dimension* of $\mathbb{K}$ with respect to $(\delta_r)_{r>0}$. We explicitly remark that Q is the homogenous dimension of $\mathbb{R}^N$ with respect to the dilations

$$D_r : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad D_r(x) = \left(rx^{(p_0)}, r^3 x^{(p_1)}, \dots, r^{2n+1} x^{(p_n)}\right).$$

Under these notations, the matrix $C(t)$ and the fundamental solution of $\mathcal{L}$ with pole at the origin can be written as follows ([14, Proposition 2.3], see also [12]):

$$C(t) = D_{\sqrt{t}} C(1) D_{\sqrt{t}}$$

and

$$\Gamma(x,t) = \begin{cases} 0 & \text{for } t \leq 0, \\[2mm] \frac{c_N}{t^{\frac{Q}{2}}} \exp\left(-\frac{1}{4}\langle C^{-1}(1)D_{\frac{1}{\sqrt{t}}}x, D_{\frac{1}{\sqrt{t}}}x\rangle\right) & \text{for } t > 0. \end{cases}$$

We observe that Γ is δ_r-homogeneous of degree $-Q$.

Throughout the paper we denote by $|\cdot|$ the Euclidean norms in $\mathbb{R}^N$, $\mathbb{R}^{p_k}$ or $\mathbb{R}$. We also denote, for $x \in \mathbb{R}^N$,

$$|x|_C^2 := \frac{1}{4}\langle C^{-1}(1)x, x\rangle.$$

For all $x \in \mathbb{R}^N$, we have

$$|E(1)x|_C^2 \geq \sigma_C^2 |x|^2 \tag{6}$$

where $4\sigma_C^2$ is the smallest eigenvalue of the positive definite matrix

$$E^T(1)C^{-1}(1)E(1).$$

We recall that the homogeneous norm $\|\cdot\| : \mathbb{R}^N \longrightarrow \mathbb{R}^+$ is a D_r-homogeneous function of degree 1 defined as follows

$$\|x\| = \sum_{i=0}^n \left|x^{(p_i)}\right|^{\frac{1}{2i+1}}, \qquad \text{for } x = \left(x^{(p_0)}, \ldots, x^{(p_n)}\right) \in \mathbb{R}^{p_0} \times \ldots \mathbb{R}^{p_n} = \mathbb{R}^N.$$

We call homogeneous cylinder of radius $r > 0$ centered at 0 the set

$$\mathcal{C}_r := \left\{x \in \mathbb{R}^N : \|x\| \leq r\right\} \times \left\{t \in \mathbb{R} : |t| \leq r^2\right\} = \delta_r(\mathcal{C}_1), \tag{7}$$

and define $\mathcal{C}_r(z_0) := z_0 \circ \mathcal{C}_r$.

Remark 2.1. The norms $\|\cdot\|$ and $|\cdot|$ can be compared as follows

$$\sigma \min\left\{|x|, |x|^{\frac{1}{2n+1}}\right\} \leq \|x\| \leq (n+1)\max\left\{|x|, |x|^{\frac{1}{2n+1}}\right\} \qquad \forall\, x \in \mathbb{R}^N, \tag{8}$$

where $\sigma = \min_{|x|=1}\|x\|$.

Indeed, on one side we simply have

$$\|x\| \leq \sum_{i=0}^n |x|^{\frac{1}{2i+1}} \leq (n+1)\max\left\{|x|, |x|^{\frac{1}{2n+1}}\right\} \qquad \forall\, x \in \mathbb{R}^N.$$

On the other hand, for any $x \neq 0$, we get

$$\frac{\|x\|}{\min\left\{|x|, |x|^{\frac{1}{2n+1}}\right\}} \geq \sum_{i=0}^n \frac{\left|x^{(p_i)}\right|^{\frac{1}{2i+1}}}{|x|^{\frac{1}{2i+1}}} = \sum_{i=0}^n \left|\left(\frac{x}{|x|}\right)^{(p_i)}\right|^{\frac{1}{2i+1}} = \left\|\frac{x}{|x|}\right\| \geq \sigma.$$

3. **Some recalls from potential theory for $\mathcal{L}$: $\mathcal{L}$-potentials and $\mathcal{L}$-capacity.** We briefly collect here some notions and results from Potential Theory applied to the operator $\mathcal{L}$.

For every open set $\Omega \subseteq \mathbb{R}^{N+1}$ we denote

$$\mathcal{L}(\Omega) := \{u \in C^\infty(\Omega) \mid \mathcal{L}u = 0\}.$$

and we call $\mathcal{L}$-*harmonic* in Ω the functions in $\mathcal{L}(\Omega)$.

We say that a bounded open set $V \subseteq \Omega$ is $\mathcal{L}$-*regular* if for every continuous function $\varphi : \partial V \longrightarrow \mathbb{R}$, there exists a unique function, h_φ^V in $\mathcal{L}(V)$, continuous in $\overline{V}$, such that

$$h_\varphi^V|_{\partial V} = \varphi.$$

Moreover, if $\varphi \geq 0$ then $h_\varphi^V \geq 0$ by the minimum principle.

A function $u : \Omega \longrightarrow]-\infty, \infty]$ is called $\mathcal{L}$-*superharmonic in* Ω if

(i) u is lower semi-continuous and $u < \infty$ in a dense subset of Ω;

(ii) for every regular set V, $\overline{V} \subseteq \Omega$, and for every $\varphi \in C(\partial V, \mathbb{R})$, $\varphi \leq u|_{\partial V}$, it follows $u \geq h_\varphi^V$ in V.

We will denote by $\overline{\mathcal{L}}(\Omega)$ the family of the $\mathcal{L}$-superharmonic functions in Ω. Since the operator $\mathcal{L}$ endows $\mathbb{R}^{N+1}$ with a structure of β-harmonic space satisfying the Doob convergence property (see [18, 1, 10]), by the Wiener resolutivity theorem, for every $f \in C(\partial\Omega)$, the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = f \end{cases}$$

has a *generalized solution in the sense of Perron–Wiener–Bauer–Brelot* given by

$$H_f^\Omega := \inf\{u \in \overline{\mathcal{L}}(\Omega) \mid \liminf_{\Omega \ni z \to \zeta} u(z) \geq f(\zeta) \quad \forall \, \zeta \in \partial\Omega\}.$$

The function H_f^Ω is $C^\infty(\Omega)$ and satisfies $\mathcal{L}u = 0$ in Ω in the classical sense. However, it is not true, in general, that H_f^Ω continuously takes the boundary values prescribed by f. A point $z_0 \in \partial\Omega$ such that

$$\lim_{\Omega \ni z \to z_0} H_f^\Omega(z) = f(z_0) \quad \text{for every } f \in C(\partial\Omega)$$

is called $\mathcal{L}$-*regular* for Ω.

For our regularity criteria we still need a few more definitions. We denote by $\mathcal{M}(\mathbb{R}^{N+1})$ the collection of all nonnegative Radon measure on $\mathbb{R}^{N+1}$ and we call

$$\Gamma_\mu(z) := \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta) \, d\mu(\zeta), \qquad z \in \mathbb{R}^{N+1},$$

the $\mathcal{L}$-*potential of* μ.

If F is a compact set of $\mathbb{R}^{N+1}$ and $\mathcal{M}(F)$ is the collection of all nonnegative Radon measure on $\mathbb{R}^{N+1}$ with support in F, the $\mathcal{L}$-*capacity of* F is defined as

$$\text{cap}\,(F) := \sup\{\mu(\mathbb{R}^{N+1}) \mid \mu \in \mathcal{M}(F), \ \Gamma_\mu \leq 1 \text{ on } \mathbb{R}^{N+1}\}.$$

We list some properties of the $\mathcal{L}$-capacity cap$(\cdot)$. For every F, F_1 and F_2 compact subsets of $\mathbb{R}^{N+1}$, we have:

(i) $\text{cap}\,(F) < \infty$;

(ii) if $F_1 \subseteq F_2$, then $\text{cap}\,(F_1) \leq \text{cap}\,(F_2)$;

(iii) $\text{cap}\,(F_1 \cup F_2) \leq \text{cap}\,(F_1) + \text{cap}\,(F_2)$;

(iv) $\text{cap}\,(z_0 \circ F) = \text{cap}\,(F)$ for every $z_0 \in \mathbb{R}^{N+1}$;

(v) $\text{cap}\,(\delta_r(F)) = r^Q \text{cap}\,(F)$ for every $r > 0$;

(vi) if $F = A \times \{\tau\}$ for some compact set $A \subset \mathbb{R}^N$, then $\text{cap}\,(F) = |A|$;

(vii) if $F \subset \mathbb{R}^N \times [a, b]$, then we have

$$\text{cap}\,(F) \geq \frac{|F|}{b-a}. \tag{9}$$

The properties $(i) - (v)$ are quite standard, and they follow from the features of Γ. We want to spend few words on the last two properties. Property (vi) was proved in [13, Proposizione 5.1] in the case of the heat operator, namely with the capacity build on the Gauss-Weierstrass kernel. It can be proved verbatim proceeding in our situation: the main tools are the facts that Γ has integral 1 over $\mathbb{R}^N$, and it reproduces the solutions of the Cauchy problems. Property (vii) appears to be new

even in the classical parabolic case (at least to the best of our knowledge), and it can be deduced readily from (vi). As a matter of fact, if a compact set F lies in a strip $\mathbb{R}^N \times [a, b]$, we have

$$
\begin{aligned}
(b-a)\operatorname{cap}(F) &= \int_a^b \operatorname{cap}(F)\,\mathrm{d}\tau \\
&\geq \int_a^b \operatorname{cap}(F \cap \{t = \tau\})\,\mathrm{d}\tau = \int_a^b |F \cap \{t = \tau\}|\,\mathrm{d}\tau = |F|.
\end{aligned}
$$

The last notions we need are the ones of *reduced function* and of *balayage*. To this aim, for any compact set F, we introduce the notation

$$
\Phi_F := \left\{ v \in \overline{\mathcal{L}}(\mathbb{R}^{N+1}) \,:\, v \geq 0 \text{ in } \mathbb{R}^{N+1},\ v \geq 1 \text{ in } F \right\}. \tag{10}
$$

Then, the reduced function W_F and the balayage V_F of 1 on F are respectively defined as

$$
W_F := \inf \{ v \mid v \in \Phi_F \},
$$

and

$$
V_F(z) = \liminf_{\zeta \longrightarrow z} W_F(\zeta), \qquad z \in \mathbb{R}^{N+1}.
$$

From general balayage theory we have that V_F is less or equal than 1 everywhere, identically 1 in the interior of F, it vanishes at infinity, is a superharmonic function on $\mathbb{R}^{N+1}$ and harmonic on $\mathbb{R}^{N+1} \setminus \partial F$. Furthermore, the following properties will be useful for us. Let F, F_1, F_2 be compact subsets of $\mathbb{R}^{N+1}$, and let $(F_n)_{n \in \mathbb{N}}$ be a sequence of compact subsets of $\mathbb{R}^{N+1}$, we have:

(i) if $F_1 \subseteq F_2 \subseteq \mathbb{R}^{N+1}$, then $V_{F_1} \leq V_{F_2}$;

(ii) $V_{F_1 \cup F_2} \leq V_{F_1} + V_{F_2}$;

(iii) if $F \subseteq \bigcup_{n \in \mathbb{N}} F_n$, then $V_F \leq \sum_{n=1}^\infty V_{F_n}$.

The first property is a consequence of the definition of balayage; for the second and the third one we refer respectively to [2, Proposition 5.3.1] and [2, Theorem 4.2.2 and Corollary 4.2.2].

Now, following the same lines of the proof of [13, Teorema 1.1], we have the existence of a unique measure $\mu_F \in \mathcal{M}(F)$ such that

$$
V_F(z) = \Gamma_{\mu_F}(z) = \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta)\,\mathrm{d}\mu_F(\zeta) \quad \forall z \in \mathbb{R}^{N+1}, \tag{11}
$$

and

$$
\mu_F(\mathbb{R}^{N+1}) = \operatorname{cap}(F).
$$

V_F is also called the $\mathcal{L}$-*equilibrium potential* of F and μ_F the $\mathcal{L}$-*equilibrium measure* of F. The proof of this fact relies on the good behavior of Γ, a representation formula of Riesz-type for $\mathcal{L}$-superharmonic functions proved in [1, Theorem 5.1], and a Maximum Principle for $\mathcal{L}$ (see [1, Proposition 2.3]).

Fix now a bounded open set Ω compactly contained in $\mathbb{R}^{N+1}$, and $z_0 = (x_0, t_0) \in \partial\Omega$. Keeping in mind the definition of homogeneous cylinders in (7), let us denote by

$$
G_r = \{(x, t) \in \mathcal{C}_r(z_0) \setminus \Omega \,:\, t \leq t_0\}. \tag{12}
$$

From general balayage theory and proceeding, e.g., as in [16, Theorem 4.6], we can characterize the regularity of the boundary point of Ω by the following condition:

the point $z_0 \in \partial\Omega$ is $\mathcal{L}$-regular if and only if

$$
\lim_{r \to 0} V_{G_r}(z_0) > 0. \tag{13}
$$

4. **A crucial estimate.** We start by recalling the following identity, whose proof can be found in [14, Remark 2.1] (see also [12]),

$$E(\lambda^2 s)D_\lambda = D_\lambda E(s) \qquad \forall \lambda > 0, \forall s \in \mathbb{R}. \tag{14}$$

In what follows we will need the following lemma.

Lemma 4.1. *For $0 > t > \tau$ we have the following matrix inequality*

$$E^T(t)C^{-1}(t-\tau)E(t) \geq C^{-1}(-\tau).$$

Proof. Since for symmetric positive definite matrices we have

$$M_1 \leq M_2 \quad \Rightarrow \quad M_1^{-1} \geq M_2^{-1}$$

(see [9, Corollary 7.7.4]) and recalling that $E^{-1}(t) = E(-t)$, it is enough to prove that

$$E(-t)C(t-\tau)E^T(-t) \leq C(-\tau). \tag{15}$$

From the very definition of the matrix C we get

$$
\begin{aligned}
E(-t)C(t-\tau)E^T(-t) &= e^{tB}\left(\int_0^{t-\tau} e^{-sB}Ae^{-sB^T}\,\mathrm{ds}\right)e^{tB^T} \\
&= \int_0^{t-\tau} e^{(t-s)B}Ae^{(t-s)B^T}\,\mathrm{ds} \\
&= \int_{-t}^{-\tau} e^{-\sigma B}Ae^{-\sigma B^T}\,\mathrm{d}\sigma.
\end{aligned}
$$

Since $-\tau > -t > 0$ and A is nonnegative definite, we have

$$\int_{-t}^{-\tau} e^{-\sigma B}Ae^{-\sigma B^T}\,\mathrm{d}\sigma \leq \int_0^{-\tau} e^{-\sigma B}Ae^{-\sigma B^T}\,\mathrm{d}\sigma = C(-\tau)$$

which proves (15) and the lemma. $\qquad\square$

A crucial role in the proof of our main theorem will be played by the ratio $\frac{\Gamma(z,\zeta)}{\Gamma(0,\zeta)}$, for $z = (x,t)$ and $\zeta = (\xi,\tau)$ with $0 > t > \tau$. We use the following notations

$$\mu = \frac{-t}{-\tau} \in (0,1), \qquad M(z) = \left|D_{\frac{1}{\sqrt{-t}}}x\right|, \qquad M(\zeta) = \left|D_{\frac{1}{\sqrt{-\tau}}}\xi\right|. \tag{16}$$

Lemma 4.2. *There exists a positive constant C such that, for any $z = (x,t), \zeta = (\xi,\tau)$ with $0 > t > \tau$ and $\mu \leq \min\{\frac{1}{2}, \frac{\sigma^2}{(n+1)^2}\}$, we have*

$$\frac{\Gamma(z,\zeta)}{\Gamma(0,\zeta)} \leq \left(\frac{1}{1-\mu}\right)^{\frac{Q}{2}} e^{C\sqrt{\mu}M(z)M(\zeta)}.$$

Proof. In our notations we can write

$$
\begin{aligned}
\frac{\Gamma(z,\zeta)}{\Gamma(0,\zeta)} &= \frac{(t-\tau)^{-\frac{Q}{2}}e^{-\left|D_{\frac{1}{\sqrt{t-\tau}}}(x-E(t-\tau)\xi)\right|_C^2}}{(-\tau)^{-\frac{Q}{2}}e^{-\left|D_{\frac{1}{\sqrt{-\tau}}}(E(-\tau)\xi)\right|_C^2}} \\
&= \left(\frac{1}{1-\mu}\right)^{\frac{Q}{2}} e^{\left|D_{\frac{1}{\sqrt{-\tau}}}(E(-\tau)\xi)\right|_C^2 - \left|D_{\frac{1}{\sqrt{t-\tau}}}(x-E(t-\tau)\xi)\right|_C^2}.
\end{aligned}
$$

Let us deal with the exponential term

$$\left| D_{\frac{1}{\sqrt{-\tau}}} \left(E(-\tau)\xi \right) \right|_C^2 - \left| D_{\frac{1}{\sqrt{t-\tau}}} \left(x - E(t-\tau)\xi \right) \right|_C^2 = \tag{17}$$

$$= \frac{1}{4} \left\langle C^{-1}(-\tau) E(-\tau)\xi, E(-\tau)\xi \right\rangle$$

$$\quad - \frac{1}{4} \left\langle C^{-1}(t-\tau) \left(x - E(t-\tau)\xi \right), \left(x - E(t-\tau)\xi \right) \right\rangle.$$

Lemma 4.1 says in particular that we have

$$\left\langle C^{-1}(-\tau) E(-\tau)\xi, E(-\tau)\xi \right\rangle - \left\langle C^{-1}(t-\tau) E(t-\tau)\xi, E(t-\tau)\xi \right\rangle \leq 0.$$

Using this in (17) we get

$$\left| D_{\frac{1}{\sqrt{-\tau}}} \left(E(-\tau)\xi \right) \right|_C^2 - \left| D_{\frac{1}{\sqrt{t-\tau}}} \left(x - E(t-\tau)\xi \right) \right|_C^2 \leq \tag{18}$$

$$\leq -\frac{1}{4} \left\langle C^{-1}(t-\tau)x, x \right\rangle + \frac{1}{2} \left\langle C^{-1}(t-\tau)x, E(t-\tau)\xi \right\rangle$$

$$\leq \frac{1}{2} \left\langle C^{-1}(t-\tau)x, E(t-\tau)\xi \right\rangle$$

$$\leq \frac{1}{2} \left(\left\langle C^{-1}(t-\tau)x, x \right\rangle \left\langle C^{-1}(t-\tau)E(t-\tau)\xi, E(t-\tau)\xi \right\rangle \right)^{\frac{1}{2}}.$$

We are going to bound $\left\langle C^{-1}(t-\tau)x, x \right\rangle$ and $\left\langle C^{-1}(t-\tau)E(t-\tau)\xi, E(t-\tau)\xi \right\rangle$ separately. We have

$$\left\langle C^{-1}(t-\tau)x, x \right\rangle = \left\langle C^{-1}\left(\frac{1}{\mu} - 1 \right) D_{\frac{1}{\sqrt{-t}}}x, D_{\frac{1}{\sqrt{-t}}}x \right\rangle$$

$$\leq \left\| C^{-1}\left(\frac{1}{\mu} - 1 \right) \right\| M^2(z),$$

where $\|A\|$ stands for the operator norm of a matrix A (i.e. its biggest eigenvalue for symmetric matrices). By (8), for any vector v with $|v| = 1$ we get

$$\min \left\{ \left| D_{\sqrt{\mu}}v \right|, \left| D_{\sqrt{\mu}}v \right|^{\frac{1}{2n+1}} \right\} \leq \frac{1}{\sigma} \sqrt{\mu} \|v\|$$

$$\leq \frac{n+1}{\sigma} \sqrt{\mu} \max \left\{ |v|, |v|^{\frac{1}{2n+1}} \right\} = \frac{n+1}{\sigma} \sqrt{\mu}.$$

From $\mu \leq \frac{\sigma^2}{(n+1)^2}$ we then deduce $\left| D_{\sqrt{\mu}}v \right| \leq \frac{n+1}{\sigma} \sqrt{\mu}$. Hence, since μ is also less than $\frac{1}{2}$,

$$\left\langle C^{-1}\left(\frac{1}{\mu} - 1 \right) v, v \right\rangle = \left\langle C^{-1}(1-\mu) D_{\sqrt{\mu}}v, D_{\sqrt{\mu}}v \right\rangle$$

$$\leq \left\| C^{-1}(1-\mu) \right\| \left| D_{\sqrt{\mu}}v \right|^2 \leq \frac{(n+1)^2}{\sigma^2} \left\| C^{-1}(1-\mu) \right\| \mu$$

$$\leq \frac{(n+1)^2}{\sigma^2} \left\| C^{-1}\left(\frac{1}{2} \right) \right\| \mu \qquad \forall |v| = 1.$$

This gives

$$\left\langle C^{-1}(t-\tau)x, x \right\rangle \leq \frac{(n+1)^2}{\sigma^2} \left\| C^{-1}\left(\frac{1}{2} \right) \right\| \mu M^2(z). \tag{19}$$

On the other hand, by the commutation property (14), we get

$$\left\langle C^{-1}(t-\tau)E(t-\tau)\xi, E(t-\tau)\xi \right\rangle =$$

$$= \left\langle C^{-1}(1-\mu)D_{\frac{1}{\sqrt{-\tau}}}E(t-\tau)\xi, D_{\frac{1}{\sqrt{-\tau}}}E(t-\tau)\xi \right\rangle$$

$$\leq \left\| C^{-1}(1-\mu) \right\| \left| D_{\frac{1}{\sqrt{-\tau}}}E(t-\tau)\xi \right|^2 = \left\| C^{-1}(1-\mu) \right\| \left| E(1-\mu)D_{\frac{1}{\sqrt{-\tau}}}\xi \right|^2$$

$$\leq \left\| C^{-1}(1-\mu) \right\| \left\| E^T(1-\mu)E(1-\mu) \right\| M^2(\zeta)$$

$$\leq \left\| C^{-1}\left(\frac{1}{2}\right) \right\| \left\| E^T(1-\mu)E(1-\mu) \right\| M^2(\zeta).$$

Since $0 < \mu \leq \frac{1}{2}$, the term $\left\| E^T(1-\mu)E(1-\mu) \right\|$ is bounded from above by a universal constant C_0^2. Thus we have

$$\left\langle C^{-1}(t-\tau)E(t-\tau)\xi, E(t-\tau)\xi \right\rangle \leq C_0^2 \left\| C^{-1}\left(\frac{1}{2}\right) \right\| M^2(\zeta). \qquad (20)$$

Plugging (19) and (20) in (18), we get

$$\left| D_{\frac{1}{\sqrt{-\tau}}}(E(-\tau)\xi) \right|_C^2 - \left| D_{\frac{1}{\sqrt{t-\tau}}}(x - E(t-\tau)\xi) \right|_C^2 \leq$$

$$\leq \frac{C_0}{2}\frac{n+1}{\sigma} \left\| C^{-1}\left(\frac{1}{2}\right) \right\| \sqrt{\mu}M(z)M(\zeta).$$

Therefore

$$\frac{\Gamma(z,\zeta)}{\Gamma(0,\zeta)} \leq \left(\frac{1}{1-\mu}\right)^{\frac{Q}{2}} e^{C\sqrt{\mu}M(z)M(\zeta)}$$

with $C = \frac{C_0}{2}\frac{n+1}{\sigma} \left\| C^{-1}\left(\frac{1}{2}\right) \right\|$. $\qquad \square$

5. **Necessary condition for regularity.** The characterization in (13), together with the following lemma, will give the necessity of (3) in Theorem 1.1.

Lemma 5.1. *For every fixed $p \in \mathbb{N}$, let us split the set G_r defined in (12) as follows*

$$G_r = G_r^p \cup G_r^{*p},$$

where

$$G_r^p = \left\{ z \in G_r \mid \Gamma(z_0, z) \geq \left(\frac{1}{\lambda}\right)^{p\log p} \right\} \cup \{z_0\},$$

$$and \quad G_r^{*p} = \left\{ z \in G_r \mid \Gamma(z_0, z) \leq \left(\frac{1}{\lambda}\right)^{p\log p} \right\}.$$

Then,

$$\lim_{r \to 0} V_{G_r}(z_0) = \lim_{r \to 0} V_{G_r^p}(z_0).$$

Proof. From the monotonicity and subadditivity properties of the balayage, we have

$$V_{G_r^p}(z_0) \leq V_{G_r}(z_0) \leq V_{G_r^p}(z_0) + V_{G_r^{*p}}(z_0).$$

Furthermore, by (11),

$$V_{G_r^{*p}}(z_0) \leq \left(\frac{1}{\lambda}\right)^{p\log p} \operatorname{cap}\left(G_r^{*p}\right).$$

On the other hand, from the monotonicity and homogeneity properties of the capacities, it follows

$$\mathrm{cap}\,(G_r^{*p}) \leq \mathrm{cap}\,(\mathcal{C}_r(z_0)) = \mathrm{cap}\,(z_0 \circ \delta_r(\mathcal{C}_1)) = r^Q \mathrm{cap}\,(\mathcal{C}_1(z_0)).$$

Hence $\mathrm{cap}\,(G_r^{*p})$ goes to zero as r goes to zero. This proves the lemma. $\square$

Proof of necessary condition in Theorem 1.1. Assume

$$\sum_{k=1}^{\infty} V_{\Omega_k^c(z_0)}(z_0) < +\infty.$$

We are going to prove the non regularity of the boundary point z_0. The assumption implies that for every $\varepsilon > 0$, there exists $p \in \mathbb{N}$ such that

$$\sum_{k=p}^{\infty} V_{\Omega_k^c(z_0)}(z_0) < \varepsilon.$$

On the other hand, with the notations of the previous lemma, for any positive r

$$G_r^p \subseteq \bigcup_{k=p}^{\infty} \Omega_k^c(z_0),$$

so that,

$$V_{G_r^p}(z_0) \leq \sum_{k=p}^{\infty} V_{\Omega_k^c(z_0)}(z_0) < \varepsilon.$$

Then, from Lemma 5.1, we get $\lim_{r \to 0} V_{G_r}(z_0) \leq \varepsilon$ for every $\varepsilon > 0$, which implies

$$\lim_{r \longrightarrow 0} V_{G_r}(z_0) = 0.$$

Hence, by (13), the boundary point z_0 is not $\mathcal{L}$-regular. $\square$

6. **Sufficient condition for regularity.** In this section we prove the *if* part of Theorem 1.1. This is the core of our main result and requires three lemmas.

Lemma 6.1. *Suppose we have a sequence of compact sets $\{F_k\}_{k \in \mathbb{N}}$ in $\mathbb{R}^{N+1}$ such that*

$$\begin{cases} F_k \cap F_h = \emptyset & \text{if } k \neq h, \\ \forall r > 0 \ \exists \bar{k} \ \text{ such that } \ F_k \subseteq G_r & \text{for } k \geq \bar{k}. \end{cases}$$

Suppose also that the following two conditions hold true:

(i)

$$\sum_{k=1}^{+\infty} V_{F_k}(z_0) = +\infty;$$

(ii)

$$\sup_{h \neq k} \sup \left\{ \frac{\Gamma(z, \zeta)}{\Gamma(z_0, \zeta)} \ : \ z \in F_h, \ \zeta \in F_k \right\} \leq M_0.$$

Then we have

$$V_{G_r}(z_0) \geq \frac{1}{2M_0} \quad \text{for every positive } r.$$

Proof. Let $A > \frac{2}{M_0}$, and fix any $r > 0$. Let us pick $m, n \in \mathbb{N}$ with $m < n$ such that

$$\bigcup_{k=m}^{n} F_k \subseteq G_r \quad \text{and} \quad \sum_{k=m}^{n} V_{F_k}(z_0) \geq A.$$

We are going to denote by $G_{m,n} = \bigcup_{k=m}^{n} F_k$ and by $W_{m,n}(z) = \sum_{k=m}^{n} V_{F_k}(z)$. We want to estimate $W_{m,n}$ on $G_{m,n}$.

Take $z \in G_{m,n}$. We have then $z \in F_h$ for some $h \in \{m, \dots, n\}$. Of course we have $V_{F_h}(z) \leq 1$. On the other hand, if $k \neq h$ we get

$$V_{F_k}(z) = \int_{F_k} \Gamma(z, \zeta) \, d\mu_k(\zeta) = \int_{F_k} \frac{\Gamma(z, \zeta)}{\Gamma(z_0, \zeta)} \Gamma(z_0, \zeta) \, d\mu_k(\zeta) \leq M_0 V_{F_k}(z_0).$$

Hence $V_{F_k} \leq M_0 V_{F_k}(z_0)$ in F_h. By the continuity of the equilibrium potentials (outside of their relative compact sets) there exists an open neighborhood O_h of F_h such that

$$V_{F_k}(z) \leq M_0 V_{F_k}(z_0) + \frac{1}{2^k} \quad \forall z \in O_h, \text{ and } \forall k \in \{m, \dots, n\}, \, k \neq h.$$

We put $O = \bigcup_h O_h$. For $z \in O$ we get

$$W_{m,n}(z) \leq 1 + M_0 \sum_{k=m}^{n} V_{F_k}(z_0) + \sum_{k=m}^{n} \frac{1}{2^k} \leq 2 + M_0 \sum_{k=m}^{n} V_{F_k}(z_0).$$

If we consider the function $v_{m,n} = \frac{1}{2 + M_0 \sum_{k=m}^{n} V_{F_k}(z_0)} W_{m,n}$, we thus get $v_{m,n} \leq 1$ in O. Moreover, the function $v_{m,n}$ is a nonnegative $\mathcal{L}$-superharmonic function in $\mathbb{R}^{N+1}$, it is $\mathcal{L}$-harmonic in $\mathbb{R}^{N+1} \setminus G_{m,n}$, and it vanishes at the infinity. Recalling the notation fixed in (10), if we take any function $u \in \Phi_{G_{m,n}}$ we have

$$\begin{cases} u - v_{m,n} \in \overline{\mathcal{L}}(\mathbb{R}^{N+1} \setminus G_{m,n}), \\ \liminf_{z \to \infty} u(z) - v_{m,n}(z) \geq 0, \\ \liminf_{z \to \zeta \in \partial G_{m,n}} u(z) - v_{m,n}(z) \geq u(\zeta) - 1 \geq 0. \end{cases}$$

The maximum principle yields that $u - v_{m,n}$ has to be nonnegative in $\mathbb{R}^{N+1} \setminus G_{m,n}$. On the other hand, $u \geq 1 \geq v_{m,n}$ in $G_{m,n}$. Therefore $u \geq v_{m,n}$ in $\mathbb{R}^{N+1}$, for every $u \in \Phi_{G_{m,n}}$. This implies that

$$V_{G_{m,n}}(z) \geq v_{m,n}(z) = \frac{W_{m,n}(z)}{2 + M_0 \sum_{k=m}^{n} V_{F_k}(z_0)} \quad \text{for all } z \in \mathbb{R}^{N+1}.$$

In particular this has to be true at $z = z_0$, i.e.

$$V_{G_{m,n}}(z_0) \geq \frac{\sum_{k=m}^{n} V_{F_k}(z_0)}{2 + M_0 \sum_{k=m}^{n} V_{F_k}(z_0)}.$$

Since the function $s \mapsto \frac{s}{2 + M_0 s}$ is increasing, we deduce

$$V_{G_{m,n}}(z_0) \geq \frac{A}{2 + M_0 A} > \frac{1}{2M_0}.$$

This concludes the proof since $V_{G_r} \geq V_{G_{m,n}}$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

In order to simplify the notations, from now on we assume $z_0 = 0 \in \partial\Omega$. This is not restrictive because of the left-invariance property. We want to choose suitably

the compact sets F_k of the previous lemma. For any fixed $\lambda \in (0,1)$, we recall that

$$\Omega_k^c(0) = \left\{ z \in \Omega^c \; : \; \left(\frac{1}{\lambda} \right)^{k \log k} \leq \Gamma(0,z) \leq \left(\frac{1}{\lambda} \right)^{(k+1)\log(k+1)} \right\}.$$

Now, we set

$$\alpha(k) = k \log k \tag{21}$$

and denote

$$T_k = \max_{(x,t)\in\Omega_k^c(0)} -t = \left(c_N \lambda^{\alpha(k)} \right)^{\frac{2}{Q}}. \tag{22}$$

We fix $q \in \mathbb{N}$ such that

$$q \geq q_0 := 4 + \frac{m}{\log\left(\frac{1}{\lambda}\right)}, \tag{23}$$

where

$$m = \max\left\{ 2, \frac{Q}{\log 6}, \frac{2\sigma_C^2}{\log 6}, \frac{Q\log 2}{\log 8}, \frac{2Q\log\left(\frac{n+1}{\sigma}\right)}{\log 8} \right\},$$

and σ_C, σ are the constants in (6) and (8). We also denote by

$$p = 1 + \left[\frac{q}{2} \right] = 1 + \text{the integer part of } \frac{q}{2}. \tag{24}$$

So $\frac{q}{2} \leq p \leq 1 + \frac{q}{2} < q - 1$. For any $k \in \mathbb{N}$ we want to consider the sets

$$\Omega_{kq}^c(0) = \left\{ z \in \Omega^c \; : \; \left(\frac{1}{\lambda} \right)^{\alpha(kq)} \leq \Gamma(0,z) \leq \left(\frac{1}{\lambda} \right)^{\alpha(kq+1)} \right\}.$$

Moreover, we put

$$\Omega_{kq}^c(0) = \left(\Omega_{kq}^c(0) \cap \{ t \geq -T_{kq}^* \} \right) \cup \left(\Omega_{kq}^c(0) \cap \{ t \leq -T_{kq}^* \} \right) := F_k^{(0)} \cup F_k \tag{25}$$

where

$$T_{kq}^* = T_{kq+p} = \left(c_N \lambda^{\alpha(kq+p)} \right)^{\frac{2}{Q}}. \tag{26}$$

First we notice that, since $kq + p < q(k+1)$, F_k lies strictly below F_{k+1}, namely

$$\min_{(x,t)\in F_h} t = -T_{hq} > -T_{kq}^* = \max_{(\xi,\tau)\in F_k} \tau \qquad \forall h, k \in \mathbb{N}, \; h > k. \tag{27}$$

Lemma 6.2. *We have*

$$\sum_{k=1}^{+\infty} V_{F_k^{(0)}}(0) < +\infty.$$

Proof. We are going to prove that $F_k^{(0)}$ is contained in a homogeneous cylinder $\mathcal{C}_{r_k}$, where r_k is such that

$$\sum_{k=1}^{+\infty} \left(\frac{1}{\lambda} \right)^{\alpha(kq+1)} r_k^Q < +\infty. \tag{28}$$

This is enough to prove the statement since

$$V_{F_k^{(0)}}(0) = \int_{F_k^{(0)}} \Gamma(0,\zeta) \, \mathrm{d}\mu_{F_k^{(0)}}(\zeta) \leq \left(\frac{1}{\lambda} \right)^{\alpha(kq+1)} \text{cap}\,(F_k^{(0)}),$$

and by monotonicity and homogeneity we have

$$\text{cap}\,(F_k^{(0)}) \leq \text{cap}\,(\mathcal{C}_{r_k}) = \text{cap}\,(\mathcal{C}_1) r_k^Q.$$

In order to prove (28), we have to find a good bound for r_k.

Fix $z = (x,t) \in F_k^{(0)}$. Since in particular $z \in \Omega_{kq}^c(0)$, we have

$$\left| D_{\frac{1}{\sqrt{-t}}}(E(-t)x) \right|_C^2 \le \log\left(\frac{c_N \lambda^{\alpha(kq)}}{(-t)^{\frac{Q}{2}}} \right).$$

On the other hand, by (14) and (6), we get

$$\left| D_{\frac{1}{\sqrt{-t}}}(E(-t)x) \right|_C^2 = \left| E(1)D_{\frac{1}{\sqrt{-t}}}x \right|_C^2 \ge \sigma_C^2 \left| D_{\frac{1}{\sqrt{-t}}}x \right|^2$$

and then

$$\left| D_{\frac{1}{\sqrt{-t}}}x \right|^2 \le \frac{1}{\sigma_C^2} \log\left(\frac{c_N \lambda^{\alpha(kq)}}{(-t)^{\frac{Q}{2}}} \right). \tag{29}$$

Therefore, from (8), we deduce

$$\frac{1}{\sqrt{-t}}\|x\| = \left\| D_{\frac{1}{\sqrt{-t}}}x \right\| \le (n+1)\max\left\{ \left| D_{\frac{1}{\sqrt{-t}}}x \right|, \left| D_{\frac{1}{\sqrt{-t}}}x \right|^{\frac{1}{2n+1}} \right\}$$

$$\le (n+1)\max\left\{ \frac{1}{\sigma_C}\log^{\frac{1}{2}}\left(\frac{c_N \lambda^{\alpha(kq)}}{(-t)^{\frac{Q}{2}}} \right), \frac{1}{\sigma_C^{\frac{1}{2n+1}}}\log^{\frac{1}{2(2n+1)}}\left(\frac{c_N \lambda^{\alpha(kq)}}{(-t)^{\frac{Q}{2}}} \right) \right\}.$$

From the choice $\alpha(k) = k\log k$ in (21) it follows that

$$k \mapsto \alpha(kq+p) - \alpha(kq) \quad \text{is monotone increasing.}$$

In particular, by (24), $\alpha(kq+p) - \alpha(kq) \ge \alpha(\frac{3}{2}q) - \alpha(q) \ge \frac{1}{2}q\log(\frac{3}{2}q) \ge \frac{1}{2}q\log 6$. Because of our choice of q (23) and T_{kq}^* (26), we have then $\alpha(kq+p) - \alpha(kq) \ge \frac{Q}{2\log(\frac{1}{\lambda})}$ and so

$$(T_{kq}^*)^{\frac{Q}{2}} = c_N \lambda^{\alpha(kq+p)} \le c_N \lambda^{\alpha(kq)} e^{-\frac{Q}{2}} \qquad \forall k.$$

This fact and the fact that for $\beta > 0$ the function $s \mapsto s\log^\beta \frac{\alpha}{s^Q}$ is increasing in the interval $(0, e^{-\beta}\alpha^{\frac{1}{Q}}]$ allow to bound the term $\|x\|$ further. Indeed, having $0 < -t \le T_{kq}^*$, we get

$$\|x\| \le (n+1)\max\left\{ \frac{\sqrt{-t}}{\sigma_C}\log^{\frac{1}{2}}\left(\frac{c_N \lambda^{\alpha(kq)}}{(-t)^{\frac{Q}{2}}} \right), \frac{\sqrt{-t}}{\sigma_C^{\frac{1}{2n+1}}}\log^{\frac{1}{2(2n+1)}}\left(\frac{c_N \lambda^{\alpha(kq)}}{(-t)^{\frac{Q}{2}}} \right) \right\}$$

$$\le (n+1)\max\left\{ \frac{\sqrt{T_{kq}^*}}{\sigma_C}\log^{\frac{1}{2}}\left(\frac{c_N \lambda^{\alpha(kq)}}{(T_{kq}^*)^{\frac{Q}{2}}} \right), \frac{\sqrt{T_{kq}^*}}{\sigma_C^{\frac{1}{2n+1}}}\log^{\frac{1}{2(2n+1)}}\left(\frac{c_N \lambda^{\alpha(kq)}}{(T_{kq}^*)^{\frac{Q}{2}}} \right) \right\}.$$

Since $\frac{1}{2}q\log 6 \ge \frac{\sigma_C^2}{\log(\frac{1}{\lambda})}$ we have also $(T_{kq}^*)^{\frac{Q}{2}} \le c_N \lambda^{\alpha(kq)} e^{-\sigma_C^2}$, which says that $\log^{\frac{1}{2}}\left(\frac{c_N \lambda^{\alpha(kq)}}{(T_{kq}^*)^{\frac{Q}{2}}} \right) \ge \sigma_C$ and implies

$$\|x\| \le \frac{(n+1)}{\sigma_C}\sqrt{T_{kq}^*}\log^{\frac{1}{2}}\left(\frac{c_N \lambda^{\alpha(kq)}}{(T_{kq}^*)^{\frac{Q}{2}}} \right).$$

Summing up, we have just proved that

$$(x,t) \in F_k^{(0)} \quad \Longrightarrow \quad \begin{cases} \|x\| \le \frac{n+1}{\sigma_C}\sqrt{T_{kq}^*}\log^{\frac{1}{2}}\left(\frac{c_N \lambda^{\alpha(kq)}}{(T_{kq}^*)^{\frac{Q}{2}}} \right) =: r_k & \text{and} \\ 0 < -t \le T_{kq}^* \le (n+1)^2 T_{kq}^* \le r_k^2, \end{cases}$$

namely

$$F_k^{(0)} \subseteq \mathcal{C}_{r_k}.$$

We are left with verifying (28) with this choice of r_k. We have thus to prove that

$$\sum_{k=1}^{+\infty} \left(\frac{1}{\lambda}\right)^{\alpha(kq+1)-\alpha(kq+p)} (\alpha(kq+p)-\alpha(kq))^{\frac{Q}{2}} < +\infty.$$

The sequences $\alpha(kq+1) - \alpha(kq+p)$ and $\alpha(kq+p) - \alpha(kq)$ (recalling (21)) are asymptotically equivalent respectively to $(1-p)\log(kq+p)$ and $p\log(kq+p)$. Hence, the series is equivalent to

$$\sum_{k=1}^{+\infty} \frac{1}{(kq+p)^{(p-1)\log\frac{1}{\lambda}}} \log^{\frac{Q}{2}}(kq+p),$$

which is convergent since $p \geq \frac{q}{2} > 1 + \frac{1}{\log(\frac{1}{\lambda})}$ by (24) and (23). This proves (28), and therefore the lemma. $\square$

Lemma 6.3. *There exists a positive constant M_0 such that*

$$\frac{\Gamma(z,\zeta)}{\Gamma(0,\zeta)} \leq M_0 \quad \forall\, z \in F_h, \forall\, \zeta \in F_k, \quad \forall\, h, k \in \mathbb{N}, \ h \neq k.$$

Proof. Fix any $h, k \in \mathbb{N}$ with $h \neq k$. If $h \leq k-1$, then $\Gamma(z,\zeta) = 0$ and the statement is trivial. Thus, suppose $h \geq k + 1$. Keeping in mind the notations fixed in (16), (22) and (26), for every $z = (x,t) \in F_h$ and $\zeta = (\xi,\tau) \in F_k$ we have

$$\mu = \frac{-t}{-\tau} \leq \frac{T_{hq}}{T_{kq}^*} = \left(\frac{\lambda^{\alpha(hq)}}{\lambda^{\alpha(kq+p)}}\right)^{\frac{2}{Q}} = \left(\frac{1}{\lambda}\right)^{\frac{2}{Q}(\alpha(kq+p)-\alpha(hq))}.$$

By (24) and the monotonicity properties we have $\alpha(hq) - \alpha(kq+p) \geq \alpha(kq+q) - \alpha(kq+p) \geq \alpha(2q) - \alpha(q+p) \geq \alpha(2q) - \alpha(\frac{3}{2}q + 1) \geq (\frac{q}{2}-1)\log(2q)$. By our choice of q (23) we have then

$$\alpha(hq) - \alpha(kq+p) \geq \left(\frac{q}{2}-1\right)\log(8) \geq \frac{Q}{2}\frac{\max\{\log 2, \log(\frac{n+1}{\sigma})^2\}}{\log(\frac{1}{\lambda})}$$

which implies $\mu \leq \min\{\frac{1}{2}, \frac{\sigma^2}{(n+1)^2}\}$. This fact allows us to exploit Lemma 4.2 and get

$$\frac{\Gamma(z,\zeta)}{\Gamma(0,\zeta)} \leq \left(\frac{1}{1-\mu}\right)^{\frac{Q}{2}} e^{C\sqrt{\mu}M(z)M(\zeta)} \leq 2^{\frac{Q}{2}} e^{C\sqrt{\mu}M(z)M(\zeta)},$$

for some structural positive constant C. To prove the statement we need to show that the term

$$\mu M^2(z) M^2(\zeta)$$

is uniformly bounded for $z \in F_h$ and $\zeta \in F_k$. By estimating as in (29) we have

$$\begin{aligned}
M^2(z) &= \left|D_{\frac{1}{\sqrt{-t}}} x\right|^2 \leq \frac{1}{\sigma_C^2} \log\left(\frac{c_N \lambda^{\alpha(hq)}}{(-t)^{\frac{Q}{2}}}\right) \\
&\leq \frac{1}{\sigma_C^2} \log\left(\frac{c_N \lambda^{\alpha(hq)}}{(T_{hq}^*)^{\frac{Q}{2}}}\right) = \frac{1}{\sigma_C^2} \log\left(\frac{1}{\lambda}\right)(\alpha(hq+p)-\alpha(hq)),
\end{aligned}$$

and analogously

$$M^2(\zeta) \leq \frac{1}{\sigma_C^2} \log\left(\frac{1}{\lambda}\right) (\alpha(kq+p) - \alpha(kq)).$$

In order to bound $\mu M^2(z) M^2(\zeta)$ we are thus going to estimate the term

$$(\alpha(kq+p) - \alpha(kq))(\alpha(hq+p) - \alpha(hq)) \left(\frac{1}{\lambda^{\frac{2}{Q}}}\right)^{(\alpha(kq+p)-\alpha(hq))}$$

$$\leq (\alpha(kq+p) - \alpha(kq))(\alpha(hq+p) - \alpha(hq)) \left(\frac{1}{\lambda^{\frac{2}{Q}}}\right)^{(\alpha(kq+p)-\alpha(kq+q-1)+\alpha(hq-1)-\alpha(hq))}$$

$$= \left((\alpha(kq+p) - \alpha(kq)) \left(\frac{1}{\lambda^{\frac{2}{Q}}}\right)^{(\alpha(kq+p)-\alpha(kq+q-1))}\right)$$

$$\cdot \left((\alpha(hq+p) - \alpha(hq)) \left(\frac{1}{\lambda^{\frac{2}{Q}}}\right)^{(\alpha(hq-1)-\alpha(hq))}\right)$$

$$=: A_k \cdot B_h.$$

Since $p < q - 1$ by (24), and $\alpha(n+s) - \alpha(n)$ is asymptotically equivalent (recalling (21)) to $s \log(n+s)$ as n goes to ∞, it is easy to check that the sequences A_k and B_h are convergent to 0. Therefore they are a fortiori bounded. This proves the lemma. $\square$

Proof of sufficient condition in Theorem 1.1. As we noticed, it is not restrictive to assume $z_0 = 0$. Then, our assumption implies

$$\sum_{k=1}^{\infty} V_{\Omega_k^c(0)}(0) = +\infty$$

for some fixed $\lambda \in (0,1)$. Take $q \in \mathbb{N}$ as in (23). There exists at least one $i \in \{0, \ldots, q-1\}$ such that

$$\sum_{k=1}^{\infty} V_{\Omega_{kq+i}^c(0)}(0) = +\infty.$$

We can assume without loss of generality that $i = 0$, i.e.

$$\sum_{k=1}^{\infty} V_{\Omega_{kq}^c(0)}(0) = +\infty.$$

Let us split the sets $\Omega_{kq}^c(0)$ as in (25). In this way we have defined the sequence of compact sets F_k. We want to check that such a sequence satisfies the hypotheses of Lemma 6.1.

First of all, from (27), we have that the F_k's are disjoint. Moreover, since $F_k \subset \Omega_{kq}^c(0)$, it is easy to see that the sets converge from below to the point 0 (e.g., using that $\Gamma(0, \cdot)$ is δ_r-homogeneous of degree $-Q$). Lemma 6.3 provide the existence of a positive constant M_0 for which condition (ii) in Lemma 6.1 holds true. The last assumption we have to verify is the condition (i). To do this, we recall that the subadditivity of the equilibrium potentials implies that

$$V_{\Omega_{kq}^c(0)} \leq V_{F_k} + V_{F_k^{(0)}}.$$

Lemma 6.2 says that $\sum_k V_{F_k^{(0)}}(0)$ is convergent. We then deduce

$$\sum_{k=1}^{+\infty} V_{F_k}(0) = +\infty,$$

which is condition (i).

Then, we can apply Lemma 6.1 and infer that $V_{G_r}(0) \geq \frac{1}{2M_0}$ for all positive r. The regularity of the point 0 is thus ensured by the characterization in (13). $\square$

7. **The Wiener-type test, and the cone condition.** In this section we want to prove Corollary 1.2, and Corollary 1.3.

First, we want to show how one can deduce the Wiener-type test of Corollary 1.2 from Theorem 1.1: it follows easily from the representation of the potentials (11).

Proof of Corollary 1.2. For every $k \in \mathbb{N}$, we denote by μ_k the $\mathcal{L}$-equilibrium measure of $\Omega_k^c(z_0)$. Then, keeping in mind the very definition of $\Omega_k^c(z_0)$, we have:

$$
\begin{aligned}
V_{\Omega_k^c(z_0)}(z_0) &= \int_{\Omega_k^c(z_0)} \Gamma(z_0, \zeta) \, d\mu_k(\zeta) \, d\zeta \\
&\leq \left(\frac{1}{\lambda}\right)^{(k+1)\log(k+1)} \mu_k(\Omega_k^c(z_0)) = \frac{\mathrm{cap}\,(\Omega_k^c(z_0))}{\lambda^{(k+1)\log(k+1)}}.
\end{aligned}
$$

Analogously,

$$V_{\Omega_k^c(z_0)}(z_0) \geq \frac{\mathrm{cap}\,(\Omega_k^c(z_0))}{\lambda^{k \log k}}.$$

Hence,

$$\sum_{k=1}^{\infty} \frac{\mathrm{cap}\,(\Omega_k^c(z_0))}{\lambda^{k \log k}} \leq \sum_{k=1}^{\infty} V_{\Omega_k^c(z_0)}(z_0) \leq \sum_{k=1}^{\infty} \frac{\mathrm{cap}\,(\Omega_k^c(z_0))}{\lambda^{(k+1)\log(k+1)}}.$$

The assertions (i) and (ii) directly follow from these inequalities, and from Theorem 1.1. $\square$

The main statement in Corollary 1.3 follows from the sufficient condition (i) we have just proved, and from (9). Indeed, we have

$$\mathrm{cap}\,(\Omega_k^c(z_0)) \geq \frac{|\Omega_k^c(z_0)|}{(c_N \lambda^{k \log k})^{\frac{2}{Q}}} \tag{30}$$

since $\Omega_k^c(z_0) \subset \mathbb{R}^N \times [t_0 - T_k, t_0]$ and we recall from (22) that $T_k^{\frac{Q}{2}} = c_N \lambda^{k \log k}$.

Finally, we have to deal with the proof of the cone condition. To this aim, we need some definitions. We call $\mathcal{L}$-cone of vertex $0 \in \mathbb{R}^{N+1}$ a set of the form

$$K_R(B) := \{(D_r(\xi), -r^2) \,:\, \xi \in B, \ 0 \leq r \leq R\}$$

for some bounded set $B \subset \mathbb{R}^N$ with non-empty interior, and for some positive R. We call $\mathcal{L}$-cone of vertex z_0 the left-translated cone

$$z_0 \circ K_R(B).$$

Definition 7.1. Let Ω be a bounded open subset of $\mathbb{R}^{N+1}$ and $z_0 \in \partial\Omega$. We say that Ω has the exterior $\mathcal{L}$-cone property at z_0 if there exists an $\mathcal{L}$-cone of vertex z_0 which is completely contained in Ω^c.

We can now complete the proof.

Proof of Corollary 1.3. As we said, from (30) we get

$$\sum_{k=1}^{\infty} \frac{\operatorname{cap}\left(\Omega_k^c(z_0)\right)}{\lambda^{k\log k}} \geq c_N^{-\frac{2}{Q}} \sum_{k=1}^{\infty} \frac{|\Omega_k^c(z_0)|}{\lambda^{\frac{Q+2}{Q}k\log k}}$$

and the first part of the proof follows. If we suppose that Ω has the exterior $\mathcal{L}$-cone property at z_0, we want to prove that the series on the r.h.s. is divergent. In particular, we are going to prove that the terms of that series are uniformly bigger than a positive constant, for k big enough.

Without loss of generality, we can assume $z_0 = 0$. Denote

$$F_r^{\theta} := \left\{ z \in \mathbb{R}^{N+1} \,:\, \frac{1}{r} \leq \Gamma(0, z) \leq \frac{\theta}{r} \right\}, \quad \text{for } r > 0, \text{ and for } \theta > 1,$$

and let $r_k = \lambda^{k\log k}$. For any $\theta > 1$ there exists $\bar{k}$ such that we have

$$\Omega_k^c(z_0) \supseteq F_{r_k}^{\theta} \cap K_R(B) \qquad \forall k \geq \bar{k}.$$

On the other hand

$$F_{r_k}^{\theta} \cap K_R(B) = \delta_{\frac{1}{r_k^{\frac{1}{Q}}}} \left(F_1^{\theta} \cap K_{R\,r_k^{-\frac{1}{Q}}}(B) \right).$$

We claim that there exist $\bar{k}_1 \geq \bar{k}$ and a non-empty open set $A \subset \mathbb{R}^{N+1}$ such that

$$A \subseteq F_1^{\theta} \cap K_{R\,r_k^{-\frac{1}{Q}}}(B) \qquad \forall k \geq \bar{k}_1. \tag{31}$$

If this is the case, we get

$$|\Omega_k^c(z_0)| \geq \left| \delta_{\frac{1}{r_k^{\frac{1}{Q}}}} \left(F_1^{\theta} \cap K_{R\,r_k^{-\frac{1}{Q}}}(B) \right) \right| = r_k^{\frac{Q+2}{Q}} |F_1^{\theta} \cap K_{R\,r_k^{-\frac{1}{Q}}}(B)| \geq r_k^{\frac{Q+2}{Q}} |A|$$

for all $k \geq \bar{k}_1$, which is exactly the desired relation

$$\frac{|\Omega_k^c(z_0)|}{\lambda^{\frac{Q+2}{Q}k\log k}} \geq |A| > 0 \qquad \forall k \geq \bar{k}_1.$$

Hence, we are left with the proof of the claim (31). Take $\bar{k}_1 \geq \bar{k}$ such that

$$\sup_{\xi \in \operatorname{int}(B)} \Gamma(0, (\xi, -1)) < \frac{R^Q}{r_k} \qquad \forall k \geq \bar{k}_1.$$

Consider

$$A := \left\{ (D_\rho(\xi), -\rho^2) \,:\, \xi \in \operatorname{int}(B), \text{ and } \frac{1}{\theta}\Gamma(0, (\xi, -1)) < \rho^Q < \Gamma(0, (\xi, -1)) \right\},$$

which is open, and non-empty since $\operatorname{int}(B) \neq \emptyset$ and $\theta > 1$. Moreover $A \subset F_1^{\theta}$ by construction, and $A \subset K_{R\,r_k^{-\frac{1}{Q}}}(B)$ for $k \geq \bar{k}_1$ because of the inequality $\rho^Q < \frac{R^Q}{r_k}$. The proof is thus complete. $\square$

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