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Proportional Lumpability

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Context - Continuous Time Markov Chains

- Continuous Time Markov Chains are the underlying semantics of many high-level formalisms for modeling, analysing and verifying stochastic systems, such as Stochastic Petri nets, Stochastic Automata Networks, Markovian process algebras
- High-level languages simplify the specification task thanks to compositionality and abstraction
- So, even very compact specifications can generate very large stochastic systems that are difficult/impossible to analyse

Context - Lumpability

- In the non-deterministic setting bisimulation allows to quotient the state space and precisely characterizes modal logic [Van Benthem Th.]

- On Markov Chains lumpability [Kemeny-Snell 1976] (probabilistic bisimulation [Larsen-Skou 1991]) plays the same role, preserving stationary quantities [Buchholz 1994] and stochastic/probabilistic modal logics [Larsen-Skou 1991, Desharnais et al 2002, Bernardo et al. 2019]

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Issue

Lumpability is too demanding

As a consequence it usually provides poor reductions

Context - Pseudo-Metrics on Paths

- Distances measuring the difference between states of probabilistic systems are introduced in [Desharnais et al. 1999]
- The distance evaluates the probabilities along paths allowing discounts
- Probabilistic bisimilar states have distance 0
- Behavioural properties have been largely investigated [van Breugel et al. 2001, Wild et al. 2019]
- Compositionality properties have been proved [Gebler et al. 2015]
- Algorithmic solutions have been proposed [Bacci et al. Concur 2019]
- Stationary distribution bounds?

Context - Quasi Lumpability and ϵ-Bisimulation

- Quasi Lumpability relates states allowing ϵ perturbations of the outgoing probabilities/rates [Franceschinis et al. 1994]
- Bounds on the stationary distributions have been proved
- Behavioural properties have been studied on ϵ-Bisimulation [Desharnais et al. 2008, Tracol et al. 2011, Abate et al. 2014, Abate et al. 2017]
- Algorithmic solutions have been proposed [Milios et al. 2012]

Approximations

Context - Quasi Lumpability and ϵ-Bisimulation

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Unfortunately

It is not possible to exactly reconstruct the stationary distribution of the original system

Proportional Lumpability

Motivation

We aim at relaxing the conditions of lumpability while allowing to derive the exact stationary indices for the original system

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Contribution

- We define the notion of Proportional Lumpability over Continuous Time Markov Chains (CTMC)
- We show that this allows to derive properties of the original systems
- We introduce the notion of Proportional Bisimulation over the stochastic process algebra PEPA and prove that it induces a proportional lumpability on the underlying semantics

Outline of the Talk

- The notions of Lumpability and Quasi Lumpability over CTMC
- The notion of Proportional Lumpability and its properties
- Proportional Lumpability over the Process Algebra PEPA
- Example
- Conclusions

CTMC

Let $X(t)$ with $t \in \mathbb{R}^+$ be a stochastic process taking values in a discrete space $\mathcal{S}$. $X(t)$ is a CTMC if it is stationary and markovian

We focus on finite, time-homogeneous, ergodic Markov Chains

Infinitesimal Generator

A CTMC is given as a matrix Q of dim. $|\mathcal{S}| \times |\mathcal{S}|$ such that:

- for $i \neq j$ the transition rate from i to j is $q(i,j) \geq 0$, i.e.,

$$Prob(X(t+h) = j | X(t) = i) = q(i,j) * h + o(h)$$

- $q(i,i) = -\sum_{j \neq i} q(i,j)$

Stationary Analysis

Stationary Distribution

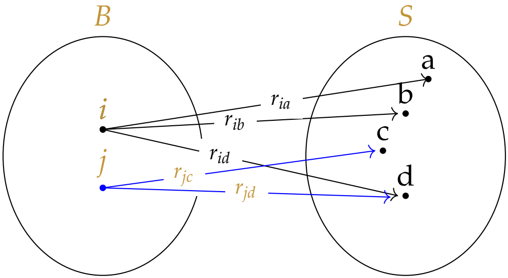
A distribution π over $\mathcal{S}$ such that $\pi(i)$ is the probability of being in i when time goes to ∞

In our setting π is the unique distribution that solves

$$\pi Q = 0$$

Stationary Performances Indices

Stationary performances indices, such as throughput, expected response time, resource utilization, can be computed from the steady state distribution π



$$r_{ia} + r_{ib} + r_{id} = r_{jc} + r_{jd}$$

Strong Lumpability

The strong lumpability $\sim$ is the largest equivalence over $\mathcal{S}$ such that $\forall B, S \in \mathcal{S}/\sim$ and $\forall i, j \in B$

$$\sum_{a \in S} q(i,a) = \sum_{a \in S} q(j,a)$$

Lumpability

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Properties

- We can safely restrict to $B \neq S$
- There always exists a unique maximum lumpability
- The stationary distribution Π of the lumped chain is the aggregation of π
- Probabilistic modal logic properties are preserved

Quasi Lumpability

Quasi Lumpability [Franceschinis et al. ’94, Milios et al. 2012]

An ϵ-quasi lumpability $\mathcal{R}$ is an equivalence over $\mathcal{S}$ such that $\forall B, S \in \mathcal{S}/\mathcal{R}$ and $\forall i, j \in B$

$$|\sum_{a \in S} q(i,a) - \sum_{a \in S} q(j,a)| \leq \epsilon$$

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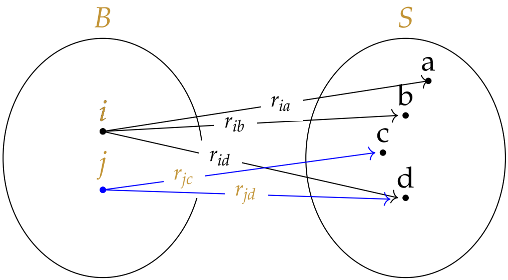
Properties

- It was originally defined splitting Q into Q^- and Q^ϵ
- Bounds on the exact stationary distribution can be computed
- Algorithms for approximating an optimal aggregation have been proposed



$$r_{ia} + r_{ib} + r_{id} = 10 \qquad r_{jc} + r_{jd} = 100$$

$$\epsilon \geq 90$$

Proportional Lumpability

Proportional Lumpability

Given $\kappa : \mathcal{S} \to \mathbb{R}^+$, a κ-proportional lumpability $\mathcal{R}$ is an equivalence over $\mathcal{S}$ such that $\forall B, S \in \mathcal{S}/\mathcal{R}$ and $\forall i, j \in B$

$$\frac{\sum_{a \in S} q(i,a)}{\kappa(i)} = \frac{\sum_{a \in S} q(j,a)}{\kappa(j)}$$

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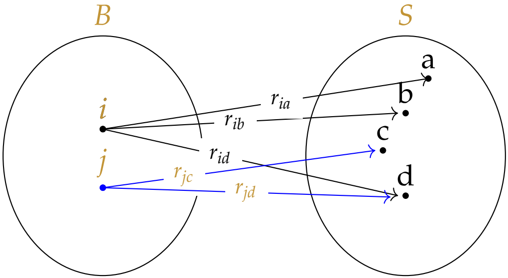
Properties

- We can safely restrict to $B \neq S$
- There exists a unique maximum κ-proportional lumpability $\sim_\kappa$
- More properties …

Proportional Lumpability – Example



$r_{ia} + r_{ib} + r_{id} = 10 \qquad r_{jc} + r_{jd} = 100$

$\kappa(i) = 1 \qquad \kappa(j) = 10$

One Function to Rule them All

An equivalence relation $\sim$ is a proportional lumpability (w.r.t. some function κ) iff $\forall B, S \in \mathcal{S}/\sim$ and $\forall i, j \in B$:

- $q_\sim(i) \neq 0$ iff $q_\sim(j) \neq 0$
- if $q_\sim(i) \neq 0$, then $\frac{q(i,S)}{q_\sim(i)} = \frac{q(j,S)}{q_\sim(j)}$

where $q_\sim(x) = \sum_{y \not\sim x} q(x, y)$

Independence on κ

One Function to Rule them All

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Consequences

- We have an algorithm for checking whether a relation is a proportional lumpability
- Still we do not know how to compute a proportional lumpability

Three is a Magic Number

An equivalence relation $\sim$ is a proportional lumpability iff $\forall B, S, T \in \mathcal{S}/\sim$ with $B \neq S, T$ and $\forall i, j \in B$:

- $q(i, T) \neq 0$ iff $q(j, T) \neq 0$
- if $q(i, T) \neq 0$, then

$$\frac{q(i, S)}{q(i, T)} = \frac{q(j, S)}{q(j, T)}$$

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- if $q(i, T) \neq 0$, then

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Remark

This is in the direction of a partition refinement algorithm for proportional lumpability

Perturbed Systems

It is any CTMC $X'(t)$ over the state space $\mathcal{S}$ having generator Q' such that $\forall i \in \mathcal{S}$ and $\forall S \in \mathcal{S}/\!\sim$

$$\sum_{a \in S, a \neq i} q'(i,a) = \frac{\sum_{a \in S, a \neq i} q(i,a)}{\kappa(i)}$$

Perturbed Systems

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Example

$X'(t)$ defined by

$$q'(i,a) = \frac{q(i,a)}{\kappa(i)} \qquad \text{for any } a \neq i$$

Proposition

The stationary distributions of $X(t)$ and $X'(t)$ are related as follows

$$\pi(i) = \frac{\pi'(i)}{K\kappa(i)}$$

where K is a normalization factor

Aggregated System

It is the CTMC $\widetilde{X}(t)$

- over the state space $\mathcal{S}/\sim$
- it has infinitesimal generator $\widetilde{Q}$ with $\widetilde{q}(B, S) = \frac{\sum_{a \in S} q(i,a)}{\kappa(i)}$ with $i \in B$

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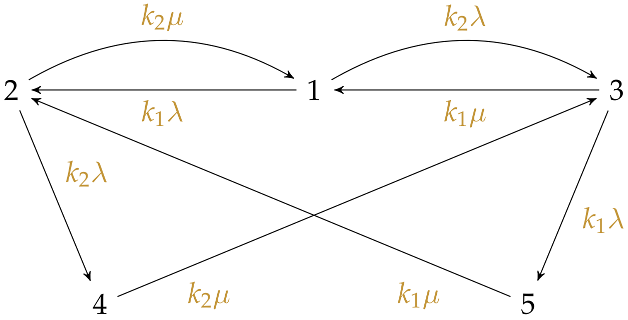
Proposition

The stationary distributions of $X(t)$ and $\widetilde{X}(t)$ are related as follows

$$\widetilde{\pi}(S) = \frac{\sum_{i \in S} \pi(i)\kappa(i)}{\widetilde{K}}$$

where $\widetilde{K}$ is a normalization factor



$$\kappa(1) = 1 \;\; \kappa(2) = k_2 \;\; \kappa(3) = k_1 \;\; \kappa(4) = k_2 \;\; \kappa(5) = k_1$$

PEPA Syntax

Let $\mathcal{A}$ be a set of actions with $\tau \in \mathcal{A}$
Let $\alpha \in \mathcal{A}$, $A \subseteq \mathcal{A}$, and $r \in \mathbb{R}$

$$S ::= \mathbf{0} \mid (\alpha, r).S \mid S + S \mid X$$
$$P ::= P \bowtie_{A} P \mid P/A \mid P \setminus A \mid S$$

Each variable X is associated to a definition $X \stackrel{\text{def}}{=} P$

PEPA Semantics

It defines Labeled Continuous Time Markov Chains

$$\frac{P \xrightarrow{(\alpha,r)} P'}{P \underset{A}{\bowtie} Q \xrightarrow{(\alpha,r)} P' \underset{A}{\bowtie} Q} \ (\alpha \notin A) \qquad \frac{Q \xrightarrow{(\alpha,r)} Q'}{P \underset{A}{\bowtie} Q \xrightarrow{(\alpha,r)} P \underset{A}{\bowtie} Q'} \ (\alpha \notin A)$$

$$\frac{P \xrightarrow{(\alpha,r_1)} P' \quad Q \xrightarrow{(\alpha,r_2)} Q'}{P \underset{A}{\bowtie} Q \xrightarrow{(\alpha,R)} P' \underset{A}{\bowtie} Q'} \quad (\alpha \in A)$$

$$\text{where} \quad R = \frac{r_1}{r_\alpha(P)} \frac{r_2}{r_\alpha(Q)} \min(r_\alpha(P), r_\alpha(Q))$$

Lumpable bisimilarity [Hillston et al. 2013, Alzetta et al. 2018]

A lumpable bisimilarity is an equivalence $\mathcal{R}$ such that for each action α, $\forall B, S \in \mathcal{C}/\mathcal{R}$, and $\forall P, Q \in B$

- either $\alpha \neq \tau$,
- or $\alpha = \tau$ and $B \neq S$,

it holds

$$\sum_{P' \in S,\ P \xrightarrow{(\alpha, r_\alpha)} P'} r_\alpha = \sum_{Q' \in S,\ Q \xrightarrow{(\alpha, r_\alpha)} Q'} r_\alpha$$

Properties

There exists a unique maximum lumpable bisimilarity $\approx_l$, it is *contextual*, *action preserving*, and induces a *lumpability*

Proportional bisimilarity

Given $\kappa : \mathcal{C} \to \mathbb{R}^+$ a κ-proportional bisimilarity is an equivalence $\mathcal{R}$ such that for each action α, $\forall B, S \in \mathcal{C}/\mathcal{R}$, and $\forall P, Q \in B$

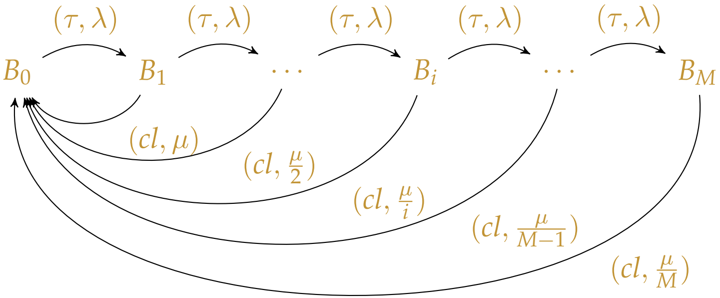
- either $\alpha \neq \tau$,
- or $\alpha = \tau$ and $B \neq S$,

it holds

$$\frac{\sum_{P' \in S,\ P \xrightarrow{(\alpha, r_\alpha)} P'} r_\alpha}{\kappa(P)} = \frac{\sum_{Q' \in S,\ Q \xrightarrow{(\alpha, r_\alpha)} Q'} r_\alpha}{\kappa(Q)}$$

Properties

There exists a unique maximum proportional bisimilarity $\approx_l^\kappa$, it induces a *proportional lumpability*


B_0
B_1
$\dots$
B_i
$\dots$
B_M
(τ, λ)
(τ, λ)
(τ, λ)
(τ, λ)
(τ, λ)
(cl, μ)
$(cl, \frac{\mu}{2})$
$(cl, \frac{\mu}{i})$
$(cl, \frac{\mu}{M-1})$
$(cl, \frac{\mu}{M})$

Conclusions

- The notion of proportional lumpability has been introduced
- It “preserves” the stationary distribution
- It can be applied for PEPA components reduction
- We are optimizing its computation and proving compositionality properties