Bridging Causal Reversibility and Time Reversibility: A Stochastic Process Algebraic Approach

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Reversible Computing

- Reversibility in mathematics: inverse function, inverse operation, ...
- More recent in informatics: seminal papers by Landauer (1961) and Bennett (1973) on IBM Journal of Research and Development.
- Landauer principle states that any *irreversible* manipulation of information such as:
  - erasure of bits
  - merging of computation paths

  must be accompanied by a corresponding *entropy increase*.
- Verified in 2012 and given a physical interpretation in 2018.

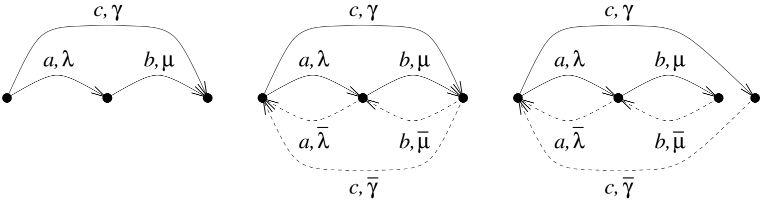
- Irreversible computations cause *heat dissipation* into circuits.
- Any reversible computation, where no information is lost, may be potentially carried out without releasing any heat.
- Low energy consumption could therefore be achieved by resorting to reversible computing.
- In addition, many applications of reversible computing:
    - Biochemical reaction modeling.
    - Parallel discrete-event simulation.
    - Robotics and control theory.
    - Fault tolerant computing systems.
    - Concurrent program debugging.

Integrating Causal and Time Reversibilities

- Two directions of computation in a reversible system:
  - Forward: coincides with the normal way of computing.
  - Backward: the effects of the forward one are undone when needed.
- Different notions of reversibility in different settings:
  - Causal reversibility is the capability of going back to a past state in a way that is *consistent with the computational history* of the system (easy for sequential systems, hard for concurrent and distributed ones).
  - Time reversibility refers to the conditions under which the stochastic behavior remains the same when the direction of time is reversed (quantitative system models, efficient performance evaluation).
- How to integrate them so as to be jointly achieved by construction? And in which setting is it convenient to investigate this?

- Process algebra constitutes a *common ground* for concurrency theory and probability theory ([LarsenSkou91, Hillston96]).
- Stochastic process algebra: each action is equipped with a positive real number expressing the *rate* at which the action is executed (unique parameter of the *exponentially distributed random duration* of the action, whose inverse is the *average duration* of the action).
- CTMC – Continuous-Time Markov Chain as underlying stochastic process (matrix or state-transition graph), whose *memoryless property* fits well with the *interleaving view* of concurrency.
- Equipped with several behavioral equivalences each accounting for both functional and performance aspects.

- Ensuring that a system is both causally reversible and time reversible is not a trivial task even in a sequential system.
- Consider $<a, \lambda>.<b, \mu>.\underline{0} + <c, \gamma>.\underline{0}$.
- The first step is to add a backward transition for each forward one.
- Then make sure that they respect the causal history of the system and finally select backward rates so that the product of the rates along any cycle is the same when changing direction [Kelly79].



RMPC – Reversible Markovian Process Calculus

- Stochastic variant of CCSK [PhillipsUlidowski07] with CSP parallel.
- Syntax of standard forward processes (future only):

$$P ::= \underline{0} \mid \langle a, \lambda \rangle . P \mid P + P \mid P \parallel_L P$$

- Syntax of non-standard reversible processes (past too):

$$R ::= P \mid \langle a, \lambda \rangle [i] . R \mid R + R \mid R \parallel_L R$$

- Executed actions decorated with communication keys so as to be able to know who synchronized with whom when undoing actions.
- $\langle a, \lambda \rangle . \langle b, \mu \rangle . \underline{0}$ is a forward process.
- $\langle a, \lambda \rangle [i] . \langle b, \mu \rangle . \underline{0}$ is a reversible process.
- $\langle a, \lambda \rangle . \langle b, \mu \rangle [j] . \underline{0} \notin \mathbb{P}$: it is *not* an admissible process because unexecuted actions (future) cannot precede executed ones (past).

- The semantics for RMPC is the labeled transition system $(\mathbb{P}, \mathcal{L}, \mapsto)$.
- The set of labels $\mathcal{L}$ is given by $\mathcal{A} \times \mathcal{R} \times \mathcal{K}$ where:
  - $\mathcal{A}$ is a countable set of actions.
  - $\mathcal{R} = \mathbb{R}_{>0}$ is a set of rates.
  - $\mathcal{K}$ is a countable set of keys.
- The transition relation $\longmapsto \subseteq \mathbb{P} \times \mathcal{L} \times \mathbb{P}$ is given by $\longrightarrow \cup \dashrightarrow$ where:
  - $\longrightarrow$ is the forward transition relation.
  - $\dashrightarrow$ is the backward transition relation.
- For each forward rule there will be a symmetrical backward rule.
- Dynamic operators are treated as static for enabling reversibility.
- *std*(R) means that R is a standard forward process.
- *key*(R) is the set of keys in R, which is $\emptyset$ when *std*(R).
- Backward rates, written $\overline{\lambda}$, are left unspecified for now.

- Semantic rules for action prefix:

$$\frac{\mathit{std}(R)}{<a,\lambda>.R \xrightarrow{<a,\lambda>[i]} <a,\lambda>[i].R} \qquad \frac{\mathit{std}(R)}{<a,\lambda>[i].R \overset{<a,\overline{\lambda}>[i]}{\dashrightarrow} <a,\lambda>.R}$$

$$\frac{R \xrightarrow{<b,\mu>[j]} R' \quad j \neq i}{<a,\lambda>[i].R \xrightarrow{<b,\mu>[j]} <a,\lambda>[i].R'} \qquad \frac{R \overset{<b,\overline{\mu}>[j]}{\dashrightarrow} R' \quad j \neq i}{<a,\lambda>[i].R \overset{<b,\overline{\mu}>[j]}{\dashrightarrow} <a,\lambda>[i].R'}$$

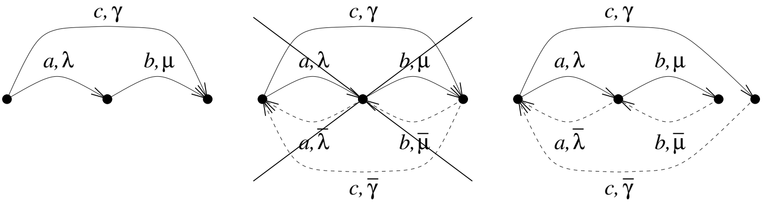
- A *fresh* key is generated and bound to the executed action.
- The prefix related to the executed action is *not discarded*.
- It becomes a key-storing part of the target process, necessary to offer again that action after coming back.
- Additional rule for performing unexecuted actions that are preceded by already executed actions (direct consequence of making prefix static).

- Semantic rules for alternative composition (up to commutativity):

$$\frac{R \xrightarrow{<a,\lambda>[i]} R' \quad \mathit{std}(S)}{R + S \xrightarrow{<a,\lambda>[i]} R' + S} \qquad \frac{R \overset{<a,\overline{\lambda}>[i]}{\dashrightarrow} R' \quad \mathit{std}(S)}{R + S \overset{<a,\overline{\lambda}>[i]}{\dashrightarrow} R' + S}$$

- The subprocess not involved in the executed action is *not discarded* but cannot proceed further (only the non-standard subprocess can).
- It becomes part of the target process, which is necessary for offering again the original choice after coming back.
- The standard subprocess constitutes a dead context that comes into play again after that all the executed actions will have been undone.
- If both subprocesses are standard, then the race policy applies: each action has an execution probability proportional to its rate.

- Consider again $<a, \lambda>.<b, \mu>.\underline{0} + <c, \gamma>.\underline{0}$.
- In addition to generating a backward transition for each forward one, the states corresponding to the two instances of $\underline{0}$ get separated:
  - $<a, \lambda>[i].<b, \mu>[j].\underline{0} + <c, \gamma>.\underline{0}$.
  - $<a, \lambda>.<b, \mu>.\underline{0} + <c, \gamma>[i].\underline{0}$.
- Thus the rightmost labeled transition system is produced (up to $\mathcal{K}$):



- We will see that there is a price to pay w.r.t. behavioral equivalences.

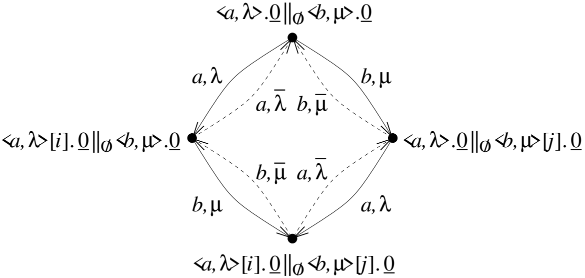
- Semantic rules for parallel composition (up to commutativity):

$$\frac{R \xrightarrow{<a,\lambda>[i]} R' \quad S \xrightarrow{<a,\mu>[i]} S' \quad a \in L}{R \parallel_L S \xrightarrow{<a,\lambda\cdot\mu>[i]} R' \parallel_L S'} \qquad \frac{R \overset{<a,\overline{\lambda}>[i]}{\dashrightarrow} R' \quad S \overset{<a,\overline{\mu}>[i]}{\dashrightarrow} S' \quad a \in L}{R \parallel_L S \overset{<a,\overline{\lambda}\cdot\overline{\mu}>[i]}{\dashrightarrow} R' \parallel_L S'}$$

$$\frac{R \xrightarrow{<a,\lambda>[i]} R' \quad a \notin L \quad i \notin \mathsf{key}(S)}{R \parallel_L S \xrightarrow{<a,\lambda>[i]} R' \parallel_L S} \qquad \frac{R \overset{<a,\overline{\lambda}>[i]}{\dashrightarrow} R' \quad a \notin L \quad i \notin \mathsf{key}(S)}{R \parallel_L S \overset{<a,\overline{\lambda}>[i]}{\dashrightarrow} R' \parallel_L S}$$

- Synchronizations enforced only on actions in L with the *same key*, necessary to enable a correct undoing of synchronizations themselves.
- A cooperation action is assumed to be exponentially timed with rate given by the *product of the rates* of the two involved actions.
- Key uniqueness across parallel composition too.

- Race policy and memoryless property of exponentially timed actions allow for the interleaving view of concurrency:



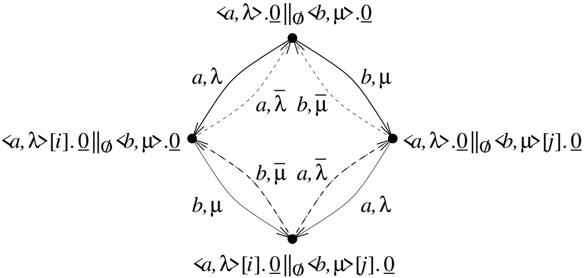
- Initially the execution probability of a is $\frac{\lambda}{\lambda+\mu}$, the one of b is $\frac{\mu}{\lambda+\mu}$, and the one of their simultaneous termination is 0.
- If a terminates first, b is still in progress in the reached state and its residual duration is still exponentially distributed with rate μ.

- The set $\mathbb{P}$ of processes *reachable from a standard one* is such that:
  - Future actions cannot precede past ones (syntax constraint).
  - Key uniqueness holds within any non-standard process:
    $<a, \lambda>[i].<b, \mu>[i].\underline{0}$ and $<a, \lambda>[i].\underline{0} \parallel_{\emptyset} <b, \mu>[i].\underline{0}$ *not* admissible.
  - In any choice at least one of the two subprocesses is standard:
    $<a, \lambda>[i].\underline{0} + <b, \mu>[j].\underline{0}$ *not* admissible.
- A prerequisite for causal and time reversibilities is the loop property: executed actions can be undone and undone actions can be redone.
- Between any two states, either there are no transitions, or there are pairs of transitions of which one is forward and the other is backward.
- The loop property holds in RMPC:
  - Let $R, R' \in \mathbb{P}$. Then $R \xrightarrow{<a,\lambda>[i]} R'$ iff $R' \overset{<a,\overline{\lambda}>[i]}{\dashrightarrow} R$.
  - Let $R, R' \in \mathbb{P}$ and $\sigma \in \mathcal{L}^*$. Then $R \xrightarrow{\sigma} R'$ iff $R' \overset{\overline{\sigma}}{\dashrightarrow} R$.

Causal Reversibility of RMPC

- Causal reversibility is the ability to backtrack:
  - correctly, without encountering previously inaccessible states;
  - flexibly, along any causally equivalent path.
- Takes place by reverting actions starting from the last performed one.
- The last performed action may not be unique in a concurrent system.
- Undo an action only if its consequences have all been undone already.
- Originally addressed in [DanosKrivine04] and [PhillipsUlidowski07].
- General technique recently proposed in [LanesePhillipsUlidowski20].
- Application of [LPU20] to our stochastic variant of CCSK [PU07] after importing the concurrent transitions notion from RCCS [DK04].

- Two coinitial transitions θ_1 and θ_2 from $R \in \mathbb{P}$ are concurrent iff they are *not in conflict*, i.e., neither of the following holds:
  - $R = \mathcal{C}[P_1 + P_2]$ with θ_k deriving from $P_k \xrightarrow{<a_k,\lambda_k>[i_k]} S_k$ for $k = 1, 2$.
  - $\theta_1 = R \xrightarrow{<a,\lambda>[i]} S_1$ and $\theta_2 = R \overset{<b,\overline{\mu}>[j]}{\dashrightarrow} S_2$ with j causing i in S_1.
- Examples of concurrent transitions (from the top and bottom states):



- According to [LPU20] causal reversibility stems from the following, along with a suitable notion of causal equivalence for computations.
- Square property: Let $R, R' \in \mathbb{P}$ and $\theta_1 = R \xmapsto{\ell_1} R_1$, $\theta_2 = R \xmapsto{\ell_2} R_2$ be two coinitial transitions. If θ_1 and θ_2 are concurrent, then there exist two cofinal transitions $\theta_2' = R_1 \xmapsto{\ell_2} R'$, $\theta_1' = R_2 \xmapsto{\ell_1} R'$.
- Backward transitions independence: Let $R' \in \mathbb{P}$. Any two coinitial backward transitions $\theta_1 = R' \overset{\ell_1}{\dashrightarrow} R_1$, $\theta_2 = R' \overset{\ell_2}{\dashrightarrow} R_2$ are concurrent.
- Past well foundedness: Let $R_0 \in \mathbb{P}$. Then there is no infinite sequence of backward transitions such that $R_i \overset{\ell_i}{\dashrightarrow} R_{i+1}$ for all $i \in \mathbb{N}$.
- The three properties are valid in RMPC.

- Abstracting from the order of concurrent transitions.
- Causal equivalence is the smallest equivalence relation $\asymp$ on computations closed under composition that satisfies:
  - $\theta\,\overline{\theta} \asymp \varepsilon$ and $\overline{\theta}\,\theta \asymp \varepsilon$.
  - $\theta_1\theta'_2 \asymp \theta_2\theta'_1$ for any coinitial concurrent $\theta_1 = R \stackrel{\ell_1}{\longmapsto} R_1$, $\theta_2 = R \stackrel{\ell_2}{\longmapsto} R_2$ and any cofinal composable $\theta'_2 = R_1 \stackrel{\ell_2}{\longmapsto} R'$, $\theta'_1 = R_2 \stackrel{\ell_1}{\longmapsto} R'$.
- Parabolic lemma: For any computation ω, there exist two forward computations ω_1 and ω_2 such that $\omega \asymp \overline{\omega_1}\,\omega_2$ and $|\omega_1| + |\omega_2| \leq |\omega|$.
- The maximum freedom of choice among transitions is reached by going backward (drawing potential energy from memory by undoing all the executed actions) and only then going forward (restarting).

- Causal consistency: Let ω_1 and ω_2 be two computations. Then $\omega_1 \asymp \omega_2$ iff ω_1 and ω_2 are both coinitial and cofinal.
- Causal equivalence characterizes a space for *admissible rollbacks*:
  - *Correctness*: they do not lead to states that are not reachable by some forward computation.
  - *Flexibility*: reverse operations can be rearranged w.r.t. the order in which the undone concurrent transitions were originally performed.
- The states reached by any backward computation could be reached by performing forward computations only.
- In conclusion, RMPC meets causal reversibility by construction.

Time Reversibility of RMPC

- A *random variable* X takes every value with a specific probability.
- A *stochastic process* describes the evolution over time of some random phenomenon through a sequence of random variables $X(t)$, one for each time instant t.
- A stochastic process $X(t)$ taking values from a discrete state space $\mathcal{S}$ for $t \in \mathbb{R}_{\geq 0}$ is a continuous-time Markov chain (CTMC) iff for all $n \in \mathbb{N}$, $t_0 < t_1 < \cdots < t_n < t_{n+1} \in \mathbb{R}_{\geq 0}$, $s_0, s_1, \ldots, s_n, s_{n+1} \in \mathcal{S}$: $\Pr\{X(t_{n+1}) = s_{n+1} \mid X(t_0) = s_0, X(t_1) = s_1, \ldots, X(t_n) = s_n\} = \Pr\{X(t_{n+1}) = s_{n+1} \mid X(t_n) = s_n\}$.
- The probability of moving from one state to another does not depend on the particular path that has been followed in the past to reach the current state, hence that path can be forgotten (*memorylessness*).

- A CTMC $X(t)$ is:
  - *Time homogeneous* iff $\Pr\{X(t+t') = s' \mid X(t) = s\}$ does not depend on the time instant t, so that $r_{s,s'} = \lim_{t' \to 0^+} \frac{\Pr\{X(t+t')=s' | X(t)=s\}}{t'}$.
  - *Irreducible* iff each of its states is reachable from every other state with probability greater than 0.
  - *Ergodic* iff it is irreducible and each of its states is positive recurrent, i.e., the CTMC will eventually return to it with probability 1 in an expected number of steps that is finite.
- Ergodicity coincides with irreducibility when the CTMC has finitely many states, as they form a finite strongly connected component.
- The sojourn time in state s is exponentially distributed with rate given by the sum r_s of the rates of the moves of s.
- The average sojourn time in s is $1/r_s$.
- The probability of moving from s to s' is $r_{s,s'}/r_s$.

- Every time-homogeneous and ergodic CTMC $X(t)$ is *stationary*, i.e., $(X(t_i + t'))_{1 \leq i \leq n}$ has the same joint distribution as $(X(t_i))_{1 \leq i \leq n}$ for all $n \in \mathbb{N}_{\geq 1}$ and $t_1 < \cdots < t_n, t' \in \mathbb{R}_{\geq 0}$.
- In this case $X(t)$ has a unique *steady-state probability distribution* $\boldsymbol{\pi} = (\pi(s))_{s \in \mathcal{S}}$ that fulfills $\pi(s) = \lim_{t \to \infty} \Pr\{X(t) = s \mid X(0) = s'\}$ for any $s' \in \mathcal{S}$ because it has reached equilibrium.
- Computed by solving the linear system of *global balance equations* $\boldsymbol{\pi} \cdot \mathbf{Q} = \mathbf{0}$ subject to $\sum_{s \in \mathcal{S}} \pi(s) = 1$ and $\pi(s) \in \mathbb{R}_{>0}$ for all $s \in \mathcal{S}$ (incoming probability flux equal to outgoing probability flux).
- The *infinitesimal generator matrix* $\mathbf{Q} = (q_{s,s'})_{s,s' \in \mathcal{S}}$ is such that $q_{s,s'} = r_{s,s'}$ for $s \neq s'$ while $q_{s,s} = -\sum_{s' \neq s} q_{s,s'}$.
- A CTMC can be represented through $\mathbf{Q}$ or as a state-transition graph in which every transition is labeled with the corresponding rate > 0.

- A CTMC $X(t)$ is time reversible iff the behavior remains the same when the direction of time is reversed.
- $(X(t_i))_{1 \le i \le n}$ has the same joint distribution as $(X(t' - t_i))_{1 \le i \le n}$ for all $n \in \mathbb{N}_{\ge 1}$ and $t_1 < \cdots < t_n, t' \in \mathbb{R}_{\ge 0}$.
- $X(t)$ and its reversed version $X(-t)$ are stochastically identical: they are stationary and share the same steady-state distribution $\boldsymbol{\pi}$.
- In order for a stationary CTMC $X(t)$ to be time reversible, it is necessary and sufficient that one of the following holds [Kelly79]:
  - The *partial balance equations* $\pi(s) \cdot q_{s,s'} = \pi(s') \cdot q_{s',s}$ are satisfied for all distinct $s, s' \in \mathcal{S}$.
  - $q_{s_1,s_2} \cdot \ldots \cdot q_{s_{n-1},s_n} \cdot q_{s_n,s_1} = q_{s_1,s_n} \cdot q_{s_n,s_{n-1}} \cdot \ldots \cdot q_{s_2,s_1}$ for all $n \in \mathbb{N}_{\ge 2}$ and distinct $s_1, \ldots, s_n \in \mathcal{S}$.
- The sum of the partial balance equations for $s \in \mathcal{S}$ yields the global balance equation $\pi(s) \cdot |q_{s,s}| = \sum_{s' \neq s} \pi(s') \cdot q_{s',s}$.

- Given $R \in \mathbb{P}$, derive its CTMC $\mathcal{M}[\![R]\!]$ from $[\![R]\!] = (\mathbb{P}, \mathcal{L}, \longmapsto, R)$.
- $<a, \lambda>.P$ has a forward transition to $<a, \lambda>[i].P$ *for each* $i \in \mathcal{K}$, while each $<a, \lambda>[i].P$ has a single backward transition to $<a, \lambda>.P$.
- When building the CTMC, reason in terms of transition bundles in order not to alter overall exit rates.
- Collect all the transitions departing from the same state and labeled with the same action, the same rate, and *different keys*, whose target states are syntactically identical up to keys in the same positions.
- $\equiv_{\mathcal{K}}$ is the smallest equivalence relation on $\mathbb{P}$ that satisfies:
  - $\mathcal{C}[<a, \lambda>[i].R] \;\equiv_{\mathcal{K}}\; \mathcal{C}[<a, \lambda>[j].R]$
    if $\mathcal{C}$ does not contain occurrences of $\|_L$ with $a \in L$
    and i and j do not occur in R and $\mathcal{C}$.
  - $\mathcal{C}[\mathcal{C}_1[<a, \lambda_1>[i].R_1] \,\|_{L_1} \ldots \|_{L_{n-1}} \mathcal{C}_n[<a, \lambda_n>[i].R_n]] \;\equiv_{\mathcal{K}}$
    $\mathcal{C}[\mathcal{C}_1[<a, \lambda_1>[j].R_1] \,\|_{L_1} \ldots \|_{L_{n-1}} \mathcal{C}_n[<a, \lambda_n>[j].R_n]]$
    if $a \in L_1 \cap \ldots \cap L_{n-1}$ for $n \geq 2$, i and j do not occur in $\mathcal{C}$, $\mathcal{C}_1$, ..., $\mathcal{C}_n$, R_1, ..., R_n, and no further a with key i occurs in $\mathcal{C}$.

- The CTMC underlying $R \in \mathbb{P}$ is the labeled transition system $\mathcal{M}[\![R]\!] = (\mathbb{P}/\equiv_{\mathcal{K}}, \mathcal{A} \times \mathcal{R}, \mapsto_{\mathcal{K}}, [R]_{\equiv_{\mathcal{K}}})$ where:
  - $\mathbb{P}/\equiv_{\mathcal{K}}$ is the quotient set of $\equiv_{\mathcal{K}}$ over $\mathbb{P}$, i.e., the set of classes of $\equiv_{\mathcal{K}}$-equivalent processes.
  - $[R]_{\equiv_{\mathcal{K}}}$ is the $\equiv_{\mathcal{K}}$-equivalence class of R, which simply is $\{R\}$ when R is standard and hence contains no keys.
  - $\mapsto_{\mathcal{K}} \subseteq (\mathbb{P}/\equiv_{\mathcal{K}}) \times (\mathcal{A} \times \mathcal{R}) \times (\mathbb{P}/\equiv_{\mathcal{K}})$ is the transition relation given by $\longrightarrow_{\mathcal{K}} \cup \dashrightarrow_{\mathcal{K}}$ where:
    - $[S]_{\equiv_{\mathcal{K}}} \xrightarrow{<a,\lambda>}_{\mathcal{K}} [S']_{\equiv_{\mathcal{K}}}$ whenever $S \xrightarrow{<a,\lambda>[i]} S'$ for some $i \in \mathcal{K}$.
    - $[S]_{\equiv_{\mathcal{K}}} \overset{<a,\overline{\lambda}>}{\dashrightarrow}_{\mathcal{K}} [S']_{\equiv_{\mathcal{K}}}$ whenever $S \overset{<a,\overline{\lambda}>[i]}{\dashrightarrow} S'$ for some $i \in \mathcal{K}$.
- Finite-state action-labeled CTMC with no keys on transitions.
- Let $R \in \mathbb{P}$. Then $\mathcal{M}[\![R]\!]$ is time homogeneous and ergodic, hence it is stationary too.

- Several time reversibility conditions, along with product-form results.
- Time reversibility based on rate equality $\overline{\lambda} = \lambda$ for all $\lambda \in \mathbb{R}_{>0}$.
- Let $R \in \mathbb{P}$, $\mathcal{S}$ be the set of states of $\mathcal{M}[\![R]\!]$, and $n = |\mathcal{S}|$.
  If every backward rate is equal to the corresponding forward rate, then the steady-state distribution $\boldsymbol{\pi}$ satisfies $\pi(s) = 1/n$ for all $s \in \mathcal{S}$.
- Let $R \in \mathbb{P}$. If every backward rate is equal to the corresponding forward rate, then $\mathcal{M}[\![R]\!]$ is time reversible.
- Time reversibility based on tree-like birth-death structures [Kelly79].
- Let $R \in \mathbb{P}$. If parallel composition does not occur in R, then $\mathcal{M}[\![R]\!]$ is a tree-like birth-death process.
- Let $R \in \mathbb{P}$. If parallel composition does not occur in R, then $\mathcal{M}[\![R]\!]$ is time reversible.

- Time reversibility based on closure w.r.t. parallel composition [MR15].
- Let $R_1, R_2 \in \mathbb{P}$. Under the assumptions of forward-backward rate eq. or tree-like birth-death structure, $\mathcal{M}[\![R_1 \parallel_L R_2]\!]$ is time reversible too.
- Let $R \in \mathbb{P}$ be such that parallel composition cannot occur within the scope of action prefix or choice. Then $\mathcal{M}[\![R]\!]$ is time reversible.
- Quite useful because systems are typically modeled as the parallel composition of a certain number of sequential processes, i.e., processes in which only action prefix and choice occur.
- Derivation of product-form results [MarinRossi15].
- Let $R_1, R_2 \in \mathbb{P}$. Under the assumptions of forward-backward rate eq. or tree-like birth-death structure, if the set of states of $\mathcal{M}[\![R_1 \parallel_L R_2]\!]$ is equal to $\mathcal{S}_{R_1} \times \mathcal{S}_{R_2}$, where $\mathcal{S}_{R_k}$ is the set of states of $\mathcal{M}[\![R_k]\!]$, then $\pi(s_1, s_2) = \pi_{R_1}(s_1) \cdot \pi_{R_2}(s_2)$ for all $(s_1, s_2) \in \mathcal{S}_{R_1} \times \mathcal{S}_{R_2}$.

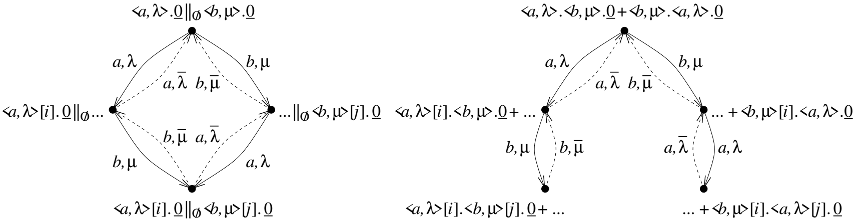
Forward and Backward Markovian Bisimilarity

- Markovian bisimilarity equates processes that can stepwise mimic each other's *functional and performance* behavior [Hillston96].
- Consider an action-labeled CTMC $(\mathcal{S}, \mathcal{A} \times \mathcal{R}, \longmapsto)$.
- Two states $s_1, s_2 \in \mathcal{S}$ are Markovian bisimilar, written $s_1 \sim_{\mathrm{MB}} s_2$, iff there exists a Markovian bisimulation $\mathcal{B}$ such that $(s_1, s_2) \in \mathcal{B}$.
- An equivalence relation $\mathcal{B}$ over the set of states $\mathcal{S}$ is a Markovian bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all actions $a \in \mathcal{A}$ and equivalence classes $C \in \mathcal{S}/\mathcal{B}$:

$$\mathit{rate}(s_1, a, C) \; = \; \mathit{rate}(s_2, a, C)$$

  where $\mathit{rate}(s, a, C) \; = \; \sum \{\!| \, \lambda \in \mathcal{R} \mid \exists s' \in C. \, s \xrightarrow{<a, \lambda>} s' \, |\!\}$.
- Compositional reasoning, sound and complete axiomatization, modal logical characterization, polynomial-time decidability.

- Following [PhillipsUlidowski07] we may adapt Markovian bisimilarity to RMPC by means of two analogous rate equality conditions: one for forward transitions and one for backward transitions.
- The $\|_{-}$ expansion law of $\sim_{\text{MB}}$ would *not* hold (true concurrency):



- The backward transitions of the bottom states would not match.
- After performing a then b, could one go back by perfoming a then b?
  - Parallel system: yes, without violating causality.
  - Sequential interleaved system: no.

- Following [DeNicolaMontanariVaandrager90], for a more easily tractable equivalence in a reversible setting it is necessary to enforce not only causality but also *history preservation*.
- When going backward, a process can only move along the path representing the history that brought the process to the current state.
- Back-and-forth bisimilarity for nondeterministic processes is defined:
  - over computations (not states) in order to preserve history;
  - with forward transitions traversable in both directions (loop property);
  - by separating outgoing/incoming transitions in the bisimulation game.
- The strong version coincides with Milner strong bisimilarity.
- The weak version coincides with Van Glabbeek and Weijland branching bisimilarity (finer than Milner weak bisimilarity).

- A sequence $\xi = (s_0, <a_1, \lambda_1>, s_1) \dots (s_{n-1}, <a_n, \lambda_n>, s_n) \in \longmapsto^*$ is called a *path* of length n from state s_0.
- *first*$(\xi) = s_0$, *last*$(\xi) = s_n$, ε is the empty path.
- *path*(s) is the set of paths from state s.
- A pair $\rho = (s, \xi)$ is called a *run* from state s iff $\xi \in$ *path*(s), in which case *path*$(\rho) = \xi$, *first*$(\rho) =$ *first*(ξ), *last*$(\rho) =$ *last*(ξ), with *first*$(\rho) =$ *last*$(\rho) = s$ when $\xi = \varepsilon$.
- *run*(s) is the set of runs from state s.
- The composition of $\rho = (s, \xi) \in$ *run*(s) and $\rho' = (s', \xi') \in$ *run*(s'), where *last*$(\rho) =$ *first*(ρ'), is given by $\rho\rho' = (s, \xi\xi') \in$ *run*(s).
- $\rho \xrightarrow{<a,\lambda>} \rho'$ iff there exists $\rho'' = (s, (s, <a, \lambda>, s'))$ with $s =$ *last*(ρ) such that $\rho' = \rho\rho''$ (note that *first*$(\rho) =$ *first*(ρ')).

- Forward and time-abstract backward Markovian bisimilarity $\sim_{\rm FTABMB}$ is defined by handling outgoing and incoming transitions differently:
  - overall rate comparison for outgoing transitions;
  - only existence verification for incoming transitions so as not to violate the law $<a, \lambda_1>.P + <a, \lambda_2>.P = <a, \lambda_1 + \lambda_2>.P$ of $\sim_{\rm MB}$.
- An equivalence relation $\mathcal{B}$ over the set of runs $\mathcal{U}$ is a forward and time-abstract backward Markovian bisimul. iff, whenever $(\rho_1, \rho_2) \in \mathcal{B}$, then for all actions $a \in \mathcal{A}$ and equivalence classes $C \in \mathcal{U}/\mathcal{B}$:

$$\begin{array}{rcl} \mathit{rate}_{\rm out}(\rho_1, a, C) & = & \mathit{rate}_{\rm out}(\rho_2, a, C) \\ \mathit{trans}_{\rm in}(\rho_1, a, C) & = & \mathit{trans}_{\rm in}(\rho_2, a, C) \end{array}$$

  where:
  - $\mathit{rate}_{\rm out}(\rho, a, C) = \sum \{\!| \lambda \in \mathcal{R} \mid \exists \rho' \in C.\, \rho \xrightarrow{<a,\lambda>} \rho' |\!\}$.
  - $\mathit{trans}_{\rm in}(\rho, a, C) = 1$ if there exists $\rho' \xrightarrow{<a,\lambda>} \rho$ with $\rho' \in C$, $\mathit{trans}_{\rm in}(\rho, a, C) = 0$ otherwise.
- $\sim_{\rm FTABMB}$ coincides with $\sim_{\rm MB}$ and hence inherits all of its properties.

Concluding Remarks and Future Work

- Bridging causal reversibility and time reversibility is feasible.
- Stochastic process algebra confirmed to be a good working tool.
- Robustness of [PhillipsUlidowski07] to support time reversibility too.
- Exploitation of [LPU20] after importing from [DK04] into [PU07].
- We conjecture that time reversibility holds over the entire RMPC.
- Importing $\|_{-}$ closure of t.r. and product form from [MarinRossi15].
- Tractability of fw.-bw. Markovian bisimilarity by extending [DMV90].
- Systems are not fully reversible when admit irreversible actions too, how to deal with piecewise reversibility?
- Recursion yields infinitely many states due to past action decoration, how to handle this?