Generalized Proportional Lumpability

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Abstract. In this paper, we address the challenge of state space explosion in the analysis of large stochastic models by advancing the lumpability approach, a state aggregation technique that exploits structural regularities in Markov chains to efficiently compute stationary performance indices. We generalize the concept of proportional lumpability, which extends the well-known notion of lumpability and allows for the exact computation of stationary performance indices, in contrast to quasi-lumpability, which only provides bounds. Proportional lumpability is achieved through a perturbation of the original Markov chain's transition rates, guided by a proportionality function. We further explore the idea of perturbing Markov chains through left and right multiplications by square matrices, introducing the concepts of left and right-perturbed Markov chains, which preserve the original model's topology. For left-perturbed Markov chains, the steady-state distribution of the original chain can be derived by multiplying the probability vector of the perturbed chain by the square matrix used to define the perturbation. In contrast, for right-perturbed Markov chains, the steady-state probability distribution remains unchanged.

Keywords: CTMCs · Aggregation techniques · Lumpability

1 Introduction

In the realm of performance evaluation for complex systems, continuous-time Markov chains (CTMCs) serve as a foundational model for a wide array of modeling frameworks, including Stochastic Petri nets [17], Stochastic Automata Networks (SAN) [21], queuing networks [26], and various Markovian process algebras [11,12]. The primary objective in these contexts is to compute stationary performance indices, such as throughput, expected response time, and resource utilization. Achieving this necessitates the prior determination of the stationary probability distribution of the underlying CTMC.

Despite the advantages offered by high-level modeling formalisms, which facilitate the specification of quantitative models through compositional properties and hierarchical structures, the inherent complexity of the models can lead to state spaces of considerable size. This complexity can render the analysis of such

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models not only challenging but, in some cases, computationally infeasible. To address the state space explosion problem, one effective strategy is to aggregate states with equivalent behaviors, based on a suitable notion of equivalence.

In this paper, we explore the lumpability approach [6,13,25] as a means to mitigate the challenges associated with large state spaces in the computation of stationary performance indices for stochastic models. The lumpability method employs a state aggregation technique that is applicable to Markov chains demonstrating structural regularity. This approach not only facilitates the efficient computation of exact performance indices when the model is lumpable but also highlights the limitations of traditional lumpability concepts. While various definitions of lumpability exist in the literature, it is well recognized that not all Markov chains are amenable to lumping, with only a minority of real-world applications exhibiting non-trivial lumpability.

To extend the applicability of lumpability, in [15] we introduced the concept of proportional lumpability, which generalizes the well-known notion of lumpability and allows for the exact computation of stationary performance indices—unlike quasi-lumpability, which only provides bounds. Proportional lumpability is achieved through a perturbation of the original Markov chain's transition rates, guided by a proportionality function.

In this work, we revisit proportional lumpability, reformulating it through matrix multiplications. We then present two matrix-based methods for perturbing a Markov chain, referred to as *left- and right-perturbations*, and provide characterizations for two specific classes of these perturbations. Notably, for left-perturbed Markov chains, the steady-state distribution of the original chain can be derived by multiplying the steady-state probability vector of the perturbed chain by the matrix used to define the perturbation. In contrast, for right-perturbed Markov chains, the steady-state distribution remains unchanged. These methods preserve the original model's topology while enabling the exact computation of stationary performance indices. Finally, we introduce the concept of generalized proportional lumpability, extending the original lumpability definition by exploring the notions of left- and right-perturbations of Markov chains. This generalization offers new perspectives for efficiently analyzing large stochastic systems.

Structure of the Paper. The paper is structured as follows: In Sect. 2, we review the theoretical background on continuous-time Markov chains and recall the concepts of lumpability and proportional lumpability. In Sect. 3, we introduce the matrix-based multiplication methods for perturbing a Markov chain, named *left- and right-perturbations*. Moreover, we introduce the concept of generalized proportional lumpability. Section 4 concludes the paper.

2 Background

In this section, we review the theoretical background on continuous-time Markov chains and introduce the concepts of ordinary lumpability and proportional lumpability.

2.1 Continuous-Time Markov Chains

A Continuous-Time Markov Chain (CTMC) is a stochastic process $X(t)$ for $t \in \mathbb{R}^+$ taking values into a discrete state space $\mathcal{S}$ such that the *Markov property* holds [13]. An ergodic CTMC possesses an *equilibrium* (or *steady-state*) *distribution*, that is the *unique* collection of positive real numbers $\pi(s)$ with $s \in \mathcal{S}$ such that

$$\pi(s) = \lim_{t\to\infty} Prob(X(t) = s \mid X(0) = s')\,.$$

Notice that the above equation for $\pi(s)$ is independent of s'. We denote by $q(s, s')$ the transition rate out of state s to state s', with $s \neq s'$, and by $q(s)$ the sum of all transition rates out of state s to any other state in the chain. A state s for which $q(s) = \infty$ is called an instantaneous state since when entered it is instantaneously left. Whereas such states are theoretically possible, we shall assume throughout that $0 < q(s) < \infty$ for all state s. The infinitesimal generator matrix $\mathbf{Q}$ of a CTMC $X(t)$ with state space $\mathcal{S}$ is the $|\mathcal{S}| \times |\mathcal{S}|$ matrix whose off-diagonal elements are the $q(s, s')$'s and whose diagonal elements are the negative sum of the extra diagonal elements of each row, i.e., $q(s, s) = -\sum_{s'\in\mathcal{S},\ s'\neq s} q(s, s')$. For the sake of simplicity, we use $q(s, s')$ to denote the components of matrix $\mathbf{Q}$. For $s \in \mathcal{S}$ and $S \subseteq \mathcal{S}$ we write $q(s, S)$ and $q(S, s)$ to denote $\sum_{s'\in S} q(s, s')$ and $\sum_{s'\in S} q(s', s)$, respectively.

Any non-trivial vector of positive real numbers $\boldsymbol{\mu}$ satisfying the system of global balance equations (GBEs) $\boldsymbol{\mu}\mathbf{Q} = \mathbf{0}$ is called *invariant measure* of the CTMC. For an irreducible CTMC $X(t)$, if $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are two invariant measures of $X(t)$, then there exists a constant $k > 0$ such that $\boldsymbol{\mu}_1 = k\boldsymbol{\mu}_2$. If the CTMC is ergodic, then there exists a unique invariant measure $\boldsymbol{\pi}$ whose components sum to unity, i.e., $\sum_{s\in\mathcal{S}} \pi(s) = 1$. In this case $\boldsymbol{\pi}$ is the *equilibrium* or *steady-state distribution* of the CTMC.

In the remainder of this paper, we will refer to finite ergodic CTMCs by their infinitesimal generators. Thus, with a slight abuse of notation, we will denote a finite ergodic CTMC over the state space $\mathcal{S}$ as $\mathbf{Q}$. To verify that a square matrix is the infinitesimal generator of a finite ergodic CTMC, the following conditions must be met.

Definition 1 (Finite Ergodic CTMC Represented as a Square Matrix). *Let $\mathbf{Q}$ be a square matrix, then $\mathbf{Q}$ is a finite ergodic CTMC if*

1. *All off-diagonal elements are non-negative.*
2. *The sum of each row is zero.*
3. *It serves as the adjacency matrix of a strongly connected weighted directed graph.*

Below, we will write that $\mathbf{Q}$ is a CTMC to indicate that $\mathbf{Q}$ is the infinitesimal generator of a finite ergodic CTMC.

2.2 Ordinary Lumpability

In performance and reliability analysis, the concept of *lumpability* offers a method for simplifying models. It enables the creation of a reduced Markov chain that is smaller than the original one, while still allowing for accurate determination of results for the initial process.

The concept of lumpability can be formally defined using equivalence relations over the state space of a Markov chain. These equivalence relations create a *partition* of the state space, where aggregation is performed by grouping equivalent states into macro-states, thereby reducing the overall size of the state space. If the partition meets the criteria for *ordinary* lumpability [2,13], then the equilibrium solution of the aggregated process can, be used to derive an exact solution of the original one.

Ordinary lumpability, also known as *strong* lumpability, was introduced in [13] and further explored in [1,6,16,25].

Definition 2 (Ordinary Lumpability). *Let* $\mathbf{Q}$ *be a CTMC over* $\mathcal{S}$ *and* $\sim$ *be an equivalence relation over* $\mathcal{S}$. *We say* $\sim$ *is an* ordinary lumpability *for* $\mathbf{Q}$ *if* $\sim$ *induces a partition on* $\mathcal{S}$ *such that for any equivalence classes* $S_i, S_j \in \mathcal{S}/\sim$ *with* $i \neq j$ *and* $s, s' \in S_i$,

$$q(s, S_j) = q(s', S_j)\,, \text{ i.e.,}$$

$$\sum_{s'' \in S_j} q(s, s'') = \sum_{s'' \in S_j} q(s', s'')\,.$$

Thus, an equivalence relation over the state space of a Markov chain exhibits ordinary lumpability if it induces a partition into equivalence classes such that any two states within the same class have identical aggregated transition rates to any other class. It is important to note that every Markov chain is ordinarily lumpable with respect to the identity relation, as well as the trivial relation where all states belong to a single equivalence class.

In [13] the authors prove that for an equivalence relation $\sim$ over the state space of a Markov chain $\mathbf{Q}$, the aggregated process is a Markov chain for every initial distribution if, and only if, $\sim$ is a ordinary lumpability for $\mathbf{Q}$. Moreover, when the ordinary lumpability condition holds the infinitesimal generator of the lumped chain can be directed computed from the original generator as expressed by the following theorem.

Theorem 1 (Aggregated CTMC for ordinary lumpability). *Let* $\mathbf{Q}$ *be a CTMC over* $\mathcal{S}$ *having equilibrium distribution* $\boldsymbol{\pi}$. *If* $\sim$ *is a ordinary lumpability for* $\mathbf{Q}$, *then the aggregated CTMC* $\widetilde{\mathbf{Q}}$ *over the state space* $\mathcal{S}/\sim$ *is defined as: for any equivalence classes* $S_i, S_j \in \mathcal{S}/\sim$ *with* $i \neq j$ *and for any* $s \in S_i$,

$$\widetilde{q}(S_i, S_j) = q(s, S_j)\,.$$

Then the equilibrium distribution $\widetilde{\boldsymbol{\pi}}$ *of* $\widetilde{\mathbf{Q}}$ *is such that for any equivalence class* $S \in \mathcal{S}/\sim$,

$$\widetilde{\pi}(S) = \sum_{s \in S} \pi(s)\,.$$

A well known characterization of ordinary lumpapability is formulated in terms of matrix equations (see, e.g., [6,13]) as outlined below. Let us first introduce the following definition.

Definition 3 (Matrices Associated to $\sim$). *Let $\sim$ be an equivalence relation over the state space $\mathcal{S}$ of a CTMC $\mathbf{Q}$. The matrices V and U associated to $\sim$ are defined as follows:*

- *V is the Boolean matrix of dimension $|\mathcal{S}| \times |\mathcal{S}/\sim|$ that assigns each state to its equivalence class, i.e., $v(s, S_i) = 1$ iff $s \in S_i$.*
- *U be the matrix of dimension $|\mathcal{S}/\sim| \times |\mathcal{S}|$ in which the row corresponding to an equivalence class S_i is the uniform distribution probability over the elements of S_i, i.e., $u(S_i, s) = 1/|S_i|$ if $s \in S_i$ and 0 otherwise.*

The matrix U can be obtained as $diag(eV)^{-1}V^T$, where e is the row vector with all elements equal to 1, $diag(eV)$ is the diagonal matrix with the ith element of the vector eV on the ith row, and V^T is the transpose of V. These choices ensure the following results [6,13].

Theorem 2 (Matrix-based characterization of ordinary lumpability). *Let $\mathbf{Q}$ be a CTMC over $\mathcal{S}$ and $\sim$ be an equivalence relation over $\mathcal{S}$. Let V and U be the matrices associated to $\sim$. The relation $\sim$ is an ordinary lumpability for $\mathbf{Q}$ if and only if*

$$\mathbf{Q}V = VU\mathbf{Q}V\,.$$

Moreover, the infinitesimal generator $\widetilde{\mathbf{Q}}$ of the aggregated CTMC can be obtained as

$$\widetilde{\mathbf{Q}} = U\mathbf{Q}V\,.$$

In general, ordinary lumpability is a stringent condition that is often difficult to achieve in practical systems. To address this, the notion of quasi-lumpability, also known as *near-lumpability*, [6,9] was introduced, allowing for an approximate aggregation of states where the reduced system remains "close enough" to a Markov process, even if it doesn't perfectly preserve the Markov property. This approach offers a method to simplify and analyze large, complex systems by relaxing the strict conditions of ordinary lumpability. It allows for the study of systems where ordinary lumpability is not achievable but an approximate aggregation remains useful, provided small errors or perturbations in transition probabilities are acceptable.

Definition 4 (Quasi-lumpability). *Let $\mathbf{Q}$ be a CTMC over $\mathcal{S}$ and $\sim$ be an equivalence relation over $\mathcal{S}$. We say $\sim$ is a* quasi-lumpability *for $\mathbf{Q}$ with respect to ϵ if $\sim$ induces a partition on $\mathcal{S}$ such that for any equivalence classes $S_i, S_j \in \mathcal{S}/\sim$ with $S_i \neq S_j$ and $s, s' \in S_i$,*

$$|q(s, S_j) - q(s', S_j)| \leq \epsilon\,.$$

However, it is important to note that in the case of quasi-lumpability, the system's behavior cannot be perfectly characterized. Instead, techniques are focused on computing bounds for the steady-state probabilities [8–10,24], offering a useful approximation but not exact results.

2.3 Proportional Lumpability

In [14], we introduced a new concept of lumpability called *proportional lumpability*. Like quasi-lumpability, it extends the traditional definition of ordinary lumpability. However, unlike the broader concept of quasi-lumpability, proportional lumpability enables us to derive an exact solution for the original process.

Definition 5 (Proportional Lumpability). *Let* $\mathbf{Q}$ *be a CTMC over* $\mathcal{S}$ *and* $\sim$ *be an equivalence relation over* $\mathcal{S}$. *We say that* $\sim$ *is a* proportional lumpability *for* $\mathbf{Q}$ *if there exists a function* κ *from* $\mathcal{S}$ *to* $\mathbb{R}^+$ *such that* $\sim$ *induces a partition on* $\mathcal{S}$ *satisfying the property that for any equivalence classes* $S_i, S_j \in \mathcal{S}/\sim$ *with* $i \neq j$ *and* $s, s' \in S_i$,

$$\frac{q(s, S_j)}{\kappa(s)} = \frac{q(s', S_j)}{\kappa(s')}.$$

We say that $\sim$ *is a* κ-proportional lumpability *for* $\mathbf{Q}$ *if* $\sim$ *is a proportional lumpability for* $\mathbf{Q}$ *with respect to the function* κ.

The following theorem [14,15] proves that proportional lumpability allows one to compute an exact solution for the original model.

Theorem 3 (Aggregated CTMC for Proportional Lumpability). *Let* $\mathbf{Q}$ *be a CTMC over* $\mathcal{S}$ *having equilibrium distribution* $\boldsymbol{\pi}$. *If* κ *is a function from* $\mathcal{S}$ *to* $\mathbb{R}^+$ *and* $\sim$ *is a* κ*-proportional lumpability for* $\mathbf{Q}$, *then the aggregated CTMC* $\widetilde{\mathbf{Q}}$ *over the state space* $\mathcal{S}/\sim$ *is defined as: for any equivalence classes* $S_i, S_j \in \mathcal{S}/\sim$ *and for any* $s \in S_i$,

$$\widetilde{q}(S_i, S_j) = \frac{q(s, S_j)}{\kappa(s)}.$$

Then, the invariant measure $\widetilde{\boldsymbol{\mu}}$ *of* $\widetilde{\mathbf{Q}}$ *is such that for any equivalence class* $S \in \mathcal{S}/\sim$,

$$\widetilde{\mu}(S) = \sum_{s \in S} \pi(s)\kappa(s). \tag{1}$$

The next Definition 6 introduces a way to perturb a proportionally lumpable CTMC in order to obtain a strongly lumpable one. In contrast with previous perturbation-based approaches, Theorem 4 gives a way to compute the stationary probabilities of a proportionally lumpable chain given those of the perturbed lumpable one. The proof of Theorem 4 is given in [15].

Definition 6 (Perturbed Markov Chains. *Let* $\mathbf{Q}$ *be a CTMC over* $\mathcal{S}$. *Let* κ *be a function from* $\mathcal{S}$ *to* $\mathbb{R}^+$. *We say that a CTMC* $\mathbf{Q}'$ *is a perturbation of* $\mathbf{Q}$ *with respect to* κ *if* $\mathbf{Q}'$ *is obtained from* $\mathbf{Q}$ *by perturbing its rates such that for all* $s, s' \in \mathcal{S}$ *with* $s \neq s'$,

$$q'(s, s') = \frac{q(s, s')}{\kappa(s)}.$$

Theorem 4 (Steady State Distribution for Proportional Lumpability). *Let* $\mathbf{Q}$ *be a CTMC over* $\mathcal{S}$ *having equilibrium distribution* $\boldsymbol{\pi}$*. Let* κ *be a function from* $\mathcal{S}$ *to* $\mathbb{R}^+$*. Then, for any perturbation* $\mathbf{Q}'$ *of the original chain* $\mathbf{Q}$ *with respect to* κ*, if* $\boldsymbol{\pi'}$ *be the equilibrium distribution of* $\mathbf{Q}'$ *and* $\bar{K} = \sum_{s \in \mathcal{S}} \pi'(s)/\kappa(s)$ *then, for all* $s \in \mathcal{S}$

$$\pi(s) = \frac{\pi'(s)}{\bar{K}\ \kappa(s)} \,.$$

In [15,19,20], we established three alternative characterizations of proportional lumpability, which are instrumental in developing an efficient algorithm to automatically verify this property. The first characterization, proven in [15], enables efficient verification of whether a partition of the state space in a Markov chain is induced by proportional lumpability. The second characterization, introduced in [19] and proven in [15], was used to design an algorithm that computes the coarsest proportional lumpability for a given Markov chain in $O(|\mathcal{S}|^4)$ time. The third characterization, proven in [20], improves this complexity, providing an algorithm that runs in $O(|\mathcal{S}|^2 \log |\mathcal{S}|)$.

3 Generalized Proportional Lumpability

In this section, we revisit the concept of proportional lumpability, reformulating it in terms of matrix multiplications. We then introduce two matrix-based multiplication methods for perturbing a Markov chain, ensuring that the steady-state probabilities of the original chain can be easily retrieved. Finally, based on these perturbations, we introduce the concept of generalized proportional lumpability.

3.1 Matrix-Based Characterization of Proportional Lumpability

We introduce the matrix-based characterization of proportional lumpability. Given a function κ from $\mathcal{S}$ to $\mathbb{R}^+$, the matrix K associated to κ is the diagonal matrix of size $|\mathcal{S}| \times |\mathcal{S}|$ in which $k(s,s) = 1/\kappa(s)$.

Theorem 5 (Matrix-Based Characterization of Proportional Lumpability). *Let* $\mathbf{Q}$ *be a CTMC over* $\mathcal{S}$ *and* $\sim$ *be an equivalence relation over* $\mathcal{S}$*. Let* V *and* U *be the matrices associated with* $\sim$*. Let* κ *be a function from* $\mathcal{S}$ *to* $\mathbb{R}^+$ *and* K *be the matrix associated to* κ*. The equivalence relation* $\sim$ *is a* κ*-proportional lumpability for* $\mathbf{Q}$ *if and only if*

$$K\mathbf{Q}V = VUK\mathbf{Q}V$$

Moreover, the aggregate CTMC $\widetilde{\mathbf{Q}}$ *over* $\mathcal{S}/\sim$ *is defined as:*

$$\widetilde{\mathbf{Q}} = UK\mathbf{Q}V$$

Proof. First observe that $K\mathbf{Q}$ is a CTMC according to Definition 1. Moreover, by Definition 6, $K\mathbf{Q}$ is a perturbation of $\mathbf{Q}$ with respect to κ such that $\sim$ is a κ-proportional lumpability for $\mathbf{Q}$ if and only if $\sim$ is an ordinary lumpability for $K\mathbf{Q}$. Hence, the proof follows from Theorem 2. □

In other words, proportional lumpability in the original chain corresponds to ordinary lumpability in the perturbed chain, which is obtained by multiplying each rate $q(s, s')$ by the inverse of the coefficient $\kappa(s)$ associated with s. This perturbed chain, defined in Definition 6, can be obtained by applying the linear transformation defined by K to the original chain $\mathbf{Q}$. This formulation enables us to easily rediscover Theorem 4. Specifically, if $\boldsymbol{\pi}'$ is the steady-state distribution of the perturbed chain $\mathbf{Q}' = K\mathbf{Q}$, then from $\boldsymbol{\pi}'K\mathbf{Q} = \mathbf{0}$, we find that normalizing $\boldsymbol{\pi}'K$ yields the steady-state distribution of $\mathbf{Q}$. Moreover, since K is invertible and $\boldsymbol{\pi}$ is the steady-state distribution of $\mathbf{Q}$, then $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$ implies $\boldsymbol{\pi}K^{-1}K\mathbf{Q} = \mathbf{0}$, which means that by normalizing the vector $\boldsymbol{\pi}K^{-1}$ we obtain the steady-state distribution $\boldsymbol{\pi}'$ of the perturbed chain $\mathbf{Q}' = K\mathbf{Q}$.

From the above considerations, it emerges that a possible generalization of the notion of proportional lumpability can be achieved by replacing the diagonal matrix K with a generic square matrix A of size $|\mathcal{S}| \times |\mathcal{S}|$, provided that $\mathbf{Q}' = A\mathbf{Q}$ still remains a CTMC with the same topology as $\mathbf{Q}$.

3.2 Left-Perturbed Markov Chains

We introduce the concept of *left-perturbation* of a CTMC $\mathbf{Q}$.

Definition 7 (Left-perturbed Markov chains). *Let* $\mathbf{Q}$ *be a CTMC over* $\mathcal{S}$ *and* A *be a matrix of size* $|\mathcal{S}| \times |\mathcal{S}|$. *The matrix* $\mathbf{Q}' = A\mathbf{Q}$ *is a* left-perturbation of $\mathbf{Q}$ with respect to A *if it satisfies the following properties:*

1. *All off-diagonal elements are non-negative.*
2. *The sum of each row is zero.*
3. *For all* $s \neq s'$, *the off-diagonal element* $q(s, s') \neq 0$ *if and only if the corresponding off-diagonal element of* $\mathbf{Q}' = A\mathbf{Q}$ *is non-zero, i.e.,* $q'(s, s') \neq 0$.

This generalization of the concept of a perturbed Markov chain consists in altering the rates $q(s, s')$ of the original chain $\mathbf{Q}$ through a linear combination of the outgoing and incoming rates associated with state s.

Notice that, when A is a diagonal matrix with positive elements on the main diagonal, then all the above conditions are guaranteed. The rates $q(s, s')$ of the original chain are perturbed as $q'(s, s') = a(s, s)q(s, s')$ and the property that the sum of each row is zero is preserved. Furthermore, the topology of the original chain is maintained, i.e., for all $s \neq s'$, $q(s, s') \neq 0$ if and only if $q'(s, s') \neq 0$.

Example 1. Consider a reliability problem for a system composed of N components. We are interested in determining the number of operational components at any given time. Thus, the state space is defined as $\mathcal{S} = \{S_i : 0 \leq i \leq N\}$, where S_i represents the state of the system with i functioning components.

In each state S_i, we assume that the time until a component fails follows an exponential distribution with rate μ_i. Additionally, each failed component can be repaired at a rate of λ. However, the system might also experience failures caused by common factors that lead to the simultaneous failure of multiple components. These common cause failures can arise from shared vulnerabilities, such as issues with the power supply, environmental factors (e.g., earthquakes, floods, humidity), or common maintenance problems. The simultaneous failure of i components due to such a common cause occurs at a rate δ_i. Figure 1 depicts the state transition diagram for the system's repair model.

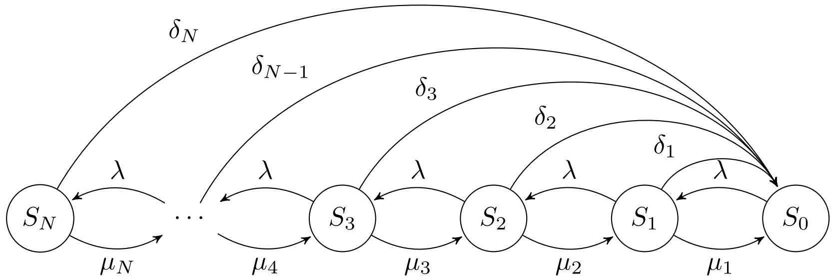

Fig. 1. CTMC for system repair model with common cause failures.

The infinitesimal generator matrix $\mathbf{Q}$ for this system is given by:

$$\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots & 0 & 0 \\ \mu_1 + \delta_1 & -(\lambda + \mu_1 + \delta_1) & \lambda & 0 & \dots & 0 & 0 \\ \delta_2 & \mu_2 & -(\lambda + \mu_2 + \delta_2) & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \delta_N & 0 & 0 & 0 & \dots & \mu_N & -(\mu_N + \delta_N) \end{pmatrix}$$

Consider the matrix A defined as

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1/(\delta_1 + \mu_1) & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1/\delta_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1/\delta_N \end{pmatrix}$$

then $A\mathbf{Q}$ is the matrix

$$A\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots & 0 & 0 \\ 1 & \frac{-(\lambda+\mu_1+\delta_1)}{\mu_1+\delta_1} & \frac{\lambda}{\mu_1+\delta_1} & 0 & \dots & 0 & 0 \\ 1 & \frac{\mu_2}{\delta_2} & \frac{-(\lambda+\mu_2+\delta_2)}{\delta_2} & \frac{\lambda}{\delta_2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots & \frac{\mu_N}{\delta_N} & \frac{-(\mu_N+\delta_N)}{\delta_N} \end{pmatrix}$$

that is, the infinitesimal generator matrix of a left-perturbation of $\mathbf{Q}$ as depicted in Fig. 2. It is easy to see that $A\mathbf{Q}$ is lumpable with respect to the relation $\sim$ over $\mathcal{S}$ given by the reflexive, symmetric and transitive closure of $\{(S'_i, S'_j) : 1 \leq i, j \leq N\}$. This relation induces two equivalence classes, $C_0 = \{S'_0\}$ and $C_1 = \{S'_1, \ldots, S'_N\}$, and the model in Fig. 2 is lumpable to the one depicted in Fig. 3. According to Definition 5, the CTMC $\mathbf{Q}$ is proportionally lumpable with respect to $\sim$ and the function κ such that $\kappa(S'_0) = 1$ and $\kappa(S'_i) = q(S'_i, S'_0)$ for $i \in \{1, \ldots, N\}$.

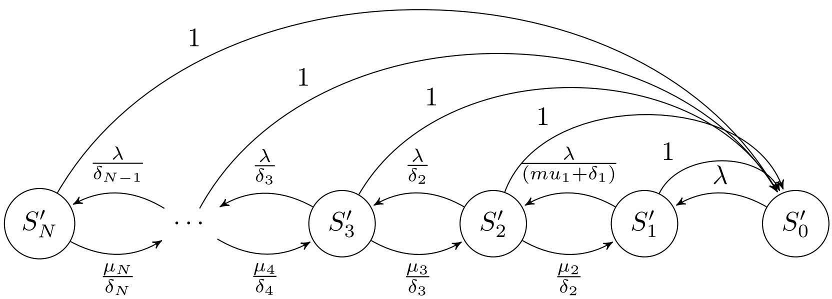

Fig. 2. Left-perturbation of the CTMC in Fig. 1.

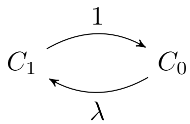

Fig. 3. Aggregation for the lumpable CTMC in Fig. 2.

Hereafter, we provide a characterization for another class of left-perturbed Markov chains.

Theorem 6 (A Class of Left-Perturbed Markov Chains). *Let* $\mathbf{Q}$ *be a CTMC over* $\mathcal{S}$ *and* A *be a matrix of size* $|\mathcal{S}| \times |\mathcal{S}|$ *such that*

1. *All the diagonal elements of* A *are positive.*
2. *All the off-diagonal elements of* A *are non-positive.*
3. *for all* $s \neq s'$, *if* $q(s, s') = 0$ *then* $a(s, s') = 0$.
4. *for all* $s \neq s'$, *if there exists* $s'' \neq s, s'$ *such that* $q(s', s'') \neq 0$ *then* $a(s, s') = 0$.

The matrix $\mathbf{Q}' = A\mathbf{Q}$ *is a* left-perturbation of $\mathbf{Q}$ with respect to A.

Proof. By definition of matrix multiplication, for all $s, s' \in \mathcal{S}$,

$$q'(s,s') = \sum_{s'' \in \mathcal{S}} a(s,s'')q(s'',s').$$

According to Definition 7, we first prove that all off-diagonal elements of $\mathbf{Q}'$ are non-negative. Indeed, for $s \neq s'$,

$$q'(s,s') = a(s,s)q(s,s') + a(s,s')q(s',s') + \sum_{\substack{s'' \in \mathcal{S} \\ s'' \neq s,s'}} a(s,s'')q(s'',s')\,.$$

By condition *4.* in the theorem statement, if some of the $q(s'',s')$ is non-zero then $a(s,s'')$ is zero, i.e., the last addend of the above summation is always zero. Therefore, from the fact that $q(s,s) = -\sum_{s' \in \mathcal{S},\ s' \neq s} q(s,s')$, we have

$$\begin{aligned} q'(s,s') &= a(s,s)q(s,s') + a(s,s')q(s',s') \\ &= a(s,s)q(s,s') - \sum_{\substack{s'' \in \mathcal{S} \\ s'' \neq s'}} a(s,s')q(s',s'') \\ &= a(s,s)q(s,s') - a(s,s')q(s',s) \end{aligned} \tag{2}$$

since, by condition *3.* above, if some of the $q(s',s'')$ is non-zero then $a(s,s')$ is zero. It is now easy to see that $q'(s,s')$ is non-negative since, by hypothesis, $a(s,s)$ is positive and $a(s,s')$ is non-positive, while by definition of infinitesimal generator, both $q(s,s')$ and $q(s',s)$ are non-negative.

We now prove that the sum of each row of $\mathbf{Q}'$ is zero, that is, for all $s \in \mathcal{S}$:

$$\sum_{s' \in \mathcal{S}} q'(s,s') = q'(s,s) + \sum_{\substack{s' \in \mathcal{S} \\ s' \neq s}} q'(s,s') = 0\,. \tag{3}$$

First observe that, by definition of matrix multiplication

$$\begin{aligned} q'(s,s) &= a(s,s)q(s,s) + \sum_{\substack{s' \in \mathcal{S} \\ s' \neq s}} a(s,s')q(s',s) \\ &= -\sum_{\substack{s' \in \mathcal{S} \\ s' \neq s}} a(s,s)q(s,s') + \sum_{\substack{s' \in \mathcal{S} \\ s' \neq s}} a(s,s')q(s',s)\,. \end{aligned}$$

Moreover, by the fact that for all $s, s' \in \mathcal{S}$,

$$q'(s,s') = a(s,s)q(s,s') - a(s,s')q(s',s),$$

we have

$$\sum_{\substack{s' \in \mathcal{S} \\ s' \neq s}} q'(s,s') = \sum_{\substack{s' \in \mathcal{S} \\ s' \neq s}} a(s,s)q(s,s') - \sum_{\substack{s' \in \mathcal{S} \\ s' \neq s}} a(s,s')q(s',s) = -q'(s,s)$$

and this concludes the proof of Eq. (3).

Finally, for all $s \neq s'$, the off-diagonal element $q(s, s') \neq 0$ if and only if $q'(s, s') \neq 0$. The proof trivially follows from Eq. (2) and the statement's conditions. Specifically,

- If $q(s, s') \neq 0$ then, by Condition *1.*, $a(s, s)q(s, s') \neq 0$ and then, by Eq. (2) and the fact that $-a(s, s')q(s', s)$ is non-negative, it holds that $q'(s, s') \neq 0$.
- If $q'(s, s') \neq 0$ then, by contradiction, assume that $q(s, s') = 0$. Hence, by Eq. (2), $q'(s, s') = q(s, s)a(s, s')$. However, by Condition *3.*, we have $a(s, s') = 0$, i.e., $q'(s, s') = 0$ which contradicts the hypothesis.

□

The steady state distribution of the original chain $\mathbf{Q}$ can be obtained from the steady state distribution of the left-pertubed Markov chain $A\mathbf{Q}$ as follows.

Theorem 7 (Steady State Distribution of Left-Perturbed Chains). *Let* $\mathbf{Q}$ *be a CTMC over* $\mathcal{S}$ *and* A *be a matrix of size* $|\mathcal{S}| \times |\mathcal{S}|$ *such that* $\mathbf{Q}' = A\mathbf{Q}$ *is a left-perturbed Markov chain of* $\mathbf{Q}$ *with respect to* A*. If* $\boldsymbol{\pi}'$ *is the steady-state distribution of* $\mathbf{Q}'$*, then* $\boldsymbol{\pi}'A$ *is an invariant measure of the original chain* $\mathbf{Q}$*, i.e., the steady state distribution* $\boldsymbol{\pi}$ *of* $\mathbf{Q}$ *is obtained by normalizing* $\boldsymbol{\pi}'A$*.*

Example 2. Consider a single-server system where customers arrive and are served one by one. However, the server might occasionally be interrupted due to technical issues or other external factors. In the normal operating state S_i, the server processes customers at the rate μ_i, and customers can continue arriving at rate λ. When an interruption occurs, the system transitions to state S'_i. In this state, the server stops, meaning no service is provided, and no customers can enter or leave the system. After the interruption is resolved, the system transitions back to S_i, and service resumes. The interruption rate γ_i controls the movement from the normal service state S_i to the interrupted state S'_i where the system is paused. This rate depends on the number of customers in the queue. The recovery rate δ governs the movement from the interrupted state S'_i back to the normal state S_i when the server resumes service and it is independent on the number of customers in the queue.

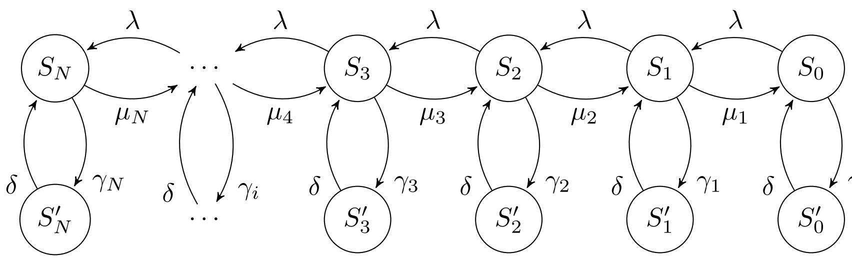

Fig. 4. CTMC for the interrupting system.

The state transition diagram for the system model is depicted in Fig. 4, and the corresponding infinitesimal generator matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{pmatrix} \mathbf{\Lambda} & \mathbf{\Gamma} \\ \mathbf{\Delta} & -\mathbf{\Delta} \end{pmatrix}$$

where the states are considered in the order $S_0, \ldots, S_N, S'_0, \ldots, S'_N$, and $\mathbf{\Lambda}$, $\mathbf{\Gamma}$, $\mathbf{\Delta}$ and $-\mathbf{\Delta}$ are $N \times N$ matrices as follows:

$$\mathbf{\Lambda} = \begin{pmatrix} -(\lambda+\gamma_0) & \lambda & 0 & \ldots & 0 & 0 \\ \mu_1 & -(\lambda+\mu_1+\gamma_1) & \lambda & \ldots & 0 & 0 \\ 0 & \mu_2 & -(\lambda+\mu_2+\gamma_2) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \mu_N & -(\mu_N+\gamma_N) \end{pmatrix}$$

$$\mathbf{\Gamma} = \begin{pmatrix} \gamma_0 & 0 & \ldots & 0 & 0 \\ 0 & \gamma_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & \gamma_{N-1} & 0 \\ 0 & 0 & \ldots & 0 & \gamma_N \end{pmatrix}$$

$$\mathbf{\Delta} = \begin{pmatrix} \delta & 0 & \ldots & 0 & 0 \\ 0 & \delta & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & \delta & 0 \\ 0 & 0 & \ldots & 0 & \delta \end{pmatrix} \qquad -\mathbf{\Delta} = \begin{pmatrix} -\delta & 0 & \ldots & 0 & 0 \\ 0 & -\delta & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & -\delta & 0 \\ 0 & 0 & \ldots & 0 & -\delta \end{pmatrix}$$

Let $\gamma > \gamma_0, \ldots, \gamma_N$. Consider the matrix A defined as

$$A = \begin{pmatrix} \mathbf{I} & \mathbf{\Gamma}' \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

where $\mathbf{0}$, $\mathbf{I}$ and $\mathbf{\Gamma}'$ are $N \times N$ matrices with $\mathbf{0}$ being the zero matrix, $\mathbf{I}$ being the identity matrix and $\mathbf{\Gamma}'$ being defined as follows:

$$\mathbf{\Gamma}' = \begin{pmatrix} (\gamma_0-\gamma)/\delta & 0 & \ldots & 0 & 0 \\ 0 & (\gamma_1-\gamma)/\delta & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & (\gamma_{N-1}-\gamma)/\delta & 0 \\ 0 & 0 & \ldots & 0 & (\gamma_N-\gamma)/\delta \end{pmatrix}$$

The perturbed Markov chain $A\mathbf{Q}$ has the form

$$A\mathbf{Q} = \begin{pmatrix} \mathbf{\Lambda}' & \mathbf{\Gamma}'' \\ \mathbf{\Delta} & -\mathbf{\Delta} \end{pmatrix}$$

with $\mathbf{\Lambda}'$ and $\mathbf{\Gamma}''$ being

$$\mathbf{\Lambda}' = \begin{pmatrix} -(\lambda+\gamma) & \lambda & 0 & \dots & 0 & 0 \\ \mu_1 & -(\lambda+\mu_1+\gamma) & \lambda & \dots & 0 & 0 \\ 0 & \mu_2 & -(\lambda+\mu_2+\gamma) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu_N & -(\mu_N+\gamma) \end{pmatrix}$$

$$\mathbf{\Gamma}'' = \begin{pmatrix} \gamma & 0 & \dots & 0 & 0 \\ 0 & \gamma & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \gamma & 0 \\ 0 & 0 & \dots & 0 & \gamma \end{pmatrix}$$

It is easy to see that $A\mathbf{Q}$ is lumpable with respect to the relation $\sim$ over $\mathcal{S}$ given by the reflexive, symmetric and transitive closure of $\{(T_i, T_j) : 0 \leq i, j \leq N\} \cup \{(T'_i, T'_j) : 0 \leq i, j \leq N\}$. This relation induces two equivalence classes, $E_0 = \{T_0, \dots, T_N\}$ and $E_1 = \{T'_0, \dots, T_N\}$, and the model in Fig. 5 is lumpable to the one depicted in Fig. 6.

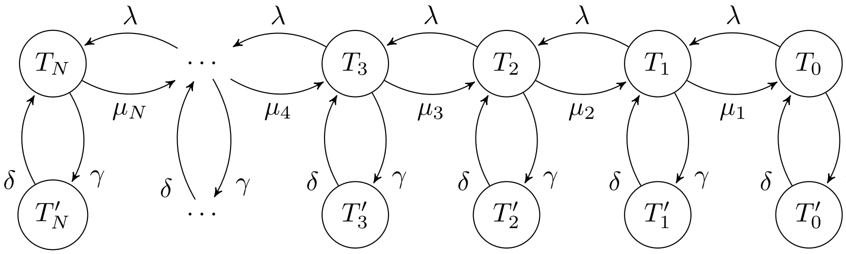

Fig. 5. Left-perturbation of the CTMC in Fig. 4.

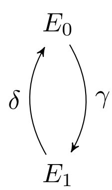

Fig. 6. Aggregation for the lumpable CTMC in Fig. 5.

3.3 Right-Perturbed Markov Chains

When transforming $\mathbf{Q}$ into $A\mathbf{Q}$, the rows of A are multiplied by the columns of $\mathbf{Q}$. An alternative approach is to apply matrix multiplication to the right of $\mathbf{Q}$, i.e., consider $\mathbf{Q}A$ to construct a perturbed Markov chain $\mathbf{Q}'$, as defined below.

Definition 8 (Right-Perturbed Markov Chains). *Let* $\mathbf{Q}$ *be a CTMC over* $\mathcal{S}$ *and* A *be a matrix of size* $|\mathcal{S}|\times|\mathcal{S}|$*. The matrix* $\mathbf{Q}' = \mathbf{Q}A$ *is a* right-perturbation of $\mathbf{Q}$ with respect to A *if it satisfies the following properties:*

1. *All off-diagonal elements are non-negative.*
2. *The sum of each row is zero.*
3. *For all* $s \neq s'$*, the off-diagonal element* $q(s,s') \neq 0$ *if and only if the corresponding off-diagonal element of* $\mathbf{Q}' = \mathbf{Q}A$ *is non-zero, i.e.,* $q'(s,s') \neq 0$*.*

In this case, even if A is diagonal, we cannot guarantee that the rows of the new matrix will sum to zero. However, if A is a diagonal matrix with all diagonal elements equal to a constant c, then the perturbation $\mathbf{Q}' = \mathbf{Q}A$ corresponds to a time scaling, or speed-up, of the Markov chain by a factor of c.

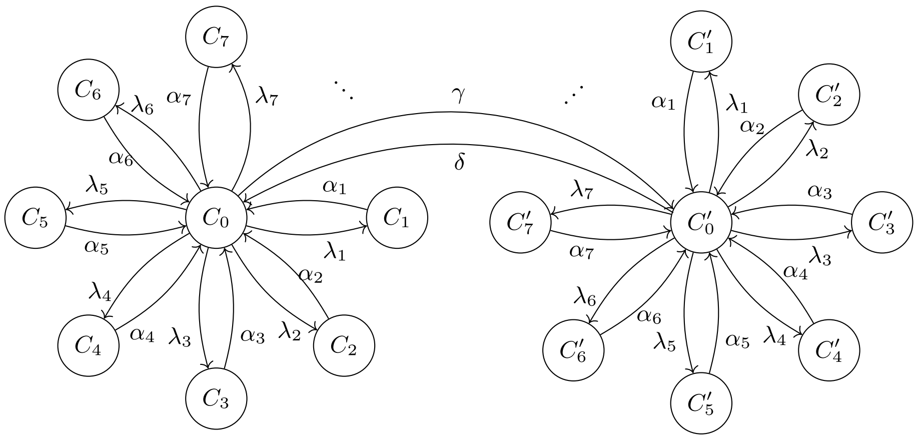

Fig. 7. CTMC of two ionic channels influencing each other.

Example 3. The study of ionic channel activity is crucial in biophysics and neuroscience [23], as it connects molecular biology with cellular physiology. Neuron spikes are generated by the opening and closing of many ionic channels. Ionic channels exhibit multiple opening levels, and their behavior is often modeled using continuous-time Markov chains [3,23].

A single ionic channel has only one observable state, known as the open state (say C_0), and N experimentally unobservable closed states (say $C_1, C_2, \ldots, C_N$), representing different opening levels. Transitions cannot occur directly between the closed states; each closed state can only transition to the open state. As a result, the closed states do not intercommunicate directly, but only through the open state. The activity of the channel can be represented by a star-graph chain in continuous time. Although individual ionic channels do not interact directly, they can influence each other indirectly through changes in membrane potential, ion concentrations (such as calcium), and feedback mechanisms. These indirect interactions are crucial for complex cellular behaviors such as neuronal firing and muscle contraction. This behavior can be modeled by transitions between the open states of individual ionic channels. Figure 7 illustrates two ionic channels influencing each other, where α_i and λ_i $(i = 1, \ldots, N)$ represent the transition rates from closed to open states, and viceversa, within the same ionic channel, while γ and δ represent the potential interactions between the two ionic channels. The infinitesimal generator matrix $\mathbf{Q}$ is defined as:

$$\mathbf{Q} = \begin{pmatrix} \boldsymbol{\Omega}_\gamma & \boldsymbol{\Psi}_\gamma \\ \boldsymbol{\Psi}_\delta & \boldsymbol{\Omega}_\delta \end{pmatrix}$$

where $\boldsymbol{\Omega}_x$ and $\boldsymbol{\Phi}_x$ for $x \in \{\gamma, \delta\}$ are as follows

$$\boldsymbol{\Omega}_x = \begin{pmatrix} -(\lambda_1 + \ldots + \lambda_N + x) & \lambda_1 & \ldots & \lambda_N \\ \alpha_1 & -\alpha_1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ \alpha_N & 0 & \ldots & -\alpha_N \end{pmatrix} \qquad \boldsymbol{\Psi}_x = \begin{pmatrix} x & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}$$

Let $\mathbf{A}$ be the square matrix

$$\mathbf{A} = \begin{pmatrix} \boldsymbol{\Omega'}_\gamma & \boldsymbol{\Psi'}_\gamma \\ \boldsymbol{\Psi}'_\delta & \boldsymbol{\Omega'}_\delta \end{pmatrix}$$

where, for $x \in \{\gamma, \delta\}$, $\boldsymbol{\Omega'}_x$ and $\boldsymbol{\Phi'}_x$ are as follows

$$\boldsymbol{\Omega'}_x = \begin{pmatrix} \frac{1}{x} & 0 & \ldots & 0 \\ \frac{1}{x} - \frac{1}{\alpha_1} & \frac{1}{\alpha_1} & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ \frac{1}{x} - \frac{1}{\alpha_N} & 0 & \ldots & \frac{1}{\alpha_N} \end{pmatrix} \qquad \boldsymbol{\Psi'}_x = \begin{pmatrix} -\frac{1}{x} & 0 & \ldots & 0 \\ -\frac{1}{x} & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ -\frac{1}{x} & 0 & \ldots & 0 \end{pmatrix}$$

then the right-perturbed Markov chain $\mathbf{Q}A$ is as

$$\mathbf{Q}A = \begin{pmatrix} \boldsymbol{\Omega''}_{\gamma,\delta} & \boldsymbol{\Psi''}_{\gamma,\delta} \\ \boldsymbol{\Psi''}_{\delta,\gamma} & \boldsymbol{\Omega''}_{\delta,\gamma} \end{pmatrix}$$

where, for $x, y \in \{\gamma, \delta\}$ and $x \neq y$, $\mathbf{\Omega''}_{x,y}$ and $\mathbf{\Phi''}_{x,y}$ are as follows

$$\mathbf{\Omega''}_{x,y} = \begin{pmatrix} -(\frac{\lambda_1}{\alpha_1} + \ldots + \frac{\lambda_N}{\alpha_N} + \frac{x+y}{y}) & \frac{\lambda_1}{\alpha_1} & \ldots & \frac{\lambda_N}{\alpha_N} \\ 1 & -1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 1 & 0 & \ldots & -1 \end{pmatrix}$$

$$\mathbf{\Psi''}_{x,y} = \begin{pmatrix} \frac{x+y}{x} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}$$

as depicted in Fig. 8. It is easy to see that $\mathbf{Q}A$ is lumpable with respect to the relation $\sim$ over $\mathcal{S}$ given by the reflexive, symmetric and transitive closure of $\{(D_i, D_j) : 1 \leq i, j \leq N\} \cup \{(D'_i, D'_j) : 1 \leq i, j \leq N\}$. This relation induces four equivalence classes, $F_1 = \{D_,1 \ldots, D_N\}$, $F_2 = \{D_0\}$, $F_3 = \{D'_0\}$ and $F_4 = \{D'_,1 \ldots, D'_N\}$, and the model in Fig. 8 is lumpable to the one depicted in Fig. 9. □

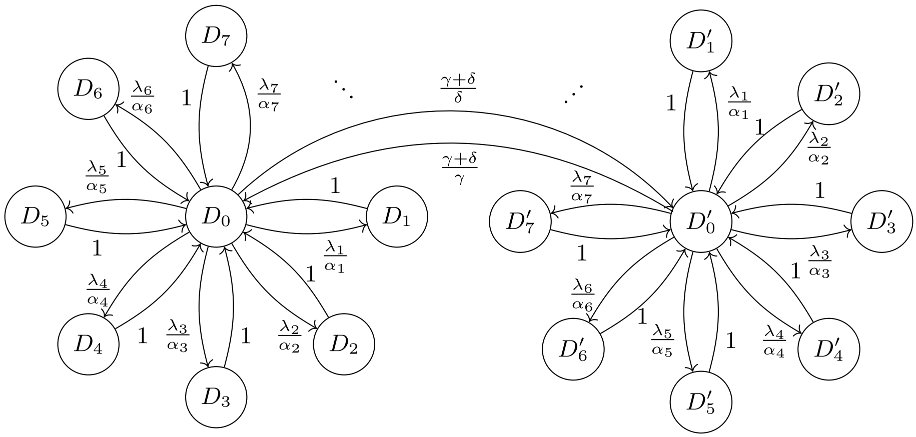

Fig. 8. Right-perturbation of the CTMC in Fig. 7.

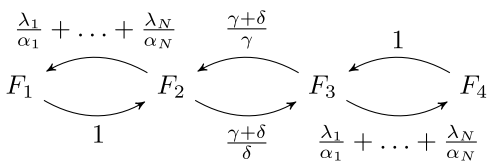

Fig. 9. Aggregation for the lumpable CTMC in Fig. 8.

The following theorem provides a characterization for a restricted class of right-perturbed Markov chains.

Theorem 8 (A Class of Right-Perturbed Markov Chains). *Let* $\mathbf{Q}$ *be a CTMC over* $\mathcal{S}$ *and* A *be a matrix of size* $|\mathcal{S}| \times |\mathcal{S}|$ *such that*

1. *All the diagonal elements of* A *are positive.*
2. *All off-diagonal elements of* A *are non-positive.*
3. *The sum of each row is zero.*
4. *for all* $s \neq s'$*, if* $q(s,s') = 0$ *then* $a(s,s') = 0$.
5. *for all* $s \neq s'$*, if there exists* $s'' \neq s, s'$ *such that* $q(s'',s) \neq 0$ *then* $a(s,s') = 0$.

The matrix $\mathbf{Q}' = \mathbf{Q}A$ *is a* right-perturbation of $\mathbf{Q}$ with respect to A.

Proof. By definition of matrix multiplication, for all $s, s' \in \mathcal{S}$,

$$q'(s,s') = \sum_{s'' \in \mathcal{S}} q(s,s'')a(s'',s').$$

According to Definition 8, we first prove that all off-diagonal elements of $\mathbf{Q}'$ are non-negative. Indeed, for $s \neq s'$,

$$q'(s,s') = q(s,s)a(s,s') + q(s,s')a(s',s') + \sum_{\substack{s'' \in \mathcal{S} \\ s'' \neq s,s'}} q(s,s'')a(s'',s').$$

By Condition *5.* in the theorem statement, if some of the $q(s,s'')$ is non-zero then $a(s'',s')$ is zero, i.e., the last addend of the above summation is always zero. Therefore,

$$q'(s,s') = q(s,s)a(s,s') + q(s,s')a(s',s'). \tag{4}$$

It is now easy to see that $q'(s,s')$ is non-negative since, by hypothesis, $a(s,s')$ is non-positive and $a(s',s')$ is positive, while by definition of infinitesimal generator, $q(s,s)$ is negative and $q(s,s')$ is non-negative.

We now prove that the sum of each row of $\mathbf{Q}'$ is zero, indeed, for all $s \in \mathcal{S}$:

$$\begin{aligned}\sum_{s' \in \mathcal{S}} q'(s,s') = \sum_{s' \in \mathcal{S}} \sum_{s'' \in \mathcal{S}} q(s,s'')a(s'',s') \\ = \sum_{s'' \in \mathcal{S}} \sum_{s' \in \mathcal{S}} q(s,s'')a(s'',s') \\ = \sum_{s'' \in \mathcal{S}} q(s,s'') \sum_{s' \in \mathcal{S}} a(s'',s') = 0\end{aligned}$$

since by Condition *3* the sum of each row of A is zero, i.e., $\sum_{s' \in \mathcal{S}} a(s'',s') = 0$.

Finally, for all $s \neq s'$, the off-diagonal element $q(s,s') \neq 0$ if and only if $q'(s,s') \neq 0$. The proof trivially follows from Eq. (4) and the statement's conditions. Specifically,

– If $q(s,s') \neq 0$ then, by Condition *1.*, $q(s,s')a(s',s') \neq 0$ and then, by Eq. (4) and the fact that $q(s,s)a(s,s')$ is non-negative, it holds that $q'(s,s') \neq 0$.

– If $q'(s,s') \neq 0$ then, by contradiction, assume that $q(s,s') = 0$. Hence, by Eq. (2), $q'(s,s') = -a(s,s')q(s',s)$. However, by Condition *4.*, we have $a(s,s') = 0$, i.e., $q'(s,s') = 0$ which contradicts the hypothesis.

□

The relationship between the steady-state distribution of the original chain $\mathbf{Q}$ and that of the right-perturbed chain $\mathbf{Q}' = \mathbf{Q}A$, is particularly intriguing in this case. Indeed, if A is invertible and $\boldsymbol{\pi}$ is the steady-state distribution of a right-perturbed Markov chain $\mathbf{Q}A$, then $\boldsymbol{\pi}\mathbf{Q}A = \mathbf{0}$ implies $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$. This means that $\mathbf{Q}$ and $\mathbf{Q}'$ have the same steady-state distribution.

Theorem 9 (Steady State Distribution for Right-Perturbed Chains). *Let $\mathbf{Q}$ be a CTMC over $\mathcal{S}$ and A be an invertible matrix of size $|\mathcal{S}| \times |\mathcal{S}|$ such that $\mathbf{Q}' = \mathbf{Q}A$ is a right-perturbed Markov chain of $\mathbf{Q}$ with respect to A. If $\boldsymbol{\pi}$ is the steady-state distribution of $\mathbf{Q}'$, then $\boldsymbol{\pi}$ is also the steady state distribution of the original chain $\mathbf{Q}$, i.e., $\mathbf{Q}$ and $\mathbf{Q}'$ have the same steady-state distribution.*

Putting the two generalizations together we obtain the following definition of *generalized proportional lumpability.*

Definition 9 (Generalized Proportional Lumpability). *Let $\mathbf{Q}$ be a CTMC over $\mathcal{S}$. Let $\sim$ be an equivalence relation over $\mathcal{S}$ and let V and U be the matrices associated to $\sim$. We say that $\sim$ is a* generalized proportional lumpability *for $\mathbf{Q}$ if there exists two invertible matrices L and R such that $\mathbf{Q}' = L\mathbf{Q}R$ satisfies the following properties:*

1. *$\mathbf{Q}'$ is the infinitesimal generator of a CTMC.*
2. *for all $s \neq s' \in \mathcal{S}$, $q'(s,s') \neq 0$ if and only if $q(s,s') \neq 0$.*
3. *$\mathbf{Q}'V = VU\mathbf{Q}'V$.*

In this case we say that $\sim$ is a generalized proportional lumpability *for $\mathbf{Q}$ with respect to L and R.*

Theorem 10 (Aggregated CTMC for Generalized Proportional Lumpability). *Let $\mathbf{Q}$ be a CTMC over $\mathcal{S}$. Let $\sim$ be an equivalence relation over $\mathcal{S}$ and let V and U be the matrices associated to $\sim$. If $\sim$ is a generalized proportional lumpability for $\mathbf{Q}$ with respect to L and R, then the aggregated CTMC $\widetilde{\mathbf{Q}}$ over $\mathcal{S}/\sim$ is defined as:*

$$\widetilde{\mathbf{Q}} = UL\mathbf{Q}RV\,.$$

The steady state distribution of the original chain $\mathbf{Q}$ can be obtained from the steady state distribution of the generalized pertubed Markov chain $L\mathbf{Q}R$ as proved below. The following theorem is a consequence of Theorems 7 and 9.

Theorem 11 (Steady State Distribution of Generalized Proportional Lumpability). *Let $\mathbf{Q}$ be a CTMC over $\mathcal{S}$ and $\sim$ be an equivalence relation over $\mathcal{S}$. The relation $\sim$ is a generalized proportional lumpability for $\mathbf{Q}$ with respect to L and R, if and only if $\sim$ is an ordinary lumpability for the CTMC $L\mathbf{Q}R$. Moreover, the steady-state distribution of $\mathbf{Q}$ is the normalized vector obtained from $\boldsymbol{\pi}'L$, where $\boldsymbol{\pi}'$ is the steady-state distribution of the CTMC $L\mathbf{Q}R$.*

4 Conclusion

In this paper, we introduced a general definition of lumpability, termed *generalized proportional lumpability*. This approach is based on a matrix-multiplication technique for perturbing a Markov chain while allowing the retrieval of the steady-state distribution of the original process. For future work, we plan to explore a similar generalization for the concept of exact lumpability and examine the relationships between generalized proportional lumpability and reversibility. Additionally, we aim to utilize this class of observable equivalences between states to broaden the definition of stochastic non-interference, as done for instance in [4,5,7,18,22] within the context of non-deterministic systems.

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References

1. Alzetta, G., Marin, A., Piazza, C., Rossi, S.: Lumping-based equivalences in Markovian automata: algorithms and applications to product-form analyses. Inf. Comput. **260**, 99–125 (2018)
2. Baarir, S., Beccuti, M., Dutheillet, C., Franceschinis, G., Haddad, S.: Lumping partially symmetrical stochastic models. Perform. Eval. **68**(1), 21–44 (2011)
3. Bhattacharya, R.N., Waymire, E.C.: Stochastic processes with applications. In: SIAM (2009)
4. Bossi, A., Focardi, R., Piazza, C., Rossi, S.: Bisimulation and unwinding for verifying possibilistic security properties. In: Zuck, L.D., Attie, P.C., Cortesi, A., Mukhopadhyay, S. (eds.) VMCAI 2003. LNCS, vol. 2575, pp. 223–237. Springer, Heidelberg (2003). https://doi.org/10.1007/3-540-36384-X_19
5. Bossi, A., Focardi, R., Piazza, C., Rossi, S.: A proof system for information flow security. In: Leuschel, M. (ed.) LOPSTR 2002. LNCS, vol. 2664, pp. 199–218. Springer, Heidelberg (2003). https://doi.org/10.1007/3-540-45013-0_16
6. Buchholz, P.: Exact and ordinary lumpability in finite Markov chains. J. Appl. Probab. **31**, 59–75 (1994)
7. Bugliesi, M., Gallina, L., Marin, A., Rossi, S., Hamadou, S.: Interference-sensitive preorders for manets. In: Ninth International Conference on Quantitative Evaluation of Systems, QEST 2012, pp. 189–198. IEEE Computer Society (2012)
8. Daly, D., Buchholz, P., Sanders, W.H.: Bound-preserving composition for Markov reward models. In: Third International Conference on the Quantitative Evaluation of Systems (QEST 2006), pp. 243–252 (2006)

9. Franceschinis, G., Muntz, R.: Bounds for Quasi-Lumpable Markov chains. Perform. Eval. **20**(1–3), 223–243 (1994)
10. Franceschinis, G., Muntz, R.: Computing bounds for the performance indices of Quasi-Lumpable stochastic well-formed nets. IEEE Trans. Software Eng. **20**(7), 516–525 (1994)
11. Hermanns, H., Herzog, U., Mertsiotakis, V.: Stochastic process algebras: between LOTOS and Markov chains. Comput. Netw. **30**, 901–924 (1998)
12. Hillston, J.: A Compositional Approach to Performance Modelling. Cambridge Press (1996)
13. Kemeny, J.G., Snell, J.L.: Finite Markov Chains. Springer (1976)
14. Marin, A., Piazza, C., Rossi, S.: Proportional lumpability. In: André, É., Stoelinga, M. (eds.) FORMATS 2019. LNCS, vol. 11750, pp. 265–281. Springer, Cham (2019). https://doi.org/10.1007/978-3-030-29662-9_16
15. Marin, A., Piazza, C., Rossi, S.: Proportional lumpability and proportional bisimilarity. Acta Informatica (2021). https://doi.org/10.1007/s00236-021-00404-y
16. Marin, A., Rossi, S.: On the relations between lumpability and reversibility. In: IEEE 22nd International Symposium on Modelling, Analysis & Simulation of Computer and Telecommunication Systems, MASCOTS 2014, pp. 427–432. IEEE Computer Society (2015)
17. Molloy, M.K.: Performance analysis using stochastic petri nets. IEEE Trans. Comput. **31**(9), 913–917 (1982)
18. Piazza, C., Pivato, E., Rossi, S.: Cops - checker of persistent security. In: Tools and Algorithms for the Construction and Analysis of Systems (TACAS 2004), pp. 144–152 (2004)
19. Piazza, C., Rossi, S.: Reasoning about proportional lumpability. In: Abate, A., Marin, A. (eds.) QEST 2021. LNCS, vol. 12846, pp. 372–390. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-85172-9_20
20. Piazza, C., Rossi, S., Smuseva, D.: Efficient algorithm for proportional lumpability and its application to selfish mining in public blockchains. Algorithms **17**(4), 159 (2024)
21. Plateau, B.: On the stochastic structure of parallelism and synchronization models for distributed algorithms. SIGMETRICS Perf. Eval. Rev. **13**(2), 147–154 (1985)
22. Rossi, S.: Model checking adaptive multilevel service compositions. In: Barbosa, L.S., Lumpe, M. (eds.) FACS 2010. LNCS, vol. 6921, pp. 106–124. Springer, Heidelberg (2012). https://doi.org/10.1007/978-3-642-27269-1_7
23. Sakmann, B.: Single-Channel Recording. Springer (2013)
24. Smith, M.J.A.: Compositional abstractions for long-run properties of stochastic systems. In: Eighth International Conference on Quantitative Evaluation of Systems, QEST, vol. 2011, pp. 223–232 (2011)
25. Sumita, U., Rieders, M.: Lumpability and time-reversibility in the aggregation-disaggregation method for large Markov chains. Commun. Stat. Stoch. Models **5**, 63–81 (1989)
26. Towsley, D.F.: Queuing network models with state-dependent routing. J. ACM **27**(2), 323–337 (1980)