Markovian Testing Equivalence and Exponentially Timed Internal Actions

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In the theory of testing for Markovian processes developed so far, exponentially timed internal actions are not admitted within processes. When present, these actions cannot be abstracted away, because their execution takes a nonzero amount of time and hence can be observed. On the other hand, they must be carefully taken into account, in order not to equate processes that are distinguishable from a timing viewpoint. In this paper, we recast the definition of Markovian testing equivalence in the framework of a Markovian process calculus including exponentially timed internal actions. Then, we show that the resulting behavioral equivalence is a congruence, has a sound and complete axiomatization, has a modal logic characterization, and can be decided in polynomial time.

1 Introduction

Markovian behavioral equivalences are a means to relate and manipulate formal models with an underlying continuous-time Markov chain (CTMC) semantics. Various proposals have appeared in the literature, which are extensions of the traditional approaches to the definition of behavioral equivalences. Markovian bisimilarity [14, 13, 5] considers two processes to be equivalent whenever they are able to mimic each other’s functional and performance behavior stepwise. Markovian testing equivalence [2] considers two processes to be equivalent whenever an external observer is not able to distinguish between them from a functional or performance viewpoint by interacting with them by means of tests and comparing their reactions. Markovian trace equivalence [19] considers two processes to be equivalent whenever they are able to perform computations with the same functional and performance characteristics.

The three Markovian behavioral equivalences mentioned above have different discriminating powers as a consequence of their different definitions. However, they are all meaningful not only from a functional standpoint [17, 11, 7], but also from a performance standpoint. In fact, Markovian bisimilarity is known to be in agreement with an exact CTMC-level aggregation called ordinary lumpability [14, 8], while Markovian testing and trace equivalences are known to be consistent with a coarser exact CTMC-level aggregation called T-lumpability [2, 3].

In this paper, we focus on the treatment of internal actions – denoted by τ as usual – that are exponentially timed. Unlike internal actions of nondeterministic processes, exponentially timed internal actions cannot be abstracted away, because their execution takes a nonzero amount of time and hence can be observed. To be precise, in [14, 6, 1] the issue of abstracting from them has been addressed, but it remains unclear whether and to what extent abstraction is possible, especially if we want to end up with a weak Markovian behavioral equivalence that induces a nontrivial, exact CTMC-level aggregation.

The definition of Markovian bisimilarity smoothly includes exponentially timed internal actions, by applying to them the same exit rate equality check that is applied to exponentially timed visible actions. Unfortunately, this is not the case with Markovian testing and trace equivalences as witnessed by the theory developed for them, which does not admit exponentially timed internal actions within processes.

QFM 2009
EPTCS ??, 2009, pp. 1–39, doi:10.4204/EPTCS.??.??

When present, these actions must be carefully taken into account in order not to equate processes that are distinguishable from a timing viewpoint. As an example, given $\lambda, \mu \in \mathbb{R}_{>0}$, processes "$<\tau,\lambda>.\underline{0}$" – which can only execute an exponentially timed internal action whose average duration is $1/\lambda$ – and "$<\tau,\mu>.\underline{0}$" – which can only execute an exponentially timed internal action whose average duration is $1/\mu$ – should not be considered equivalent if $\lambda > \mu$, as the durations of their actions are sampled from different exponential probability distributions. Moreover, if they were considered equivalent, then congruence with respect to alternative and parallel composition would not hold.

With the definition of Markovian testing equivalence given in [2] – which compares the probabilities of passing the same test within the same average time upper bound – there is no way to distinguish between the two processes above, as they pass with probability 1 the test comprising only the success state and with probability 0 any other test, independent of the fixed average time upper bound. In this paper, we show that a simple way to distinguish between the two processes above consists of imposing an additional constraint on the length of the successful computations to take into account.

For instance, if we take a test comprising only the success state, the two processes above pass the test with probability 1 for every average time upper bound if we restrict ourselves to successful computations of length 0. However, if we move to successful computations of length 1 and we use $1/\lambda$ as average time upper bound, it turns out that $<\tau,\lambda>.\underline{0}$ reaches success with probability 1 – as it has enough time on average to perform its only action – whereas $<\tau,\mu>.\underline{0}$ does not – as it has not enough time on average to perform its only action by the deadline. A similar idea applies to Markovian trace equivalence.

After introducing a Markovian process calculus that includes exponentially timed internal actions (Sect. 2), we present a new definition of Markovian testing equivalence that embodies the idea illustrated above (Sect. 3). Then, we show that (i) it coincides with the equivalence defined in [2] when exponentially timed internal actions are absent, (ii) its discriminating power does not change if we introduce exponentially timed internal actions within tests, and (iii) it inherits the fully abstract characterization studied in [2] (Sect. 4). Furthemore, we show that it is a congruence with respect to typical dynamic and static operators (Sect. 5) and has a sound and complete axiomatization for nonrecursive processes (Sect. 6), thus overcoming the limitation to dynamic operators of analogous results contained in [2]. Finally, we show that it has a modal logic characterization (Sect. 7), which is based on the same modal language as [4], and that it can be decided in polynomial time (Sect. 8).

2 Markovian Process Calculus

In this section, we present a process calculus in which every action has associated with it a rate that uniquely identifies its exponentially distributed duration. The definition of the syntax and of the semantics for the resulting Markovian process calculus – MPC for short – is followed by the introduction of some notations related to process terms and their computations that will be used in the rest of the paper.

2.1 Durational Actions and Behavioral Operators

In MPC, an exponentially timed action is represented as a pair $<a,\lambda>$. The first element, a, is the name of the action, which is τ in the case that the action is internal, otherwise it belongs to a set $Name_{\mathrm{v}}$ of visible action names. The second element, $\lambda \in \mathbb{R}_{>0}$, is the rate of the exponentially distributed random variable RV quantifying the duration of the action, i.e., $\Pr\{RV \leq t\} = 1 - \mathrm{e}^{-\lambda \cdot t}$ for $t \in \mathbb{R}_{>0}$. The average duration of the action is equal to the reciprocal of its rate, i.e., $1/\lambda$. If several exponentially timed actions are enabled, the race policy is adopted: the action that is executed is the fastest one.

The sojourn time associated with a process term P is thus the minimum of the random variables quantifying the durations of the exponentially timed actions enabled by P. Since the minimum of several exponentially distributed random variables is exponentially distributed and its rate is the sum of the rates of the original variables, the sojourn time associated with P is exponentially distributed with rate equal to the sum of the rates of the actions enabled by P. Therefore, the average sojourn time associated with P is the reciprocal of the sum of the rates of the actions it enables. The probability of executing one of those actions is given by the action rate divided by the sum of the rates of all the considered actions.

Passive actions of the form $<a, *_w>$ are also included in MPC, where $w \in \mathbb{R}_{>0}$ is the weight of the action. The duration of a passive action is undefined. When several passive actions are enabled, the reactive preselection policy is adopted. This means that, within every set of enabled passive actions having the same name, each such action is given an execution probability equal to the action weight divided by the sum of the weights of all the actions in the set. Instead, the choice among passive actions having different names is nondeterministic. Likewise, the choice between a passive action and an exponentially timed action is nondeterministic.

MPC comprises a CSP-like parallel composition operator [7] relying on an asymmetric synchronization discipline [5], according to which an exponentially timed action can synchronize only with a passive action having the same name. In other words, the synchronization between two exponentially timed actions is forbidden. Following the terminology of [12], the adopted synchronization discipline mixes generative and reactive probabilistic aspects. Firstly, among all the enabled exponentially timed actions, the proposal of an action name is generated after a selection based on the rates of those actions. Secondly, the enabled passive actions that have the same name as the proposed one react by means of a selection based on their weights. Thirdly, the exponentially timed action winning the generative selection and the passive action winning the reactive selection synchronize with each other. The rate of the synchronization is given by the rate of the selected exponentially timed action multiplied by the execution probability of the selected passive action, thus complying with the bounded capacity assumption [14].

We denote by $Act = Name \times Rate$ the set of actions of MPC, where $Name = Name_{\mathrm{v}} \cup \{\tau\}$ is the set of action names – ranged over by a, b – and $Rate = \mathbb{R}_{>0} \cup \{*_w \mid w \in \mathbb{R}_{>0}\}$ is the set of action rates – ranged over by $\tilde{\lambda}, \tilde{\mu}$. We then denote by *Relab* a set of relabeling functions $\varphi : Name \rightarrow Name$ that preserve action visibility, i.e., such that $\varphi^{-1}(\tau) = \{\tau\}$. Finally, we denote by *Var* a set of process variables ranged over by X, Y.

Definition 2.1 The set of process terms of the process language $\mathscr{PL}$ is generated by the following syntax:

P	$::=$	$\underline{0}$	inactive process
	$\mid$	$<a, \lambda>.P$	exponentially timed action prefix
	$\mid$	$<a, *_w>.P$	passive action prefix
	$\mid$	$P + P$	alternative composition
	$\mid$	$P \parallel_S P$	parallel composition
	$\mid$	P / H	hiding
	$\mid$	$P[\varphi]$	relabeling
	$\mid$	X	process variable
	$\mid$	$rec X : P$	recursion

where $a \in Name$, $\lambda, w \in \mathbb{R}_{>0}$, $S, H \subseteq Name_{\mathrm{v}}$, $\varphi \in Relab$, and $X \in Var$. We denote by $\mathbb{P}$ the set of closed and guarded process terms of $\mathscr{PL}$. ■

2.2 Operational Semantics

The semantics for MPC can be defined in the usual operational style, with an important difference with respect to the nondeterministic case. A process term like $<a,\lambda>.\underline{0} + <a,\lambda>.\underline{0}$ is not the same as $<a,\lambda>.\underline{0}$, because the average sojourn time associated with the latter, i.e., $1/\lambda$, is twice the average sojourn time associated with the former, i.e., $1/(\lambda+\lambda)$. In order to assign distinct semantic models to terms like the two considered above, we have to take into account the multiplicity of each transition, intended as the number of different proofs for the transition derivation. The semantic model $[\![P]\!]$ for a process term $P \in \mathbb{P}$ is thus a labeled multitransition system, whose multitransition relation is contained in the smallest multiset of elements of $\mathbb{P} \times \mathit{Act} \times \mathbb{P}$ satisfying the operational semantic rules of Table 1 ($\{\_ \hookrightarrow \_\}$ denotes syntactical replacement; $\{\mid , \mid\}$ are multiset parentheses).

We observe that exponential distributions fit well with the interleaving view of parallel composition. Due to their memoryless property, the execution of an exponentially timed action can be thought of as being started in the last state in which the action is enabled. Due to their infinite support, the probability that two concurrent exponentially timed actions terminate simultaneously is zero.

The CTMC underlying a process term $P \in \mathbb{P}$ can be derived from $[\![P]\!]$ iff this labeled multitransition system has no passive transitions, in which case we say that P is performance closed. We denote by $\mathbb{P}_{\mathrm{pc}}$ the set of performance closed process terms of $\mathbb{P}$.

2.3 Exit Rates of Process Terms

The exit rate of a process term $P \in \mathbb{P}$ is the rate at which P can execute actions of a certain name $a \in \mathit{Name}$ that lead to a certain destination $D \subseteq \mathbb{P}$ and is given by the sum of the rates of those actions due to the race policy. We consider a two-level definition of exit rate, with level 0 corresponding to exponentially timed actions and level -1 corresponding to passive actions:

$$\boxed{\mathit{rate}_{\mathrm{e}}(P,a,l,D) \;=\; \begin{cases} \sum\{\mid \lambda \in \mathbb{R}_{>0} \mid \exists P' \in D.\, P \xrightarrow{a,\lambda} P' \mid\} & \text{if } l = 0 \\ \sum\{\mid w \in \mathbb{R}_{>0} \mid \exists P' \in D.\, P \xrightarrow{a,*_w} P' \mid\} & \text{if } l = -1 \end{cases}}$$

where each summation is taken to be zero whenever its multiset is empty.

By summing up the rates of all the actions of a certain level l that P can execute, we obtain the total exit rate of P at level l:

$$\boxed{\mathit{rate}_{\mathrm{t}}(P,l) \;=\; \sum_{a \in \mathit{Name}} \mathit{rate}_{\mathrm{o}}(P,a,l)}$$

where:

$$\boxed{\mathit{rate}_{\mathrm{o}}(P,a,l) \;=\; \mathit{rate}_{\mathrm{e}}(P,a,l,\mathbb{P})}$$

is the overall exit rate of P with respect to a at level l.

If P is performance closed, then $\mathit{rate}_{\mathrm{t}}(P,0)$ coincides with the reciprocal of the average sojourn time associated with P. Instead, $\mathit{rate}_{\mathrm{o}}(P,a,-1)$ coincides with $\mathit{weight}(P,a)$.

2.4 Probability and Duration of Computations

A computation of a process term $P \in \mathbb{P}$ is a sequence of transitions that can be executed starting from P. The length of a computation is given by the number of transitions occurring in it. We denote by $\mathscr{C}_{\mathrm{f}}(P)$ the multiset of finite-length computations of P. We say that two distinct computations are independent of each other if neither is a proper prefix of the other one. In the following, we concentrate on finite

$$(\textsc{Pre}_1)\quad \frac{}{<a,\lambda>.P \xrightarrow{a,\lambda} P} \qquad\qquad (\textsc{Pre}_2)\quad \frac{}{<a,*_w>.P \xrightarrow{a,*_w} P}$$

$$(\textsc{Alt}_1)\quad \frac{P_1 \xrightarrow{a,\tilde{\lambda}} P'}{P_1 + P_2 \xrightarrow{a,\tilde{\lambda}} P'} \qquad\qquad (\textsc{Alt}_2)\quad \frac{P_2 \xrightarrow{a,\tilde{\lambda}} P'}{P_1 + P_2 \xrightarrow{a,\tilde{\lambda}} P'}$$

$$(\textsc{Par}_1)\quad \frac{P_1 \xrightarrow{a,\tilde{\lambda}} P'_1 \qquad a \notin S}{P_1 \,\|_S\, P_2 \xrightarrow{a,\tilde{\lambda}} P'_1 \,\|_S\, P_2} \qquad\qquad (\textsc{Par}_2)\quad \frac{P_2 \xrightarrow{a,\tilde{\lambda}} P'_2 \qquad a \notin S}{P_1 \,\|_S\, P_2 \xrightarrow{a,\tilde{\lambda}} P_1 \,\|_S\, P'_2}$$

$$(\textsc{Syn}_1)\quad \frac{P_1 \xrightarrow{a,\lambda} P'_1 \qquad P_2 \xrightarrow{a,*_w} P'_2 \qquad a \in S}{P_1 \,\|_S\, P_2 \xrightarrow{a,\lambda \cdot \frac{w}{weight(P_2,a)}} P'_1 \,\|_S\, P'_2}$$

$$(\textsc{Syn}_2)\quad \frac{P_1 \xrightarrow{a,*_w} P'_1 \qquad P_2 \xrightarrow{a,\lambda} P'_2 \qquad a \in S}{P_1 \,\|_S\, P_2 \xrightarrow{a,\lambda \cdot \frac{w}{weight(P_1,a)}} P'_1 \,\|_S\, P'_2}$$

$$(\textsc{Syn}_3)\quad \frac{P_1 \xrightarrow{a,*_{w_1}} P'_1 \qquad P_2 \xrightarrow{a,*_{w_2}} P'_2 \qquad a \in S}{P_1 \,\|_S\, P_2 \xrightarrow{a,*_{norm(w_1,w_2,a,P_1,P_2)}} P'_1 \,\|_S\, P'_2}$$

$$(\textsc{Hid}_1)\quad \frac{P \xrightarrow{a,\tilde{\lambda}} P' \qquad a \in H}{P/H \xrightarrow{\tau,\tilde{\lambda}} P'/H} \qquad\qquad (\textsc{Hid}_2)\quad \frac{P \xrightarrow{a,\tilde{\lambda}} P' \qquad a \notin H}{P/H \xrightarrow{a,\tilde{\lambda}} P'/H}$$

$$(\textsc{Rel})\quad \frac{P \xrightarrow{a,\tilde{\lambda}} P'}{P[\varphi] \xrightarrow{\varphi(a),\tilde{\lambda}} P'[\varphi]}$$

$$(\textsc{Rec})\quad \frac{P\{rec X : P \hookrightarrow X\} \xrightarrow{a,\tilde{\lambda}} P'}{rec X : P \xrightarrow{a,\tilde{\lambda}} P'}$$

$$weight(P,a) \;=\; \sum\{\!|\, w \in \mathbb{R}_{>0} \mid \exists P' \in \mathbb{P}.\, P \xrightarrow{a,*_w} P' \,|\!\}$$

$$norm(w_1,w_2,a,P_1,P_2) \;=\; \frac{w_1}{weight(P_1,a)} \cdot \frac{w_2}{weight(P_2,a)} \cdot (weight(P_1,a) + weight(P_2,a))$$

Table 1: Operational semantic rules for MPC

multisets of independent, finite-length computations. Below we define the probability and the duration of a computation $c \in \mathscr{C}_{\mathrm{f}}(P)$ for $P \in \mathbb{P}_{\mathrm{pc}}$, using $\_ \circ \_$ for sequence concatenation and $|\_|$ for sequence length.

The probability of executing c is the product of the execution probabilities of the transitions of c:

$$\boxed{prob(c) \;=\; \begin{cases} 1 & \text{if } |c| = 0 \\ \frac{\lambda}{rate_{\mathrm{t}}(P,0)} \cdot prob(c') & \text{if } c \equiv P \xrightarrow{a,\lambda} c' \end{cases}}$$

We also define the probability of executing a computation in $C \subseteq \mathscr{C}_{\mathrm{f}}(P)$ as:

$$\boxed{prob(C) \;=\; \sum_{c \in C} prob(c)}$$

whenever C is finite and all of its computations are independent of each other.

The stepwise average duration of c is the sequence of average sojourn times in the states traversed by c:

$$\boxed{time_{\mathrm{a}}(c) \;=\; \begin{cases} \varepsilon & \text{if } |c| = 0 \\ \frac{1}{rate_{\mathrm{t}}(P,0)} \circ time_{\mathrm{a}}(c') & \text{if } c \equiv P \xrightarrow{a,\lambda} c' \end{cases}}$$

where ε is the empty stepwise average duration. We also define the multiset of computations in $C \subseteq \mathscr{C}_{\mathrm{f}}(P)$ whose stepwise average duration is not greater than $\theta \in (\mathbb{R}_{>0})^*$ as:

$$\boxed{C_{\leq \theta} \;=\; \{\!|\, c \in C \mid |c| \leq |\theta| \wedge \forall i = 1, \dots, |c|.\, time_{\mathrm{a}}(c)[i] \leq \theta[i] \,|\!\}}$$

Moreover, we denote by C^l the multiset of computations in $C \subseteq \mathscr{C}_{\mathrm{f}}(P)$ whose length is equal to $l \in \mathbb{N}$.

We conclude by observing that the average duration of a finite-length computation has been defined as the sequence of average sojourn times in the states traversed by the computation. The same quantity could have been defined as the sum of the same basic ingredients, but this would not have been appropriate as explained in [19, 2].

3 Redefining Markovian Testing Equivalence

The basic idea behind testing equivalence is to infer information about the behavior of process terms by interacting with them by means of tests and comparing their reactions. In a Markovian setting, we are not only interested in verifying whether tests are passed or not, but also in measuring the probability with which they are passed and the time taken to pass them. Therefore, we have to restrict ourselves to $\mathbb{P}_{\mathrm{pc}}$.

As in the nondeterministic setting, the most convenient way to represent a test is through a process term, which interacts with any process term under test by means of a parallel composition operator that enforces synchronization on the set $Name_{\mathrm{v}}$ of all visible action names. Due to the adoption of an asymmetric synchronization discipline, a test can comprise only passive visible actions, so that the composite term inherits performance closure from the process term under test.

From a testing viewpoint, in any of its states a process term under test generates the proposal of an action to be executed by means of a race among the exponentially timed actions enabled in that state. If the name of the proposed action is τ, then the process term advances by itself. Otherwise, the test either reacts by participating in the interaction with the process term through a passive action having the same name as the proposed exponentially timed action, or blocks the interaction if it has no passive actions with the proposed name.

Markovian testing equivalence relies on comparing the process term probabilities of performing successful test-driven computations within arbitrary sequences of average amounts of time. Due to the presence of these average time upper bounds, for the test representation we can restrict ourselves to nonrecursive process terms. In other words, the expressiveness provided by finite-state labeled multitransition systems with an acyclic structure is enough for tests.

In order not to interfere with the quantitative aspects of the behavior of process terms under test, we avoid the introduction of a success action ω. The successful completion of a test is formalized in the text syntax by replacing $\underline{0}$ with a zeroary operator s denoting a success state. Ambiguous tests including several summands among which at least one equal to s are avoided through a two-level syntax.

Definition 3.1 The set $\mathbb{T}_{\mathrm{R}}$ of reactive tests is generated by the following syntax:

$$\boxed{\begin{array}{rcl} T & ::= & \mathrm{s} \mid T' \\ T' & ::= & <a, *_w>.T \mid T' + T' \end{array}}$$

where $a \in Name_{\mathrm{v}}$ and $w \in \mathbb{R}_{>0}$. ■

Definition 3.2 Let $P \in \mathbb{P}_{\mathrm{pc}}$ and $T \in \mathbb{T}_{\mathrm{R}}$. The interaction system of P and T is process term $P \,\|_{Name_{\mathrm{v}}}\, T \in \mathbb{P}_{\mathrm{pc}}$ and we say that:

- A configuration is a state of $[\![P \,\|_{Name_{\mathrm{v}}}\, T]\!]$, which is formed by a process and a test projection.
- A configuration is successful iff its test projection is s.
- A test-driven computation is a computation of $[\![P \,\|_{Name_{\mathrm{v}}}\, T]\!]$.
- A test-driven computation is successful iff it traverses a successful configuration.

We denote by $\mathscr{SC}(P,T)$ the multiset of successful computations of $P \,\|_{Name_{\mathrm{v}}}\, T$. ■

If a process term $P \in \mathbb{P}_{\mathrm{pc}}$ under test has no exponentially timed τ-actions as it was in [2], then for all reactive tests $T \in \mathbb{T}_{\mathrm{R}}$ it turns out that: (i) all the computations in $\mathscr{SC}(P,T)$ have a finite length due to the restrictions imposed on the test syntax; (ii) all the computations in $\mathscr{SC}(P,T)$ are independent of each other because of their maximality; (iii) the multiset $\mathscr{SC}(P,T)$ is finite because P and T are finitely branching. Thus, all definitions of Sect. 2.4 are applicable to $\mathscr{SC}(P,T)$ and also to $\mathscr{SC}_{\leq\theta}(P,T)$ for any sequence $\theta \in (\mathbb{R}_{>0})^*$ of average amounts of time.

In order to cope with the possible presence of exponentially timed τ-actions within P in such a way that all the properties above hold – especially independence – we have to consider subsets of $\mathscr{SC}_{\leq\theta}(P,T)$ including all successful test-driven computations of the same length. This is also necessary to distinguish among process terms comprising only exponentially timed τ-actions – like $<\tau,\lambda>.\underline{0}$ and $<\tau,\mu>.\underline{0}$, with $\lambda > \mu$, mentioned in Sect. 1 – as there is a single test, s, that those process terms can pass. The only option is to compare them after executing the same number of τ-actions.

Since no element of $\mathscr{SC}_{\leq\theta}(P,T)$ can be longer than $|\theta|$, we should consider every possible subset $\mathscr{SC}^{l}_{\leq\theta}(P,T)$ for $0 \leq l \leq |\theta|$. However, it is enough to consider $\mathscr{SC}^{|\theta|}_{\leq\theta}(P,T)$, as shorter successful test-driven computations can be taken into account when imposing prefixes of θ as average time upper bounds. Therefore, the novelty with respect to [2] is simply the presence of the additional constraint $|\theta|$.

Definition 3.3 Let $P_1, P_2 \in \mathbb{P}_{\mathrm{pc}}$. We say that P_1 is Markovian testing equivalent to P_2, written $P_1 \sim_{\mathrm{MT}} P_2$, iff for all reactive tests $T \in \mathbb{T}_{\mathrm{R}}$ and sequences $\theta \in (\mathbb{R}_{>0})^*$ of average amounts of time:

$$prob(\mathscr{SC}^{|\theta|}_{\leq\theta}(P_1,T)) = prob(\mathscr{SC}^{|\theta|}_{\leq\theta}(P_2,T))$$

■

Note that we have not defined a may equivalence and a must equivalence as in the nondeterministic case [11]. The reason is that in this Markovian framework the possibility and the necessity of passing a test are not sufficient to discriminate among process terms, as they are qualitative concepts. What we have considered here is a single quantitative notion given by the probability of passing a test (within an average time upper bound); hence, the definition of a single equivalence. This quantitative notion subsumes both the possibility of passing a test – which can be encoded as the probability of passing the test being greater than zero – and the necessity of passing a test – which can be encoded as the probability of passing the test being equal to one.

Although we could have defined Markovian testing equivalence as the kernel of a Markovian testing preorder, this has not been done. The reason is that such a preorder would have boiled down to an equivalence relation, because for each reactive test passed by P_1 within θ with a probability less than the probability with which P_2 passes the same test within θ, in general it is possible to find a dual reactive test for which the relation between the two probabilities is inverted.

Another important difference with respect to the nondeterministic case is that the presence of average time upper bounds makes it possible to decide whether a test is passed or not even if the process term under test can execute infinitely many exponentially timed τ-actions. In other words, τ-divergence does not need to be taken into account.

4 Basic Properties and Characterizations

First of all, we observe that, whenever exponentially timed τ-actions are absent, the new Markovian testing equivalence $\sim_{\rm MT}$ coincides with the old one defined in [2], which we denote by $\sim_{\rm MT,old}$. In the following, we use $\mathbb{P}_{\rm pc,v}$ to refer to the process terms of $\mathbb{P}_{\rm pc}$ that contain no exponentially timed τ-actions.

Proposition 4.1 Let $P_1, P_2 \in \mathbb{P}_{\rm pc,v}$. Then $P_1 \sim_{\rm MT} P_2 \iff P_1 \sim_{\rm MT,old} P_2$. ■

Then, we have two alternative characterizations of $\sim_{\rm MT}$, which provide further justifications for the way in which the equivalence has been defined. The first one establishes that the discriminating power does not change if we consider a set $\mathbb{T}_{\rm R,lib}$ of tests with the following more liberal syntax:

$$T \ ::= \ \mathrm{s} \mid <a, *_w>.T \mid T + T$$

provided that by successful configuration we mean a configuration whose test projection includes s as top-level summand. Let us denote by $\sim_{\rm MT,lib}$ the resulting variant of Markovian testing equivalence.

Proposition 4.2 Let $P_1, P_2 \in \mathbb{P}_{\rm pc}$. Then $P_1 \sim_{\rm MT,lib} P_2 \iff P_1 \sim_{\rm MT} P_2$. ■

The second characterization establishes that the discriminating power does not change if we consider a set $\mathbb{T}_{{\rm R},\tau}$ of tests capable of moving autonomously by executing exponentially timed τ-actions:

$$\begin{array}{rcl} T & ::= & \mathrm{s} \mid T' \\ T' & ::= & <a, *_w>.T \mid <\tau, \lambda>.T \mid T' + T' \end{array}$$

Let us denote by $\sim_{{\rm MT},\tau}$ the resulting variant of Markovian testing equivalence.

Proposition 4.3 Let $P_1, P_2 \in \mathbb{P}_{\rm pc}$. Then $P_1 \sim_{{\rm MT},\tau} P_2 \iff P_1 \sim_{\rm MT} P_2$. ■

Finally, we have two further alternative characterizations of $\sim_{\rm MT}$ coming from [2]. The first one establishes that the discriminating power does not change if we consider the (more accurate) probability distribution of passing tests within arbitrary sequences of amounts of time, rather than the (easier to work with) probability of passing tests within arbitrary sequences of average amounts of time.

The second characterization fully abstracts from comparing process term behavior in response to tests. This is achieved by considering traces that are extended at each step with the set of visible action names permitted by the environment at that step (not to be confused with a ready set). A consequence of the structure of extended traces is the identification of a set $\mathbb{T}_{\mathrm{R,c}}$ of canonical reactive tests, which is generated by the following syntax:

$$\boxed{T \;::=\; \mathrm{s} \mid {<}a,*_1{>}.T + \sum_{b \in \mathscr{E} - \{a\}} {<}b,*_1{>}.{<}\mathrm{z},*_1{>}.\mathrm{s}}$$

where $a \in \mathscr{E}$, $\mathscr{E} \subseteq Name_{\mathrm{v}}$ finite, the summation is absent whenever $\mathscr{E} = \{a\}$, and z is a visible action name representing failure that can occur within tests but not within process terms under test. Similar to the case of probabilistic testing equivalence [9, 10], each of these canonical reactive tests admits a single computation leading to success, whose intermediate states can have additional computations each leading to failure in one step. We point out that the canonical reactive tests are name deterministic, in the sense that the names of the passive actions occurring in any of their branches are all distinct.

5 Congruence Property

Markovian testing equivalence is a congruence with respect to all MPC operators. In particular, unlike [2], we have a full congruence result with respect to parallel composition.

Theorem 5.1 Let $P_1, P_2 \in \mathbb{P}_{\mathrm{pc}}$. Whenever $P_1 \sim_{\mathrm{MT}} P_2$, then:

1. ${<}a,\lambda{>}.P_1 \sim_{\mathrm{MT}} {<}a,\lambda{>}.P_2$ for all ${<}a,\lambda{>} \in Act$.
2. $P_1 + P \sim_{\mathrm{MT}} P_2 + P$ and $P + P_1 \sim_{\mathrm{MT}} P + P_2$ for all $P \in \mathbb{P}_{\mathrm{pc}}$.
3. $P_1 \parallel_S P \sim_{\mathrm{MT}} P_2 \parallel_S P$ and $P \parallel_S P_1 \sim_{\mathrm{MT}} P \parallel_S P_2$ for all $P \in \mathbb{P}$ and $S \subseteq Name_{\mathrm{v}}$ s.t. $P_1 \parallel_S P, P_2 \parallel_S P \in \mathbb{P}_{\mathrm{pc}}$.
4. $P_1/H \sim_{\mathrm{MT}} P_2/H$ for all $H \subseteq Name_{\mathrm{v}}$.
5. $P_1[\varphi] \sim_{\mathrm{MT}} P_2[\varphi]$ for all $\varphi \in Relab$. ■

It is worth stressing that the additional constraint on the length of successful test-driven computations present in Def. 3.3 is fundamental for achieving congruence with respect to alternative and parallel composition. As an example, if it were ${<}\tau,\lambda{>}.\underline{0} \sim_{\mathrm{MT}} {<}\tau,\mu{>}.\underline{0}$ for $\lambda > \mu$, then we would have ${<}\tau,\lambda{>}.\underline{0} + {<}a,\gamma{>}.\underline{0} \not\sim_{\mathrm{MT}} {<}\tau,\mu{>}.\underline{0} + {<}a,\gamma{>}.\underline{0}$. In fact, when the average time upper bound is high enough, the probability of passing ${<}a,*_1{>}.\mathrm{s}$ is $\frac{\gamma}{\lambda+\gamma}$ for the first term, whereas it is $\frac{\gamma}{\mu+\gamma}$ for the second term. We also mention that Props. 4.2 and 4.3 are exploited in the congruence proof for static operators.

6 Sound and Complete Axiomatization

Markovian testing equivalence has a sound and complete axiomatization over the set $\mathbb{P}_{\mathrm{pc,nrec}}$ of nonrecursive process terms of $\mathbb{P}_{\mathrm{pc}}$, given by the set $\mathscr{A}_{\mathrm{MT}}$ of equational laws of Table 2.

Apart from the usual laws for the alternative composition operator and for the unary static operators, unlike the axiomatization of [2] we now have laws dealing with concurrency. In particular, axiom $\mathscr{A}_{\mathrm{MT},5}$ concerning the parallel composition of $P \equiv \sum_{i \in I} {<}a_i,\tilde{\lambda}_i{>}.P_i$ and $Q \equiv \sum_{j \in J} {<}b_j,\tilde{\mu}_j{>}.Q_j$ – where I and J are nonempty finite index sets and each summation on the right-hand side of the axiom is taken to be $\underline{0}$ whenever its set of summands is empty – is the expansion law when enforcing generative-reactive and reactive-reactive synchronizations. This axiom applies to non-performance-closed process terms too; e.g., the last addendum on its right-hand side is related to reactive-reactive synchronizations.

$$(\mathscr{A}_{\text{MT},1}) \qquad P_1 + P_2 = P_2 + P_1$$

$$(\mathscr{A}_{\text{MT},2}) \qquad (P_1 + P_2) + P_3 = P_1 + (P_2 + P_3)$$

$$(\mathscr{A}_{\text{MT},3}) \qquad P + \underline{0} = P$$

$$(\mathscr{A}_{\text{MT},4}) \qquad \sum_{i \in I} <a, \lambda_i>. \sum_{j \in J_i} <b_{i,j}, \mu_{i,j}>.P_{i,j} = <a, \sum_{k \in I} \lambda_k>. \sum_{i \in I} \sum_{j \in J_i} <b_{i,j}, \frac{\lambda_i}{\sum_{k \in I} \lambda_k} \cdot \mu_{i,j}>.P_{i,j}$$

if: I is a finite index set with $|I| \geq 2$;
for all $i \in I$, index set J_i is finite and its summation is $\underline{0}$ if $J_i = \emptyset$;
for all $i_1, i_2 \in I$ and $b \in Name$:

$$\sum_{j \in J_{i_1}} \{\!|\, \mu_{i_1,j} \mid b_{i_1,j} = b \,|\!\} = \sum_{j \in J_{i_2}} \{\!|\, \mu_{i_2,j} \mid b_{i_2,j} = b \,|\!\}$$

$$\begin{aligned}
(\mathscr{A}_{\text{MT},5}) \qquad \sum_{i \in I} <a_i, \tilde{\lambda}_i>.P_i \parallel_S \sum_{j \in J} <b_j, \tilde{\mu}_j>.Q_j = {} & \\
& \sum_{k \in I, a_k \notin S} <a_k, \tilde{\lambda}_k>. \left(P_k \parallel_S \sum_{j \in J} <b_j, \tilde{\mu}_j>.Q_j \right) + \\
& \sum_{h \in J, b_h \notin S} <b_h, \tilde{\mu}_h>. \left(\sum_{i \in I} <a_i, \tilde{\lambda}_i>.P_i \parallel_S Q_h \right) + \\
& \sum_{k \in I, a_k \in S, \tilde{\lambda}_k \in \mathbb{R}_{>0}} \; \sum_{h \in J, b_h = a_k, \tilde{\mu}_h = *_{w_h}} <a_k, \tilde{\lambda}_k \cdot \frac{w_h}{weight(Q, b_h)}>.(P_k \parallel_S Q_h) + \\
& \sum_{h \in J, b_h \in S, \tilde{\mu}_h \in \mathbb{R}_{>0}} \; \sum_{k \in I, a_k = b_h, \tilde{\lambda}_k = *_{v_k}} <b_h, \tilde{\mu}_h \cdot \frac{v_k}{weight(P, a_k)}>.(P_k \parallel_S Q_h) + \\
& \sum_{k \in I, a_k \in S, \tilde{\lambda}_k = *_{v_k}} \; \sum_{h \in J, b_h = a_k, \tilde{\mu}_h = *_{w_h}} <a_k, *_{norm(v_k, w_h, a_k, P, Q)}>.(P_k \parallel_S Q_h)
\end{aligned}$$

$$(\mathscr{A}_{\text{MT},6}) \qquad \sum_{i \in I} <a_i, \tilde{\lambda}_i>.P_i \parallel_S \underline{0} = \sum_{k \in I, a_k \notin S} <a_k, \tilde{\lambda}_k>.P_k$$

$$(\mathscr{A}_{\text{MT},7}) \qquad \underline{0} \parallel_S \sum_{j \in J} <b_j, \tilde{\mu}_j>.Q_j = \sum_{h \in J, b_h \notin S} <b_h, \tilde{\mu}_h>.Q_h$$

$$(\mathscr{A}_{\text{MT},8}) \qquad \underline{0} \parallel_S \underline{0} = \underline{0}$$

$$(\mathscr{A}_{\text{MT},9}) \qquad \underline{0}/H = \underline{0}$$

$$(\mathscr{A}_{\text{MT},10}) \qquad (<a, \tilde{\lambda}>.P)/H = <\tau, \tilde{\lambda}>.(P/H) \qquad \text{if } a \in H$$

$$(\mathscr{A}_{\text{MT},11}) \qquad (<a, \tilde{\lambda}>.P)/H = <a, \tilde{\lambda}>.(P/H) \qquad \text{if } a \notin H$$

$$(\mathscr{A}_{\text{MT},12}) \qquad (P_1 + P_2)/H = P_1/H + P_2/H$$

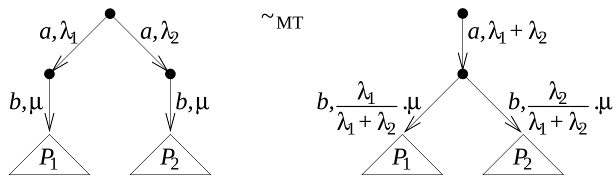
$$(\mathscr{A}_{\text{MT},13}) \qquad \underline{0}[\varphi] = \underline{0}$$

$$(\mathscr{A}_{\text{MT},14}) \qquad (<a, \tilde{\lambda}>.P)[\varphi] = <\varphi(a), \tilde{\lambda}>.(P[\varphi])$$

$$(\mathscr{A}_{\text{MT},15}) \qquad (P_1 + P_2)[\varphi] = P_1[\varphi] + P_2[\varphi]$$

Table 2: Equational laws for $\sim_{\text{MT}}$

Like in [2], the law characterizing $\sim_{\rm MT}$ is the axiom schema $\mathscr{A}_{\rm MT,4}$, which in turn subsumes the law $<a,\lambda_1>.P + <a,\lambda_2>.P = <a,\lambda_1+\lambda_2>.P$ characterizing Markovian bisimilarity. The simplest instance of axiom schema $\mathscr{A}_{\rm MT,4}$ is depicted below:



As emphasized by the figure above, $\sim_{\rm MT}$ allows choices to be deferred in the case of branches that start with the same action name (see the two a-branches on the left-hand side) and are followed by sets of actions having the same names and total rates (see $\{<b,\mu>\}$ after each of the two a-branches).

Theorem 6.1 Let $P_1, P_2 \in \mathbb{P}_{\rm pc,nrec}$. Then $\mathscr{A}_{\rm MT} \vdash P_1 = P_2 \iff P_1 \sim_{\rm MT} P_2$. ■

7 Modal Logic Characterization

Markovian testing equivalence has a modal logic characterization that, as in [4], is based on a modal language comprising true, disjunction, and diamond. A constraint is imposed on formulas of the form $\phi_1 \vee \phi_2$, which does not reduce the expressive power as it is consistent with the name-deterministic nature of branches within canonical reactive tests (see Sect. 4).

Definition 7.1 The set of formulas of the modal language $\mathscr{ML}_{\rm MT}$ is generated by the following syntax:

$$\boxed{\begin{array}{rcl} \phi & ::= & \text{true} \mid \phi' \\ \phi' & ::= & \langle a \rangle \phi \mid \phi' \vee \phi' \end{array}}$$

where $a \in \mathit{Name}_{\rm v}$ and each formula of the form $\phi_1 \vee \phi_2$ satisfies:

$$\mathit{init}(\phi_1) \cap \mathit{init}(\phi_2) \;=\; \emptyset$$

with $\mathit{init}(\phi)$ being defined by induction on the syntactical structure of ϕ as follows:

$$\begin{array}{rcl} \mathit{init}(\text{true}) & = & \emptyset \\ \mathit{init}(\langle a \rangle \phi) & = & \{a\} \\ \mathit{init}(\phi_1 \vee \phi_2) & = & \mathit{init}(\phi_1) \cup \mathit{init}(\phi_2) \end{array}$$ ■

Probabilistic and temporal information do not decorate any operator of the modal language, but come into play through a quantitative interpretation function inspired by [16] that replaces the usual boolean satisfaction relation. This interpretation function measures the probability that a process term satisfies a formula quickly enough on average. The constraint imposed by Def. 7.1 on disjunctions guarantees that their subformulas exercise independent computations of the process term, thus ensuring the correct calculation of the probability of satisfying the overall formula. In order to manage exponentially timed τ-actions, unlike [4] the length of the computations satisfying the formula has to be taken into account as well.

Definition 7.2 The interpretation function $[\![.]\!]^{\cdot}_{\rm MT}$ of $\mathscr{ML}_{\rm MT}$ over $\mathbb{P}_{\rm pc} \times (\mathbb{R}_{>0})^*$ is defined by letting:

$$\boxed{[\![\phi]\!]^{|\theta|}_{\rm MT}(P,\theta) \;=\; \begin{cases} 0 & \text{if } |\theta| = 0 \wedge \phi \not\equiv \text{true or} \\ & \quad |\theta| > 0 \wedge \mathit{rate}_{\rm o}(P, \mathit{init}(\phi) \cup \{\tau\}, 0) = 0 \\ 1 & \text{if } |\theta| = 0 \wedge \phi \equiv \text{true} \end{cases}}$$

otherwise by induction on the syntactical structure of ϕ and on the length of θ as follows:

$$[\![\mathrm{true}]\!]_{\mathrm{MT}}^{|t\circ\theta|}(P, t\circ\theta) \;=\; \begin{cases} \sum\limits_{P \xrightarrow{\tau,\lambda} P'} \frac{\lambda}{rate_{\mathrm{o}}(P,\tau,0)} \cdot [\![\mathrm{true}]\!]_{\mathrm{MT}}^{|\theta|}(P',\theta) & \text{if } \frac{1}{rate_{\mathrm{o}}(P,\tau,0)} \leq t \\ 0 & \text{if } \frac{1}{rate_{\mathrm{o}}(P,\tau,0)} > t \end{cases}$$

$$[\![\langle a\rangle\phi]\!]_{\mathrm{MT}}^{|t\circ\theta|}(P, t\circ\theta) \;=\; \begin{cases} \sum\limits_{P \xrightarrow{a,\lambda} P'} \frac{\lambda}{rate_{\mathrm{o}}(P,\{a,\tau\},0)} \cdot [\![\phi]\!]_{\mathrm{MT}}^{|\theta|}(P',\theta) + & \\ \quad \sum\limits_{P \xrightarrow{\tau,\lambda} P'} \frac{\lambda}{rate_{\mathrm{o}}(P,\{a,\tau\},0)} \cdot [\![\langle a\rangle\phi]\!]_{\mathrm{MT}}^{|\theta|}(P',\theta) & \text{if } \frac{1}{rate_{\mathrm{o}}(P,\{a,\tau\},0)} \leq t \\ 0 & \text{if } \frac{1}{rate_{\mathrm{o}}(P,\{a,\tau\},0)} > t \end{cases}$$

$$\begin{aligned} [\![\phi_1 \vee \phi_2]\!]_{\mathrm{MT}}^{|t\circ\theta|}(P, t\circ\theta) \;=\;& p_1 \cdot [\![\phi_1]\!]_{\mathrm{MT}}^{|t_1\circ\theta|}(P_{\textit{no-init-}\tau}, t_1\circ\theta) + p_2 \cdot [\![\phi_2]\!]_{\mathrm{MT}}^{|t_2\circ\theta|}(P_{\textit{no-init-}\tau}, t_2\circ\theta) \\ &+ \sum\limits_{P \xrightarrow{\tau,\lambda} P'} \frac{\lambda}{rate_{\mathrm{o}}(P, init(\phi_1\vee\phi_2)\cup\{\tau\},0)} \cdot [\![\phi_1 \vee \phi_2]\!]_{\mathrm{MT}}^{|\theta|}(P',\theta) \end{aligned}$$

where $P_{\textit{no-init-}\tau}$ is P devoid of all of its computations starting with a τ-transition – which is assumed to be $\underline{0}$ whenever all the computations of P start with a τ-transition – and for $j \in \{1,2\}$:

$$p_j \;=\; \frac{rate_{\mathrm{o}}(P, init(\phi_j),0)}{rate_{\mathrm{o}}(P, init(\phi_1\vee\phi_2)\cup\{\tau\},0)} \qquad t_j \;=\; t + \left(\frac{1}{rate_{\mathrm{o}}(P, init(\phi_j),0)} - \frac{1}{rate_{\mathrm{o}}(P, init(\phi_1\vee\phi_2)\cup\{\tau\},0)}\right) \qquad \blacksquare$$

In the definition above, p_j represents the probability with which P performs actions whose name is in $init(\phi_j)$ rather than actions whose name is in $init(\phi_k)\cup\{\tau\}$, $k = 3-j$, given that P can perform actions whose name is in $init(\phi_1\vee\phi_2)\cup\{\tau\}$. These probabilities are used as weights for the correct account of the probabilities with which P satisfies only ϕ_1 or ϕ_2 in the context of the satisfaction of $\phi_1\vee\phi_2$. If such weights were omitted, then the fact that $\phi_1\vee\phi_2$ offers a set of initial actions at least as large as the ones offered by ϕ_1 alone and by ϕ_2 alone would be ignored, thus leading to a potential overestimate of the probability of satisfying $\phi_1\vee\phi_2$.

Similarly, t_j represents the extra average time granted to P for satisfying only ϕ_j. This extra average time is equal to the difference between the average sojourn time in P when only actions whose name is in $init(\phi_j)$ are enabled and the average sojourn time in P when also actions whose name is in $init(\phi_k)\cup\{\tau\}$, $k = 3-j$, are enabled. Since the latter cannot be greater than the former due to the race policy – more enabled actions means less time spent on average in a state – considering t instead of t_j in the satisfaction of ϕ_j in isolation would lead to a potential underestimate of the probability of satisfying $\phi_1\vee\phi_2$ within the given average time upper bound, as P may satisfy $\phi_1\vee\phi_2$ within $t\circ\theta$ even if P satisfies neither ϕ_1 nor ϕ_2 taken in isolation within $t\circ\theta$.

Theorem 7.3 $P_1 \sim_{\mathrm{MT}} P_2 \iff \forall\phi\in\mathscr{ML}_{\mathrm{MT}}.\forall\theta\in(\mathbb{R}_{>0})^*.\,[\![\phi]\!]_{\mathrm{MT}}^{|\theta|}(P_1,\theta) = [\![\phi]\!]_{\mathrm{MT}}^{|\theta|}(P_2,\theta)$. ■

8 Verification Algorithm

Markovian testing equivalence can be decided in polynomial time. The reason is that Markovian testing equivalence coincides with Markovian ready equivalence and, given two process terms, their underlying CTMCs in which action names have not been discarded from transition labels are Markovian ready equivalent iff the corresponding embedded DTMCs in which transitions have been labeled with suitably augmented names are related by probabilistic ready equivalence. The latter equivalence is decidable in polynomial time [15] through a reworking of the algorithm for probabilistic language equivalence [18].

Following [19], the transformation of a name-labeled CTMC into the corresponding embedded name-labeled DTMC is carried out by simply turning the rate of each transition into the corresponding execution probability. Then, we need to encode the total exit rate of each state of the original name-labeled CTMC inside the names of all transitions departing from that state in the associated embedded DTMC.

Acknowledgment: This work has been funded by MIUR-PRIN project *PaCo – Performability-Aware Computing: Logics, Models, and Languages*.

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A Appendix

This appendix contains the proofs of the results shown in Sects. 4, 5, 6, 7, and 8. Some of these results are based on a necessary condition for establishing whether two process terms are Markovian testing equivalent, which we now recall after introducing the notion of trace associated with a computation.

Definition A.1 Let $P \in \mathbb{P}$ and $c \in \mathscr{C}_{\mathrm{f}}(P)$. The concrete trace associated with the execution of c is the sequence of action names labeling the transitions of c:

$$\boxed{\mathit{trace}_{\mathrm{c}}(c) \;=\; \begin{cases} \varepsilon & \text{if } |c| = 0 \\ a \circ \mathit{trace}_{\mathrm{c}}(c') & \text{if } c \equiv P \xrightarrow{a,\tilde{\lambda}} c' \end{cases}}$$

where ε is the empty trace. We denote by $\mathit{trace}(c)$ the visible part of $\mathit{trace}_{\mathrm{c}}(c)$, i.e., the subsequence of $\mathit{trace}_{\mathrm{c}}(c)$ obtained by removing all the occurrences of τ. ■

The above mentioned necessary condition requires that for each computation of any of the two terms there exists a computation of the other term with the same concrete trace and stepwise average duration, such that any pair of corresponding states traversed by the two computations have the same overall exit rates with respect to all action names.

Proposition A.2 Let $P_1, P_2 \in \mathbb{P}_{\mathrm{pc}}$. Whenever $P_1 \sim_{\mathrm{MT}} P_2$, then for all $c_k \in \mathscr{C}_{\mathrm{f}}(P_k)$, $k \in \{1,2\}$, there exists $c_h \in \mathscr{C}_{\mathrm{f}}(P_h)$, $h \in \{1,2\} - \{k\}$, such that:

$$\begin{aligned} \mathit{trace}_{\mathrm{c}}(c_k) &= \mathit{trace}_{\mathrm{c}}(c_h) \\ \mathit{time}_{\mathrm{a}}(c_k) &= \mathit{time}_{\mathrm{a}}(c_h) \end{aligned}$$

and for all $a \in \mathit{Name}$ and $i \in \{0, \ldots, |c_k|\}$:

$$\mathit{rate}_{\mathrm{o}}(P_k^i, a, 0) \;=\; \mathit{rate}_{\mathrm{o}}(P_h^i, a, 0)$$

with P_k^i (resp. P_h^i) being the i-th state traversed by c_k (resp. c_h).

Proof A reworking of the proof of Prop. 4.6/Cor. 4.7 of [2] that proceeds by induction on $|c_k|$:

- Let $|c_k| = 0$. Then it trivially exists $c_h \in \mathscr{C}_{\mathrm{f}}(P_h)$ such that:

  $$\begin{aligned} \mathit{trace}_{\mathrm{c}}(c_k) &= \varepsilon = \mathit{trace}_{\mathrm{c}}(c_h) \\ \mathit{time}_{\mathrm{a}}(c_k) &= \varepsilon = \mathit{time}_{\mathrm{a}}(c_h) \end{aligned}$$

  with c_k and c_h being unique and $P_k^0 \equiv P_k \sim_{\mathrm{MT}} P_h \equiv P_h^0$.
  Suppose that for some $a \in \mathit{Name}$:

  $$\mathit{rate}_{\mathrm{o}}(P_k^0, a, 0) \;>\; \mathit{rate}_{\mathrm{o}}(P_h^0, a, 0)$$

  If $a = \tau$, for $T' \equiv \mathrm{s}$ and $\theta' = 1/\mathit{rate}_{\mathrm{o}}(P_k^0, \tau, 0)$ we would have:

  $$\mathit{prob}(\mathscr{SC}^1_{\leq \theta'}(P_k, T')) \;=\; 1 \;\neq\; 0 \;=\; \mathit{prob}(\mathscr{SC}^1_{\leq \theta'}(P_h, T'))$$

  which contradicts $P_k \sim_{\mathrm{MT}} P_h$; hence, it must be $\mathit{rate}_{\mathrm{o}}(P_k^0, \tau, 0) = \mathit{rate}_{\mathrm{o}}(P_h^0, \tau, 0)$.
  If $a \neq \tau$, for $T'' \equiv <a, *_1>.\mathrm{s}$ and $\theta'' = 1/(\mathit{rate}_{\mathrm{o}}(P_k^0, a, 0) + \mathit{rate}_{\mathrm{o}}(P_k^0, \tau, 0))$ we would have:

  $$\mathit{prob}(\mathscr{SC}^1_{\leq \theta''}(P_k, T'')) \;>\; 0 \;=\; \mathit{prob}(\mathscr{SC}^1_{\leq \theta''}(P_h, T''))$$

  which again contradicts $P_k \sim_{\mathrm{MT}} P_h$; hence, it must be $\mathit{rate}_{\mathrm{o}}(P_k^0, a, 0) = \mathit{rate}_{\mathrm{o}}(P_h^0, a, 0)$ also for all $a \in \mathit{Name}_{\mathrm{v}}$.

- Let $|c_k| = n > 0$ and assume that the result holds for all computations in $\mathscr{C}_{\mathrm{f}}(P_k)$ of length less than n. Suppose $c_k \equiv c_k' \xrightarrow{a',\lambda} P_k^n$. Since c_k' belongs to $\mathscr{C}_{\mathrm{f}}(P_k)$ and has length equal to $n-1$, by the induction hypothesis there exists $c_h' \in \mathscr{C}_{\mathrm{f}}(P_h)$ such that:

  $$\begin{aligned} \mathit{trace}_{\mathrm{c}}(c_k') &= \mathit{trace}_{\mathrm{c}}(c_h') \\ \mathit{time}_{\mathrm{a}}(c_k') &= \mathit{time}_{\mathrm{a}}(c_h') \end{aligned}$$

and for all $a \in Name$ and $i \in \{0, \dots, n-1\}$:
$$rate_{\mathrm{o}}(P_k^i, a, 0) \;=\; rate_{\mathrm{o}}(P_h^i, a, 0)$$
As a consequence, we have:
$$\begin{array}{rcl} trace_{\mathrm{c}}(c_k) & = & trace_{\mathrm{c}}(c'_k) \circ a' \;= \\ & = & trace_{\mathrm{c}}(c'_h) \circ a' \;=\; trace_{\mathrm{c}}(c_h) \end{array}$$
and:
$$\begin{array}{rcl} time_{\mathrm{a}}(c_k) & = & time_{\mathrm{a}}(c'_k) \circ \frac{1}{rate_{\mathrm{t}}(P_k^{n-1}, 0)} \;= \\ & = & time_{\mathrm{a}}(c'_h) \circ \frac{1}{rate_{\mathrm{t}}(P_h^{n-1}, 0)} \;=\; time_{\mathrm{a}}(c_h) \end{array}$$
where $c_h \equiv c'_h \xrightarrow{a',\mu} P_h^n$ belongs to $\mathscr{C}_{\mathrm{f}}(P_h)$, otherwise – i.e., if c'_h could not be extended with an a'-transition in P_h – a test whose only trace coincides with $trace(c_k)$ would be enough to distinguish P_k from P_h when considering successful test-driven computations of length n.

It remains to establish whether $rate_{\mathrm{o}}(P_k^n, a, 0) = rate_{\mathrm{o}}(P_h^n, a, 0)$ for all $a \in Name$. Unlike the base case of the induction, c_k and c_h above are not necessarily unique – with respect to their concrete trace, their stepwise average duration, and the overall exit rates of their traversed states except for the last one – in $\mathscr{C}_{\mathrm{f}}(P_k)$ and $\mathscr{C}_{\mathrm{f}}(P_h)$, respectively. Since we are focusing on a specific c_k, we now show that if for each $c_{h,j} \in \mathscr{C}_{\mathrm{f}}(P_h)$ having the same characteristics as c_h above there exists $a_j \in Name$ such that:
$$rate_{\mathrm{o}}(P_k^n, a_j, 0) \;\neq\; rate_{\mathrm{o}}(P_{h,j}^n, a_j, 0)$$
then we can build a test that distinguishes P_k from P_h.

In fact, consider a set of computations of length n with the same concrete trace and stepwise average duration as c_k and c_h above that intersects both $\mathscr{C}_{\mathrm{f}}(P_k)$ and $\mathscr{C}_{\mathrm{f}}(P_h)$. We say that this set is maximal iff it comprises all and only such computations with corresponding states – including the last one – that pairwise enable actions with the same names and have the same total exit rate. Since $P_k \sim_{\mathrm{MT}} P_h$, the computations of a maximal set belonging to $\mathscr{C}_{\mathrm{f}}(P_k)$ have the same probability as the computations of the maximal set belonging to $\mathscr{C}_{\mathrm{f}}(P_h)$, as can be seen by taking a test that enables at each step all the visible actions occurring in P_k and P_h and reaches success only along a trace coinciding with the trace characterizing the maximal set (the successful test-driven computations to consider are those of length n for increasing average time upper bounds of length n).

We also say that a maximal set (of computations of length n with the same concrete trace and stepwise average duration as c_k and c_h above) is rate matching iff for each computation of the set belonging to $\mathscr{C}_{\mathrm{f}}(P_k)$ there exists a computation of the set belonging to $\mathscr{C}_{\mathrm{f}}(P_h)$ such that their corresponding states – including the last one – pairwise have the same overall exit rates with respect to all action names, and vice versa. Since $P_k \sim_{\mathrm{MT}} P_h$, for all $a \in Name$ the probability of performing a computation of a rate-matching maximal set belonging to $\mathscr{C}_{\mathrm{f}}(P_k)$ extended with an a-transition is the same as the probability of performing a computation of the rate-matching maximal set belonging to $\mathscr{C}_{\mathrm{f}}(P_k)$ extended with an a-transition.

After removing every rate-matching maximal set from the sets of computations (belonging to $\mathscr{C}_{\mathrm{f}}(P_k)$ and $\mathscr{C}_{\mathrm{f}}(P_h)$) of length n with the same concrete trace and stepwise average duration as c_k and c_h above, at least one of the two sets C_k and C_h of remaining computations will be nonempty because of the assumption that c_k is not matched by any $c_{h,j}$. There are two cases.

In the first case, there exist some remaining computations in the same set, say C_k, such that the last state of each of them has the same sum r of the overall exit rates with respect to a nonempty subset $\{b_1, \dots, b_m\}$ of $Name$, while the last state of each of the other remaining computations has a lower sum of the overall exit rates with respect to the same action names. If there are several such groups of remaining computations, we take one whose nonempty subset of $Name$ giving rise to r

is minimal. In this case, we build a test T whose only trace coincides with $trace(c_k)$ extended with a choice among m passive transitions labeled with $b_1,\dots,b_m$ each leading to s. If $\tau \in \{b_1,\dots,b_m\}$, then there will be only $m-1$ branches at the end of T.

In the second case, there is no nonempty subset of *Name* giving rise to a maximum sum of overall exit rates in the last state of remaining computations belonging all to C_k or all to C_h. We then take the non-rate-matching maximal set such that the last state of each of its computations has the maximum total exit rate r. If the last state of the computations of several non-rate-matching maximal sets has the same total exit rate r, we take one set in which the last state of its computations enables actions whose set of names is minimal. Let $\{b_1,\dots,b_m\}$ be the set of names of the actions enabled by the last state of each of these computations, where $m > 1$ otherwise the considered maximal set would be rate matching. In this case, we build a test T whose only trace coincides with $trace(c_k)$ extended with a choice among m passive transitions labeled with $b_1,\dots,b_m$ – with the branches being $m-1$ if $\tau \in \{b_1,\dots,b_m\}$ – such that only some of them leads to s, while the others lead to $<\mathrm{z},*_1>$.s. Those leading to s have to be chosen on the basis of the different overall exit rates with respect to $b_1,\dots,b_m$ exhibited by the last state of the computations of the considered non-rate-matching maximal set, so that the one-step extended versions of the computations of the set belonging to, say, C_k get a higher probability than the one-step extended versions of the computations of the set belonging to C_h. The existence of a maximal set allowing for such a choice is guaranteed by the validity of the non-rate-matching property within the maximal set and the absence of a nonempty subset of *Name* giving rise to a maximum sum of overall exit rates in the last state of remaining computations belonging all to C_k or all to C_h.

In each of the two cases, for some suitable θ of length n we would have:

$$prob(\mathscr{SC}^{n+1}_{\leq \theta \circ \frac{1}{r}}(P_k,T)) \;=\; p_k + q_k \;>\; p_h + q_h \;=\; prob(\mathscr{SC}^{n+1}_{\leq \theta \circ \frac{1}{r}}(P_h,T))$$

where $p_h = 0$ in the first case and $q_k, q_h \geq 0$ are the possible contributions of rate-matching maximal sets (whose stepwise average duration does not exceed θ and whose computations have last states such that the sum of their overall exit rates with respect to $b_1,\dots,b_m$ does not exceed r), with $q_k = q_h$ by virtue of $P_k \sim_{\mathrm{MT}} P_h$. Since the above inequality contradicts $P_k \sim_{\mathrm{MT}} P_h$, for at least one $c_{h,j}$ it must be:

$$rate_{\mathrm{o}}(P^n_k, a, 0) \;=\; rate_{\mathrm{o}}(P^n_{h,j}, a, 0)$$

for all $a \in Name$. ■

Corollary A.3 Let $P_1, P_2 \in \mathbb{P}_{\mathrm{pc}}$. Whenever $P_1 \sim_{\mathrm{MT}} P_2$, then for all $a \in Name$:

$$rate_{\mathrm{o}}(P_1, a, 0) \;=\; rate_{\mathrm{o}}(P_2, a, 0)$$ ■

The condition expressed in Prop. A.2 is necessary but not sufficient. The following two process terms:

$$<a,\lambda_1>.<b,\mu>.\underline{0} + <a,\lambda_2>.<c,\gamma>.\underline{0}$$
$$<a,\lambda'_1>.<b,\mu>.\underline{0} + <a,\lambda'_2>.<c,\gamma>.\underline{0}$$

satisfy the condition when $\lambda_1 + \lambda_2 = \lambda'_1 + \lambda'_2$, but are not Markovian testing equivalent if $\lambda_1 \neq \lambda'_1$, $\lambda_2 \neq \lambda'_2$, and $b \neq c$ or $\mu \neq \gamma$.

A.1 Proofs of Results of Sect. 4

Proof of Prop. 4.1. Let us preliminarily observe that, given $P \in \mathbb{P}_{\mathrm{pc,v}}$, $T \in \mathbb{T}_{\mathrm{R}}$, and $\theta \in (\mathbb{R}_{>0})^*$, due to the absence of actions of the form $<\tau,\lambda>$ within P, every successful test-driven computation is maximal, i.e., it cannot be further extended. Denoting by $\theta|n$ the prefix of θ of length n, it thus holds:

$$prob(\mathscr{SC}_{\leq \theta}(P,T)) \;=\; \sum_{n=0}^{|\theta|} prob(\mathscr{SC}^n_{\leq \theta|n}(P,T))$$

Given $P_1, P_2 \in \mathbb{P}_{\mathrm{pc,v}}$, we now proceed in two steps:

- $\Rightarrow$ Supposing $P_1 \sim_{\mathrm{MT}} P_2$, from the initial observation we immediately derive that for all $T \in \mathbb{T}_{\mathrm{R}}$ and $\theta \in (\mathbb{R}_{>0})^*$:

$$\begin{aligned} prob(\mathcal{SC}_{\le\theta}(P_1,T)) &= \sum_{n=0}^{|\theta|} prob(\mathcal{SC}^n_{\le\theta|n}(P_1,T)) = \\ &= \sum_{n=0}^{|\theta|} prob(\mathcal{SC}^n_{\le\theta|n}(P_2,T)) = prob(\mathcal{SC}_{\le\theta}(P_2,T)) \end{aligned}$$

  hence $P_1 \sim_{\mathrm{MT,old}} P_2$.

- $\Leftarrow$ Supposing $P_1 \sim_{\mathrm{MT,old}} P_2$, let us prove that for all $T \in \mathbb{T}_{\mathrm{R}}$ and $\theta \in (\mathbb{R}_{>0})^*$:

$$prob(\mathcal{SC}^{|\theta|}_{\le\theta}(P_1,T)) = prob(\mathcal{SC}^{|\theta|}_{\le\theta}(P_2,T))$$

  by proceeding by induction on $|\theta|$:

  - Let $|\theta| = 0$. From the initial observation and $P_1 \sim_{\mathrm{MT,old}} P_2$, we immediately derive:

$$\begin{aligned} prob(\mathcal{SC}^0_{\le\varepsilon}(P_1,T)) &= prob(\mathcal{SC}_{\le\varepsilon}(P_1,T)) = \\ &= prob(\mathcal{SC}_{\le\varepsilon}(P_2,T)) = prob(\mathcal{SC}^0_{\le\varepsilon}(P_2,T)) \end{aligned}$$

  - Let $|\theta| = n > 0$ and assume that for all $m = 0, \ldots, n-1$:

$$prob(\mathcal{SC}^m_{\le\theta|m}(P_1,T)) = prob(\mathcal{SC}^m_{\le\theta|m}(P_2,T))$$

    from which it follows:

$$\sum_{m=0}^{n-1} prob(\mathcal{SC}^m_{\le\theta|m}(P_1,T)) = \sum_{m=0}^{n-1} prob(\mathcal{SC}^m_{\le\theta|m}(P_2,T))$$

    From the initial observation and $P_1 \sim_{\mathrm{MT,old}} P_2$, we then obtain:

$$\begin{aligned} prob(\mathcal{SC}^{|\theta|}_{\le\theta}(P_1,T)) &= prob(\mathcal{SC}_{\le\theta}(P_1,T)) - \sum_{m=0}^{n-1} prob(\mathcal{SC}^m_{\le\theta|m}(P_1,T)) = \\ &= prob(\mathcal{SC}_{\le\theta}(P_2,T)) - \sum_{m=0}^{n-1} prob(\mathcal{SC}^m_{\le\theta|m}(P_2,T)) = prob(\mathcal{SC}^{|\theta|}_{\le\theta}(P_2,T)) \end{aligned}$$

■

Proof of Prop. 4.2. We preliminarily observe that for all $P \in \mathbb{P}_{\mathrm{pc}}$, $T \in \mathbb{T}_{\mathrm{R}} - \{\mathrm{s}\}$, and $\theta \in (\mathbb{R}_{>0})^*$ it holds:

$$prob(\mathcal{SC}^{|\theta|}_{\le\theta}(P,\mathrm{s}+T)) = prob(\mathcal{SC}^{|\theta|}_{\le\theta}(P,\mathrm{s})) + prob(\mathcal{SC}^{|\theta|}_{\le\theta}(P,T))$$

In fact, for $|\theta| = 0$ we have $prob(\mathcal{SC}^{|\theta|}_{\le\theta}(P,\mathrm{s}+T)) = prob(\mathcal{SC}^{|\theta|}_{\le\theta}(P,\mathrm{s})) = 1$ and $prob(\mathcal{SC}^{|\theta|}_{\le\theta}(P,T)) = 0$, while for $|\theta| > 0$ we have $prob(\mathcal{SC}^{|\theta|}_{\le\theta}(P,\mathrm{s})) = 0$ and $prob(\mathcal{SC}^{|\theta|}_{\le\theta}(P,\mathrm{s}+T)) = prob(\mathcal{SC}^{|\theta|}_{\le\theta}(P,T))$.

Given $P_1, P_2 \in \mathbb{P}_{\mathrm{pc}}$, we now proceed in two steps:

- $\Rightarrow$ It follows from the fact that $\mathbb{T}_{\mathrm{R}} \subset \mathbb{T}_{\mathrm{R,lib}}$.

- $\Leftarrow$ In order to avoid trivial cases, we consider a test $T \in \mathbb{T}_{\mathrm{R,lib}} - \mathbb{T}_{\mathrm{R}}$ and we derive from it a set of tests $break_{\mathrm{s}}(T) \subseteq \mathbb{T}_{\mathrm{R}}$ by proceeding by induction on the syntactical structure of T as follows:

$$break_{\mathrm{s}}(T) = \begin{cases} \{\mathrm{s}\} & \text{if } T \equiv \mathrm{s} \\ \{<a,*_w>.T'' \mid T'' \in break_{\mathrm{s}}(T')\} & \text{if } T \equiv <a,*_w>.T' \\ break_{\mathrm{s}}(T') \cup \{\mathrm{s}\} & \text{if } T \equiv T' + \mathrm{s} \text{ or } T \equiv \mathrm{s} + T' \\ \{T'_1 + T'_2 \mid T'_1 \in break_{\mathrm{s}}(T_1) - \{\mathrm{s}\}, & \text{if } T \equiv T_1 + T_2 \text{ and } T_1 \not\equiv \mathrm{s} \not\equiv T_2 \text{ and} \\ \qquad T'_2 \in break_{\mathrm{s}}(T_2) - \{\mathrm{s}\}\} \cup \{\mathrm{s}\} & \quad \mathrm{s} \in break_{\mathrm{s}}(T_1) \cup break_{\mathrm{s}}(T_2) \\ \{T'_1 + T'_2 \mid T'_1 \in break_{\mathrm{s}}(T_1), & \text{if } T \equiv T_1 + T_2 \text{ and} \\ \qquad T'_2 \in break_{\mathrm{s}}(T_2)\} & \quad \mathrm{s} \notin break_{\mathrm{s}}(T_1) \cup break_{\mathrm{s}}(T_2) \end{cases}$$

From the initial observation and $P_1 \sim_{\rm MT} P_2$, it follows that for all $\theta \in (\mathbb{R}_{>0})^*$ it holds:

$$\begin{aligned} prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_1, T)) &= \sum_{T' \in break_{\rm s}(T)} prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_1, T')) = \\ &= \sum_{T' \in break_{\rm s}(T)} prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_2, T')) = prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_2, T)) \end{aligned}$$

from which we can conclude that $P_1 \sim_{\rm MT,lib} P_2$. ■

Proof of Prop. 4.3. We proceed in two steps:

- $\Rightarrow$ It follows from the fact that $\mathbb{T}_{\rm R} \subset \mathbb{T}_{{\rm R},\tau}$.
- $\Leftarrow$ In order to avoid trivial cases, we consider a test $T \in \mathbb{T}_{{\rm R},\tau} - \mathbb{T}_{\rm R}$ and we derive from it a test $remove_\tau(T) \in \mathbb{T}_{\rm R,lib}$ by proceeding by induction on the syntactical structure of T as follows:

$$remove_\tau(T) = \begin{cases} {\rm s} & \text{if } T \equiv {\rm s} \\ <a, *_w>.remove_\tau(T') & \text{if } T \equiv <a, *_w>.T' \text{ (with } a \in Name_{\rm v}) \\ remove_\tau(T') & \text{if } T \equiv <\tau, \lambda>.T' \\ remove_\tau(T_1) + remove_\tau(T_2) & \text{if } T \equiv T_1 + T_2 \end{cases}$$

Since $remove_\tau(T) \in \mathbb{T}_{\rm R,lib}$, from the proof of Prop. 4.2 and $P_1 \sim_{\rm MT} P_2$ we obtain that for all $\theta \in (\mathbb{R}_{>0})^*$ it holds:

$$\begin{aligned} prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_1, remove_\tau(T))) &= \sum_{T' \in break_{\rm s}(remove_\tau(T))} prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_1, T')) = \\ &= \sum_{T' \in break_{\rm s}(remove_\tau(T))} prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_2, T')) = prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_2, remove_\tau(T))) \end{aligned}$$

Due to $P_1 \sim_{\rm MT} P_2$, for each $T' \in break_{\rm s}(remove_\tau(T))$ we have:

$$\begin{aligned} prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_1, T')) &= \sum_{c' \in \mathcal{C}_{\rm s}(T')} prob(\mathcal{ESC}^{|\theta|}_{\leq\theta}(P_1, T', c')) = \\ &= \sum_{c' \in \mathcal{C}_{\rm s}(T')} prob(\mathcal{ESC}^{|\theta|}_{\leq\theta}(P_2, T', c')) = prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_2, T')) \end{aligned}$$

where $\mathcal{C}_{\rm s}(T')$ is the multiset of computations of T' ending with s and $\mathcal{ESC}(P_k, T', c')$, $k \in \{1, 2\}$, is the multiset of successful computations of P_k driven by T' that exercise c'. Moreover, for each $c' \in \mathcal{C}_{\rm s}(T')$ we have:

$$prob(\mathcal{ESC}^{|\theta|}_{\leq\theta}(P_1, T', c')) = prob(\mathcal{ESC}^{|\theta|}_{\leq\theta}(P_2, T', c'))$$

When moving from T to $remove_\tau(T)$, for all $c \in \mathcal{C}_{\rm s}(T)$ it holds that the computations of P_k, $k \in \{1, 2\}$, exercising c are the same as those exercising $remove_\tau(c)$, which is the computation obtained from c by removing all of its τ-transitions. Therefore, it is possible to establish a bijective correspondence between $\mathcal{ESC}(P_k, T, c)$ and $\mathcal{ESC}(P_k, remove_\tau(T), remove_\tau(c))$ and hence with $\mathcal{ESC}(P_k, T', remove_\tau(c))$ for each $T' \in break_{\rm s}(remove_\tau(T))$ such that $remove_\tau(c) \in \mathcal{C}_{\rm s}(T')$.

Since $P_1 \sim_{\rm MT} P_2$, by virtue of Prop. A.2 we have that for each computation of P_1 (resp. P_2) exercising $remove_\tau(c)$ there exists a computation of P_2 (resp. P_1) exercising $remove_\tau(c)$, such that both computations have the same concrete trace, have the same stepwise average duration, and traverse states having pairwise the same overall exit rates with respect to the various action names.

As a consequence, when from $remove_\tau(c)$ we go back to c, i.e., when we reintroduce the exponentially timed τ-actions that had been removed, each of these actions comes into play in a pair of corresponding states of P_1 and P_2 having the same overall exit rates with respect to the various action names. No reintroduced test action of the form $<\tau, \lambda>$ can discriminate between those two states, because it does not disable any of their actions but simply increases their total exit rates – which are equal – by the same value λ.

Given $T' \in \mathit{break}_{\mathrm{s}}(\mathit{remove}_{\tau}(T))$ such that $\mathit{remove}_{\tau}(c) \in \mathscr{C}_{\mathrm{s}}(T')$, from:

$$\mathit{prob}(\mathscr{E}\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P_1, T', \mathit{remove}_{\tau}(c))) \,=\, \mathit{prob}(\mathscr{E}\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P_2, T', \mathit{remove}_{\tau}(c)))$$

and the bijective correspondence between $\mathscr{E}\mathscr{S}\mathscr{C}(P_k, T, c)$ and $\mathscr{E}\mathscr{S}\mathscr{C}(P_k, T', \mathit{remove}_{\tau}(c))$, $k \in \{1, 2\}$, it then follows:

$$\mathit{prob}(\mathscr{E}\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P_1, T, c)) \,=\, \mathit{prob}(\mathscr{E}\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P_2, T, c))$$

and hence:

$$\begin{aligned} \mathit{prob}(\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P_1, T)) \;&=\; \sum_{c \in \mathscr{C}_{\mathrm{s}}(T)} \mathit{prob}(\mathscr{E}\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P_1, T, c)) \;=\; \\ &=\; \sum_{c \in \mathscr{C}_{\mathrm{s}}(T)} \mathit{prob}(\mathscr{E}\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P_2, T, c)) \;=\; \mathit{prob}(\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P_2, T)) \end{aligned}$$

from which we can conclude that $P_1 \sim_{\mathrm{MT},\tau} P_2$. ■

As regards the first of the two further alternative characterizations coming from [2], we have the following. When $P \in \mathbb{P}_{\mathrm{pc}}$, the stepwise duration of c is defined as the sequence of random variables quantifying the sojourn times in the states traversed by c:

$$\boxed{\mathit{time}_{\mathrm{d}}(c) \,=\, \begin{cases} \varepsilon & \text{if } |c| = 0 \\ \mathit{Exp}_{\mathit{rate}_{\mathrm{t}}(P,0)} \circ \mathit{time}_{\mathrm{d}}(c') & \text{if } c \equiv P \xrightarrow{a,\lambda} c' \end{cases}}$$

where ε is the empty stepwise duration while $\mathit{Exp}_{\mathit{rate}_{\mathrm{t}}(P,0)}$ is the exponentially distributed random variable with rate $\mathit{rate}_{\mathrm{t}}(P, 0) \in \mathbb{R}_{>0}$. We also define the probability distribution of executing a computation in $C \subseteq \mathscr{C}_{\mathrm{f}}(P)$ within a sequence $\theta \in (\mathbb{R}_{>0})^*$ of time units as:

$$\boxed{\mathit{prob}_{\mathrm{d}}(C, \theta) \,=\, \sum_{c \in C}^{|c| \leq |\theta|} \mathit{prob}(c) \cdot \prod_{i=1}^{|c|} \Pr\{\mathit{time}_{\mathrm{d}}(c)[i] \leq \theta[i]\}}$$

whenever C is finite and all of its computations are independent of each other. In the definition above, $\Pr\{\mathit{time}_{\mathrm{d}}(c)[i] \leq \theta[i]\} = 1 - \mathrm{e}^{-\theta[i]/\mathit{time}_{\mathrm{a}}(c)[i]}$ is the cumulative distribution function of the exponentially distributed random variable $\mathit{time}_{\mathrm{d}}(c)[i]$, whose expected value is $\mathit{time}_{\mathrm{a}}(c)[i]$.

Definition A.4 Let $P_1, P_2 \in \mathbb{P}_{\mathrm{pc}}$. We say that P_1 is Markovian distribution-testing equivalent to P_2, written $P_1 \sim_{\mathrm{MT,d}} P_2$, iff for all reactive tests $T \in \mathbb{T}_{\mathrm{R}}$ and sequences $\theta \in (\mathbb{R}_{>0})^*$ of amounts of time:

$$\mathit{prob}_{\mathrm{d}}(\mathscr{S}\mathscr{C}^{|\theta|}(P_1, T), \theta) \,=\, \mathit{prob}_{\mathrm{d}}(\mathscr{S}\mathscr{C}^{|\theta|}(P_2, T), \theta)$$

■

Proposition A.5 Let $P_1, P_2 \in \mathbb{P}_{\mathrm{pc}}$. Whenever $P_1 \sim_{\mathrm{MT,d}} P_2$, then for all $c_k \in \mathscr{C}_{\mathrm{f}}(P_k)$, $k \in \{1, 2\}$, there exists $c_h \in \mathscr{C}_{\mathrm{f}}(P_h)$, $h \in \{1, 2\} - \{k\}$, such that:

$$\begin{aligned} \mathit{trace}_{\mathrm{c}}(c_k) \;&=\; \mathit{trace}_{\mathrm{c}}(c_h) \\ \mathit{time}_{\mathrm{d}}(c_k) \;&=\; \mathit{time}_{\mathrm{d}}(c_h) \end{aligned}$$

and for all $a \in \mathit{Name}$ and $i \in \{0, \ldots, |c_k|\}$:

$$\mathit{rate}_{\mathrm{o}}(P_k^i, a, 0) \,=\, \mathit{rate}_{\mathrm{o}}(P_h^i, a, 0)$$

with P_k^i (resp. P_h^i) being the i-th state traversed by c_k (resp. c_h).

Proof Similar to the proof of Prop. A.2. ■

Proposition A.6 Let $P_1, P_2 \in \mathbb{P}_{\rm pc}$. Then $P_1 \sim_{\rm MT,d} P_2 \iff P_1 \sim_{\rm MT} P_2$.

Proof A reworking of the proof of Thm. 4.20/Cor. 4.21 of [2] that proceeds as follows. Recalled that $time_{\rm a}(\_)[.]$ is the expected value of random variable $time_{\rm d}(\_)[.]$, given $T \in \mathbb{T}_{\rm r}$ and $\theta \in (\mathbb{R}_{>0})^*$ the result follows from the fact that the first (resp. last) equality below implies all the subsequent (resp. preceding) ones under $P_1 \sim_{\rm MT,d} P_2$ (resp. $P_1 \sim_{\rm MT} P_2$):

$$\begin{aligned}
\sum_{c_1 \in \mathscr{SC}^{|\theta|}(P_1,T)} prob(c_1) \cdot \prod_{i=1}^{|\theta|} \Pr\{time_{\rm d}(c_1)[i] \le \theta[i]\} &= \sum_{c_2 \in \mathscr{SC}^{|\theta|}(P_2,T)} prob(c_2) \cdot \prod_{i=1}^{|\theta|} \Pr\{time_{\rm d}(c_2)[i] \le \theta[i]\} \\
\sum_{c_1 \in \mathscr{SC}^{|\theta|}(P_1,T)} prob(c_1) \cdot \prod_{i=1}^{|\theta|} \tfrac{{\rm d}\Pr\{time_{\rm d}(c_1)[i] \le \theta[i]\}}{{\rm d}\theta[i]} &= \sum_{c_2 \in \mathscr{SC}^{|\theta|}(P_2,T)} prob(c_2) \cdot \prod_{i=1}^{|\theta|} \tfrac{{\rm d}\Pr\{time_{\rm d}(c_2)[i] \le \theta[i]\}}{{\rm d}\theta[i]} \\
\sum_{c_1 \in \mathscr{SC}^{|\theta|}(P_1,T)} prob(c_1) \cdot \prod_{i=1}^{|\theta|} \theta[i] \cdot \tfrac{{\rm d}\Pr\{time_{\rm d}(c_1)[i] \le \theta[i]\}}{{\rm d}\theta[i]} &= \sum_{c_2 \in \mathscr{SC}^{|\theta|}(P_2,T)} prob(c_2) \cdot \prod_{i=1}^{|\theta|} \theta[i] \cdot \tfrac{{\rm d}\Pr\{time_{\rm d}(c_2)[i] \le \theta[i]\}}{{\rm d}\theta[i]} \\
\sum_{c_1 \in \mathscr{SC}^{|\theta|}(P_1,T)} prob(c_1) \cdot \prod_{i=1}^{|\theta|} \int_0^\infty \theta[i] \cdot \tfrac{{\rm d}\Pr\{time_{\rm d}(c_1)[i] \le \theta[i]\}}{{\rm d}\theta[i]}\, {\rm d}\theta[i] &= \sum_{c_2 \in \mathscr{SC}^{|\theta|}(P_2,T)} prob(c_2) \cdot \prod_{i=1}^{|\theta|} \int_0^\infty \theta[i] \cdot \tfrac{{\rm d}\Pr\{time_{\rm d}(c_2)[i] \le \theta[i]\}}{{\rm d}\theta[i]}\, {\rm d}\theta[i] \\
\sum_{c_1 \in \mathscr{SC}^{|\theta|}(P_1,T)} prob(c_1) \cdot \prod_{i=1}^{|\theta|} time_{\rm a}(c_1)[i] &= \sum_{c_2 \in \mathscr{SC}^{|\theta|}(P_2,T)} prob(c_2) \cdot \prod_{i=1}^{|\theta|} time_{\rm a}(c_2)[i] \\
\sum_{c_1 \in \mathscr{SC}^{|\theta|}(P_1,T)} prob(c_1) \cdot \prod_{i=1}^{|\theta|} \Pr\{time_{\rm a}(c_1)[i] \le \theta[i]\} &= \sum_{c_2 \in \mathscr{SC}^{|\theta|}(P_2,T)} prob(c_2) \cdot \prod_{i=1}^{|\theta|} \Pr\{time_{\rm a}(c_2)[i] \le \theta[i]\} \\
\sum_{c_1 \in \mathscr{SC}^{|\theta|}_{\le \theta}(P_1,T)} prob(c_1) &= \sum_{c_2 \in \mathscr{SC}^{|\theta|}_{\le \theta}(P_2,T)} prob(c_2)
\end{aligned}$$

In fact, supposing that all the computations in $\mathscr{SC}^{|\theta|}(P_1,T)$ and in $\mathscr{SC}^{|\theta|}(P_2,T)$ with the same duration are counted only once with their total probability – which implies that $\mathscr{SC}^{|\theta|}(P_1,T)$ and $\mathscr{SC}^{|\theta|}(P_2,T)$ can be viewed as sets rather than multisets – the double implication is established below in two steps:

$\Rightarrow$ Assume that the computations (with different durations) in $\mathscr{SC}^{|\theta|}(P_1,T)$ and in $\mathscr{SC}^{|\theta|}(P_2,T)$ are such that the products of the corresponding $|\theta|$ elements in the first equality – which expresses $P_1 \sim_{\rm MT,d} P_2$ – are all different. If this were not the case, without loss of generality we could focus on any two maximal subsets of $\mathscr{SC}^{|\theta|}(P_1,T)$ and $\mathscr{SC}^{|\theta|}(P_2,T)$ satisfying this constraint.
Then the first equality is of the form:

$$\sum_{j=1}^{n} p_{1,j} \cdot D_{1,j}(\theta) \;=\; \sum_{j=1}^{n} p_{2,j} \cdot D_{2,j}(\theta)$$

where:

- $p_{1,j}, p_{2,j} \in \mathbb{R}_{]0,1]}$ for $1 \le j \le n$.
- $\sum_{j=1}^{n} p_{1,j} \le 1$ and $\sum_{j=1}^{n} p_{2,j} \le 1$.
- $D_{1,j}$ and $D_{2,j}$ are strictly increasing nonlinear continuous functions from $(\mathbb{R}_{>0})^*$ to $\mathbb{R}_{]0,1]}$ for $1 \le j \le n$.
- All functions $D_{1,j}$'s are different from each other.
- All functions $D_{2,j}$'s are different from each other.
- Due to Prop. A.5, $D_{1,j} = D_{2,j} \equiv D_j$ for $1 \le j \le n$ because $P_1 \sim_{\rm MT,d} P_2$.

Therefore, if we rewrite the form of the first equality as follows:

$$\sum_{j=1}^{n} (p_{1,j} - p_{2,j}) \cdot D_j(\theta) \;=\; 0$$

we get a homogeneous linear system composed of uncountably many equations whose unknowns are the $(p_{1,j} - p_{2,j})$'s. Since the values belonging to the j-th column, $1 \leq j \leq n$, of the coefficient matrix of the system are all positive and taken from the strictly increasing nonlinear function D_j, with such D_j's being all different from each other, the rows of the coefficient matrix are all linearly independent. Thus, the system admits only the solution zero, i.e., $p_{1,j} = p_{2,j}$ for $1 \leq j \leq n$. As a consequence, when proving the implication of the equalities from the first one to the sixth one, we can exploit the fact that "=" is a congruence with respect to addition and multiplication, which allows us to substitute equals for equals within the same context "$\sum prob(\_) \cdot \prod$" of both sides of the equalities.

From the sixth equality, it then follows the last one, i.e., $P_1 \sim_{\mathrm{MT}} P_2$, because for $k \in \{1,2\}$ it holds:

$$\prod_{i=1}^{|\theta|} \Pr\{time_{\mathrm{a}}(c_k)[i] \leq \theta[i]\} \;=\; \begin{cases} 1 & \text{if } c_k \in \mathscr{SC}^{|\theta|}_{\leq \theta}(P_k, T) \\ 0 & \text{if } c_k \notin \mathscr{SC}^{|\theta|}_{\leq \theta}(P_k, T) \end{cases}$$

$\Leftarrow$ Using the same argument as the one at the end of the previous step, we derive that $P_1 \sim_{\mathrm{MT}} P_2$, i.e., the last equality, implies the sixth one. The latter equality is of the form:

$$\sum_{j=1}^{n} p_{1,j} \cdot v_{1,j}(\theta) \;=\; \sum_{j=1}^{n} p_{2,j} \cdot v_{2,j}(\theta)$$

where:

- $p_{1,j}, p_{2,j} \in \mathbb{R}_{]0,1]}$ for $1 \leq j \leq n$.
- $\sum_{j=1}^{n} p_{1,j} \leq 1$ and $\sum_{j=1}^{n} p_{2,j} \leq 1$.
- $v_{1,j}$ and $v_{2,j}$ are functions from $(\mathbb{R}_{>0})^*$ to $\{0,1\}$ for $1 \leq j \leq n$.
- Due to Prop. A.2, $v_{1,j} = v_{2,j} \equiv v_j$ for $1 \leq j \leq n$ because $P_1 \sim_{\mathrm{MT}} P_2$.

Therefore, if we rewrite the form of the sixth equality as follows:

$$\sum_{j=1}^{n} (p_{1,j} - p_{2,j}) \cdot v_j(\theta) \;=\; 0$$

and we choose increasing values of θ, we obtain $p_{1,j} = p_{2,j}$ for $1 \leq j \leq n$. As a consequence, when proving the implication of the equalities from the sixth one to the first one – which expresses $P_1 \sim_{\mathrm{MT,d}} P_2$ – we can exploit the fact that "=" is a congruence with respect to addition and multiplication, which allows us to substitute equals for equals within the same context "$\sum prob(\_) \cdot \prod$" of both sides of the equalities. ∎

We now address the second of the two further alternative characterizations coming from [2].

Definition A.7 An element ξ of $(Name_{\mathrm{v}} \times 2^{Name_{\mathrm{v}}})^*$ is an extended trace iff:

- either ξ is the empty sequence ε;
- or $\xi \equiv (a_1, \mathscr{E}_1) \circ (a_2, \mathscr{E}_2) \circ \ldots \circ (a_n, \mathscr{E}_n)$ for some $n \in \mathbb{N}_{>0}$, with $a_i \in \mathscr{E}_i$ and $\mathscr{E}_i$ finite for each $i = 1, \ldots, n$.

We denote by $\mathscr{ET}$ the set of extended traces. ∎

Definition A.8 Let $\xi \in \mathscr{ET}$. The trace associated with ξ is defined by induction on the length of ξ through the following $(Name_{\mathrm{v}})^*$-valued function:

$$\boxed{trace_{\mathrm{et}}(\xi) \;=\; \begin{cases} \varepsilon & \text{if } |\xi| = 0 \\ a \circ trace_{\mathrm{et}}(\xi') & \text{if } \xi \equiv (a, \mathscr{E}) \circ \xi' \end{cases}}$$

where ε is the empty trace. ∎

Definition A.9 Let $P \in \mathbb{P}_{\mathrm{pc}}$, $c \in \mathscr{C}_{\mathrm{f}}(P)$, and $\xi \in \mathscr{E}\mathscr{T}$. We say that c is compatible with ξ iff:

$$trace(c) \;=\; trace_{\mathrm{et}}(\xi)$$

We denote by $\mathscr{C}\mathscr{C}(P,\xi)$ the multiset of computations in $\mathscr{C}_{\mathrm{f}}(P)$ that are compatible with ξ. ■

Given $P \in \mathbb{P}_{\mathrm{pc}}$, $\xi \in \mathscr{E}\mathscr{T}$, and $c \in \mathscr{C}\mathscr{C}(P,\xi)$, we have to consider the probability and the duration of c with respect to ξ, which are defined by taking into account the action names permitted at each step by ξ. The probability of executing c with respect to ξ is defined as:

$$\boxed{prob_{\xi}(c) \;=\; \begin{cases} 1 & \text{if } |c| = 0 \\ \frac{\lambda}{rate_{\mathrm{o}}(P,\mathscr{E}\cup\{\tau\},0)} \cdot prob_{\xi'}(c') & \text{if } c \equiv P \xrightarrow{a,\lambda} c' \\ & \quad \text{with } \xi \equiv (a,\mathscr{E}) \circ \xi' \\ \frac{\lambda}{rate_{\mathrm{o}}(P,\mathscr{E}\cup\{\tau\},0)} \cdot prob_{\xi}(c') & \text{if } c \equiv P \xrightarrow{\tau,\lambda} c' \\ & \quad \text{with } \xi \equiv (a,\mathscr{E}) \circ \xi' \\ \frac{\lambda}{rate_{\mathrm{o}}(P,\tau,0)} \cdot prob_{\xi}(c') & \text{if } c \equiv P \xrightarrow{\tau,\lambda} c' \wedge \xi \equiv \varepsilon \end{cases}}$$

We also define the probability of executing a computation in $C \subseteq \mathscr{C}\mathscr{C}(P,\xi)$ with respect to ξ as:

$$\boxed{prob_{\xi}(C) \;=\; \sum_{c \in C} prob_{\xi}(c)}$$

whenever C is finite and all of its computations are independent of each other.
The stepwise average duration of c with respect to ξ is defined as:

$$\boxed{time_{\mathrm{a},\xi}(c) \;=\; \begin{cases} \varepsilon & \text{if } |c| = 0 \\ \frac{1}{rate_{\mathrm{o}}(P,\mathscr{E}\cup\{\tau\},0)} \circ time_{\mathrm{a},\xi'}(c') & \text{if } c \equiv P \xrightarrow{a,\lambda} c' \\ & \quad \text{with } \xi \equiv (a,\mathscr{E}) \circ \xi' \\ \frac{1}{rate_{\mathrm{o}}(P,\mathscr{E}\cup\{\tau\},0)} \circ time_{\mathrm{a},\xi}(c') & \text{if } c \equiv P \xrightarrow{\tau,\lambda} c' \\ & \quad \text{with } \xi \equiv (a,\mathscr{E}) \circ \xi' \\ \frac{1}{rate_{\mathrm{o}}(P,\tau,0)} \circ time_{\mathrm{a},\xi}(c') & \text{if } c \equiv P \xrightarrow{\tau,\lambda} c' \wedge \xi \equiv \varepsilon \end{cases}}$$

where ε is the empty stepwise average duration. We also define the multiset of computations in $C \subseteq \mathscr{C}\mathscr{C}(P,\xi)$ whose stepwise average duration with respect to ξ is not greater than $\theta \in (\mathbb{R}_{>0})^*$ as:

$$\boxed{C_{\leq\theta,\xi} \;=\; \{\!| c \in C \mid |c| \leq |\theta| \wedge \forall i = 1,\dots,|c|.\, time_{\mathrm{a},\xi}(c)[i] \leq \theta[i] \,|\!\}}$$

Moreover, as before we denote by C^l the multiset of computations in $C \subseteq \mathscr{C}\mathscr{C}(P,\xi)$ whose length is equal to $l \in \mathbb{N}$.

Definition A.10 Let $P_1, P_2 \in \mathbb{P}_{\mathrm{pc}}$. We say that P_1 is Markovian extended-trace equivalent to P_2, written $P_1 \sim_{\mathrm{MTr,e}} P_2$, iff for all extended traces $\xi \in \mathscr{E}\mathscr{T}$ and sequences $\theta \in (\mathbb{R}_{>0})^*$ of average amounts of time:

$$prob_{\xi}(\mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta,\xi}(P_1,\xi)) \;=\; prob_{\xi}(\mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta,\xi}(P_2,\xi))$$

■

Theorem A.11 Let $P_1, P_2 \in \mathbb{P}_{\rm pc}$. Then $P_1 \sim_{\rm MTr,e} P_2 \iff P_1 \sim_{\rm MT} P_2$.

Proof A reworking of the proof of Thm. 4.35 of [2] that proceeds in two steps:

$\Rightarrow$ Let us denote by $\mathscr{C}_{\rm s}(T)$ the multiset of computations of $T \in \mathbb{T}_{\rm R}$ ending with s and by $\mathscr{ESC}(P, T, c)$ the multiset of successful computations of $P \in \mathbb{P}_{\rm pc}$ driven by T that exercise $c \in \mathscr{C}_{\rm s}(T)$. Given an arbitrary $\theta \in (\mathbb{R}_{>0})^*$, for $k \in \{1, 2\}$ it holds:

$$prob(\mathscr{SC}^{|\theta|}_{\leq \theta}(P_k, T)) \, = \sum_{c \in \mathscr{C}_{\rm s}(T)} prob(\mathscr{ESC}^{|\theta|}_{\leq \theta}(P_k, T, c))$$

Let us define the extended trace associated with $c \in \mathscr{C}_{\rm s}(T)$ as follows:

$$trace_{\rm e}(c) \, = \begin{cases} \varepsilon & \text{if } |c| = 0 \\ (a, \{b \in Name_{\rm v} \mid weight(T, b) > 0\}) \circ trace_{\rm e}(c') & \text{if } c \equiv T \xrightarrow{a, *_w} c' \end{cases}$$

We now prove that for all $\theta \in (\mathbb{R}_{>0})^*$, $P \in \mathbb{P}_{\rm pc}$, $T \in \mathbb{T}_{\rm R}$, and $c \in \mathscr{C}_{\rm s}(T)$:

$$prob(\mathscr{ESC}^{|\theta|}_{\leq \theta}(P, T, c)) \, = \, prob_{\rm r}(c) \cdot prob_{trace_{\rm e}(c)}(\mathscr{CC}^{|\theta|}_{\leq \theta, trace_{\rm e}(c)}(P, trace_{\rm e}(c)))$$

where the reactive probability of c is defined as follows:

$$prob_{\rm r}(c) \, = \begin{cases} 1 & \text{if } |c| = 0 \\ \frac{w}{weight(T, a)} \cdot prob_{\rm r}(c') & \text{if } c \equiv T \xrightarrow{a, *_w} c' \end{cases}$$

There are two cases:

- Let $|c| = 0$. Then $c \equiv T \equiv$ s, $trace_{\rm e}(c) \equiv \varepsilon$, and $prob_{\rm r}(c) = 1$. There are two subcases. If $|\theta| = 0$, then:

  $$prob(\mathscr{ESC}^{|\theta|}_{\leq \theta}(P, T, c)) \, = 1 = \, prob_{\rm r}(c) \cdot prob_{trace_{\rm e}(c)}(\mathscr{CC}^{|\theta|}_{\leq \theta, trace_{\rm e}(c)}(P, trace_{\rm e}(c)))$$

  If $|\theta| > 0$, then the only computations that exercise c or are compatible with $trace_{\rm e}(c)$ are those composed of τ-transitions performed by P, hence:

  $$prob(\mathscr{ESC}^{|\theta|}_{\leq \theta}(P, T, c)) \, = \, prob_{\rm r}(c) \cdot prob_{trace_{\rm e}(c)}(\mathscr{CC}^{|\theta|}_{\leq \theta, trace_{\rm e}(c)}(P, trace_{\rm e}(c)))$$

- Let $|c| > 0$ with $c \equiv T \xrightarrow{a, *_w} c'$, $a \in Name_{\rm v}$, and T' being the first state of c'. Let us proceed by induction on $|\theta|$:

  * Let $|\theta| = 0$. Then $\mathscr{ESC}^{|\theta|}_{\leq \theta}(P, T, c) = \emptyset = \mathscr{CC}^{|\theta|}_{\leq \theta, trace_{\rm e}(c)}(P, trace_{\rm e}(c))$ and hence:

    $$prob(\mathscr{ESC}^{|\theta|}_{\leq \theta}(P, T, c)) \, = 0 = \, prob_{\rm r}(c) \cdot prob_{trace_{\rm e}(c)}(\mathscr{CC}^{|\theta|}_{\leq \theta, trace_{\rm e}(c)}(P, trace_{\rm e}(c)))$$

  * Let $|\theta| > 0$ with $\theta \equiv t \circ \theta'$, $t \in \mathbb{R}_{>0}$, and assume that for all $\bar{P} \in \mathbb{P}_{\rm pc}$, $\bar{T} \in \mathbb{T}_{\rm R}$, and $\bar{c} \in \mathscr{C}_{\rm s}(\bar{T})$:

    $$prob(\mathscr{ESC}^{|\theta'|}_{\leq \theta'}(\bar{P}, \bar{T}, \bar{c})) \, = \, prob_{\rm r}(\bar{c}) \cdot prob_{trace_{\rm e}(\bar{c})}(\mathscr{CC}^{|\theta'|}_{\leq \theta', trace_{\rm e}(\bar{c})}(\bar{P}, trace_{\rm e}(\bar{c})))$$

    For some $r \in \mathbb{R}_{\geq 0}$, it holds:

    $$rate_{\rm t}(P \parallel_{Name_{\rm v}} T, 0) \, = \, r \, = \sum_{b \in \mathscr{E} \cup \{\tau\}} rate_{\rm o}(P, b, 0)$$

    where $\mathscr{E} = \{b \in Name_{\rm v} \mid weight(T, b) > 0\}$. There are two subcases. If $r = 0$ or $\frac{1}{r} > t$, then:

    $$prob(\mathscr{ESC}^{|\theta|}_{\leq \theta}(P, T, c)) \, = 0 = \, prob_{\rm r}(c) \cdot prob_{trace_{\rm e}(c)}(\mathscr{CC}^{|\theta|}_{\leq \theta, trace_{\rm e}(c)}(P, trace_{\rm e}(c)))$$

    If $r > 0$ and $\frac{1}{r} \leq t$, then:

    $$\begin{aligned} prob(\mathscr{ESC}^{|\theta|}_{\leq \theta}(P, T, c)) \, = & \sum_{P \xrightarrow{a, \lambda} P'} \frac{\lambda \cdot w / weight(T, a)}{r} \cdot prob(\mathscr{ESC}^{|\theta'|}_{\leq \theta'}(P', T', c')) \\ & + \sum_{P \xrightarrow{\tau, \lambda} P'} \frac{\lambda}{r} \cdot prob(\mathscr{ESC}^{|\theta'|}_{\leq \theta'}(P', T, c)) \end{aligned}$$

From the induction hypothesis, it follows:

$$\begin{aligned}
prob(\mathscr{E}\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P,T,c)) \;=\;& \sum_{P \xrightarrow{a,\lambda} P'} \frac{\lambda}{r} \cdot \frac{w}{weight(T,a)} \cdot prob_{\mathrm{r}}(c') \cdot prob_{trace_{\mathrm{e}}(c')}(\mathscr{C}\mathscr{C}^{|\theta'|}_{\leq\theta',trace_{\mathrm{e}}(c')}(P',trace_{\mathrm{e}}(c'))) \\
&+ \sum_{P \xrightarrow{\tau,\lambda} P'} \frac{\lambda}{r} \cdot prob_{\mathrm{r}}(c) \cdot prob_{trace_{\mathrm{e}}(c)}(\mathscr{C}\mathscr{C}^{|\theta'|}_{\leq\theta',trace_{\mathrm{e}}(c)}(P',trace_{\mathrm{e}}(c))) \;= \\
=\;& \sum_{P \xrightarrow{a,\lambda} P'} \frac{\lambda}{r} \cdot prob_{\mathrm{r}}(c) \cdot prob_{trace_{\mathrm{e}}(c')}(\mathscr{C}\mathscr{C}^{|\theta'|}_{\leq\theta',trace_{\mathrm{e}}(c')}(P',trace_{\mathrm{e}}(c'))) \\
&+ \sum_{P \xrightarrow{\tau,\lambda} P'} \frac{\lambda}{r} \cdot prob_{\mathrm{r}}(c) \cdot prob_{trace_{\mathrm{e}}(c)}(\mathscr{C}\mathscr{C}^{|\theta'|}_{\leq\theta',trace_{\mathrm{e}}(c)}(P',trace_{\mathrm{e}}(c))) \;= \\
=\;& prob_{\mathrm{r}}(c) \cdot (\sum_{P \xrightarrow{a,\lambda} P'} \frac{\lambda}{r} \cdot prob_{trace_{\mathrm{e}}(c')}(\mathscr{C}\mathscr{C}^{|\theta'|}_{\leq\theta',trace_{\mathrm{e}}(c')}(P',trace_{\mathrm{e}}(c'))) \\
&+ \sum_{P \xrightarrow{\tau,\lambda} P'} \frac{\lambda}{r} \cdot prob_{trace_{\mathrm{e}}(c)}(\mathscr{C}\mathscr{C}^{|\theta'|}_{\leq\theta',trace_{\mathrm{e}}(c)}(P',trace_{\mathrm{e}}(c)))) \;= \\
=\;& prob_{\mathrm{r}}(c) \cdot prob_{trace_{\mathrm{e}}(c)}(\mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta,trace_{\mathrm{e}}(c)}(P,trace_{\mathrm{e}}(c)))
\end{aligned}$$

Since $P_1 \sim_{\mathrm{MTr,e}} P_2$, for all $\theta \in (\mathbb{R}_{>0})^*$ and $c \in \mathscr{C}_{\mathrm{s}}(T)$, $T \in \mathbb{T}_{\mathrm{R}}$, we have:

$$prob_{trace_{\mathrm{e}}(c)}(\mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta,trace_{\mathrm{e}}(c)}(P_1,trace_{\mathrm{e}}(c))) \;=\; prob_{trace_{\mathrm{e}}(c)}(\mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta,trace_{\mathrm{e}}(c)}(P_2,trace_{\mathrm{e}}(c)))$$

As a consequence, it holds:

$$prob(\mathscr{E}\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P_1,T,c)) \;=\; prob(\mathscr{E}\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P_2,T,c))$$

from which it follows:

$$prob(\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P_1,T)) \;=\; prob(\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P_2,T))$$

and hence $P_1 \sim_{\mathrm{MT}} P_2$.

$\Leftarrow$ Let us define the test associated with $\xi \in \mathscr{E}\mathscr{T}$ as follows:

$$test(\xi) \triangleq \begin{cases} \mathrm{s} & \text{if } |\xi| = 0 \\ <a,*_1>.test(\xi') + \sum\limits_{b\in\mathscr{E}-\{a\}} <b,*_1>.<\mathrm{z},*_1>.\mathrm{s} & \text{if } \xi \equiv (a,\mathscr{E}) \circ \xi' \end{cases}$$

where the summation is absent whenever $\mathscr{E} = \{a\}$ and z is a visible action name representing failure that can occur within tests but not within process terms under test. We denote by $\mathbb{T}_{\mathrm{R,c}}$ the resulting set of tests.

We now prove that for all $\theta \in (\mathbb{R}_{>0})^*$, $P \in \mathbb{P}_{\mathrm{pc}}$, and $\xi \in \mathscr{E}\mathscr{T}$:

$$prob(\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P,test(\xi))) \;=\; prob_{\xi}(\mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta,\xi}(P,\xi))$$

There are two cases:

- Let $|\xi| = 0$. Then $\xi \equiv \varepsilon$ and $test(\xi) \equiv \mathrm{s}$. There are two subcases.
  If $|\theta| = 0$, then:
  $$prob(\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P,test(\xi))) \;=1=\; prob_{\xi}(\mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta,\xi}(P,\xi))$$
  If $|\theta| > 0$, then the only computations that exercise $test(\xi)$ or are compatible with ξ are those composed of τ-transitions performed by P, hence:
  $$prob(\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P,test(\xi))) \;=\; prob_{\xi}(\mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta,\xi}(P,\xi))$$
- Let $|\xi| > 0$ with $\xi \equiv (a,\mathscr{E}) \circ \xi'$, $a \in Name_{\mathrm{v}}$. Let us proceed by induction on $|\theta|$:
  * Let $|\theta| = 0$. Then $\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P,test(\xi)) = \emptyset = \mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta,\xi}(P,\xi)$ and hence:
    $$prob(\mathscr{S}\mathscr{C}^{|\theta|}_{\leq\theta}(P,test(\xi))) \;=0=\; prob_{\xi}(\mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta,\xi}(P,\xi))$$
  * Let $|\theta| > 0$ with $\theta \equiv t \circ \theta'$, $t \in \mathbb{R}_{>0}$, and assume that for all $\bar{P} \in \mathbb{P}_{\mathrm{pc}}$ and $\bar{\xi} \in \mathscr{E}\mathscr{T}$:
    $$prob(\mathscr{S}\mathscr{C}^{|\theta'|}_{\leq\theta'}(\bar{P},test(\bar{\xi}))) \;=\; prob_{\bar{\xi}}(\mathscr{C}\mathscr{C}^{|\theta'|}_{\leq\theta',\bar{\xi}}(\bar{P},\bar{\xi}))$$

For some $r \in \mathbb{R}_{\ge 0}$, it holds:
$$rate_{\rm t}(P \,\|_{Name_{\rm v}}\, test(\xi), 0) \;=\; r \;=\; \sum_{b \in \mathscr{E} \cup \{\tau\}} rate_{\rm o}(P, b, 0)$$
There are two subcases.
If $r = 0$ or $\frac{1}{r} > t$, then:
$$prob(\mathscr{SC}^{|\theta|}_{\le \theta}(P, test(\xi))) \;= 0 =\; prob_{\xi}(\mathscr{CC}^{|\theta|}_{\le \theta, \xi}(P, \xi))$$
If $r > 0$ and $\frac{1}{r} \le t$, then:
$$\begin{aligned} prob(\mathscr{SC}^{|\theta|}_{\le \theta}(P, test(\xi))) \;&=\; \sum_{P \xrightarrow{a, \lambda} P'} \frac{\lambda}{r} \cdot prob(\mathscr{SC}^{|\theta'|}_{\le \theta'}(P', test(\xi'))) \\ &\quad + \sum_{P \xrightarrow{\tau, \lambda} P'} \frac{\lambda}{r} \cdot prob(\mathscr{SC}^{|\theta'|}_{\le \theta'}(P', test(\xi))) \end{aligned}$$
From the induction hypothesis, it follows:
$$\begin{aligned} prob(\mathscr{SC}^{|\theta|}_{\le \theta}(P, test(\xi))) \;&=\; \sum_{P \xrightarrow{a, \lambda} P'} \frac{\lambda}{r} \cdot prob_{\xi'}(\mathscr{CC}^{|\theta'|}_{\le \theta', \xi'}(P', \xi')) \\ &\quad + \sum_{P \xrightarrow{\tau, \lambda} P'} \frac{\lambda}{r} \cdot prob_{\xi}(\mathscr{CC}^{|\theta'|}_{\le \theta', \xi}(P', \xi)) \;= \\ &=\; prob_{\xi}(\mathscr{CC}^{|\theta|}_{\le \theta, \xi}(P, \xi)) \end{aligned}$$

Since $P_1 \sim_{\rm MT} P_2$, for all $T \in \mathbb{T}_{\rm R,c}$ and $\theta \in (\mathbb{R}_{>0})^*$ we have:
$$prob(\mathscr{SC}^{|\theta|}_{\le \theta}(P_1, T)) \;=\; prob(\mathscr{SC}^{|\theta|}_{\le \theta}(P_2, T))$$
which is equivalent to say that for all $\xi \in \mathscr{ET}$ and $\theta \in (\mathbb{R}_{>0})^*$:
$$prob(\mathscr{SC}^{|\theta|}_{\le \theta}(P_1, test(\xi))) \;=\; prob(\mathscr{SC}^{|\theta|}_{\le \theta}(P_2, test(\xi)))$$
As a consequence, it holds:
$$prob_{\xi}(\mathscr{CC}^{|\theta|}_{\le \theta, \xi}(P_1, \xi)) \;=\; prob_{\xi}(\mathscr{CC}^{|\theta|}_{\le \theta, \xi}(P_2, \xi))$$
and hence $P_1 \sim_{\rm MTr,e} P_2$. ■

Corollary A.12 Let $P_1, P_2 \in \mathbb{P}_{\rm pc}$. Then $P_1 \sim_{\rm MT} P_2$ iff for all $T \in \mathbb{T}_{\rm R,c}$ and $\theta \in (\mathbb{R}_{>0})^*$:
$$prob(\mathscr{SC}^{|\theta|}_{\le \theta}(P_1, T)) \;=\; prob(\mathscr{SC}^{|\theta|}_{\le \theta}(P_2, T))$$
■

A.2 Proofs of Results of Sect. 5

Proof of Thm. 5.1. In the case of the action prefix operator and of the alternative composition operator, it is a reworking of the proof of Thm. 4.53 of [2] that proceeds as follows:

- Assuming $P_1 \sim_{\rm MT} P_2$ for $P_1, P_2 \in \mathbb{P}_{\rm pc}$, let us demonstrate that for all $<a, \lambda> \in Act$, $T \in \mathbb{T}_{\rm R}$, and $\theta \in (\mathbb{R}_{>0})^*$:
$$prob(\mathscr{SC}^{|\theta|}_{\le \theta}(<a, \lambda>.P_1, T)) \;=\; prob(\mathscr{SC}^{|\theta|}_{\le \theta}(<a, \lambda>.P_2, T))$$
There are two cases.
  - Let $T \equiv {\rm s}$. Then there are two subcases.
If $|\theta| = 0$, then:
$$prob(\mathscr{SC}^{|\theta|}_{\le \theta}(<a, \lambda>.P_1, T)) \;= 1 =\; prob(\mathscr{SC}^{|\theta|}_{\le \theta}(<a, \lambda>.P_2, T))$$
If $|\theta| > 0$ with $\theta \equiv t \circ \theta'$, $t \in \mathbb{R}_{>0}$, we have two further subcases.
    * If $a \in Name_{\rm v}$ or $\frac{1}{\lambda} > t$, then:
$$prob(\mathscr{SC}^{|\theta|}_{\le \theta}(<a, \lambda>.P_1, T)) \;= 0 =\; prob(\mathscr{SC}^{|\theta|}_{\le \theta}(<a, \lambda>.P_2, T))$$

- If $a = \tau$ and $\frac{1}{\lambda} \leq t$, then:
  $$\begin{aligned} prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(<a,\lambda>.P_1,T)) &= prob(\mathcal{SC}^{|\theta'|}_{\leq\theta'}(P_1,T)) = \\ &= prob(\mathcal{SC}^{|\theta'|}_{\leq\theta'}(P_2,T)) = prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(<a,\lambda>.P_2,T)) \end{aligned}$$
  because $P_1 \sim_{\mathrm{MT}} P_2$.

– Let $T \not\equiv \mathrm{s}$. Then there are two subcases.
If $|\theta| = 0$, then:
$$prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(<a,\lambda>.P_1,T)) = 0 = prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(<a,\lambda>.P_2,T))$$
If $|\theta| > 0$ with $\theta \equiv t \circ \theta'$, $t \in \mathbb{R}_{>0}$, we have three further subcases:

- If $\frac{1}{\lambda} > t$ or $a \in Name_{\mathrm{v}}$ and $weight(T,a) = 0$, then:
  $$prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(<a,\lambda>.P_1,T)) = 0 = prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(<a,\lambda>.P_2,T))$$
- If $\frac{1}{\lambda} \leq t$ and $a \in Name_{\mathrm{v}}$ and $weight(T,a) > 0$, then:
  $$\begin{aligned} prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(<a,\lambda>.P_1,T)) &= \sum_{T \xrightarrow{a,*_w} T'} \frac{w}{weight(T,a)} \cdot prob(\mathcal{SC}^{|\theta'|}_{\leq\theta'}(P_1,T')) = \\ &= \sum_{T \xrightarrow{a,*_w} T'} \frac{w}{weight(T,a)} \cdot prob(\mathcal{SC}^{|\theta'|}_{\leq\theta'}(P_2,T')) = prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(<a,\lambda>.P_2,T)) \end{aligned}$$
  because $P_1 \sim_{\mathrm{MT}} P_2$.
- If $\frac{1}{\lambda} \leq t$ and $a = \tau$, then:
  $$\begin{aligned} prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(<a,\lambda>.P_1,T)) &= prob(\mathcal{SC}^{|\theta'|}_{\leq\theta'}(P_1,T)) = \\ &= prob(\mathcal{SC}^{|\theta'|}_{\leq\theta'}(P_2,T)) = prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(<a,\lambda>.P_2,T)) \end{aligned}$$
  because $P_1 \sim_{\mathrm{MT}} P_2$.

• Assuming $P_1 \sim_{\mathrm{MT}} P_2$ for $P_1, P_2 \in \mathbb{P}_{\mathrm{pc}}$, let us demonstrate that for all $P \in \mathbb{P}_{\mathrm{pc}}$, $T \in \mathbb{T}_{\mathrm{R}}$, and $\theta \in (\mathbb{R}_{>0})^*$:
$$prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_1 + P,T)) = prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_2 + P,T))$$
There are two cases:

– Let $T \equiv \mathrm{s}$ and $|\theta| = 0$. Then:
$$prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_1 + P,T)) = 1 = prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_2 + P,T))$$

– Let $T \not\equiv \mathrm{s}$ or $|\theta| > 0$. For $k \in \{1,2\}$, we have:
$$prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_k + P,T)) = \begin{cases} p_k \cdot prob(\mathcal{SC}^{|\theta_k|}_{\leq\theta_k}(P_k,T)) + p'_k \cdot prob(\mathcal{SC}^{|\theta'_k|}_{\leq\theta'_k}(P,T)) & \text{if } r_k > 0 \wedge r > 0 \\ prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P_k,T)) & \text{if } r_k > 0 \wedge r = 0 \\ prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P,T)) & \text{if } r_k = 0 \wedge r > 0 \\ 0 & \text{if } r_k = 0 \wedge r = 0 \end{cases}$$
where:
$$\begin{aligned} r_k &= rate_{\mathrm{t}}(P_k \,\|_{Name_{\mathrm{v}}}\, T, 0) & r &= rate_{\mathrm{t}}(P \,\|_{Name_{\mathrm{v}}}\, T, 0) \\ p_k &= \tfrac{r_k}{r_k + r} & p'_k &= \tfrac{r}{r_k + r} \\ \theta_k[i] &= \begin{cases} \theta[i] + (\frac{1}{r_k} - \frac{1}{r_k + r}) & \text{if } i = 1 \\ \theta[i] & \text{if } i > 1 \end{cases} & \theta'_k[i] &= \begin{cases} \theta[i] + (\frac{1}{r} - \frac{1}{r_k + r}) & \text{if } i = 1 \\ \theta[i] & \text{if } i > 1 \end{cases} \end{aligned}$$
In fact, in the case in which both $P_k \,\|_{Name_{\mathrm{v}}}\, T$ and $P \,\|_{Name_{\mathrm{v}}}\, T$ can perform at least one action – deriving from a synchronization with T or from a τ-action of the subprocess under test – p_k (resp. p'_k) is the probability that the first transition in a successful T-driven computation is originated by P_k (resp. P). Similarly, θ_k (resp. θ'_k) takes into account the extra average time that is available to P_k (resp. P) in the context of $P_k + P$ when executing the first transition of a successful T-driven computation.

Since $P_1 \sim_{\rm MT} P_2$, from Cor. A.3 we derive that $r_1 = r_2$ and hence $p_1 = p_2$, $p'_1 = p'_2$, $\theta_1 = \theta_2$, and $\theta'_1 = \theta'_2$. From $P_1 \sim_{\rm MT} P_2$ and $\theta_1 = \theta_2$, we also derive:

$$\begin{aligned} prob(\mathcal{SC}^{|\theta_1|}_{\le \theta_1}(P_1, T)) &= prob(\mathcal{SC}^{|\theta_2|}_{\le \theta_2}(P_2, T)) \\ prob(\mathcal{SC}^{|\theta|}_{\le \theta}(P_1, T)) &= prob(\mathcal{SC}^{|\theta|}_{\le \theta}(P_2, T)) \end{aligned}$$

As a consequence:

$$prob(\mathcal{SC}^{|\theta|}_{\le \theta}(P_1 + P, T)) = prob(\mathcal{SC}^{|\theta|}_{\le \theta}(P_2 + P, T))$$

With regard to the three static operators, assuming $P_1 \sim_{\rm MT} P_2$ for $P_1, P_2 \in \mathbb{P}_{\rm pc}$ we proceed as follows:

- In the case of the parallel composition operator, the congruence result follows from the fact that, for $k \in \{1,2\}$ and for all $P \in \mathbb{P}$ and $S \subseteq Name_{\rm v}$ such that $P_k \,\|_S\, P \in \mathbb{P}_{\rm pc}$, $T \in \mathbb{T}_{\rm R}$, and $\theta \in (\mathbb{R}_{>0})^*$, $prob(\mathcal{SC}^{|\theta|}_{\le \theta}(P_k \,\|_S\, P, T))$ is equal to $prob(\mathcal{SC}^{|\theta|}_{\le \theta}(P_k, \overline{T}))$ for some $\overline{T} \in \mathbb{T}_{\rm R}$ derived from T.
  To this purpose, we observe that $P_k \,\|_S\, P$ can synchronize with T on $a \in Name_{\rm v}$ iff:
  - either $a \notin S$ and P_k can synchronize with T on a or P can synchronize (through an exponentially timed action) with T on a;
  - or $a \in S$ and P_k can synchronize with T on a and P can synchronize (through a passive action) with T on a.

  With respect to the set of actions enabled by P_k and its derivatives, the context $\_ \,\|_S\, P$ can:
  - restrict the set by disabling exponentially timed actions on P_k side whose name belongs to S; restrictions can be managed in $\overline{T}$ by simply introducing suitable passive actions representing failure, whose visible name z cannot occur in process terms under test;
  - enlarge the set by enabling exponentially timed actions on P side whose name does not belong to S; no such enlargement will ever be caught by the derivatives of P_k that do not enable those actions, unless we introduce suitable exponentially timed τ-actions within the derivatives of $\overline{T}$ that can be reached after executing at most $|\theta|$ actions.

  Restrictions and enlargements of the set of actions enabled by P_k and its derivatives are dealt with by placing T in the context $\_ \,\|_S\, P$ through a family of functions $combine_{S,P,n}(.)$ yielding $\overline{T}$. For $|\theta| = n = 0$, we let $combine_{S,P,n}(T) = T$, whereas for $|\theta| = n > 0$ we have three cases:
  - If $T \equiv {\rm s}$, then:
  $$combine_{S,P,n}(T) \;=\; {\rm s} + \sum_{P \xrightarrow{\tau,\lambda} P'} <\tau, \lambda>.combine_{S,P',n-1}({\rm s})$$
  where the second summand is absent if P cannot execute exponentially timed τ-actions.
  - If $T \not\equiv {\rm s}$ and T can perform actions whose names do not belong to S, or P can perform exponentially timed visible actions whose names do not belong to S such that T can perform passive actions with the same names, or both P and T can perform passive actions with the same names belonging to S, or P can execute exponentially timed τ-actions, then:
  $$\begin{aligned} combine_{S,P,n}(T) \;=\; & \sum_{a \notin S \wedge T \xrightarrow{a,*_w} T'} <a, *_w>.combine_{S,P,n-1}(T') \;+ \\ & \sum_{a \notin S \cup \{\tau\} \wedge P \xrightarrow{a,\lambda} P' \wedge T \xrightarrow{a,*_w} T'} <\tau, \lambda \cdot \frac{w}{weight(T,a)}>.combine_{S,P',n-1}(T') \;+ \\ & \sum_{a \in S \wedge P \xrightarrow{a,*_v} P' \wedge T \xrightarrow{a,*_w} T'} <a, *_{norm(v,w,a,P,T)}>.combine_{S,P',n-1}(T') \;+ \\ & \sum_{P \xrightarrow{\tau,\lambda} P'} <\tau, \lambda>.combine_{S,P',n-1}(T) \end{aligned}$$

– If $T \not\equiv \mathrm{s}$ and none of the conditions of the previous case hold (i.e., if T can perform only actions whose names belong to S with P enabling no passive visible actions with those names and P cannot execute exponentially timed τ-actions), then:

$$combine_{S,P,n}(T) \, = \, <\mathrm{z}, *_1>.\mathrm{s}$$

As can be noted, subterms of the resulting test $\overline{T}$ equivalent to T may belong to $\mathbb{T}_{\mathrm{R,lib}} \cup \mathbb{T}_{\mathrm{R},\tau}$, but at that point we can exploit the proofs of Props. 4.2 and 4.3 to derive a set of tests that is equivalent to $\overline{T}$, each element of which belongs to $\mathbb{T}_{\mathrm{R}}$.

- In the case of the hiding operator, the congruence result follows from the fact that, for $k \in \{1,2\}$ and for all $H \subseteq Name_{\mathrm{v}}$, $T \in \mathbb{T}_{\mathrm{R}}$, and $\theta \in (\mathbb{R}_{>0})^*$, $prob(\mathscr{SC}^{|\theta|}_{\leq \theta}(P_k/H, T))$ is equal to $prob(\mathscr{SC}^{|\theta|}_{\leq \theta}(P_k, \overline{T}))$ for some $\overline{T} \in \mathbb{T}_{\mathrm{R}}$ derived from T.
To this purpose, on the test side we observe that every computation of T that comprises an action whose name belongs to H cannot be exercised by P_k/H, and hence cannot lead P_k/H to success. When deriving $\overline{T}$, that action must not be removed from the test – in order not to alter the quantitative T-driven behavior of P_k with respect to the quantitative T-driven behavior of P_k/H, in which the τ-action corresponding to that action can be executed anyway – but must lead P_k to fail immediately afterwards – which is achieved by introducing a passive action whose visible name z cannot occur inside processes under test. Test $\overline{T}$ is thus yielded by a family of functions $hide'_H(.)$ defined by induction on the syntactical structure of T as follows:

$$hide'_H(T) \, = \, \begin{cases} \mathrm{s} & \text{if } T \equiv \mathrm{s} \\ <a, *_w>.<\mathrm{z}, *_1>.\mathrm{s} & \text{if } T \equiv <a, *_w>.T' \text{ and } a \in H \\ <a, *_w>.hide'_H(T') & \text{if } T \equiv <a, *_w>.T' \text{ and } a \notin H \\ hide'_H(T_1) + hide'_H(T_2) & \text{if } T \equiv T_1 + T_2 \end{cases}$$

On the process side, we observe that each derivative of $hide'_H(T)$ different from $<\mathrm{z}, *_1>.\mathrm{s}$ that does not enable actions whose names belong to H may erroneously block a derivative of P_k enabling some of those actions, in the sense that the corresponding derivative of T would not block the corresponding derivative of P_k/H as the latter would move autonomously by performing τ-actions. When building $\overline{T}$, we thus need to extend each such derivative of $hide'_H(T)$ by offering all the possible sequences of length at most $|\theta|$ of actions whose names belong to H, with each such action followed by the derivative itself. More precisely, we define a family of functions $hide''_{H',n}(.)$ with H' being the set of names in H for which there are no enabled actions. For $H' = \emptyset$ or $|\theta| = n = 0$ or a derivative T' of $hide'_H(T)$ equal to $<\mathrm{z}, *_1>.\mathrm{s}$, we let $hide''_{H',n}(T') = T'$, otherwise:

$$hide''_{H',n}(T') \, = \, T' + \sum_{a \in H'} <a, *_1>.hide''_{H',n-1}(T')$$

Since s does not enable any action whose name is in H, the resulting test $\overline{T}$ equivalent to T may belong to $\mathbb{T}_{\mathrm{R,lib}}$, but at that point we can exploit the proof of Prop. 4.2 to derive a set of tests that is equivalent to $\overline{T}$, each element of which belongs to $\mathbb{T}_{\mathrm{R}}$.

- In the case of the relabeling operator, the congruence result follows from the fact that, for $k \in \{1,2\}$ and for all $\varphi \in Relab$, $T \in \mathbb{T}_{\mathrm{R}}$, and $\theta \in (\mathbb{R}_{>0})^*$, $prob(\mathscr{SC}^{|\theta|}_{\leq \theta}(P_k[\varphi], T))$ is equal to $prob(\mathscr{SC}^{|\theta|}_{\leq \theta}(P_k, \overline{T}))$ for some $\overline{T} \in \mathbb{T}_{\mathrm{R}}$ derived from T.
In fact, observed that $P_k[\varphi]$ can perform $<a, \lambda>$ iff P_k can perform $<b, \lambda>$ for $b \in \varphi^{-1}(a)$, we have that $P_k[\varphi]$ can synchronize with T on $a \in Name_{\mathrm{v}}$ iff P_k can synchronize with $unrelabel_\varphi(T)$ on $b \in \varphi^{-1}(a)$, where $unrelabel_\varphi(T)$ – which yields $\overline{T}$ – is defined by induction on the syntactical

structure of T as follows:

$$unrelabel_{\varphi}(T) = \begin{cases} \text{s} & \text{if } T \equiv \text{s} \\ <\text{z}, *_1>.\text{s} & \text{if } T \equiv <a, *_w>.T' \text{ and } \varphi^{-1}(a) = \emptyset \\ \sum_{b \in \varphi^{-1}(a)} <b, *_w>.unrelabel_{\varphi}(T') & \text{if } T \equiv <a, *_w>.T' \text{ and } \varphi^{-1}(a) \neq \emptyset \\ unrelabel_{\varphi}(T_1) + unrelabel_{\varphi}(T_2) & \text{if } T \equiv T_1 + T_2 \end{cases}$$

where z is a visible action name that cannot occur inside process terms under test. ■

A.3 Proofs of Results of Sect. 6

Proof of Thm. 6.1. A reworking of the proofs of Thms. 4.54 and 4.57 of [2] that proceeds in two steps:

$\Rightarrow$ Since $\sim_{\text{MT}}$ is an equivalence relation and a congruence with respect to all the operators of MPC, in any deduction based on $\mathscr{A}_{\text{MT}}$ it is correct to use reflexivity, symmetry, transitivity, and substitutivity with respect to all the operators of MPC.
As far as the set of specific axioms is concerned, apart from $\mathscr{A}_{\text{MT},4}$ it is trivial to prove their soundness with respect to $\sim_{\text{MT}}$. In particular, we observe that the five summands on the right-hand side of $\mathscr{A}_{\text{MT},5}$ are in full accordance with the operational semantic rules for the parallel composition operator.
With regard to $\mathscr{A}_{\text{MT},4}$, it suffices to observe what follows:

- Both terms occurring in $\mathscr{A}_{\text{MT},4}$ can initially execute only a-actions.
- The average time to execute them is $1 / \sum_{k \in I} \lambda_k$ in both terms.
- If $J_i = \emptyset$ for all $i \in I$, then the a-derivative term is $\underline{0}$ with probability 1 both on the left and on the right, so no test can distinguish between the two original terms.
- If $J_i \neq \emptyset$ for all $i \in I$, then the a-derivative term is $\sum_{j \in J_i} <b_{i,j}, \mu_{i,j}>.P_{i,j}$ with probability $\lambda_i / \sum_{k \in I} \lambda_k$ on the left, while it is $\sum_{i \in I} \sum_{j \in J_i} <b_{i,j}, \lambda_i / \sum_{k \in I} \lambda_k \cdot \mu_{i,j}>.P_{i,j}$ with probability 1 on the right. Not even at this point can a test make a distinction for the following reasons:
  - ∗ All the a-derivative terms can initially execute the same set of action names $\{b_1, \dots, b_n\}$, where the b_h-actions, $1 \leq h \leq n$, have the same total rate μ_h in all the a-derivative terms.
  - ∗ The a-b_h-derivative term is $P_{i,j}$ with the same probability $(\lambda_i / \sum_{k \in I} \lambda_k) \cdot (\mu_{i,j} / \sum_{1 \leq h \leq n} \mu_h)$ both on the left and on the right.
  - ∗ Since the b_h-actions have the same total rate μ_h in all the a-derivative terms, the denominator of the second fraction above changes in the same way on the left and on the right depending on the actions that are enabled by a specific test.

$\Leftarrow$ We say that a nonrecursive process term $P \in \mathbb{P}$ is in testing-minimal sum normal form (tmsnf) iff:

- either $P \equiv \underline{0}$;
- or $P \equiv \sum_{i \in I} <a_i, \tilde{\lambda}_i>.P_i$ with I finite and nonempty, P initially minimal with respect to $\mathscr{A}_{\text{MT},4}$, and P_i in tmsnf for all $i \in I$.

By initial minimality of P with respect to $\mathscr{A}_{\text{MT},4}$, we mean that no subset of summands of P matches the left-hand side term of $\mathscr{A}_{\text{MT},4}$. From the definition, it follows that the initial minimality holds with respect to $\mathscr{A}_{\text{MT},3}$ as well.
We also introduce the size of a nonrecursive process term as an upper bound to the length of its longest computation, which is inductively defined as follows:

$$\begin{array}{rcl}
size(\underline{0}) & = & 0 \\
size(<a,\tilde{\lambda}>.P) & = & 1+size(P) \\
size(P_1+P_2) & = & \max(size(P_1),size(P_2)) \\
size(P_1 \parallel_S P_2) & = & size(P_1)+size(P_2) \\
size(P/L) & = & size(P) \\
size(P[\varphi]) & = & size(P)
\end{array}$$

We now prove that for each nonrecursive process term $P \in \mathbb{P}$ there exists a nonrecursive process term $Q \in \mathbb{P}$ in tmsnf such that $\mathscr{A}_{\mathrm{MT}} \vdash P = Q$, by proceeding by induction on the syntactical structure of P:

- If $P \equiv \underline{0}$, the result follows by taking $Q \equiv \underline{0}$ (which is in tmsnf) and using reflexivity.
- If $P \equiv <a,\tilde{\lambda}>.P'$, then by the induction hypothesis there exists Q' in tmsnf such that $\mathscr{A}_{\mathrm{MT}} \vdash P' = Q'$. From substitutivity with respect to action prefix, we obtain that $\mathscr{A}_{\mathrm{MT}} \vdash <a,\tilde{\lambda}>.P' = <a,\tilde{\lambda}>.Q'$, from which the result follows as $<a,\tilde{\lambda}>.Q'$ is in tmsnf.
- If $P \equiv P_1 + P_2$, then by the induction hypothesis there exist Q_1 and Q_2 in tmsnf such that $\mathscr{A}_{\mathrm{MT}} \vdash P_1 = Q_1$ and $\mathscr{A}_{\mathrm{MT}} \vdash P_2 = Q_2$. From substitutivity with respect to alternative composition, we obtain that $\mathscr{A}_{\mathrm{MT}} \vdash P_1 + P_2 = Q_1 + Q_2$. There are two cases.
  If $Q_1 + Q_2$ is in tmsnf, then we are done.
  If $Q_1 + Q_2$ is not in tmsnf (because it is not initially minimal with respect to $\mathscr{A}_{\mathrm{MT},3}$ or $\mathscr{A}_{\mathrm{MT},4}$), the result follows after as many applications of $\mathscr{A}_{\mathrm{MT},3}$ and $\mathscr{A}_{\mathrm{MT},4}$ as needed – possibly preceded by applications of $\mathscr{A}_{\mathrm{MT},1}$ and $\mathscr{A}_{\mathrm{MT},2}$ – by virtue of substitutivity with respect to alternative composition as well as transitivity.
- If $P \equiv P_1 \parallel_S P_2$, then by the induction hypothesis there exist Q_1 and Q_2 in tmsnf such that $\mathscr{A}_{\mathrm{MT}} \vdash P_1 = Q_1$ and $\mathscr{A}_{\mathrm{MT}} \vdash P_2 = Q_2$. From substitutivity with respect to alternative composition, we obtain that $\mathscr{A}_{\mathrm{MT}} \vdash P_1 \parallel_S P_2 = Q_1 \parallel_S Q_2$. Let us proceed by induction on $size(Q_1 \parallel_S Q_2)$:
  * If $size(Q_1 \parallel_S Q_2) = 0$, then $Q_1 \equiv Q_2 \equiv \underline{0}$ and hence the result follows from $\mathscr{A}_{\mathrm{MT},8}$ and transitivity.
  * Let $size(Q_1 \parallel_S Q_2) = n > 0$ and assume that the result holds for every pair of nonrecursive process terms Q_1' and Q_2' in tmsnf such that $size(Q_1' \parallel_S Q_2') < n$. There are two cases.
    If exactly one between Q_1 and Q_2 is $\underline{0}$, then the result follows from $\mathscr{A}_{\mathrm{MT},6}$ or $\mathscr{A}_{\mathrm{MT},7}$ and transitivity.
    If neither Q_1 nor Q_2 is $\underline{0}$, we rewrite $Q_1 \parallel_S Q_2$ by means of $\mathscr{A}_{\mathrm{MT},5}$. Since the size of each parallel composition $Q_1' \parallel_S Q_2'$ occurring in one of the summands of the resulting process term is less than n, by the induction hypothesis each such $Q_1' \parallel_S Q_2'$ can be rewritten into a nonrecursive process term in tmsnf. The result then follows after as many applications of $\mathscr{A}_{\mathrm{MT},3}$ and $\mathscr{A}_{\mathrm{MT},4}$ as needed (possibly preceded by applications of $\mathscr{A}_{\mathrm{MT},1}$ and $\mathscr{A}_{\mathrm{MT},2}$) by virtue of substitutivity with respect to alternative composition as well as transitivity.
- If $P \equiv P'/H$, then by the induction hypothesis there exists Q' in tmsnf such that $\mathscr{A}_{\mathrm{MT}} \vdash P' = Q'$. From substitutivity with respect to hiding, we obtain that $\mathscr{A}_{\mathrm{MT}} \vdash P'/H = Q'/H$. Let us proceed by induction on $size(Q'/H)$:
  * If $size(Q'/H) = 0$, then $Q' \equiv \underline{0}$ and hence the result follows from $\mathscr{A}_{\mathrm{MT},9}$ and transitivity.
  * Let $size(Q'/H) = n > 0$ and assume that the result holds for every nonrecursive process term Q'' in tmsnf such that $size(Q''/H) < n$. In this case, we distribute $\_/H$ among all the summands of Q' by means of repeated applications of $\mathscr{A}_{\mathrm{MT},12}$, then we apply $\mathscr{A}_{\mathrm{MT},10}$ or $\mathscr{A}_{\mathrm{MT},11}$ to each summand augmented with $\_/H$. Since the size of each Q''/H

occurring in one of the summands of the resulting process term is less than n, by the induction hypothesis each such Q''/H can be rewritten into a nonrecursive process term in tmsnf. The result then follows after as many applications of $\mathscr{A}_{\mathrm{MT},4}$ as needed (possibly preceded by applications of $\mathscr{A}_{\mathrm{MT},1}$ and $\mathscr{A}_{\mathrm{MT},2}$) by virtue of substitutivity with respect to alternative composition as well as transitivity.

- If $P \equiv P'[\varphi]$, then by the induction hypothesis there exists Q' in tmsnf such that $\mathscr{A}_{\mathrm{MT}} \vdash P'{=}Q'$. From substitutivity with respect to relabeling, we obtain that $\mathscr{A}_{\mathrm{MT}} \vdash P'[\varphi] = Q'[\varphi]$. Let us proceed by induction on $size(Q'[\varphi])$:
  * If $size(Q'[\varphi]) = 0$, then $Q' \equiv \underline{0}$ and hence the result follows from $\mathscr{A}_{\mathrm{MT},13}$ and transitivity.
  * Let $size(Q'[\varphi]) = n > 0$ and assume that the result holds for every nonrecursive process term Q'' in tmsnf such that $size(Q''[\varphi]) < n$. In this case, we distribute $\_[\varphi]$ among all the summands of Q' by means of repeated applications of $\mathscr{A}_{\mathrm{MT},15}$, then we apply $\mathscr{A}_{\mathrm{MT},14}$ to each summand augmented with $\_[\varphi]$. Since the size of each $Q''[\varphi]$ occurring in one of the summands of the resulting process term is less than n, by the induction hypothesis each such $Q''[\varphi]$ can be rewritten into a nonrecursive process term in tmsnf. The result then follows after as many applications of $\mathscr{A}_{\mathrm{MT},4}$ as needed (possibly preceded by applications of $\mathscr{A}_{\mathrm{MT},1}$ and $\mathscr{A}_{\mathrm{MT},2}$) by virtue of substitutivity with respect to alternative composition as well as transitivity.

Given $P_1, P_2 \in \mathbb{P}_{\mathrm{pc,nrec}}$ such that $P_1 \sim_{\mathrm{MT}} P_2$, we prove that $\mathscr{A}_{\mathrm{MT}} \vdash P_1 = P_2$ by assuming without loss of generality that both P_1 and P_2 are in tmsnf. In fact, if this were not the case, we could derive $Q_1, Q_2 \in \mathbb{P}_{\mathrm{pc,nrec}}$ in tmsnf such that $\mathscr{A}_{\mathrm{MT}} \vdash P_1 = Q_1$ and $\mathscr{A}_{\mathrm{MT}} \vdash P_2 = Q_2$ (hence $P_1 \sim_{\mathrm{MT}} Q_1$ and $P_2 \sim_{\mathrm{MT}} Q_2$ due to the soundness of the axioms with respect to $\sim_{\mathrm{MT}}$), with $Q_1 \sim_{\mathrm{MT}} Q_2$ (because it also holds $P_1 \sim_{\mathrm{MT}} P_2$ and $\sim_{\mathrm{MT}}$ is a transitive relation). So, if we proved that $\mathscr{A}_{\mathrm{MT}} \vdash Q_1 = Q_2$, it would then follow $\mathscr{A}_{\mathrm{MT}} \vdash P_1 = P_2$ by transitivity.
Let us proceed by induction on the syntactical structure of P_1 in tmsnf:

- If $P_1 \equiv \underline{0}$, from $P_1 \sim_{\mathrm{MT}} P_2$ and P_2 in tmsnf it follows that $P_2 \equiv \underline{0}$, hence the result by reflexivity.
- If $P_1 \equiv \sum_{i \in I_1} <\!a_i, \lambda_i\!>.P_{1,i}$ with I_1 finite and nonempty, from $P_1 \sim_{\mathrm{MT}} P_2$ and P_2 in tmsnf it follows that $P_2 \equiv \sum_{j \in I_2} <\!b_j, \mu_j\!>.P_{2,j}$ with I_2 finite and nonempty. By virtue of Cor. A.3, from $P_1 \sim_{\mathrm{MT}} P_2$ we derive that:
$$\{a_i \mid i \in I_1\} \;=\; \{b_j \mid j \in I_2\} \;\equiv\; \{c_1, \dots, c_n\}$$
with:
$$rate_{\mathrm{o}}(P_1, c_k, 0) \;=\; rate_{\mathrm{o}}(P_2, c_k, 0)$$
for each $k = 1, \dots, n$. We can then concentrate on a generic c_k and on the two sets of summands of P_1 and P_2 enabling c_k-actions:
$$\begin{array}{rcl} S_{k,1} & = & \{<\!a_i, \lambda_i\!>.P_{1,i} \mid i \in I_1 \wedge a_i = c_k\} \\ S_{k,2} & = & \{<\!b_j, \mu_j\!>.P_{2,j} \mid j \in I_2 \wedge b_j = c_k\} \end{array}$$
which satisfy the following two properties:
  1. $\sum_{P \in S_{k,1}} rate_{\mathrm{o}}(P, c_k, 0) \;=\; \sum_{P \in S_{k,2}} rate_{\mathrm{o}}(P, c_k, 0)$.
  2. The derivative terms $P_{1,i}$ (resp. $P_{2,j}$) occurring in $S_{k,1}$ (resp. $S_{k,2}$) are all inequivalent with respect to $\sim_{\mathrm{MT}}$ due to the initial minimality of P_1 (resp. P_2) with respect to $\mathscr{A}_{\mathrm{MT},4}$. In fact, due to such an initial minimality, taken two derivative terms in the same summand set, it must be the case that their sets of initial action names are different or the total exit rate with respect to one of these initial action names is different in the two derivative terms, thus violating the necessary condition for $\sim_{\mathrm{MT}}$ stated by Cor. A.3.

Let us prove that for each summand $<a_i,\lambda_i>.P_{1,i} \in S_{k,1}$ there exists exactly one summand $<b_j,\mu_j>.P_{2,j} \in S_{k,2}$ such that $\lambda_i = \mu_j$ and $P_{1,i} \sim_{\mathrm{MT}} P_{2,j}$, by proceeding by induction on $|S_{k,1}|$ (the reverse can be proved in the same way):

- If $|S_{k,1}| = 1$, then $S_{k,1}$ contains a single summand, say $<a_i,\lambda_i>.P_{1,i}$. As a consequence, $S_{k,2}$ must contain a single summand as well, say $<b_j,\mu_j>.P_{2,j}$, and it must be $\lambda_i = \mu_j$ and $P_{1,i} \sim_{\mathrm{MT}} P_{2,j}$ because $P_1 \sim_{\mathrm{MT}} P_2$ (e.g., P_1 and P_2 cannot be distinguished by tests starting with a c_k-action if $c_k \neq \tau$). The reason why $S_{k,2}$ cannot contain several summands starting with a c_k-action is that their inequivalent derivatives would either violate the necessary condition for $\sim_{\mathrm{MT}}$ stated by Cor. A.3, thus contradicting $P_1 \sim_{\mathrm{MT}} P_2$, or satisfy that necessary condition, thus contradicting the initial minimality of P_2 with respect to $\mathscr{A}_{\mathrm{MT},4}$.
- Let $|S_{k,1}| > 1$ and assume that the result holds for any two proper subsets of $S_{k,1}$ and $S_{k,2}$ satisfying properties 1 and 2. Let $S^d_{k,1}$ be the set of the summands of $S_{k,1}$ whose derivative terms have – among all the derivative terms occurring in $S_{k,1}$ – the maximum total exit rate δ with respect to an action name d. By virtue of property 2, d can be chosen in such a way that $S^d_{k,1} \neq S_{k,1}$. Then the derivative term of each summand of $S^d_{k,1}$ passes with probability 1 the test $<d,*_1>.\mathrm{s}$ (or simply s if $d = \tau$) within the minimum average time $1/\delta$ when considering successful test-driven computations of length 1, hence P_1 passes with probability $\sum_{P \in S^d_{k,1}} rate_{\mathrm{o}}(P,c_k,0) / \sum_{P \in S_{k,1}} rate_{\mathrm{o}}(P,c_k,0)$ the test $<c_k,*_1>.<d,*_1>.\mathrm{s}$ (or simply $<c_k,*_1>.\mathrm{s}$ if $c_k \neq \tau$ and $d = \tau$, $<d,*_1>.\mathrm{s}$ if $c_k = \tau$ and $d \neq \tau$, or s if $c_k = d = \tau$) within the minimum average time sequence $1/\sum_{P \in S_{k,1}} rate_{\mathrm{o}}(P,c_k,0) \circ 1/\delta$ when considering successful test-driven computations of length 2. Since $P_1 \sim_{\mathrm{MT}} P_2$, also P_2 must pass the same test in the same way as P_1, hence there must exist a subset $S^d_{k,2}$ of $S_{k,2}$ whose derivative terms all have the maximum total exit rate δ with respect to d, with $S^d_{k,2} \neq S_{k,2}$ and $\sum_{P \in S^d_{k,1}} rate_{\mathrm{o}}(P,c_k,0) = \sum_{P \in S^d_{k,2}} rate_{\mathrm{o}}(P,c_k,0)$.

  Since $S^d_{k,1}$ and $S^d_{k,2}$ are proper subsets of $S_{k,1}$ and $S_{k,2}$ satisfying properties 1 and 2, by the induction hypothesis it follows that for each summand $<a_i,\lambda_i>.P_{1,i} \in S^d_{k,1}$ there exists exactly one summand $<b_j,\mu_j>.P_{2,j} \in S^d_{k,2}$ such that $\lambda_i = \mu_j$ and $P_{1,i} \sim_{\mathrm{MT}} P_{2,j}$.

  Likewise, since $S'_{k,1} = S_{k,1} - S^d_{k,1}$ and $S'_{k,2} = S_{k,2} - S^d_{k,2}$ are proper subsets of $S_{k,1}$ and $S_{k,2}$ satisfying properties 1 and 2, by the induction hypothesis it follows that for each summand $<a_i,\lambda_i>.P_{1,i} \in S'_{k,1}$ there exists exactly one summand $<b_j,\mu_j>.P_{2,j} \in S'_{k,2}$ such that $\lambda_i = \mu_j$ and $P_{1,i} \sim_{\mathrm{MT}} P_{2,j}$. Thus, the result follows for the whole $S_{k,1}$ and $S_{k,2}$.

As a consequence, a bijective correspondence can be established between any pair $(S_{k,1}, S_{k,2})$. For each pair of corresponding summands $<a_i,\lambda_i>.P_{1,i}$ and $<b_j,\mu_j>.P_{2,j}$, since $P_{1,i} \sim_{\mathrm{MT}} P_{2,j}$ and both subterms $P_{1,i}$ and $P_{2,j}$ are in tmsnf, by the induction hypothesis it follows that $\mathscr{A}_{\mathrm{MT}} \vdash P_{1,i} = P_{2,j}$. Thus $\mathscr{A}_{\mathrm{MT}} \vdash <a_i,\lambda_i>.P_{1,i} = <b_j,\mu_j>.P_{2,j}$ by substitutivity with respect to action prefix ($a_i = b_j$ and $\lambda_i = \mu_j$) and hence $\mathscr{A}_{\mathrm{MT}} \vdash \sum_{i \in I_1} <a_i,\lambda_i>.P_{1,i} = \sum_{j \in I_2} <b_j,\mu_j>.P_{2,j}$ by substitutivity with respect to alternative composition. ■

A.4 Proofs of Results of Sect. 7

Proof of Thm. 7.3. A reworking of the proof of Thm. 5.4 of [4] that proceeds as follows. We denote by $init(T)$ the set of names of actions enabled by $T \in \mathbb{T}_{\mathrm{R}}$. We also denote by $\mathbb{T}_{\mathrm{R,det}}$ the set of name-deterministic reactive tests (which is a superset of $\mathbb{T}_{\mathrm{R,c}}$), i.e., the set of reactive tests in which every subterm of the form $T_1 + T_2$ satisfies $init(T_1) \cap init(T_2) = \emptyset$. The result follows from the bijective correspondence between classes of tests in $\mathbb{T}_{\mathrm{R,det}}$ differing only for their action weights and formulas of $\mathscr{ML}_{\mathrm{MT}}$, which is established below in two steps:

- Firstly, we prove that for all $T \in \mathbb{T}_{\mathrm{R,det}}$ there exists $\phi_T \in \mathscr{ML}_{\mathrm{MT}}$ such that $init(\phi_T) = init(T)$ and for all $P \in \mathbb{P}_{\mathrm{pc}}$ and $\theta \in (\mathbb{R}_{>0})^*$:
$$[\![\phi_T]\!]^{|\theta|}_{\mathrm{MT}}(P, \theta) \;=\; prob(\mathscr{SC}^{|\theta|}_{\leq\theta}(P, T))$$
by proceeding by induction on the syntactical structure of T:
  - Let $T \equiv \mathrm{s}$ and take $\phi_T \equiv \mathrm{true}$. We prove that for all $P \in \mathbb{P}_{\mathrm{pc}}$ and $\theta \in (\mathbb{R}_{>0})^*$:
$$[\![\mathrm{true}]\!]^{|\theta|}_{\mathrm{MT}}(P, \theta) \;=\; prob(\mathscr{SC}^{|\theta|}_{\leq\theta}(P, \mathrm{s}))$$
by proceeding by induction on $|\theta|$:
    * If $|\theta| = 0$, then:
$$[\![\mathrm{true}]\!]^{|\theta|}_{\mathrm{MT}}(P, \theta) \;= 1 =\; prob(\mathscr{SC}^{|\theta|}_{\leq\theta}(P, \mathrm{s}))$$
    * Let $|\theta| > 0$ with $\theta = t \circ \theta'$, $t \in \mathbb{R}_{>0}$, and assume that for all $\bar{P} \in \mathbb{P}_{\mathrm{pc}}$:
$$[\![\mathrm{true}]\!]^{|\theta'|}_{\mathrm{MT}}(\bar{P}, \theta') \;=\; prob(\mathscr{SC}^{|\theta'|}_{\leq\theta'}(\bar{P}, \mathrm{s}))$$
There are two cases.
If $rate_{\mathrm{o}}(P, \tau, 0) = 0$ or $\frac{1}{rate_{\mathrm{o}}(P,\tau,0)} > t$, then:
$$[\![\mathrm{true}]\!]^{|\theta|}_{\mathrm{MT}}(P, \theta) \;= 0 =\; prob(\mathscr{SC}^{|\theta|}_{\leq\theta}(P, \mathrm{s}))$$
If $rate_{\mathrm{o}}(P, \tau, 0) > 0$ and $\frac{1}{rate_{\mathrm{o}}(P,\tau,0)} \leq t$, then:
$$\begin{aligned} [\![\mathrm{true}]\!]^{|\theta|}_{\mathrm{MT}}(P, \theta) \;&=\; \sum_{P \xrightarrow{\tau,\lambda} P'} \frac{\lambda}{rate_{\mathrm{o}}(P,\tau,0)} \cdot [\![\mathrm{true}]\!]^{|\theta'|}_{\mathrm{MT}}(P', \theta') \;= \\ &=\; \sum_{P \xrightarrow{\tau,\lambda} P'} \frac{\lambda}{rate_{\mathrm{o}}(P,\tau,0)} \cdot prob(\mathscr{SC}^{|\theta'|}_{\leq\theta'}(P', \mathrm{s})) \;=\; prob(\mathscr{SC}^{|\theta|}_{\leq\theta}(P, \mathrm{s})) \end{aligned}$$
by the induction hypothesis.
  - Let $T \equiv \,<a, *_w>.T'$. From the induction hypothesis, it follows that there exists $\phi_{T'} \in \mathscr{ML}_{\mathrm{MT}}$ such that $init(\phi_{T'}) = init(T')$ and for all $\hat{P} \in \mathbb{P}_{\mathrm{pc}}$ and $\hat{\theta} \in (\mathbb{R}_{>0})^*$:
$$[\![\phi_{T'}]\!]^{|\hat{\theta}|}_{\mathrm{MT}}(\hat{P}, \hat{\theta}) \;=\; prob(\mathscr{SC}^{|\hat{\theta}|}_{\leq\hat{\theta}}(\hat{P}, T'))$$
Take $\phi_T \equiv \langle a \rangle \phi_{T'}$. We prove that for all $P \in \mathbb{P}_{\mathrm{pc}}$ and $\theta \in (\mathbb{R}_{>0})^*$:
$$[\![\phi_T]\!]^{|\theta|}_{\mathrm{MT}}(P, \theta) \;=\; prob(\mathscr{SC}^{|\theta|}_{\leq\theta}(P, T))$$
by proceeding by induction on $|\theta|$:
    * If $|\theta| = 0$, then:
$$[\![\phi_T]\!]^{|\theta|}_{\mathrm{MT}}(P, \theta) \;= 0 =\; prob(\mathscr{SC}^{|\theta|}_{\leq\theta}(P, T))$$
    * Let $|\theta| > 0$ with $\theta = t \circ \theta'$, $t \in \mathbb{R}_{>0}$, and assume that for all $\bar{P} \in \mathbb{P}_{\mathrm{pc}}$:
$$[\![\phi_T]\!]^{|\theta'|}_{\mathrm{MT}}(\bar{P}, \theta') \;=\; prob(\mathscr{SC}^{|\theta'|}_{\leq\theta'}(\bar{P}, T))$$
There are two cases.
If $rate_{\mathrm{o}}(P, \{a, \tau\}, 0) = 0$ or $\frac{1}{rate_{\mathrm{o}}(P,\{a,\tau\},0)} > t$, then:
$$[\![\phi_T]\!]^{|\theta|}_{\mathrm{MT}}(P, \theta) \;= 0 =\; prob(\mathscr{SC}^{|\theta|}_{\leq\theta}(P, T))$$

If $rate_{\rm o}(P,\{a,\tau\},0) > 0$ and $\frac{1}{rate_{\rm o}(P,\{a,\tau\},0)} \leq t$, then:

$$\begin{aligned}
[\![\phi_T]\!]^{|\theta|}_{\rm MT}(P,\theta) \;=\; & \sum_{P \xrightarrow{a,\lambda} P'} \frac{\lambda}{rate_{\rm o}(P,\{a,\tau\},0)} \cdot [\![\phi_{T'}]\!]^{|\theta'|}_{\rm MT}(P',\theta') \\
& + \sum_{P \xrightarrow{\tau,\lambda} P'} \frac{\lambda}{rate_{\rm o}(P,\{a,\tau\},0)} \cdot [\![\phi_T]\!]^{|\theta'|}_{\rm MT}(P',\theta') \quad = \\
= \; & \sum_{P \xrightarrow{a,\lambda} P'} \frac{\lambda}{rate_{\rm o}(P,\{a,\tau\},0)} \cdot prob(\mathcal{SC}^{|\theta'|}_{\leq\theta'}(P',T')) \\
& + \sum_{P \xrightarrow{\tau,\lambda} P'} \frac{\lambda}{rate_{\rm o}(P,\{a,\tau\},0)} \cdot prob(\mathcal{SC}^{|\theta'|}_{\leq\theta'}(P',T)) \;=\; prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P,T))
\end{aligned}$$

by the induction hypotheses.

– Let $T \equiv T_1 + T_2$. From the induction hypothesis, it follows that there exist $\phi_{T_1}, \phi_{T_2} \in \mathcal{ML}_{\rm MT}$ such that $init(\phi_{T_1}) = init(T_1), init(\phi_{T_2}) = init(T_2)$ and for all $\hat{P} \in \mathbb{P}_{\rm pc}$ and $\hat{\theta} \in (\mathbb{R}_{>0})^*$:

$$\begin{aligned}
[\![\phi_{T_1}]\!]^{|\hat{\theta}|}_{\rm MT}(\hat{P},\hat{\theta}) \;&=\; prob(\mathcal{SC}^{|\hat{\theta}|}_{\leq\hat{\theta}}(\hat{P},T_1)) \\
[\![\phi_{T_2}]\!]^{|\hat{\theta}|}_{\rm MT}(\hat{P},\hat{\theta}) \;&=\; prob(\mathcal{SC}^{|\hat{\theta}|}_{\leq\hat{\theta}}(\hat{P},T_2))
\end{aligned}$$

Take $\phi_T \equiv \phi_{T_1} \vee \phi_{T_2}$, which satisfies $init(\phi_{T_1}) \cap init(\phi_{T_2}) = \emptyset$ because T is name deterministic and hence $init(T_1) \cap init(T_2) = \emptyset$. We prove that for all $P \in \mathbb{P}_{\rm pc}$ and $\theta \in (\mathbb{R}_{>0})^*$:

$$[\![\phi_T]\!]^{|\theta|}_{\rm MT}(P,\theta) \;=\; prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P,T))$$

by proceeding by induction on $|\theta|$:

$*$ If $|\theta| = 0$, then:

$$[\![\phi_T]\!]^{|\theta|}_{\rm MT}(P,\theta) \;= 0 =\; prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P,T))$$

$*$ Let $|\theta| > 0$ with $\theta = t \circ \theta'$, $t \in \mathbb{R}_{>0}$, and assume that for all $\bar{P} \in \mathbb{P}_{\rm pc}$:

$$[\![\phi_T]\!]^{|\theta'|}_{\rm MT}(\bar{P},\theta') \;=\; prob(\mathcal{SC}^{|\theta'|}_{\leq\theta'}(\bar{P},T))$$

There are two cases.

If $rate_{\rm o}(P, init(\phi_T) \cup \{\tau\}, 0) = 0$ or $\frac{1}{rate_{\rm o}(P, init(\phi_T) \cup \{\tau\}, 0)} > t$, then:

$$[\![\phi_T]\!]^{|\theta|}_{\rm MT}(P,\theta) \;= 0 =\; prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P,T))$$

If $rate_{\rm o}(P, init(\phi_T) \cup \{\tau\}, 0) > 0$ and $\frac{1}{rate_{\rm o}(P, init(\phi_T) \cup \{\tau\}, 0)} \leq t$, then after posing for $j \in \{1,2\}$:

$$\begin{aligned}
p_j \;&=\; \frac{rate_{\rm o}(P, init(\phi_{T_j}), 0)}{rate_{\rm o}(P, init(\phi_T) \cup \{\tau\}, 0)} \;=\; \frac{rate_{\rm o}(P, init(T_j), 0)}{rate_{\rm o}(P, init(T) \cup \{\tau\}, 0)} \\
t_j \;&=\; t + \left(\frac{1}{rate_{\rm o}(P, init(\phi_{T_j}), 0)} - \frac{1}{rate_{\rm o}(P, init(\phi_T) \cup \{\tau\}, 0)}\right) \;=\; t + \left(\frac{1}{rate_{\rm o}(P, init(T_j), 0)} - \frac{1}{rate_{\rm o}(P, init(T) \cup \{\tau\}, 0)}\right)
\end{aligned}$$

we have:

$$\begin{aligned}
[\![\phi_T]\!]^{|\theta|}_{\rm MT}(P,\theta) \;=\; & p_1 \cdot [\![\phi_{T_1}]\!]^{|t_1 \circ \theta'|}_{\rm MT}(P_{no\text{-}init\text{-}\tau}, t_1 \circ \theta') \\
& + p_2 \cdot [\![\phi_{T_2}]\!]^{|t_2 \circ \theta'|}_{\rm MT}(P_{no\text{-}init\text{-}\tau}, t_2 \circ \theta') \\
& + \sum_{P \xrightarrow{\tau,\lambda} P'} \frac{\lambda}{rate_{\rm o}(P, init(\phi_T) \cup \{\tau\}, 0)} \cdot [\![\phi_T]\!]^{|\theta'|}_{\rm MT}(P',\theta') \quad = \\
= \; & p_1 \cdot prob(\mathcal{SC}^{|t_1 \circ \theta'|}_{\leq t_1 \circ \theta'}(P_{no\text{-}init\text{-}\tau}, T_1)) \\
& + p_2 \cdot prob(\mathcal{SC}^{|t_2 \circ \theta'|}_{\leq t_2 \circ \theta'}(P_{no\text{-}init\text{-}\tau}, T_2)) \\
& + \sum_{P \xrightarrow{\tau,\lambda} P'} \frac{\lambda}{rate_{\rm o}(P, init(T) \cup \{\tau\}, 0)} \cdot prob(\mathcal{SC}^{|\theta'|}_{\leq\theta'}(P',T)) \;=\; prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P,T))
\end{aligned}$$

by the induction hypotheses.

- Secondly, we prove that for all $\phi \in \mathcal{ML}_{\rm MT}$ there exists $T_\phi \in \mathbb{T}_{\rm R,det}$ such that $init(T_\phi) = init(\phi)$ and for all $P \in \mathbb{P}_{\rm pc}$ and $\theta \in (\mathbb{R}_{>0})^*$:

$$prob(\mathcal{SC}^{|\theta|}_{\leq\theta}(P,T_\phi)) \;=\; [\![\phi]\!]^{|\theta|}_{\rm MT}(P,\theta)$$

by proceeding by induction on the syntactical structure of ϕ. The proof is completely symmetrical with respect to the proof of the first step, in the sense that the roles of formulas and tests are exchanged: the former are given, the latter are built on the basis of the former. ■

A.5 Proofs of Results of Sect. 8

The result exploited in that section is related to Markovian ready equivalence.

Definition A.13 Let $P \in \mathbb{P}_{\mathrm{pc}}$, $c \in \mathscr{C}_{\mathrm{f}}(P)$, and $\alpha \in (Name_{\mathrm{v}})^*$. We say that c is compatible with α iff:

$$trace(c) = \alpha$$

We denote by $\mathscr{C}\mathscr{C}(P,\alpha)$ the multiset of computations in $\mathscr{C}_{\mathrm{f}}(P)$ that are compatible with trace α. ■

Definition A.14 Let $P \in \mathbb{P}_{\mathrm{pc}}$, $c \in \mathscr{C}_{\mathrm{f}}(P)$, and $\rho \equiv (\alpha, R) \in (Name_{\mathrm{v}})^* \times 2^{Name_{\mathrm{v}}}$. We say that computation c is compatible with the ready pair ρ iff $c \in \mathscr{C}\mathscr{C}(P,\alpha)$ and the set of names of visible actions that can be performed by the last state reached by c coincides with the ready set R. We denote by $\mathscr{R}\mathscr{C}\mathscr{C}(P,\rho)$ the multiset of computations in $\mathscr{C}_{\mathrm{f}}(P)$ that are compatible with ρ. ■

Definition A.15 Let $P_1, P_2 \in \mathbb{P}_{\mathrm{pc}}$. We say that P_1 is Markovian ready equivalent to P_2, written $P_1 \sim_{\mathrm{MR}} P_2$, iff for all ready pairs $\rho \in (Name_{\mathrm{v}})^* \times 2^{Name_{\mathrm{v}}}$ and sequences $\theta \in (\mathbb{R}_{>0})^*$ of average amounts of time:

$$prob(\mathscr{R}\mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta}(P_1,\rho)) = prob(\mathscr{R}\mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta}(P_2,\rho))$$

■

Proposition A.16 Let $P_1, P_2 \in \mathbb{P}_{\mathrm{pc}}$. Then $P_1 \sim_{\mathrm{MR}} P_2 \Longleftrightarrow P_1 \sim_{\mathrm{MT}} P_2$.

Proof A reworking of the proof of Prop. 5.42 of [2] that proceeds in two steps:

- $\Rightarrow$ We prove the contrapositive, so we assume that $P_1 \not\sim_{\mathrm{MT}} P_2$. Then by virtue of Thm. A.11 there exist $\xi \in \mathscr{E}\mathscr{T}$ and $\theta \in (\mathbb{R}_{>0})^*$ such that:

  $$prob_{\xi}(\mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta,\xi}(P_1,\xi)) \neq prob_{\xi}(\mathscr{C}\mathscr{C}^{|\theta|}_{\leq\theta,\xi}(P_2,\xi))$$

  Let us consider an extended trace $\overline{\xi}$ with minimal length among those satisfying the above inequality, together with a corresponding sequence $\overline{\theta}$ of average amounts of time with minimal length. There are two cases:

  - If $\overline{\xi} \equiv \varepsilon$, then for $\alpha \equiv \varepsilon$ and $|\overline{\theta}| > 0$ the inequality above can be rewritten as follows:

    $$prob(\mathscr{C}\mathscr{C}^{|\overline{\theta}|}_{\leq\overline{\theta}}(P_1,\alpha)) \neq prob(\mathscr{C}\mathscr{C}^{|\overline{\theta}|}_{\leq\overline{\theta}}(P_2,\alpha))$$

    which in turn can be rewritten as follows:

    $$\sum_{R \in 2^{Name_{\mathrm{v}}}} prob(\mathscr{R}\mathscr{C}\mathscr{C}^{|\overline{\theta}|}_{\leq\overline{\theta}}(P_1,(\alpha,R))) \neq \sum_{R \in 2^{Name_{\mathrm{v}}}} prob(\mathscr{C}\mathscr{C}^{|\overline{\theta}|}_{\leq\overline{\theta}}(P_2,(\alpha,R)))$$

    As a consequence, there must be at least one $R \in 2^{Name_{\mathrm{v}}}$, say $\hat{R}$, for which it holds:

    $$prob(\mathscr{R}\mathscr{C}\mathscr{C}^{|\overline{\theta}|}_{\leq\overline{\theta}}(P_1,(\alpha,\hat{R}))) \neq prob(\mathscr{C}\mathscr{C}^{|\overline{\theta}|}_{\leq\overline{\theta}}(P_2,(\alpha,\hat{R})))$$

    hence $P_1 \not\sim_{\mathrm{MR}} P_2$.
  - If $\overline{\xi} \equiv \overline{\xi}' \circ (a, \mathscr{E})$ with $trace_{\mathrm{et}}(\overline{\xi}') = \alpha'$, then by virtue of the minimality of the length of $\overline{\xi}$ we have that for all $\theta' \in (\mathbb{R}_{>0})^*$:

    $$prob_{\xi_{\alpha'}}(\mathscr{C}\mathscr{C}^{|\theta'|}_{\leq\theta',\xi_{\alpha'}}(P_1,\xi_{\alpha'})) = prob_{\xi_{\alpha'}}(\mathscr{C}\mathscr{C}^{|\theta'|}_{\leq\theta',\xi_{\alpha'}}(P_2,\xi_{\alpha'}))$$

    where $\xi_{\alpha'}$ is an extended trace obtained from α' by including at each step the set of visible action names occurring in P_1 or P_2. Therefore:

    $$prob(\mathscr{C}\mathscr{C}^{|\theta'|}_{\leq\theta'}(P_1,\alpha')) = prob(\mathscr{C}\mathscr{C}^{|\theta'|}_{\leq\theta'}(P_2,\alpha'))$$

    There are two subcases:

∗ If the last states reached by the computations in $\mathscr{CC}^{|\theta'|}_{\leq\theta'}(P_1,\alpha')$ and the last states reached by the computations in $\mathscr{CC}^{|\theta'|}_{\leq\theta'}(P_2,\alpha')$ result in the same family of ready sets, then by virtue of the initial inequality there must exist a visible action name in the family of ready sets, say $\hat{a}$, such that for some $\hat{\theta} \in (\mathbb{R}_{>0})^*$ not lexicographically less than $\overline{\theta}$:

$$prob(\mathscr{CC}^{|\hat{\theta}|}_{\leq\hat{\theta}}(P_1,\alpha'\circ\hat{a})) \neq prob(\mathscr{CC}^{|\hat{\theta}|}_{\leq\hat{\theta}}(P_2,\alpha'\circ\hat{a}))$$

which can be rewritten as follows:

$$\sum_{R\in 2^{Name_{\mathrm{v}}}} prob(\mathscr{RCC}^{|\hat{\theta}|}_{\leq\hat{\theta}}(P_1,(\alpha'\circ\hat{a},R))) \neq \sum_{R\in 2^{Name_{\mathrm{v}}}} prob(\mathscr{RCC}^{|\hat{\theta}|}_{\leq\hat{\theta}}(P_2,(\alpha'\circ\hat{a},R)))$$

As a consequence, there must be at least one $R \in 2^{Name_{\mathrm{v}}}$, say $\hat{R}$, for which it holds:

$$prob(\mathscr{RCC}^{|\hat{\theta}|}_{\leq\hat{\theta}}(P_1,(\alpha'\circ\hat{a},\hat{R}))) \neq prob(\mathscr{RCC}^{|\hat{\theta}|}_{\leq\hat{\theta}}(P_2,(\alpha'\circ\hat{a},\hat{R})))$$

hence $P_1 \not\sim_{\mathrm{MR}} P_2$.

∗ If the last states reached by the computations in $\mathscr{CC}^{|\theta'|}_{\leq\theta'}(P_1,\alpha')$ and the last states reached by the computations in $\mathscr{CC}^{|\theta'|}_{\leq\theta'}(P_2,\alpha')$ result in two different families of ready sets, then there is at least one ready set, say $\hat{R}$, that is possessed by only one of the two sets of computations, say the former. Therefore, for some $\hat{\theta} \in (\mathbb{R}_{>0})^*$ it holds:

$$prob(\mathscr{RCC}^{|\hat{\theta}|}_{\leq\hat{\theta}}(P_1,(\alpha',\hat{R}))) > 0 = prob(\mathscr{RCC}^{|\hat{\theta}|}_{\leq\hat{\theta}}(P_2,(\alpha',\hat{R})))$$

hence $P_1 \not\sim_{\mathrm{MR}} P_2$.

$\Leftarrow$ By virtue of Prop. A.2, from $P_1 \sim_{\mathrm{MT}} P_2$ it follows that for all $c_k \in \mathscr{C}_{\mathrm{f}}(P_k)$, $k \in \{1,2\}$, there exists $c_h \in \mathscr{C}_{\mathrm{f}}(P_h)$, $h \in \{1,2\} - \{k\}$, such that:

$$\begin{aligned} trace_{\mathrm{c}}(c_k) &= trace_{\mathrm{c}}(c_h) \\ time_{\mathrm{a}}(c_k) &= time_{\mathrm{a}}(c_h) \end{aligned}$$

and for all $a \in Name$:

$$rate_{\mathrm{o}}(P^{\mathrm{last}}_k,a,0) = rate_{\mathrm{o}}(P^{\mathrm{last}}_h,a,0)$$

with P^{last}_k (resp. P^{last}_h) being the last state reached by c_k (resp. c_h). Therefore, given an arbitrary $\alpha \in (Name_{\mathrm{v}})^*$, for all $c_k \in \mathscr{CC}(P_k,\alpha)$, $k \in \{1,2\}$, there exists $c_h \in \mathscr{CC}(P_h,\alpha)$, $h \in \{1,2\} - \{k\}$, such that:

$$time_{\mathrm{a}}(c_k) = time_{\mathrm{a}}(c_h)$$

and for all $a \in Name$:

$$rate_{\mathrm{o}}(P^{\mathrm{last}}_k,a,0) = rate_{\mathrm{o}}(P^{\mathrm{last}}_h,a,0)$$

with the equality above meaning that every pair of matching computations (i.e., with the same trace and the same stepwise average duration) end up in states with the same ready set.

Let us consider the computations of $\mathscr{CC}(P_1,\alpha)$ and $\mathscr{CC}(P_2,\alpha)$ on the basis of their extended stepwise average duration, which is given by their stepwise average duration concatenated with the inverse of the total exit rate of their last state, or simply by their stepwise average duration whenever the total exit rate of their last state is zero. This results in two disjoint partitions of $\mathscr{CC}(P_1,\alpha) \cup \mathscr{CC}(P_2,\alpha)$ whose classes intersect both multisets: each class of the first partition collects all the matching computations with the same extended stepwise average duration ending in states with zero total exit rate, while each class of the second partition collects all the matching computations with the same extended stepwise average duration ending in states with nonzero total exit rate. We denote by $\hat{\theta}_1,\ldots,\hat{\theta}_{\hat{n}}$, $\hat{n} \in \mathbb{N}$, the extended stepwise average durations listed in increasing order resulting from the classes of the first partition and by $\overline{\theta}_1,\ldots,\overline{\theta}_{\overline{n}}$, $\overline{n} \in \mathbb{N}$, the extended stepwise average durations listed in increasing order resulting from the classes of the second partition, where $\hat{n}+\overline{n} > 0$. In turn, every class of the second partition will be formed by several groups of matching computations, with each group being characterized by a different ready set.

Let us examine the class of matching computations of the second partition whose associated extended stepwise average duration is the minimum one, i.e., $\bar{\theta}_1 \equiv \bar{\theta}'_1 \circ \bar{t}_1$ with $\bar{t}_1 \in \mathbb{R}_{>0}$. Assuming that $\bar{R}_{1,1}, \dots, \bar{R}_{1,\bar{m}_1}$ be the ready sets characterizing the groups of matching computations of the considered class listed in order of nondecreasing size, let us focus on the smallest one, i.e., $\bar{R}_{1,1}$. There are two cases:

- If $\bar{R}_{1,1} = \emptyset$ (which means that only invisible actions can be executed in the last state of the considered computations), we take a test $\bar{T}_{1,1}$ composed of a sequence terminated by s of passive visible actions whose names and order are the same as those of the actions occurring in α, which at every step also enables passive actions with all the other visible names occurring in P_1 or P_2 each followed by $<\mathrm{z}, *_1>$.s (with z being the usual visible action name admitted within tests but not within process terms under test). From $P_1 \sim_{\mathrm{MT}} P_2$, we derive:
$$prob(\mathcal{SC}^{|\bar{\theta}_1|}_{\leq \bar{\theta}_1}(P_1, \bar{T}_{1,1})) = prob(\mathcal{SC}^{|\bar{\theta}_1|}_{\leq \bar{\theta}_1}(P_2, \bar{T}_{1,1}))$$
where – due to the structure of $\bar{T}_{1,1}$ – for $k \in \{1,2\}$ it holds:
$$prob(\mathcal{SC}^{|\bar{\theta}_1|}_{\leq \bar{\theta}_1}(P_k, \bar{T}_{1,1})) = prob(\mathcal{RCC}^{|\bar{\theta}'_1|}_{\leq \bar{\theta}'_1}(P_k, (\alpha, \emptyset))) - prob(\mathcal{MCC}^{|\bar{\theta}'_1|}_{\leq \bar{\theta}'_1}(P_k, \alpha))$$
with $\mathcal{MCC}(P_k, \alpha)$ being the multiset of maximal (i.e., terminating in a state without outgoing transitions) computations in $\mathcal{C}_{\mathrm{f}}(P)$ that are compatible with α. Since $P_1 \sim_{\mathrm{MT}} P_2$ implies $P_1 \sim_{\mathrm{MTr}} P_2$ and the latter is equivalent to $P_1 \sim_{\mathrm{MTr,c}} P_2$, it holds $prob(\mathcal{MCC}^{|\bar{\theta}'_1|}_{\leq \bar{\theta}'_1}(P_1, \alpha)) = prob(\mathcal{MCC}^{|\bar{\theta}'_1|}_{\leq \bar{\theta}'_1}(P_2, \alpha))$ and hence:
$$prob(\mathcal{RCC}^{|\bar{\theta}'_1|}_{\leq \bar{\theta}'_1}(P_1, (\alpha, \emptyset))) = prob(\mathcal{RCC}^{|\bar{\theta}'_1|}_{\leq \bar{\theta}'_1}(P_2, (\alpha, \emptyset)))$$
- If $\bar{R}_{1,1} \neq \emptyset$, we take a test $\bar{T}_{1,1}$ composed of a sequence terminated by $\sum_{a \in \bar{R}_{1,1}} <a, *_1>$.s of passive visible actions whose names and order are the same as those of the actions occurring in α, which at every nonfinal step also enables passive actions with all the other visible names occurring in P_1 or P_2 each followed by $<\mathrm{z}, *_1>$.s. From $P_1 \sim_{\mathrm{MT}} P_2$, we derive:
$$prob(\mathcal{SC}^{|\bar{\theta}_1|}_{\leq \bar{\theta}_1}(P_1, \bar{T}_{1,1})) = prob(\mathcal{SC}^{|\bar{\theta}_1|}_{\leq \bar{\theta}_1}(P_2, \bar{T}_{1,1}))$$
where – due to the structure of $\bar{T}_{1,1}$ – for $k \in \{1,2\}$ it holds:
$$prob(\mathcal{SC}^{|\bar{\theta}_1|}_{\leq \bar{\theta}_1}(P_k, \bar{T}_{1,1})) = prob(\mathcal{RCC}^{|\bar{\theta}'_1|}_{\leq \bar{\theta}'_1}(P_k, (\alpha, \bar{R}_{1,1})))$$
and hence:
$$prob(\mathcal{RCC}^{|\bar{\theta}'_1|}_{\leq \bar{\theta}'_1}(P_1, (\alpha, \bar{R}_{1,1}))) = prob(\mathcal{RCC}^{|\bar{\theta}'_1|}_{\leq \bar{\theta}'_1}(P_2, (\alpha, \bar{R}_{1,1})))$$

If we focus on a generic group of matching computations of the first class of the second partition, say the one whose ready set is $\bar{R}_{1,j}$ (which cannot be empty) with $2 \leq j \leq \bar{m}_1$, we take a test $\bar{T}_{1,j}$ composed of a sequence terminated by $\sum_{a \in \bar{R}_{1,j}} <a, *_1>$.s of passive visible actions whose names and order are the same as those of the actions occurring in α, which at every nonfinal step also enables passive actions with all the other visible names occurring in P_1 or P_2 each followed by $<\mathrm{z}, *_1>$.s. From $P_1 \sim_{\mathrm{MT}} P_2$, we derive:
$$prob(\mathcal{SC}^{|\bar{\theta}_1|}_{\leq \bar{\theta}_1}(P_1, \bar{T}_{1,j})) = prob(\mathcal{SC}^{|\bar{\theta}_1|}_{\leq \bar{\theta}_1}(P_2, \bar{T}_{1,j}))$$
where – due to the structure of $\bar{T}_{1,j}$ – for $k \in \{1,2\}$ it holds:
$$prob(\mathcal{SC}^{|\bar{\theta}_1|}_{\leq \bar{\theta}_1}(P_k, \bar{T}_{1,j})) = prob(\mathcal{RCC}^{|\bar{\theta}'_1|}_{\leq \bar{\theta}'_1}(P_k, (\alpha, \bar{R}_{1,j}))) + \bar{q}_{k,1,<j}$$
In the equality above, $\bar{q}_{k,1,<j}$ represents the contribution of the groups of matching computations of the first class characterized by ready sets $\bar{R}_{1,1}, \dots, \bar{R}_{1,j-1}$, with the contribution of group j', $1 \leq j' < j$, being either $prob(\mathcal{SC}^{|\bar{\theta}_1|}_{\leq \bar{\theta}_1}(P_k, \bar{T}_{1,j'}))$ or zero depending on whether $\bar{R}_{1,j'} \subset \bar{R}_{1,j}$ or not.

Therefore, from $P_1 \sim_{\rm MT} P_2$ it follows $\overline{q}_{1,1,<j} = \overline{q}_{2,1,<j}$ and hence:

$$prob(\mathscr{RCC}^{|\overline{\theta}'_1|}_{\leq \overline{\theta}'_1}(P_1,(\alpha,\overline{R}_{1,j}))) = prob(\mathscr{RCC}^{|\overline{\theta}'_1|}_{\leq \overline{\theta}'_1}(P_2,(\alpha,\overline{R}_{1,j})))$$

If we focus on a generic class of the second partition, say the one whose associated extended stepwise average duration is $\overline{\theta}_i \equiv \overline{\theta}'_i \circ \overline{t}_i$ with $\overline{t}_i \in \mathbb{R}_{>0}$, $2 \leq i \leq \overline{n}$, and whose groups of matching computations are characterized by the ready sets $\overline{R}_{i,1}, \dots, \overline{R}_{i,\overline{m}_i}$ listed in order of nondecreasing size, we can reason in a similar way for the first group and for a generic group of the considered class by taking suitable tests $\overline{T}_{i,j}$, $1 \leq j \leq \overline{m}_i$, built in the same way as tests $\overline{T}_{1,j}$. The only difference is that, in the equalities relating the probability of $\mathscr{SC}$-multisets with the probability of $\mathscr{RCC}$-multisets, we have to take into account the contribution of the classes of the second partition whose associated extended stepwise average durations are $\overline{\theta}_1, \dots, \overline{\theta}_{i-1}$. For $k \in \{1,2\}$, the contribution of group j', $1 \leq j' \leq \overline{m}_{i'}$, of class i', $1 \leq i' < i$, to group j of class i is either $prob(\mathscr{SC}^{|\overline{\theta}_i|}_{\leq \overline{\theta}_i}(P_k,\overline{T}'_{i',j'}))$ or zero depending on whether $\overline{R}_{i',j'} \cap \overline{R}_{i,j} \neq \emptyset$ or not, where $\overline{T}'_{i',j'}$ is obtained from $\overline{T}_{i',j'}$ by enabling at the final step only passive visible actions whose name belongs to $\overline{R}_{i',j'} \cap \overline{R}_{i,j}$. From $P_1 \sim_{\rm MT} P_2$ it follows $prob(\mathscr{SC}^{|\overline{\theta}_i|}_{\leq \overline{\theta}_i}(P_1,\overline{T}'_{i',j'})) = prob(\mathscr{SC}^{|\overline{\theta}_i|}_{\leq \overline{\theta}_i}(P_2,\overline{T}'_{i',j'}))$ and hence:

$$prob(\mathscr{RCC}^{|\overline{\theta}'_i|}_{\leq \overline{\theta}'_i}(P_1,(\alpha,\overline{R}_{i,j}))) = prob(\mathscr{RCC}^{|\overline{\theta}'_i|}_{\leq \overline{\theta}'_i}(P_2,(\alpha,\overline{R}_{i,j})))$$

We finally consider the classes of the first partition, each of which is formed by a single group of matching computations characterized by the empty ready set as the total exit rate of their final state is zero. Let us examine a generic class of matching computations of the first partition, say the one whose associated extended stepwise average duration is $\hat{\theta}_l$, $1 \leq l \leq \hat{n}$. If we take a test $\hat{T}$ composed of a sequence terminated by s of passive visible actions whose names and order are the same as those of the actions occurring in α, which at every step also enables passive actions with all the other visible names occurring in P_1 or P_2 each followed by $<\!{\rm z}, *_1\!>$.s, then from $P_1 \sim_{\rm MT} P_2$ we derive:

$$prob(\mathscr{SC}^{|\hat{\theta}_l|}_{\leq \hat{\theta}_l}(P_1,\hat{T})) = prob(\mathscr{SC}^{|\hat{\theta}_l|}_{\leq \hat{\theta}_l}(P_2,\hat{T}))$$

where – due to the structure of $\hat{T}$ – for $k \in \{1,2\}$ it holds:

$$prob(\mathscr{SC}^{|\hat{\theta}_l|}_{\leq \hat{\theta}_l}(P_k,\hat{T})) = prob(\mathscr{CC}^{|\hat{\theta}_l|}_{\leq \hat{\theta}_l}(P_k,\alpha)) = prob(\mathscr{RCC}^{|\hat{\theta}_l|}_{\leq \hat{\theta}_l}(P_k,(\alpha,\emptyset))) + \sum_{\overline{R}_{i,j} \neq \emptyset} prob(\mathscr{RCC}^{|\hat{\theta}_l|}_{\leq \hat{\theta}_l}(P_k,(\alpha,\overline{R}_{i,j})))$$

Since $\sum_{\overline{R}_{i,j} \neq \emptyset} prob(\mathscr{RCC}^{|\hat{\theta}_l|}_{\leq \hat{\theta}_l}(P_1,(\alpha,\overline{R}_{i,j}))) = \sum_{\overline{R}_{i,j} \neq \emptyset} prob(\mathscr{RCC}^{|\hat{\theta}_l|}_{\leq \hat{\theta}_l}(P_2,(\alpha,\overline{R}_{i,j})))$, it holds:

$$prob(\mathscr{RCC}^{|\hat{\theta}_l|}_{\leq \hat{\theta}_l}(P_1,(\alpha,\emptyset))) = prob(\mathscr{RCC}^{|\hat{\theta}_l|}_{\leq \hat{\theta}_l}(P_2,(\alpha,\emptyset)))$$

Due to the generality of α and the consideration of all the possible ready sets after α occurring in P_1 or P_2 together with their threshold stepwise average durations, we can conclude that $P_1 \sim_{\rm MR} P_2$. ■

We conclude by showing the algorithm (based on Prop. A.16) for checking whether $P_1 \sim_{\rm MT} P_2$:

1. Transform $[\![P_1]\!]$ and $[\![P_2]\!]$ into their equivalent discrete-time versions:
   (a) Divide the rate of each transition by the total exit rate of its source state.
   (b) Augment the name of each transition with the total exit rate of its source state.
2. Compute the equivalence relation $\mathscr{R}$ that equates any two states of the discrete-time versions of $[\![P_1]\!]$ and $[\![P_2]\!]$ whenever the two sets of augmented action names labeling the transitions departing from the two states coincide.
3. For each equivalence class R induced by $\mathscr{R}$, consider R as the set of accepting states and check whether the discrete-time versions of $[\![P_1]\!]$ and $[\![P_2]\!]$ are probabilistic language equivalent.

4. Return yes/no depending on whether all the checks performed in the previous step have been successful or at least one of them has failed.

Each iteration of step 3 above requires the application of the algorithm for probabilistic language equivalence. Denoted by $NameReal_{P_1,P_2}$ the set of augmented action names labeling the transitions of the discrete-time versions of $[\![P_1]\!]$ or $[\![P_2]\!]$, the algorithm visits in breadth-first order the tree containing a node for each element of $(NameReal_{P_1,P_2})^*$ and studies the linear independence of the state probability vectors associated with a finite subset of the tree nodes:

1. Create an empty set V of state probability vectors.
2. Create a queue whose only element is the empty string ε.
3. While the queue is not empty:
   (a) Remove the first element from the queue, say string ς.
   (b) If the state probability vector of the discrete-time versions of $[\![P_1]\!]$ and $[\![P_2]\!]$ after reading ς does not belong to the vector space generated by V, then:
      i. For each $a \in NameReal_{P_1,P_2}$, add $\varsigma \circ a$ to the queue.
      ii. Add the state probability vector to V.
4. Build a three-valued state vector u whose generic element is:
   (a) 0 if it corresponds to a nonaccepting state.
   (b) 1 if it corresponds to an accepting state of the discrete-time version of $[\![P_1]\!]$.
   (c) -1 if it corresponds to an accepting state of the discrete-time version of $[\![P_2]\!]$.
5. For each $v \in V$, check whether $v \cdot u^{\mathrm{T}} = 0$.
6. Return yes/no depending on whether all the checks performed in the previous step have been successful or at least one of them has failed.

The time complexity of the algorithm is $O(n^5)$, where n is the total number of states of $[\![P_1]\!]$ and $[\![P_2]\!]$.