A Process Algebra Approach to Fuzzy Reasoning

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Outline...

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1. Fuzzy Logic: a Short Introduction
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   - Fuzzy Sets
   - Fuzziness & Probability
2. Examples
3. A Fuzzy Process Algebra
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   - FCCS
4. A Fuzzy Modal Logic
   - FHML
5. Behavioural Equivalencies
   - Fuzzy Bisimulation
6. Logical Characterization of Fuzzy Bisimulation
7. Conclusions

Fuzzy Logic

Logical System

- Multi-valued logic
- Membership function
- Linguistic Variables and value
- Operators

Theory of classes with unsharp boundaries

- Fuzzy Rules
- Fuzzification and Defuzzification

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Fuzzy Sets

- Classes of objects in which transit from membership to not membership *gradually* takes place;
- Denoted by means of a generalised *membership function* $\mu_A : U \rightarrow [0, 1]$;
- Standard operations on sets can be generalised to Fuzzy Sets:

Complement, t-norm and t-conorm

$$\mu_{\overline{A}}(x) = c(\mu_A(x))$$

$$\mu_{A \cap B}(a) = i(\mu_A(a), \mu_B(a))$$

$$\mu_{A \cup B}(a) = u(\mu_A(a), \mu_B(a))$$

An example: Max-Min

$$c(x) = 1 - x$$

$$i(x, y) = \min[x, y]$$

$$u(x, y) = \max[x, y]$$

- Commutativity, associativity, idempotence and absorption law are verifiable.

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Axioms

Axioms for $c(\cdot)$

$$c(1) = 0 \quad c(0) = 1 \quad \frac{x \leq y}{c(y) \leq c(x)} \quad c(c(x)) = x$$

Axioms for $u(\cdot,\cdot)$

$$u(0,0) = 0 \quad u(0,x) = x \quad u(1,x) = 1 \quad u(x,y) = u(y,x)$$

$$\frac{x_1 \leq x_2 \quad y_1 \leq y_2}{u(x_1,y_1) \leq u(x_2,y_2)} \qquad u(u(x,y),z) = u(x,u(y,z))$$

Axioms for $i(\cdot,\cdot)$

$$i(1,1) = 1 \quad i(0,x) = 0 \quad i(1,x) = x \quad i(x,y) = i(y,x)$$

$$\frac{x_1 \leq x_2 \quad y_1 \leq y_2}{i(x_1,y_1) \leq i(x_2,y_2)} \qquad i(i(x,y),z) = i(x,i(y,z))$$

Example: Poison drink

"Drink me" is a poison drink:

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Probability

"Drink me" is toxic with a probability of 0, 1.

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Example: Temperature Control System (TCS)

- TCS is composed of an *air conditioner system* (AC) and a *temperature sensor*;
- TCS regulates the AC power following values read from the sensor;
- *Room temperature* can be modeled by considering different states identifying different range of values:

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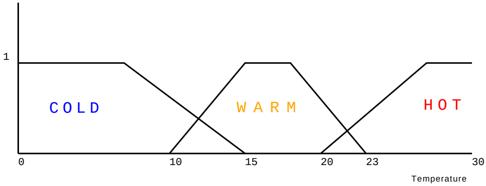

Example: Temperature Control System (TCS)

Fuzzy inference System

IF	temperature	IS	*warm*	THEN	*don't modify* AC power
IF	temperature	IS	*hot*	THEN	*increase* AC power
IF	temperature	IS	*cold*	THEN	*decrease* AC power

⋮

Outline...

Fuzzy Labelled Transition Systems

Definition ($\mathcal{L}$-FLTS)

Let $\mathcal{L} = \langle L, c, i, u \rangle$, a Fuzzy Labelled Transition System $\mathcal{F}$ for $\mathcal{L}$ ($\mathcal{L}$-FLTS) is a tuple $\langle Q, A, \chi_{\rightarrow} \rangle$ where:

- Q is a set whose elements are called states
- A is a finite set whose elements represent actions
- $\chi_{\rightarrow} : (Q \times A \times Q) \rightarrow L$ is the total *membership function*.

- $q_0 \xrightarrow{\alpha}_{\varepsilon} q_1$ denotes the transition from state q_0 to state q_1 with action α and membership degree ε to the automaton.
- Next states are selected nondeterministically.
- The membership degree associated to each transition is used to give a measure to computations.

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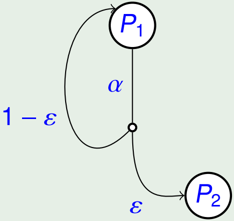
Fuzzy Labelled Transition Systems

Probabilistic LTS



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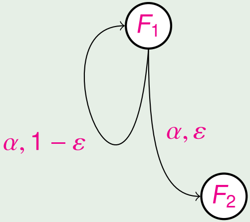

If we consider the PLTS, the total probability of computations leading to state P_2 is 1.

If we consider the FLTS, the membership degree of the computations leading to state F_2, when considering lattice L, is ε.

This intuitively means that state F_2 is reachable by performing a transition whose membership degree is ε.

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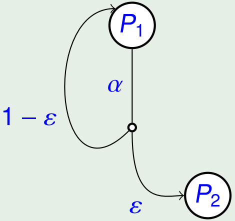

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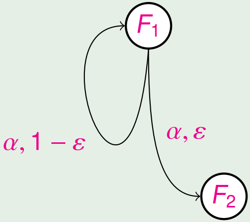

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If we consider the FLTS, the membership degree of the computations leading to state F_2, when considering lattice L, is ε.
This intuitively means that state F_2 is reachable by performing a transition whose membership degree is ε.

Fuzzy CCS

Syntax

$$
\begin{aligned}
Q &::= nil \mid X \mid \sum_{i \in I} (act_i, x_i).Q_i \\
P &::= Q \mid P_1 \mid P_2 \mid P \backslash A \mid P[f] \\
act &::= \bar{a} \mid a \mid \tau
\end{aligned}
$$

- The fundamental difference from CCS is the introduction of an attribute "x_i", that we define *action execution degree*. It is a fuzzy value able to represent, on a quality level, action behaviour.
- Operational semantics of FCCS processes is defined in term of FLTS. This approach permits modeling situations like: "the transition takes place *rarely*" or "the transition occurs *frequently*".
- Semantics for FCCS processes is defined by considering function $\mathcal{N}$ that associates to each process P and transition α the fuzzy set $\mathcal{P}$ of processes reachable from P with α.

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$\mathcal{N}$ext Function

$$\mathcal{N}(\langle\alpha,\varepsilon\rangle.P,\beta) = \begin{cases} \{P:\varepsilon\} & \textit{if } \alpha=\beta \\ \emptyset & \textit{else} \end{cases}$$

$$\mathcal{N}(P+Q,\alpha) = \mathcal{N}(P,\alpha) \cup \mathcal{N}(Q,\alpha)$$

$$\mathcal{N}(P\backslash L,\alpha) = \begin{cases} \mathcal{N}(P,\alpha)\backslash L & \textit{if}\alpha \notin L \\ \\ \emptyset & \textit{else} \end{cases}$$

$$\mathcal{N}(P[f],\alpha) = \bigcup_{\beta:f(\beta)=\alpha} \mathcal{N}(P,\beta)[f]$$

$$\mathcal{N}(P|Q,\alpha) = \begin{cases} [\mathcal{N}(P,\alpha)|Q] \cup [P|\mathcal{N}(Q,\alpha)] & \textit{if}\alpha \neq \tau \\ \\ [\mathcal{N}(P,\tau)|Q] \cup [P|\mathcal{N}(Q,\tau)] \cup & \textit{if}\alpha = \tau \\ [\bigcup_{\alpha\in\Lambda}(\mathcal{N}(P,\alpha)|\mathcal{N}(Q,\bar{\alpha}))] \cup & \\ [\bigcup_{\alpha\in\Lambda}(\mathcal{N}(P,\bar{\alpha})|\mathcal{N}(Q,\alpha))] & \end{cases}$$

The TCS implemented with FCCS

FCCS

$$SYS_{st} \triangleq (SENS_{st}|H|N|C)\backslash\{hot, warm, cold, inc, dec, noop\}$$

$$H \triangleq \langle hot, 1\rangle.\langle \overline{inc}, 1\rangle.H$$

$$N \triangleq \langle warm, 1\rangle.\langle \overline{noop}, 1\rangle.N$$

$$C \triangleq \langle cold, 1\rangle.\langle \overline{dec}, 1\rangle.C$$

$$\begin{aligned} SENS_t \triangleq\ & \langle \overline{hot}, \mu_{hot}(t)\rangle.AC_t \\ & +\langle \overline{warm}, \mu_{warm}(t)\rangle.AC_t \\ & +\langle \overline{cold}, \mu_{cold}(t)\rangle.AC_t \\ & +\langle temp_t, 1\rangle.SENS_t \end{aligned}$$

$$\begin{aligned} AC_t \triangleq\ & \langle inc, 1\rangle.SENS_{t+1} \\ & +\langle dec, 1\rangle.SENS_{t-1} \\ & +\langle noop, 1\rangle.SENS_t \end{aligned}$$

Outline. . .

Fuzzy HML

Syntax

$$\varphi ::= tt \mid \neg\,\varphi \mid \varphi \bowtie \varepsilon \mid \varphi_1 \,\wedge\, \varphi_2 \mid \langle\alpha\rangle\,\varphi \mid X \mid \nu X.\varphi$$

where $\varepsilon \in L$.

- The intersection between a L-fuzzy set and its complement could be not empty. In general, $\mathcal{M}_{\mathcal{L},\mathcal{F}}[\![\neg\varphi]\!] \neq c\,(\mathcal{M}_{\mathcal{L},\mathcal{F}}[\![\varphi]\!])$.
- FHML semantics is defined in term of functions $\mathcal{M}_{\mathcal{L},\mathcal{F}}$ and $\mathcal{M}^{\smile}_{\mathcal{L},\mathcal{F}}$ that, for each formula φ, return the L-Fuzzy Set that gives the measure of satisfaction and unsatisfaction of φ.
- This approach permits defining semantics of FHML in a general way, without considering any special constraint on the underlying L-Fuzzy Sets.

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Formulae Satisfaction Semantics

$$\mathcal{M}_{\mathcal{L},\mathcal{F}}[\![tt]\!]\delta(p) = 1$$

$$\mathcal{M}_{\mathcal{L},\mathcal{F}}[\![\neg\varphi]\!]\delta(p) = \tilde{\mathcal{M}}_{\mathcal{L},\mathcal{F}}[\![\varphi]\!]\delta(p)$$

$$\mathcal{M}_{\mathcal{L},\mathcal{F}}[\![\varphi_1 \wedge \varphi_2]\!]\delta(p) = i\,(\mathcal{M}_{\mathcal{L},\mathcal{F}}[\![\varphi_1]\!]\delta(p), \mathcal{M}_{\mathcal{L},\mathcal{F}}[\![\varphi_2]\!]\delta(p))$$

$$\mathcal{M}_{\mathcal{L},\mathcal{F}}[\![\varphi \bowtie \varepsilon]\!]\delta(p) = \begin{cases} 1 & \text{if } \mathcal{M}_{\mathcal{L},\mathcal{F}}[\![\varphi]\!]\delta(p) \bowtie \varepsilon \\ 0 & \text{else} \end{cases}$$

$$\mathcal{M}_{\mathcal{L},\mathcal{F}}[\![\langle\alpha\rangle\varphi]\!]\delta(p) = u_{q\in Q}\left(i\left(\chi_{\rightarrow}(p,\alpha,q), \mathcal{M}_{\mathcal{L},\mathcal{F}}[\![\varphi]\!]\delta\,(q)\right)\right)$$

$$\mathcal{M}_{\mathcal{L},\mathcal{F}}[\![X]\!]\delta = \delta(X)$$

$$\mathcal{M}_{\mathcal{L},\mathcal{F}}[\![\nu X.\varphi]\!]\delta = \cup\left\{\chi \mid \chi \leq \mathfrak{F}_X^{\delta,\varphi}(\chi)\right\}$$

where $\mathfrak{F}_X^{\delta,\varphi} = \mathcal{M}_{\mathcal{L},\mathcal{F}}[\![\varphi]\!]\delta[\chi/X]$

Formulae Unsatisfaction Semantics

$$\mathcal{M}^{\smile}_{\mathcal{L},\mathcal{F}}[\![tt]\!]\delta(p) = 0$$

$$\mathcal{M}^{\smile}_{\mathcal{L},\mathcal{F}}[\![\neg\varphi]\!]\delta(p) = \mathcal{M}_{\mathcal{L},\mathcal{F}}[\![\varphi]\!]\delta(p)$$

$$\mathcal{M}^{\smile}_{\mathcal{L},\mathcal{F}}[\![\varphi_1 \wedge \varphi_2]\!]\delta(p) = u\left(\mathcal{M}^{\smile}_{\mathcal{L},\mathcal{F}}[\![\varphi_1]\!]\delta(p), \mathcal{M}^{\smile}_{\mathcal{L},\mathcal{F}}[\![\varphi_2]\!]\delta(p)\right)$$

$$\mathcal{M}^{\smile}_{\mathcal{L},\mathcal{F}}[\![\varphi \bowtie \varepsilon]\!]\delta(p) = \begin{cases} 1 & \textit{if } \mathcal{M}^{\smile}_{\mathcal{L},\mathcal{F}}[\![\varphi]\!]\delta(p) \not\bowtie \varepsilon \\ 0 & \textit{else} \end{cases}$$

$$\mathcal{M}^{\smile}_{\mathcal{L},\mathcal{F}}[\![\langle\alpha\rangle\varphi]\!]\delta(p) = i_{q\in Q}\left(u\left(i\left(\chi_{\rightarrow}(p,\alpha,q), \mathcal{M}^{\smile}_{\mathcal{L},\mathcal{F}}[\![\varphi]\!]\delta(q)\right), c\left(\chi_{\rightarrow}(p,\alpha,q)\right)\right)\right)$$

$$\mathcal{M}^{\smile}_{\mathcal{L},\mathcal{F}}[\![X]\!]\delta = \delta(X)$$

$$\mathcal{M}^{\smile}_{\mathcal{L},\mathcal{F}}[\![\nu X.\varphi]\!]\delta = \cap\left\{\chi \mid \chi \geq^{\smile} \mathfrak{F}^{\delta,\varphi}_{X}(\chi)\right\}$$

where $^{\smile}\mathfrak{F}^{\delta,\varphi}_{X} = \mathcal{M}^{\smile}_{\mathcal{L},\mathcal{F}}[\![\varphi]\!]\delta[\chi/X]$

Example

FHML can be used for specifying that system process SYS_{st} can always reach a configuration where the room temperature is between 18 and 20 degrees.
The following formula states that a configuration where temperature is between 18 and 20 degrees is eventually reached:

$$\varphi = \mu X.\langle \overline{temp_{18}} \rangle tt \vee \langle \overline{temp_{19}} \rangle tt \vee \langle \overline{temp_{20}} \rangle tt \vee \langle \tau \rangle X$$

While formula:

$$\nu Y.\varphi \wedge [\tau] Y$$

states that φ is always satisfied.

Outline. . .

Fuzzy Bisimulation

Definition of Fuzzy Bisimulation is straightforward and it is somehow reminiscent of Stochastic Bisimulation.

Definition (Fuzzy Bisimulation)

Let $\langle S, A, \chi_{\rightarrow} \rangle$ a Fuzzy LTS. An equivalence relation $R \subseteq S \times S$ is a *fuzzy bisimulation* if and only if for each p and q in S, for each equivalent class C of R in S, and for each transition label α:

$$\chi_{\rightarrow}(p, \alpha, C) = \chi_{\rightarrow}(q, \alpha, C)$$

where:

$$\chi_{\rightarrow}(p, \alpha, C) = \underline{\bigvee}_{p' \in C} \chi_{\rightarrow}(p, \alpha, p')$$

Definition (Fuzzy Bisimilarity)

Let $\langle S, A, \chi_{\rightarrow} \rangle$ be a Fuzzy LTS. We say that $p, q \in Q$ are bisimilar ($p \sim_F q$), if there exists a fuzzy bisimulation R such that pRq.

Outline...

Logical Characterization of Fuzzy Bisimulation

- Formulae satisfaction induces an equivalence on the interpretation model.
- Two states of a Fuzzy LTS are equivalent if (and only if) they satisfy the same set of formulae.

Theorem

Let $\mathcal{F} = \langle S, A, \chi_{\rightarrow} \rangle$ be finite-branching. For each $p, q \in S$,

$$p \sim_F q \quad \Leftrightarrow \quad \forall \varphi,\ \mathcal{M}[\![\varphi]\!](p) = \varepsilon \Leftrightarrow \mathcal{M}[\![\varphi]\!](q) = \varepsilon$$
$$\text{and } \mathcal{M}^{\smile}[\![\varphi]\!](p) = \varepsilon \Leftrightarrow \mathcal{M}^{\smile}[\![\varphi]\!](p) = \varepsilon$$

Outline...

Summing up

- Human perception of real world abounds with concepts without strictly defined constraint.
- Fuzzy inference systems lack in compositionality and in methods for property verification.
- FCCS allows formal representation of systems behaviour in term of imprecision measure.
- Fuzzy measures can represent costs in the described system.