RECENT AND NEW PERSPECTIVES IN NONLINEAR ANALYSIS

November 3-4, 2022

Aula Magna
Collegio Raffaello
Piazza della Repubblica, 13 Urbino

Book of Abstracts

Maximum principle (with and without Patrizia Pucci) in some elliptic problems

Lucio Boccardo
Sapienza Università di Roma, Italy

How much positive are solutions of Dirichlet problems (with degenerate, noncoercive, . . . differential operators) with positive data?

Some results for perturbed (p, q)–quasilinear elliptic problems

Anna Maria Candela
Università degli Studi di Bari Aldo Moro, Italy

Let us consider the (p, q)–quasilinear elliptic problem

$$\begin{cases} -\Delta_p u - \Delta_q u \; = \; \lambda_\infty |u|^{q-2} u + f(x,u) + \varepsilon \; h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < q < +\infty$, $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $f(x, u)/|u|^{q-1}$ goes to 0 at infinity, $\varepsilon \in \mathbb{R}$ and $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a given function.

Under the only hypothesis that $h(x, u)$ is continuous, this perturbed problem may lose its variational structure. Anyway, suitable assumptions on $f(x, u)$ and appropriate procedures and estimates allow us to prove the existence of one solution, eventually more than one if all the terms are odd, if ε is small enough when λ_∞ is not an eigenvalue of $-\Delta_q$ in $W_0^{1,q}(\Omega)$ but interacts in a suitable way with sequences of pseudo–eigenvalues of such an operator.

These results are part of joint works with Rossella Bartolo and Addolorata Salvatore.

On the Leray problem and the Poiseuille flow

Michel Chipot
University of Zürich, Switzerland

The goal of this talk is to present a simple proof of existence and uniqueness of the solution of the Leray problem for high viscosities and homogeneous or nonhomogeneous boundary conditions.

Furthermore we address the issue of uniqueness of the Poiseuille flow in a pipe.

Unique continuation failure for an elliptic equation with a singular absorption term proposed for the pioneering modeling of electron beams

Jesus Ildefonso Diaz
Universidad Complutense de Madrid, Spain

Given $a > 0$ the main goal of the lecture is to find sufficient conditions on a function $j : \mathbb{R} \to [0, +\infty)$, with

$$(1) \qquad \begin{cases} j(x) > 0 \quad \text{if } x \in (-a,a), \ j \in L^1_{loc}(-a,a), \\ j(x) = 0 \qquad\qquad\qquad\quad \text{if } x \notin [-a,a], \end{cases}$$

in order to get the solvability of the singular nonlinear boundary value problem

$$(2) \qquad P_{\infty,a,j} = \begin{cases} -\Delta u + \frac{j(x)}{\sqrt{u}} = 0 & x \in (-\infty,+\infty), \ y \in (0,1), \\ u(x,0) = 0 & x \in (-\infty,+\infty), \\ u(x,1) = 1 & x \in (-\infty,+\infty), \\ \lim_{|x|\to+\infty} u(x,y) = y & y \in (0,1), \end{cases}$$

with the additional conditions

$$(3) \qquad AC_{\infty,a} = \begin{cases} \frac{\partial u}{\partial y}(x,0) = 0 & x \in (-a,a), \\ u(x,y) > 0 & x \in (-\infty,+\infty), \ y \in (0,1). \end{cases}$$

The study of the overdetermined problem (2), (3) was initiated, in the one-dimensional case (formally corresponding to the case $a = +\infty$), in the early part of the last century (by C.D. Child on 1903, and by I. Langmuir in a series of papers starting on 1904). An elegant study of the onedimensional case is due to H. Brezis (personal communication, 2004). The above formulation corresponds to a dimensionless formulation of space-charge-limited flows in diodes (the cathode is here represented by the subset $[-a,a] \times \{0\}$ and the additional condition $\frac{\partial u}{\partial y}(x,0) = 0$ for $x \in (-a,a)$ represents the vanishing of the electric field on it). It arises in many relevant applications (for instance in cancer treatments). The above formulation (2), (3) was proposed in A. Rokhlenko and J.L. Lebowitz, Phys. Rev. Lett. (2003). Notice that the additional conditions imply a failure of the "unique continuation property".

In the lecture we will prove, and make precise, a conjecture by A. Rokhlenko (Journal of Applied Physics (2006)): if $j(x)$ behaves as $A/|x \pm a|^\beta$, for some $\beta \in (0,1/2)$ and $A > 0$, near the boundary of the cathode, $x = \pm a$, then there exists a weak solution $u(x,y)$ of (2), satisfying (3), and u behaves (near the cathode $[-a,a] \times \{0\}$) as y^α for some $\alpha \in (1,4/3)$ with $\alpha = 4/3 - 2\beta$. The proof is made by constructing suitable super and subsolutions for several auxiliary problems and matching the corresponding solutions with a H^1 criterion. In particular, the global bifurcation diagram (in terms of the parameter λ) associated to the auxiliary problem

$$(4) \qquad \begin{cases} -U''(s) + \frac{C}{\sqrt{U(s)}} = \lambda U(s) \quad s \in (-R,R), \\ U(\pm R) = 0, \end{cases}$$

is completely characterized, where the positive constants C and R are given. We show: i) there is a bifurcation from the infinity for λ near $\lambda_1(R)$ (the first eigenvalue of the linear problem with $C = 0$), ii) the bifurcation curve is strictly decreasing (which implies the uniqueness of nonnegative solutions) and iii) the curve is not C^1 for a suitable value $\lambda = \lambda^* > \lambda_1(R)$ corresponding to a "flat solution" (i.e. the solution U is such that $U'(\pm R) = 0$ and $U(s) > 0$ for any $s \in (-R,R)$). This extends several results in the previous literature (see, e.g., Díaz-Hernández-Mancebo (2009) and its references).

A universal heat semigroup characterization of Sobolev and BV spaces in Carnot groups

Nicola Garofalo
Università degli Studi di Padova, Italy

In sub-Riemannian geometry there exist, in general, no known explicit representations of the heat kernels, and these functions fail to have any symmetry. In particular, they are not a function of the control distance, nor they are, in general, spherically symmetric in any of the layers of the Lie algebra. Despite this, the heat kernels possess two unexpected 'one-dimensional' symmetries. In this talk I present some notable consequences of these properties in a Carnot group of arbitrary step. I also discuss a dimensionless heat based version of the famous Bourgain-Brezis-Mironescu phenomenon. This is joint work with Giulio Tralli.

On the harmonic characterization of the spheres: rigidity and stability results

Ermanno Lanconelli
Alma Mater Studiorum - Università di Bologna, Italy

Let D be a bounded open set of $\mathbb{R}^n$ with finite perimeter and let x_0 be a point of D. Assume that $u(x_0)$ equals the average of u on ∂D for every harmonic functions u in D continuous up to the boundary. In this case one says that D is a pseudoshere centered at x_0.

In general the pseudosferes are not spheres: in 1937 Keldysch and Lavrentieff constructed a pseudosphere in $\mathbb{R}^2$ which is not a circle. As a consequence, the following problem naturally arose: when a pseudosphere is a sphere? Or, roughly speaking: is it possible to characterize the Euclidean spheres via the Gauss mean value property for harmonic function? The answer is yes.

The most general result in this direction was obtained by Lewis and Vogel in 2002: they proved that a pseudosphere is a sphere if the surface measure of its boundary has at most an Euclidean growth.

Preiss and Toro, in 2007, proved the stability of Lewis and Vogel's result. Namely: a bounded domain D, whose boundary has the Lewis and Vogel regularity property, is geometrically close to a sphere centered at x_0 if the Poisson kernel of D with pole at x_0 is almost constant.

In collaboration with Giovanni Cupini we proved that analogous rigidity and stability results hold true if the domain D has finite perimeter and only satisfies the following property: in at least one point of ∂D closest to x_0 the boundary of D is Lyapunov-Dini regular. To this end we have introduced and studied what we called the harmonic gap of ∂D with respect to x_0.

We defined this harmonic gap by using the Poisson kernel of the biggest ball centered at x_0 and contained in D.

Local Lipschitz continuity for elliptic equations under general growth conditions

Paolo Marcellini
Università degli Studi di Firenze, Italy

We give some *existence* and *interior regularity results* for weak solutions of elliptic equations in divergence form of the type

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i\left(x, u\left(x\right), Du\left(x\right)\right) = b\left(x, u\left(x\right), Du\left(x\right)\right)\ ,$$

in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$. The vector field $\left(a^i\left(x,s,\xi\right)\right)_{i=1,2,\ldots,n}$ satisfies some *general growth conditions* with respect to the gradient variable $\xi \in \mathbb{R}^n$, the so-called *$p,q-$growth conditions*. The novelties with respect to the mathematical literature on this topic are the general growth conditions and the explicit dependence of the differential equation on u, other than on its gradient Du and on the x variable.

The seminar will also examine the problem of the existence and multiplicity of weak solutions of the Dirichlet problems associated with the above elliptic differential equation, in the context of $p,q-$growth.

Some details can be found in the recent article:
P. Marcellini: Local Lipschitz continuity for $p,q-$PDEs with explicit $u-$dependence, Nonlinear Analysis, on-line 2022, `https://doi.org/10.1016/j.na.2022.113066`.

The Brouwer fixed point theorem and periodic solutions

Jean Mawhin
Université Catholique de Louvain, Belgium

The Brouwer fixed point theorem (1911) tells that any continuous self mapping of a n-cell into itself has at least one fixed point.

The first application ot this theorem to the existence of periodic solutions of an ordinary differential equations was given by Lefschetz in 1942, and followed by many other ones.

We state and prove, using the Brouwer fixed point theorem, an existence theorem for periodic solutions of an ordinary differential system containing as special cases earlier results of Krasnosel'skii (1968) and Gustafsson-Schmitt (1974). We then show that this existence theorem implies in turn the Brouwer fixed point theorem. This is a joint work with José Angel Cid.

Hopf, Caccioppoli and Schauder, reloaded

Rosario Mingione
Università degli Studi di Parma, Italy

So called Schauder estimates are in fact a contribution, at various stages, of Hopf, Caccioppoli and Schauder, between the end of the 20s and the beginning of the 30s. Later on, they were extended, with various degrees of precision, to nonlinear uniformly elliptic equations. I will present the solution to the longstanding open problems of proving estimates of such kind in the nonuniformly elliptic case and for minima of non-differentiable functionals (again considered in the nonuniformly elliptic case). From joint work with Cristiana De Filippis.

Superharmonic functions in a half space: representation, comparison principles and Liouville theorems

Enzo Mitidieri
Università degli Studi di Trieste, Italy

We prove a representation formula for superharmonic functions on the half space $\mathbb{R}^N_+ := \mathbb{R}^{N-1}\times]0,+\infty[$. As a result, we derive various comparison principles, qualitative properties of solutions of integral and differential inequalities, as well as related Liouville theorems.

Overdetermined problems and constant mean curvature surfaces in cones

Filomena Pacella
Sapienza Università di Roma, Italy

We present some recent results about the characterization of domains inside a cone which admit a solution for a partial overdetermined problem. A parallel question is to study constant mean curvature surfaces with boundary in cones. The relation between the two problems and the connection with an isoperimetric inequality will be discussed as well as the role of the convexity of the cone and the construction of counterexamples.

Two striking results in the analysis of double phase problems

Vicentiu Radulescu
University of Craiova and Romanian Academy, Romania

In this talk, I shall report on some recent results obtained jointly with N. Papageorgiou, C. Alves and D. Repovs. In the first part, I will discuss an interesting discontinuity property of the spectrum in the case of isotropic double phase equations with Dirichlet boundary condition. Next, I shall consider the anisotropic setting and I will discuss three sufficient conditions for the existence of solutions in the case of double phase with multiple 'subcritical-critical-supercritical' regimes.

Multiplicity via non-convexity

Biagio Ricceri
Università di Catania, Italy

Here is a typical result that will be presented in my lecture:

THEOREM A - *Let X be a topological space, E a real normed space and $S \subset E^*$ a convex set weakly-star dense in E^*. Moreover, let $I : X \to \mathbf{R}$ and $\psi : X \to E$ be such that $\psi(X)$ is not convex and $I + \eta \circ \psi$ is lower semicontinuous and inf-compact for all $\eta \in S$.*
Then, there exists $\tilde{\eta} \in S$ such that $I + \tilde{\eta} \circ \psi$ has at least two global minima in X.

Among the applications of Theorem A there is the following:

THEOREM B - *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function, let $\rho > 0$ and let $\omega : [0, \rho[\to [0, +\infty[$ be a continuous increasing function such that $\lim_{\xi\to\rho^-} \int_0^\xi \omega(x)dx = +\infty$. Consider $C^0([0, 1]) \times C^0([0, 1])$ endowed with the norm*

$$\|(\alpha, \beta)\| = \int_0^1 |\alpha(t)|dt + \int_0^1 |\beta(t)|dt \ .$$

Then, the following assertions are equivalent:

(a) the restriction of f to $\left[-\frac{\sqrt{\rho}}{2}, \frac{\sqrt{\rho}}{2}\right]$ is not constant;

(b) for every convex set $S \subseteq C^0([0,1]) \times C^0([0,1])$ dense in $C^0([0,1]) \times C^0([0,1])$, there exists $(\alpha, \beta) \in S$ such that the problem

$$\begin{cases} -\omega\left(\int_0^1 |u'(t)|^2 dt\right) u'' = \beta(t) f(u) + \alpha(t) & \text{in } [0,1] \\ u(0) = u(1) = 0 \\ \int_0^1 |u'(t)|^2 dt < \rho \end{cases}$$

has at least two classical solutions.

Einstein-type structures: two rigidity results

Marco Rigoli
Università degli Studi di Milano Statale, Italy

Under the notion of an Einstein-type structure on a Riemannian manifold we recover many geometric settings which have been deeply investigated in recent years. For instance, Ricci solitons, Ricci-harmonic solitons, Quasi-Einstein manifold... and others coming from General Relativity as Vacuum static spaces or directly from the theory of Einstein manifolds, as the Critical Point Equation or from the theory of Critical Metrics. We present two rigidity results, the first involving the Kth curvature operators, that show under appropriate conditions, that the starting Riemannian manifold has to be a sphere.

Higher order boundary Harnack principle on nodal domains via degenerate equations

Susanna Terracini
Università degli Studi di Torino, Italy

The ratio v/u of two solutions to a second order elliptic equation in divergence form solves a degenerate elliptic equation if u and v share the zero set; that is, $Z(u) \subseteq Z(v)$. The coefficients of the degenerate equation vanish on the nodal set as u^2. Developing a Schauder theory for such equations, we prove $C^{k,\alpha}$-regularity of the ratio from one side of the regular part of the nodal set in the spirit of the higher order boundary Harnack principle established by De Silva and Savin in [4]. Then, by a gluing lemma, the estimates extend across the regular part of the nodal set. Eventually, using conformal mapping in dimension $n = 2$, we provide local gradient estimates for the ratio which hold also across the singular part of the nodal set and depends on the highest value attained by the Almgren frequency function.

References

[1] S. Terracini, G. Tortone and S. Vita, *Higher order boundary Harnack prin- ciple on nodal domains via degenerate equations*, preprint, 2022.
[2] Y. Sire, S. Terracini and S. Vita. *Liouville type theorems and regularity of solutions to degenerate or singular problems part I: even solutions.* Comm. Partial Differential Equations, 46-2 (2021), 310-361.
[3] Y. Sire, S. Terracini and S. Vita. *Liouville type theorems and regularity of solutions to degenerate or singular problems part II: odd solutions*, Mathematics in Engineering, 3-1 (2021), 1-50.
[4] D. De Silva and O. Savin. *A note on higher regularity boundary Harnack inequality*, DCDS-A, 35(12), (2015) 6155-6163.