SCHOOL ON
NONLINEAR ELLIPTIC PROBLEMS

organized by

Giovanni Molica Bisci, Simone Secchi and Raffaella Servadei

at the

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Aims and scope

Nonlinear PDEs can be used to describe a wide variety of phenomena arising in different contexts such as geometry, physics, mechanics, engineering and, more recently, life sciences, just to name a few.

The aim of the school is to present some recent results and future trends on Nonlinear Elliptic Problems and their applications, by leading together experts in this field.

The courses organized within the school are addressed to Ph.D. students as well as Post-Doctoral and active researchers interested mostly in Nonlinear Analysis, Partial Differential Equations and their many applications.

Courses

The influence of fractional diffusion in Allen-Cahn and KPP type equations

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In this mini-course I will start explaining basic ideas concerning fractional Laplacians as well as the essential tools to treat nonlinear equations involving these operators. The main part of the course will present results on stationary solutions of fractional equations (mainly of bistable or Allen-Cahn type) and on front propagation with fractional diffusion for Fisher-KPP, combustion, and bistable reactions. The course will focus in the following papers (see arXiv):

- In several works in collaboration with Y. Sire and E. Cinti, we study the existence, uniqueness, and qualitative properties of layer or heteroclinic solutions to fractional elliptic equations involving a bistable nonlinearity. A Hamiltonian quantity and sharp energy estimates play here a central role, and we will also present some of their more recent applications given by other authors;
- I will explain the results in several articles with J.-M. Roquejoffre and A.-C. Coulon where we establish the exponential in time propagation of fronts for the Fisher-KPP equation with fractional diffusion, both in homogeneous and in periodic media;
- Finally I will describe very recent results with N. Consul and J.V. Mand on traveling fronts for the following fractional-diffusion type problem: the classical homogeneous heat equation in a half-plane with a boundary Neumann condition of bistable or combustion type.

On higher order p-Kirchhoff problems

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In the course we present some recent existence theorems for nontrivial stationary solutions of problems involving involving the p-polyharmonic Kirchhoff operator in bounded domains. The p-polyharmonic operators Δ_p^L were recently introduced in [F. Colasuonno and P. Pucci, Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations, *Nonlinear Anal.* **74** (2011) 5962-5974] for all orders L and independently, in the same volume of the journal, in [V.F. Lubyshev, Multiple solutions of an even-order nonlinear problem with convex-concave nonlinearity, *Nonlinear Anal.* **74** (2011) 1345-1354] only for L even. The results are then extended to non-degenerate $p(x)$-polyharmonic Kirchhoff operators.

Several useful properties of the underlying functional solution space $[W_0^{L,p}(\Omega)]^d$, endowed with the natural norm arising from the variational structure of the problem, are also proved both in the homogeneous case $p \equiv Const$ and in the non-homogeneous case $p = p(x)$.

In the latter some sufficient conditions on the variable exponent p are given to prove the positivity of the infimum λ_1 of the Rayleigh quotient for the $p(x)$-polyharmonic operator $\Delta_{p(x)}^L$. Other related problems will be also presented, as well as open problems.

Geometric aspects in competition-diffusion problems

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In this mini-course we will present some recent results and various open problems related to the entire solutions and to the De Giorgi conjecture for competition-diffusion systems.

In order to understand the interactions of the species in competition, we will consider the problem of the classification of the entire solutions of systems with polynomial growth (also in the case of non-local diffusion). The problem is related to the structure of the multiple cluster points of the limiting profile of the segregated species, to their regularity and to the rate convergence of the competition-diffusion systems.

Talks

Asymptotically p–linear problems for the p–Laplacian

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Motivated by the results in [4, 5], we study in [1] the multiplicity of weak solutions of the quasilinear elliptic problem

$$(P) \qquad \begin{cases} -\Delta_p u \; = \; g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $1 < p < +\infty$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, Ω is an open bounded domain of $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and g behaves as $|u|^{p-2}u$ at infinity.

In a similar setting, we consider in [2] the equation

$$-\Delta_p u + V(x)|u|^{p-2}u \; = \; g(x,u), \quad x \in \mathbf{R^N},$$

where V is a potential satisfying the assumptions in [3], so that a suitable embedding theorem for weighted Sobolev spaces holds. Both the non–resonant and the resonant case are analyzed.

References

[1] R. Bartolo, A.M. Candela and A. Salvatore, *p–Laplacian problems with nonlinearities interacting with the spectrum*, NoDEA **20**, 1701–1721 (2013).

[2] R. Bartolo, A.M. Candela and A. Salvatore, *Multiplicity results for a class of asymptotically p–linear equations on* $\mathbf{R}^N$, in progress.

[3] V. Benci and D. Fortunato, *Discreteness conditions of the spectrum of Schrödinger operators*, J. Math. Anal. Appl. **64**, 695–700 (1978).

[4] G. Li and H.S. Zhou, *Multiple solutions to p–Laplacian problems with asymptotic nonlinearity as* u^{p-1} *at infinity*, J. London Math. Soc. **65**, 123-138 (2002).

[5] K. Perera and A. Szulkin, *p–Laplacian problems where the nonlinearity crosses an eigenvalue*, Discrete Contin. Dyn. Syst. **13**, 743–753 (2005).

Unique continuation properties and essential self-adjointness for relativistic Schrödinger operators with singular potentials

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We discuss unique continuation properties for a class of relativistic fractional elliptic equations with singular potentials, as a consequence of precise asymptotics of solutions. We also treat the problem of essential self-adjointness of the corresponding relativistic Schrödinger operator providing an explicit sufficient and necessary condition on the coefficient of the singular potential for essential self-adjointness.

A Kirchhoff type problem in a non–local setting

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In a recent work written in collaboration with Enrico Valdinoci we studied the following problem

$$(P)\begin{cases} -M\left(\displaystyle\int_{\mathbb{R}^{2n}}|u(x)-u(y)|^2K(x-y)dx\,dy\right)\,\mathcal{L}_K u=\lambda f(x,u)+|u|^{2^*-2}\,u & \text{in } \Omega\\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega\,,\end{cases}$$

where $n>2s$ with $s\in(0,1)$, $2^*=2n/(n-2s)$, λ is a positive parameter, $\Omega\subset\mathbb{R}^n$ is an open bounded set, M is a continuous and positive function, f satisfies suitable growth conditions and $\mathcal{L}_K$ is a non-local operator defined as follows:

$$\mathcal{L}_K u(x)=\int_{\mathbb{R}^n}(u(x+y)+u(x-y)-2u(x))K(y)dy,$$

for all $x\in\mathbb{R}^n$. Here, the kernel $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ is a measurable function with the property that

$$\begin{aligned}&\text{there exists } \theta>0 \text{ and } s\in(0,1) \text{ such that}\\ &\theta\,|x|^{-(n+2s)}\le K(x)\le\theta^{-1}\,|x|^{-(n+2s)}\,\text{for any } x\in\mathbb{R}^n\setminus\{0\}\,.\end{aligned}$$

A typical example for K is given by $K(x)=|x|^{-(n+2s)}$. In this case problem (P) becomes

$$\begin{cases} M\left(\displaystyle\int_{\mathbb{R}^{2n}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dx\,dy\right)(-\Delta)^s u=\lambda f(x,u)+|u|^{2^*-2}\,u & \text{in } \Omega\\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega,\end{cases}$$

where $-(-\Delta)^s$ is the fractional Laplace operator.

Aim of this talk will be to present an existence result of non–negative weak solutions of problem (P). The proof of this result, reported in [1], is based on a variational method. Here, the choice of the functional space plays a very important role. For this, we will also mention a forthcoming paper [2], where it is explained why our choice is the best possible.

Moreover, we will give an interesting physical motivation of problem (P) (for a detailed description we refer to [1, Appendix]).

References

[1] A. Fiscella and E. Valdinoci, *A critical Kirchhoff type problem involving a nonlocal operator*, Nonlinear Anal., 94, 156–170 (2014).

[2] A. Fiscella, R. Servadei and E. Valdinoci, *Density properties for fractional Sobolev spaces*, in preparation.

Weyl-type law for fractional p-eigenvalue problems

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Weyl's law deals with the asymptotic behavior of variational eigenvalues (λ_k) for a problem of the type

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ $(p > 1)$. In the linear case $p = 2$, such law (proved by Weyl [5] around 1912) states that $\sharp\{k \in \mathbb{N} : \lambda_k < \lambda\} \sim (2\pi)^{-N}\omega_N|\Omega|\lambda^{N/2}$, while nonlinear versions of García Azorero & Peral Alonso [3] and Friedlander [2] yield the estimates

$$C_1|\Omega|\lambda^{N/p} \leq \sharp\{k \in \mathbb{N} : \lambda_k < \lambda\} \leq C_2|\Omega|\lambda^{N/p}, \quad \lambda > 0 \text{ large}$$

for some positive constants $C_i = C_i(p, N)$.

We deal with an eigenvalue problem for the fractional p-Laplacian with order of differentiation $0 < s < 1$

$$(P) \qquad \begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega\,. \end{cases}$$

The first eigenvalue of (P) was studied by Lindgren & Lindqvist [4] and Franzina & Palatucci [1]. By means of the cohomological index and the Krasnoselskii genus we define a non-decreasing sequence (λ_k) of variational eigenvalues for (P) and prove the following asymptotic estimates:

$$\sharp\{k \in \mathbb{N} : \lambda_k < \lambda\} \geq C_1|\Omega|^{\frac{sp}{Np-N+sp}}\lambda^{\frac{N}{Np-N+sp}}, \quad \lambda > 0 \text{ large},$$

and if $sp > N$

$$\sharp\{k \in \mathbb{N} : \lambda_k < \lambda\} \leq C_2|\Omega|^{\frac{sp}{sp-N}}\lambda^{\frac{N}{sp-N}}, \quad \lambda > 0 \text{ large},$$

for some positive constants $C_i = C_i(s, p, N)$. The study of (λ_k) is made particularly delicate by the non-local nature of the operator $(-\Delta)_p^s$, which forbids additivity of the genus and co-genus of sublevel sets of the related energy functional, with respect to decompositions of the domain. This is why we do not obtain the optimal estimate, involving a power $\lambda^{N/sp}$.

References

[1] G. Franzina and G. Palatucci, *Fractional p-eigenvalues*, Riv. Mat. Univ. Parma, to appear.

[2] L. Friedlander, *Asymptotic behavior of the eigenvalues of the p-Laplacian*, Comm. Partial Differential Equations **14**, 1059–1069 (1989).

[3] J. García Azorero and I. Peral Alonso, *Comportement asymptotique des valeurs propres du p-laplacien*, C. R. Acad. Sci. Paris Sér. I Math. **307**, 75–78 (1988).

[4] E. Lindgren and P. Lindqvist, *Fractional eigenvalues*, Calc. Var., to appear.

[5] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. **71**, 441–479 (1912).

Adams' inequality with the exact growth condition in $\mathbb{R}^4$

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Adams' inequality is the complete generalization of the Trudinger-Moser inequality to the case of Sobolev spaces involving higher order derivatives. In this talk we discuss the optimal growth rate of the exponential-type function in Adams' inequality when the problem is considered in the whole space $\mathbb{R}^4$. This is a joint work with Nader Masmoudi.

Monotonicity and 1*-dimensional symmetry for solutions of an elliptic system arising in Bose-Einstein condensation*

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We discuss monotonicity and 1-dimensional symmetry for positive solutions with algebraic growth of the semilinear elliptic system

$$(P) \qquad \begin{cases} -\Delta u = -uv^2 & \text{in } \mathbb{R}^N \\ -\Delta v = -u^2 v & \text{in } \mathbb{R}^N, \end{cases}$$

for every dimension $N \geq 2$. This system appears in the analysis of phase-separation phenomena for Bose-Einstein condensates with multiple states.

Our main result is the proof of the validity of the following conjecture, proposed by H. Berestycki, T. C. Lin, J. Wei and C. Zhao, for solutions with algebraic growth:

Conjecture. Let $N \geq 2$, let (u, v) be a solution of (P) such that

$$\lim_{x_N \to +\infty} u(x', x_N) = +\infty \qquad \lim_{x_N \to -\infty} u(x', x_N) = 0,$$
$$\lim_{x_N \to +\infty} v(x', x_N) = 0 \qquad \lim_{x_N \to -\infty} v(x', x_N) = +\infty,$$

the limit being uniform in $x' \in \mathbb{R}^{N-1}$. Then (u, v) is 1-dimensional and

$$\frac{\partial u}{\partial x_N} > 0 \quad \text{and} \quad \frac{\partial v}{\partial x_N} < 0$$

in $\mathbb{R}^N$.

This conjecture is, in our setting, the counterpart of the Gibbons' conjecture (a variant of the De Giorgi's conjecture) for the Allen-Cahn equation.
This is a joint work with Prof.Alberto Farina (Université de Picardie 'Jules Verne' de Amiens).

Finite time blow up and global solutions for fourth order damped wave equations

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In this talk, we consider a class of fourth order wave equations with linear damping term and superlinear source term, which is viewed as a mathematical model for dynamical suspension bridges. We first show that the uniqueness and existence of local solutions to the problem, and then the necessary and sufficient conditions for global existence and finite time blow-up of the local solution are given. Moreover, the potential well depth is estimated.