Front propagation for reaction equations with fractional diffusion

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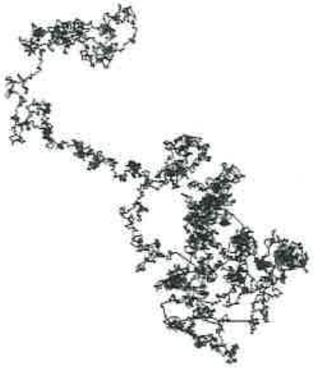
Levy processes and fractional Laplacians

$-\Delta$: Brownian motion



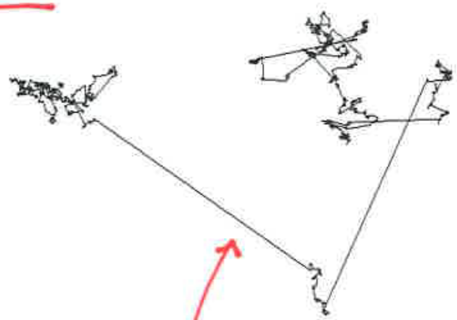
Levy processes & Fractional Laplacians,
type of "anomalous diffusions" in:

- Dislocation of crystals
(boundary reactions: the Peierls-Nabarro Problem)

- Micro-magnetism (thin films)
- Mathematical finance (American options,...)
- Quasigeostrophic equations
- The Signorini problem ("thin obstacle problem")

- Fluids, biology (front propagation, travelling waves)

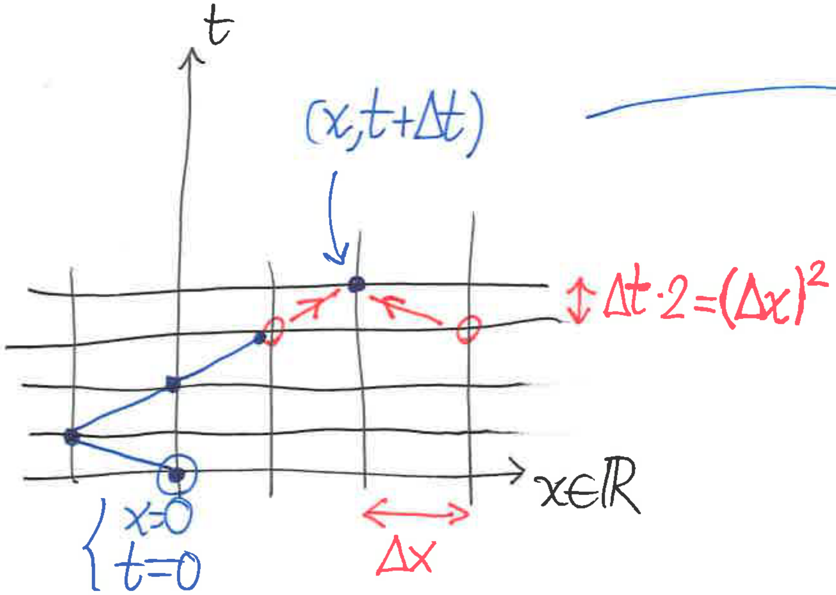
$(-\Delta)^s$, $0 < s < 1$: Levy processes



PURE
JUMP

$(-\Delta)+(-\Delta)^{1/2}$, e.g.

The heat equation & the Central Limit Theorem



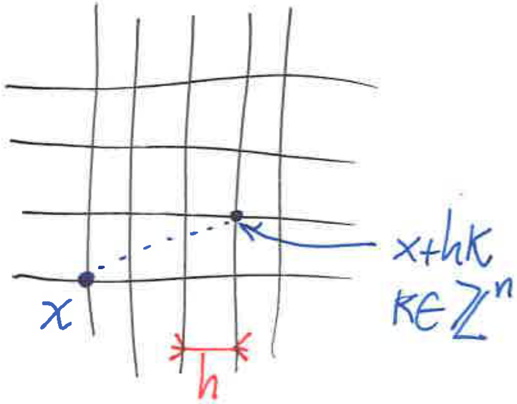
Probability:

$$u(x,t+\Delta t) = \frac{1}{2}\left(u(x-\Delta x,t) + u(x+\Delta x,t)\right)$$

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} = \frac{u(x-\Delta x,t)+u(x+\Delta x,t)-2u(x,t)}{|\Delta x|^2}$$

$$\begin{cases} \boxed{\partial_t u = \Delta_x u} \\ u(t=0)=\delta_0 \end{cases}$$

The long-jump random walk and the fractional Laplacian



$$u(x,t+\tau)=\sum_{k\in\mathbb{Z}^n}\underbrace{\mathcal{K}(k)}_{\text{prob(jump } (x\leftrightarrow x+hk))}\,u(x+hk,t)$$

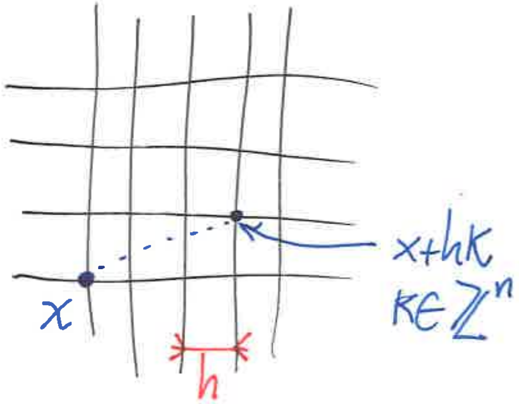
$$\Downarrow$$

$$u(x,t+\tau)-u(x,t)=\sum_{k\in\mathbb{Z}^n}\mathcal{K}(k)\left\{u(x+hk,t)-u(x,t)\right\}$$

$\tau=h^{2s}$ & $\mathcal{K}(y)=|y|^{-n-2s}$ $\longrightarrow$

$$\frac{u(x,t+\tau)-u(x)}{\tau}=h^n\sum_{k\in\mathbb{Z}^n}\mathcal{K}(hk)\left\{u(x+hk,t)-u(x,t)\right\}$$

The long jump random walk and the fractional Laplacian



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$\xrightarrow{h\downarrow 0}$ $h^{2s}=\tau\downarrow 0$

$$\boxed{\partial_t u = ct\cdot PV\int_{\mathbb{R}^n}\frac{u(x+y,t)-u(x,t)}{|y|^{n+2s}}\,dy =: ct\cdot(-\Delta)_x^s u}$$

$$=\frac{ct}{2}\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy.$$

(2^{nd} order differences)

The fractional Laplacian, $0<s<1$

$u : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$(-\Delta)^s u(x) := C_{n,s} \, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(\bar{x})}{|x-\bar{x}|^{n+2s}} \, d\bar{x}$$

$$\int_{\mathbb{R}^n} u \cdot (-\Delta)^s u = \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 \approx \|u\|_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} d\bar{x} \, \frac{|u(x)-u(\bar{x})|^2}{|x-\bar{x}|^{n+2s}} + \|u\|_{L^2(\mathbb{R}^n)}$$

$$\Updownarrow$$

$$\widehat{(-\Delta)^s u} = |\xi|^{2s} \hat{u}$$

(Fourier transform)

The half Laplacian (square root of Laplacian)

$u : \mathbb{R}^n \longrightarrow \mathbb{R}$

$(-\Delta)^{1/2} u \quad : \quad (-\Delta)^{1/2} \circ (-\Delta)^{1/2} = -\Delta$

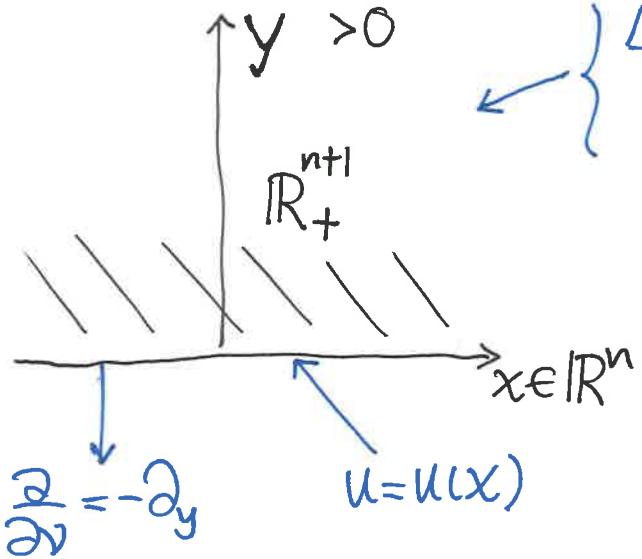
elliptic nonlocal operator of "first order".

Fourier transform: $\widehat{(-\Delta)^{1/2} u} = |\xi|\, \hat{u}$

a local (boundary reaction) representation:

$$\left\{ \begin{array}{ll} \Delta_{x,y} v = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ v = u & \text{on } \{y=0\} \end{array} \right.$$

$$\rightsquigarrow \boxed{(-\Delta)^{1/2} u\,(x) = -\partial_y \mathbf{v}(x,0)}$$



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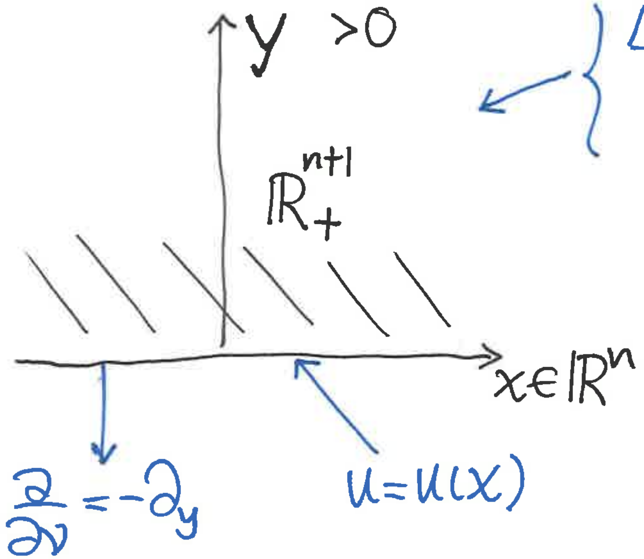


$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ v = u & \text{on } \{y=0\} \end{cases} \qquad \Delta_{x,y}$$

$$\rightsquigarrow \quad \boxed{(-\Delta)^{1/2} u(x) = -\partial_y v(x,0)}$$

since $(-\Delta)^{1/2} \circ (-\Delta)^{1/2} u =$
$= -\partial_y(-\partial_y v) = v_{yy} = -\Delta_x v(x,0)$
$= -\Delta_x u$

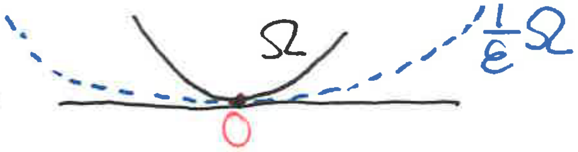
$$(-\Delta)^{1/2} u = h(x) \text{ in } \mathbb{R}^n \iff \begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \frac{\partial v}{\partial \nu} = h(x) & \text{on } \partial\mathbb{R}^{n+1}_+ \end{cases}$$

Phase transitions: boundary reactions

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_{\partial\Omega} G(u) \longrightarrow \begin{cases} \Delta u_\varepsilon = 0 & \text{in } \Omega \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = \dfrac{1}{\varepsilon} f(u_\varepsilon) & \text{on } \partial\Omega \end{cases} \qquad (P_\varepsilon)$$

$\varepsilon > 0, \quad \Omega \subset \mathbb{R}^n$ bounded

BLOW-UP



$\varepsilon \downarrow 0$

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \ (= \mathbb{R}^2_+) \\ -u_y = f(u) & \text{on } \{y=0\} \end{cases}$$

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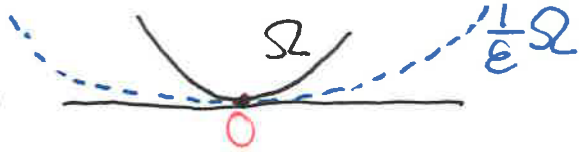

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Peierls-Nabarro problem

$f(u) = c\cdot\sin(\pi u)$

$\Updownarrow$

$$\boxed{(-\Delta)^{1/2} u = f(u) \quad \text{in } \mathbb{R}}$$

[Cabré, Solà-Morales '05]

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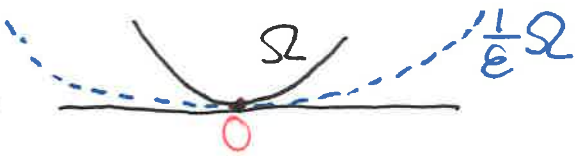

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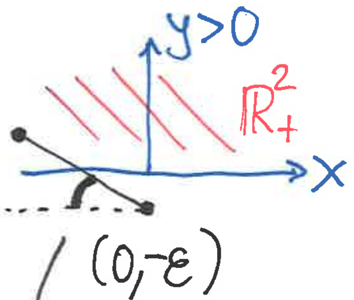

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[Cabré, Sola-Morales '05]

Explicit sol'ns $-1 \nearrow 1$ = primitive of heat kernel $\approx \int_{-\infty} \frac{1}{|x|^2}$

$\|$

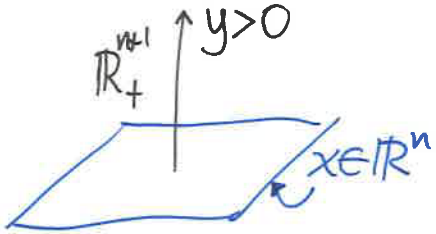
$$\underbrace{u(x,y) = \frac{2}{\pi}\arctan\frac{x}{y+\varepsilon}}_{\text{fast transition at } (0,0)}$$

The extension problem [Caffarelli-Silvestre 2007]

$0<s<1$

$u:\mathbb{R}^n\to\mathbb{R}$

$$\rightsquigarrow \begin{cases} \operatorname{div}(y^{1-2s}\nabla v)=0 & \text{in } \mathbb{R}^{n+1}_+ \\ v(x,0)=u(x) & \text{on } \partial\mathbb{R}^{n+1}_+=\mathbb{R}^n. \end{cases}$$



$v=v(x,y)$.

Thm [Caff-Silv]

$$-\lim_{y\downarrow 0} y^{1-2s} v_y = \frac{\partial v}{\partial y^s}(x,0) = \tilde{c}_{n,s}(-\Delta)^s u(x).$$

The extension problem [Caffarelli-Silvestre 2007]

$0<s<1$

$u:\mathbb{R}^n \to \mathbb{R}$ $\rightsquigarrow$

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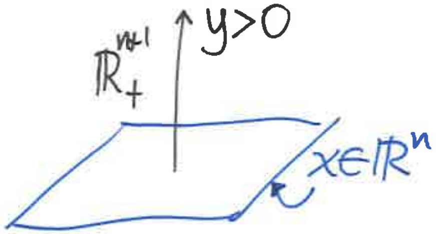

$v = v(x,y).$

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$$\boxed{-\lim_{y\downarrow 0} y^{1-2s} v_y = \frac{\partial v}{\partial y^s}(x,0) = \tilde{c}_{n,s}\,(-\Delta)^s u\,(x).}$$

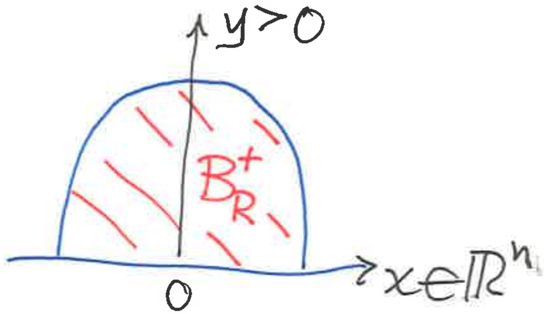
Semilinear pb:

$$\boxed{(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^n}$$

$$\Updownarrow$$

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -y^{1-2s} v_y \big|_{y=0} = f(v) & \text{on } \partial\mathbb{R}^{n+1}_+ = \{y=0\}. \end{cases}$$

Energy:



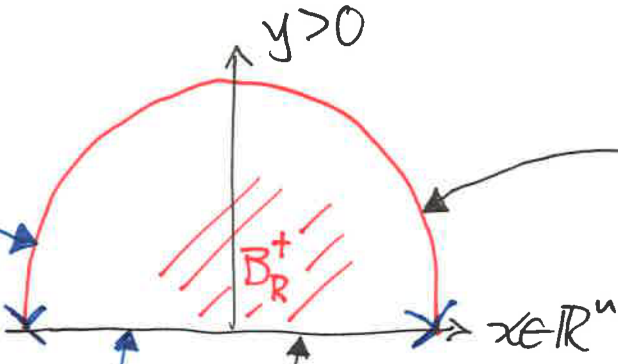
$$E_{B_R^+}(v) = \iint_{B_R^+} dx\,dy\, \frac{y^{1-2s}}{2}|\nabla v|^2 + \int_{\{|x|<R\}} dx\; G(v(x,0)).$$

Minimizers v :

$$E_{B_R^+}(v) \leq E_{B_R^+}(w)$$

$w \equiv u$ on

w free on



$\approx H^{1/2}$ weighted norm

$$\iint_{B_R^+} dx\,dy \quad y^{1-2s}|\nabla v|^2$$

$\approx H^s$-norm of trace

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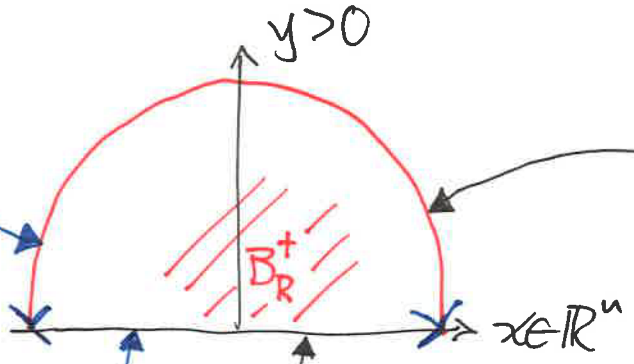


$\approx H^{1/2}$ weighted norm

$\approx H^s$-norm of trace

- **Thm** [C.-Cinti 2010]

Sharp energy estimates for minimizers of $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$:

$$\boxed{E_{B_R^+}(v)} \approx C \begin{cases} R^{n-2s} & \text{if } 0<s<1/2, \\ R^{n-1}\log R & \text{if } s=1/2, \\ R^{n-1} & \text{if } 1/2<s\leq 1. \end{cases}$$

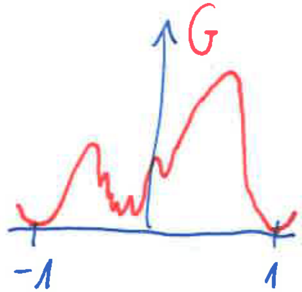
• **Thm** $\forall f$, global minimizers of $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$ are $\boxed{1\text{-D}}$ if

& $0 < s < 1$

• $n=3$ & $1/2 \leq s \leq 1$ [C.-Cinti '10]

Examples of global minimizers: monotone sol'ns ($u_{x_n} > 0$) with $u(x', x_n) \xrightarrow[x_n \to \pm\infty]{} \pm 1$

The equation $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$, $0<s<1$.



- Thm [C.& Solà-Morales, '05, $s=1/2$] [C.-Sire '10, $0<s<1$]

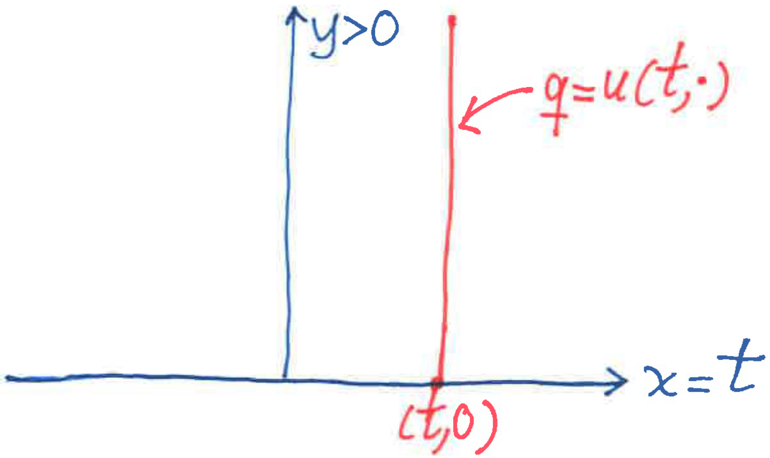
$\exists$ sol'n $u \nearrow_{-1}^{1}$ in $\mathbb{R}$ $\iff$ $\exists$ such u for $s=1$ $\iff$ $\begin{cases} G'(\pm 1)=0 \ \& \\ G(s)>G(1)=G(-1) \text{ in } (-1,1) \end{cases}$

$\oplus$ Hamiltonian equalities (in $\mathbb{R}$).

$(-\Delta)^s u = c \cdot f(u)$ in $\mathbb{R}^1$ has HAMILTONIAN STRUCTURE

$$\Updownarrow$$

$$(*) \quad \begin{cases} \operatorname{div}\left(y^{1-2s}\nabla v\right) = 0 & \text{in } \mathbb{R}^2_+ \\ 2(1-s)\lim\limits_{y\downarrow 0} -y^{1-2s} v_y = f(v) & \text{in } \mathbb{R} \end{cases}$$



$$\begin{cases} q = v(x,\cdot) = v(t,\cdot) \\ p = q' = v_x(t,\cdot) \end{cases}$$

Energy $\longrightarrow$ $L(q,p) = \frac{1}{2}\|p\|_s^2 + W(q)$ $\quad\longleftarrow\quad \|w\|_s^2 := \int_0^{+\infty} y^{1-2s}|w(y)|^2\,dy$

$$W(q) = \frac{1}{2}\|\partial_y q\|_s^2 + \frac{1}{2(1-s)}G(q(0))$$

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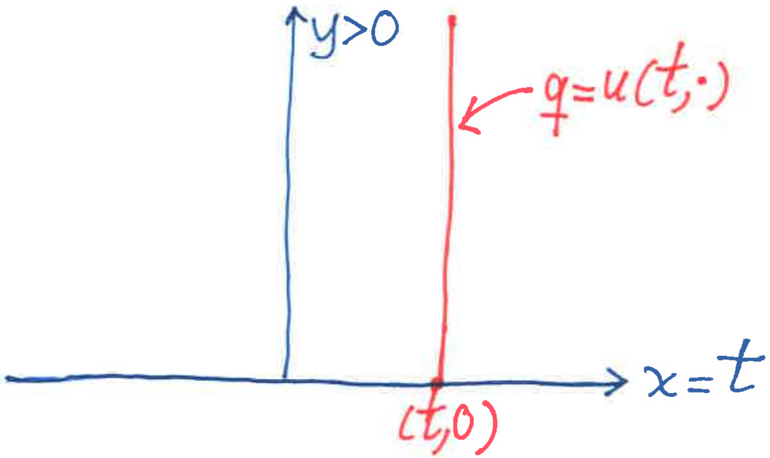

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Hamiltonian : $H(q,p) = \frac{1}{2}\|p\|_s^2 - W(q)$

$$= \int_0^{+\infty} \frac{y^{1-2s}}{2}\left\{ v_x^2(t,y) - v_y^2(t,y)\right\} dy - \frac{1}{2(1-s)}\, G(v(t,0))$$

$$\Rightarrow \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} p \\ W'(q) \end{pmatrix} = \begin{pmatrix} H_p \\ -H_q \end{pmatrix}$$

Hamiltonian identity & estimate

- **Thm** [C.-Sire '10, $0<s<1$]
$n=1$, u layer ($u \nearrow \pm 1$) soln of $(-\Delta)^s u = f(u)$, $\forall f$
v = s-extension of u. Then:

$$2(1-s)\int_0^{+\infty} \frac{z^{1-2s}}{2}\left\{v_x^2(x,z) - v_y^2(x,z)\right\} dz = G(v(x,0)) - G(1) \quad \forall x \in \mathbb{R}$$

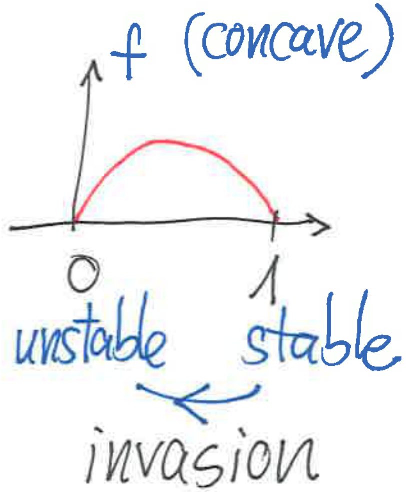
&

$$2(1-s)\int_0^{y} \frac{z^{1-2s}}{2}\left\{v_x^2(x,z) - v_y^2(x,z)\right\} dz < G(v(x,0)) - G(1) \quad \begin{cases} \forall x \in \mathbb{R} \\ \forall y \geq 0 \end{cases}.$$

Open pb for $n>1$!

- **Front propagation: monostable KPP nonlinearities**

$$\begin{cases} u_t - \Delta u = f(u) & (= u(1-u) = u - u^2) \quad \text{in } \mathbb{R}\times(0,\infty) \\ u(t=0) = u_0(x) \in [0,1] & \text{on } \mathbb{R} \end{cases}$$



Travelling wave solutions:

$u(x,t) = \phi(x+ct)$ $\exists$ for all $\boxed{c \geq c^* = 2\sqrt{f'(0)}}$

• Front propagation: monostable KPP nonlinearities

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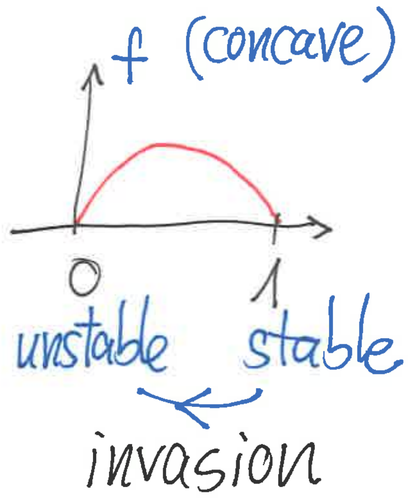


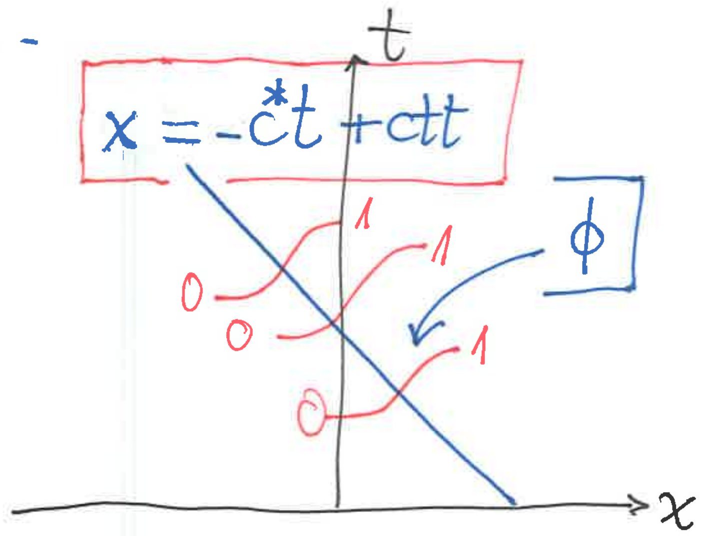
Travelling wave solutions:

$u(x,t) = \phi(x+ct)$ $\exists$ for all $\boxed{c \geq c^* = 2\sqrt{f'(0)}}$

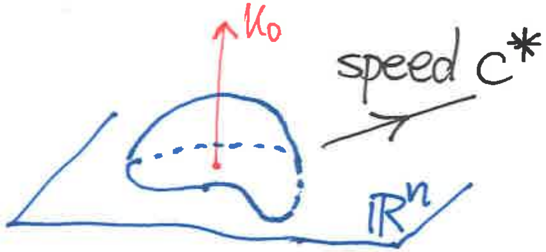
Initial condition: $u_0(x) =$ Heaviside

Thm (Kolmogorov-Petrovski-Piskunov '37)

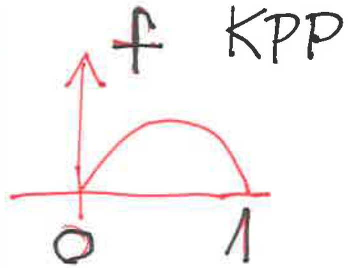
$$\boxed{\lim_{t \to +\infty} u(x+ct, t) = \begin{cases} 0 & \text{if } c > c^* \\ 1 & \text{if } c < c^* \end{cases} \quad \forall x \in \mathbb{R}}$$



Also,

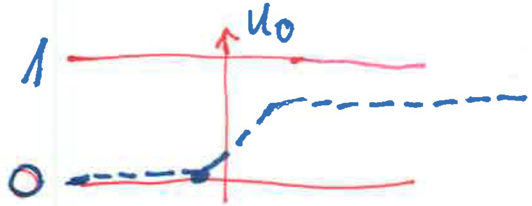


• Front propagation for KPP fractional diffusions



[Cabré-Roquejoffre '09]

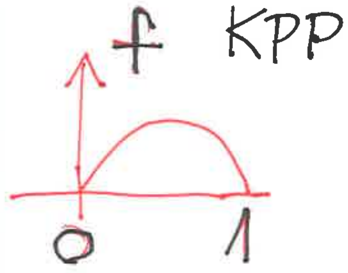
$$\begin{cases} \boxed{u_t + (-\Delta)^{\alpha} u = f(u)} \quad \text{in } \mathbb{R}\times(0,\infty), \quad \boxed{0<\alpha<1} \\ u(t=0) \text{ nondecreasing } \& \ \mathbb{R}_- \cap \operatorname{supp}(u(0,\cdot)) \text{ compact} \end{cases}$$



• Thm [C-R '09] $\nexists$ travelling waves & $\forall x$

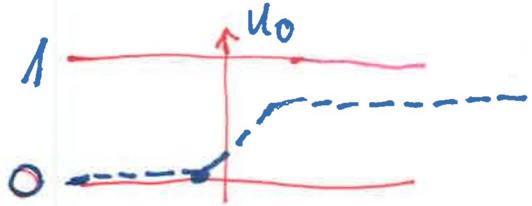
$$\lim_{t\to+\infty} u(\underbrace{x+e^{\sigma t}}_{\uparrow}, t) = \begin{cases} 0 & \text{if } \sigma > \sigma^{**} = \dfrac{f'(0)}{2\alpha} \\ 1 & \text{if } \sigma < \sigma^{**} \end{cases}$$

- Front propagation for KPP fractional diffusions



[Cabré–Roquejoffre '09]

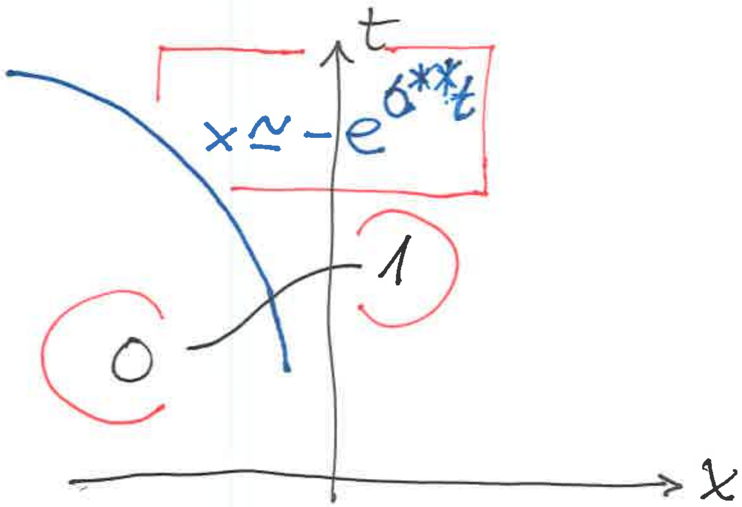
$$\begin{cases} \boxed{u_t + (-\Delta)^{\alpha} u = f(u)} \quad \text{in } \mathbb{R}\times(0,\infty), \quad \boxed{0<\alpha<1} \\ u(t=0) \text{ nondecreasing } \& \ \mathbb{R}_- \cap \operatorname{supp}(u(0,\cdot)) \text{ compact} \end{cases}$$



- Thm [C-R '09] $\nexists$ travelling waves & $\forall x$

$$\boxed{\lim_{t\to+\infty} u(\underbrace{x+e^{\sigma t}}_{\uparrow}, t) = \begin{cases} 0 & \text{if } \sigma > \sigma^{**} = \dfrac{f'(0)}{2\alpha} \\ 1 & \text{if } \sigma < \sigma^{**} \end{cases}}$$

The front travels exponentially fast:



Announced in Physics (no math proof) by:

[Mancinelli-Vergni-Vulpiani '03]

& [del-Castillo-Negrete, Carreras, Lynch '03]

Initial conditions with compact support in $\mathbb{R}^n$

$$\begin{cases} u_t + (-\Delta)^\alpha u = f(u) & \text{in } (0,\infty)\times\mathbb{R}^n \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}^n,\ 0 \leq u_0 \leq 1 \end{cases}$$

- **Thm** [C.-Roquejoffre '09] Let $\sigma^* := \dfrac{f'(0)}{n+2\alpha}$. Then:

(a) $\sigma > \sigma^* \Rightarrow u(x,t) \to 0$ unif. in $\{|x| \geq e^{\sigma t}\}$ as $t \to +\infty$.

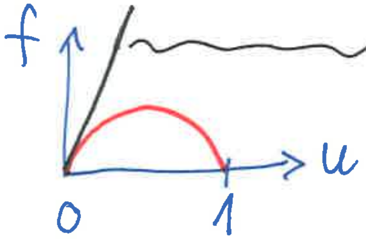
(b) $\sigma < \sigma^* \Rightarrow u(x,t) \to 1$ unif. in $\{|x| \leq e^{\sigma t}\}$ as $t \to +\infty$.

- Note: $n=1 \Rightarrow$ $\sigma^* = \dfrac{f'(0)}{1+2\alpha} < \dfrac{f'(0)}{2\alpha} = \sigma^{**}$

σ^{**}: increasing initial data

σ^*: compactly supported initial data

- **Heuristics** :



Linearization at the front:

$$v_t + (-\Delta)^\alpha v = f'(0)\, v$$

$$\downarrow$$

Solution $= v(t,x) = e^{f'(0)t} \int_{\mathbb{R}^n} \underbrace{p_\alpha(t,y)}\, u_0(x-y)\,dy$

Fractional heat kernel : $\boxed{p_\alpha(t,x) \approx} \dfrac{1}{t^{\frac{n}{2\alpha}}\left(1+\left|\frac{x}{t^{1/2\alpha}}\right|^{n+2\alpha}\right)} \approx c \cdot \dfrac{t}{|x|^{n+2\alpha}}$

$|x|$ large

Solution remains bdd, $\in (0,1)$ $\Longrightarrow$ $e^{f'(0)t} \cdot \dfrac{t}{|x|^{n+2\alpha}} \approx 1$

$$\Updownarrow$$

$$|x| \simeq \underbrace{t^{\frac{1}{n+2\alpha}}}_{\text{WRONG factor:}}\; e^{\frac{f'(0)}{n+2\alpha} t}.$$

WRONG factor: Correct

Fractional diffusion in periodic media

$$\begin{cases} u_t + (-\Delta)^\alpha u = \mu(x)\, u - u^2, \quad x \in \mathbb{R}^n, t > 0 \quad (*) \\ \mu(x) \geq \min \mu > 0, \ \mu \text{ periodic} \\ \lambda_1 = \text{principal periodic eigenvalue of } (-\Delta)^\alpha - \mu(x)\, Id \end{cases}$$

- $\lambda_1 \geq 0 \Rightarrow u \xrightarrow[t\to\infty]{} 0 \quad \forall u_0$
- $\lambda_1 < 0 \Rightarrow u \xrightarrow[t\to\infty]{} u_+$ = the stationary sol'n of $(*)$

$\leftarrow \lambda_1 = -1$ if $\mu \equiv 1$.

$\sigma^* = \frac{1}{n+2\alpha}$

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- **Thm** [C.-Coulon-Roquejoffre '12] Assume $\boxed{\lambda_1 < 0}$, $u_0 \geq 0$, $u_0 \not\equiv 0$ with compact support. Then,

$$\forall \lambda \in (0, \min \mu) \quad \{x \in \mathbb{R}^n : u(t,x) = \lambda\} \subset \left\{ c_\lambda e^{\frac{|\lambda_1|}{n+2\alpha} t} \leq |x| \leq \frac{1}{c_\lambda} e^{\frac{|\lambda_1|}{n+2\alpha} t} \right\}$$

for t large.

Open pb

$\lambda \in (0, \min u_+)$?

Heuristics predicted wrong

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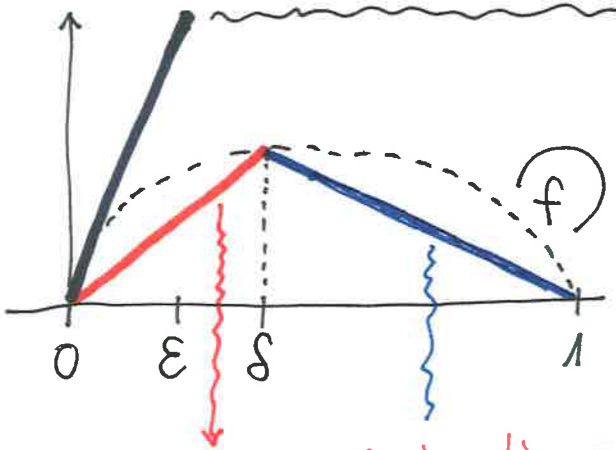
for t large.

Open pb $\lambda \in (0, \min u_+)$?

Great difference with $\alpha = 1$, where speed is cst but depends on every direction of the periodic media (Freidlin-Gärtner formula)

Heuristics predicted wrong

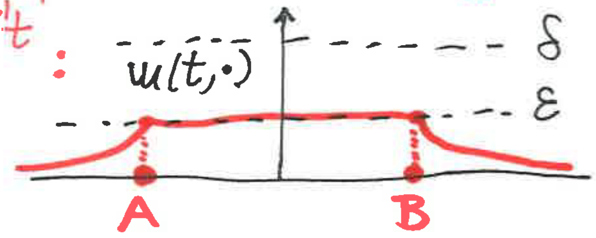
• Proofs (homogeneous media, $\mu \equiv 1$)

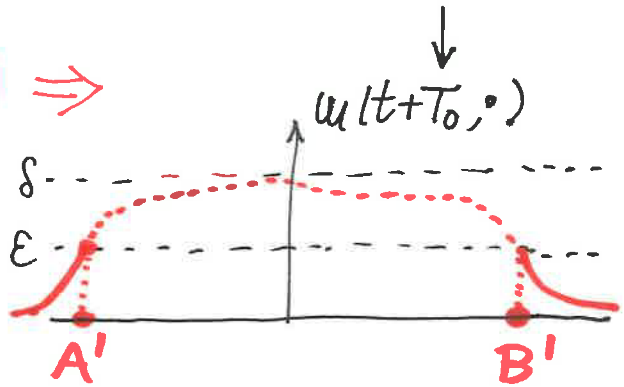


$*$ SUPERSOLUTION $\Rightarrow$ $u \to 0$ if $|x| \geq e^{\delta t}$

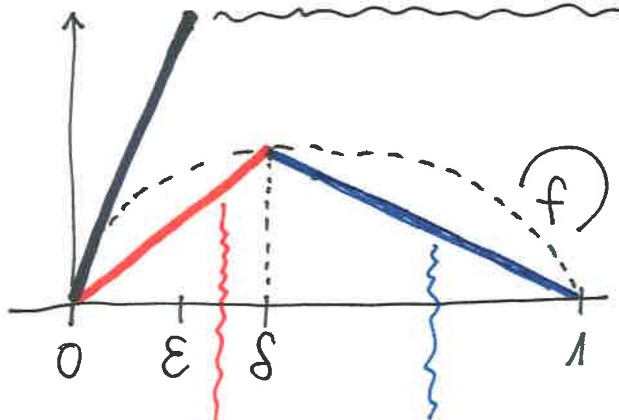
$*$ SUBSOLUTION (using linear pb $w_t + (-\Delta)^\alpha w = \frac{f(\delta)}{\delta} w$) $\Rightarrow$

$\Rightarrow$ $u \geq \varepsilon$ for $|x| \geq e^{\delta' t}$:





• Proofs (homogeneous media, $\mu \equiv 1$)



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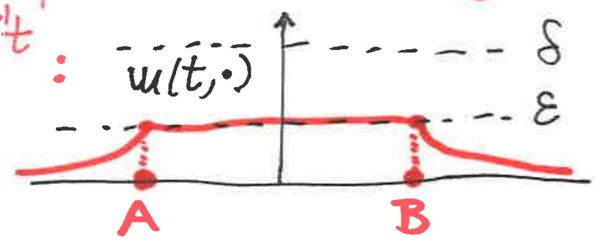

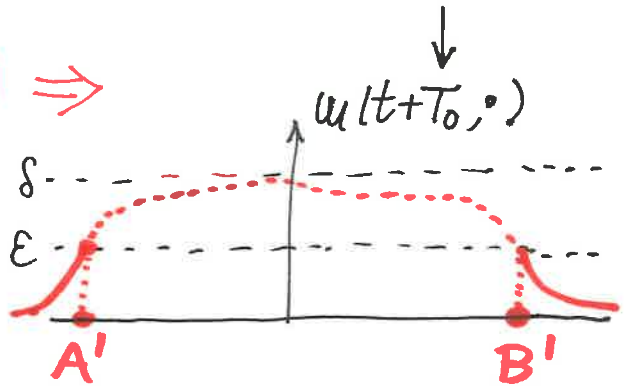

* <u>SUBSOLUTION</u> $\Rightarrow$ $u \to 1$ for $|x| \leq e^{\delta'' t}$:

linear equation with $\tilde{w}_0 := 1 + C|x|^\gamma \longrightarrow 0$

& $\geq 1-u$

- **Proof** (heterogeneous periodic media, $\mu(x) \geq \min \mu > 0$)

$$u(t,x) = \phi_1(x)\,\underbrace{v(t,x)}\,, \qquad \phi_1 : \text{periodic } 1^{st} \text{ eigenfunction}$$

$$\left((-\Delta)^\alpha \phi_1 - \mu(x)\phi_1 = \lambda_1 \phi_1 \right)$$

$$w(t,y) := v\left(t, e^{\frac{|\lambda_1|}{N+2\alpha} t}\, y\right)$$

$\hookrightarrow$ w approx. soln of a transport equation

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$\hookrightarrow$ w approx. soln of a transport equation

ANSATZ :

$$\boxed{\tilde{u}(t,x) := \phi_1(x) \frac{a}{|\lambda_1|^{-1} + b(t)\,|x|^{n+2\alpha}}}$$

$\tilde{u}$ sub / super soln $\Leftarrow$ $b(t) \approx e^{-|\lambda_1| t}$ solves ODE's.

∎