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On higher order p–Kirchhoff problems

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Basic References

- **G. Autuori, F. Colasuonno, P. P.**, *On the existence of stationary solutions for higher order p–Kirchhoff problems via variational methods*, Commun. Contemp. Math., pages 42.
- **G. Autuori, F. Colasuonno, P. P.**, *Lifespan estimates for solutions of polyharmonic Kirchhoff systems*, Math. Models Methods Appl. Sci. 22, Issue 02 - February 2012 - 1150009, pages 36.
- **F. Colasuonno, P. P.**, *Multiplicity of solutions for $p(x)$–polyharmonic Kirchhoff equations*, Nonlinear Anal. 74 (2011), 5962–5974.
- **G. Autuori, P. P.**, *Local asymptotic stability for polyharmonic Kirchhoff systems*, Appl. Anal., Special Volume dedicated to Prof. P.L. Butzer, 90 (2011), 493-514.

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- **M. Delitala, P. P., M.C. Salvatori**, *From methods of the mathematical kinetic theory for active particles to modelling Virus mutations*, Math. Models Methods Appl. Sci., 21 (2011), 843-870.
- **G. Autuori, P. P.**, *Asymptotic stability for Kirchhoff systems in variable exponent Sobolev spaces*, Complex Var. Elliptic Equ., Special Volume dedicated to Prof. V.V. Zhikov, 56 (2011), 715-753.
- **G. Autuori, P. P.**, *Kirchhoff systems with dynamic boundary conditions*, Nonlinear Anal., 73 (2010), 1952-1965.
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- **G. Autuori, P. P.**, *Kirchhoff systems with nonlinear source and boundary damping terms*, **Commun. Pure Appl. Anal., 9 (2010), 1161-1188.**
- **P. P., V. Radulescu, Remarks of eigenvalue problems for nonlinear polyharmonic equations, Comptes Rendus, Sér. Math., 348 (2010), 161-164.**
- **G. Autuori, P. P., M.C. Salvatori**, ***Asymptotic stability for Anisotropic Kirchhoff Systems***, **J. Math. Anal. Appl., 352 (2009), 149-165.**
- **G. Autuori, P. P., M.C. Salvatori**, ***Asymptotic stability for Nonlinear Kirchhoff Systems***, **Nonlinear Anal. Real World Appl., 10 (2009), 889-909.**

Related recent papers

- **P. P., M.C. Salvatori**, *On an initial value problem modeling evolution and selection in living systems*, to appear in Discrete Contin. Dyn. Syst. S., pages 15.
- **P. P., S. Saldi**, *Multiple solutions for an eigenvalue problem involving non–local elliptic p–Laplacian operator – Dedicated to Professor Ermanno Lanconelli*, submitted for publication, pages 16.
- **P. P., S. Saldi**, *Critical stationary Kirchhoff equations in $\mathbb{R}^N$ involving nonlocal operators*, in preparation, pages 17.
- **P. P., Q. Zhang**, *Existence of entire solutions for a class of variable exponent elliptic equations*, in preparation, pages 39.

Model Problem

First, consider the eigenvalue p–Kirchhoff Dirichlet problem

$$\begin{cases} M(\|u\|^p)\Delta_p^L u = \lambda\{\gamma\|u\|_{p,w}^{p(\gamma-1)}w(x)|u|^{p-2}u + f(x,u)\} \text{ in } \Omega, \\ D^\alpha u_k\big|_{\partial\Omega} = 0 \text{ for all } \alpha, \text{ with } |\alpha| \le L-1,\ k=1,\dots,d, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \ge 1$, $u = (u_1,\dots,u_d) = u(x)$, $d \ge 1$, $p > 1$, $L = 1, 2, \dots,$ $\lambda \in \mathbb{R}$, α is a multi–index, $\gamma \in [1, p_L^*/p)$ and p_L^* is the critical Sobolev exponent

$$p_L^* = \begin{cases} \dfrac{np}{n-Lp}, & \text{if } \quad n > Lp, \\ \infty, & \text{if } 1 \le n \le Lp. \end{cases} \tag{2}$$

The vectorial p–polyharmonic operator Δ_p^L **is defined by**

$$\Delta_p^L\varphi = \begin{cases} \mathcal{D}_L\big(|\mathcal{D}_L\varphi|^{p-2}\mathcal{D}_L\varphi\big), & \text{if } L = 2j, \\ -\mathrm{div}\left\{\Delta^{j-1}\left(|\mathcal{D}_L\varphi|^{p-2}\mathcal{D}_L\varphi\right)\right\}, & \text{if } L = 2j-1, \end{cases} \quad \text{for } j = 1, 2,$$

Model Problem

for all $\varphi = (\varphi_1, \dots, \varphi_d) \in [C_0^\infty(\Omega)]^d$, where $\mathcal{D}_L$ denotes the vectorial operator

$$\mathcal{D}_L\varphi = \begin{cases} (\Delta^j\varphi_1, \dots, \Delta^j\varphi_d), & \textbf{if } L = 2j, \\ (D\Delta^{j-1}\varphi_1, \dots, D\Delta^{j-1}\varphi_d), & \textbf{if } L = 2j-1, \end{cases} \textbf{ for } j = 1, 2, \dots. \tag{3}$$

For all $x \in \Omega$ the vector $\mathcal{D}_L\varphi(x)$ has dimension d if L is even or dn if L is odd and, in both cases, the dimension is simply denoted by N. The vectorial p–polyharmonic operator Δ_p^L in the weak sense is

$$\langle \Delta_p^L u, \varphi \rangle = \int_\Omega |\mathcal{D}_L u|^{p-2} \mathcal{D}_L u \cdot \mathcal{D}_L \varphi \, dx$$

for all u, $\varphi \in [W_0^{L,p}(\Omega)]^d$.

References

When $d = 1$ the scalar p–polyharmonic operator Δ_p^L was first introduced in

[CP] **F. Colasuonno, P. P.**, *Multiplicity of solutions for $p(x)$–polyharmonic Kirchhoff equations*, Nonlinear Anal. 74 (2011), 5962-5974.

for all $L \geq 1$ and $p > 1$. In the scalar case Δ_p^2 is exactly the well–known *p–biharmonic operator* $\Delta_p^2\psi = \Delta(|\Delta\psi|^{p-2}\Delta\psi)$ for all $\psi \in C_0^\infty(\Omega)$ defined in the pioneering paper

[KN] **A. Kratochvíl, J. Necǎs**, *The discreteness of the spectrum of a nonlinear Sturm–Liouville equation of fourth order*, Comment. Math. Univ. Carolinæ12 (1971), 639-653,

see also

[DO] **P. Drábek and M. Ôtani**, *Global bifurcation result for the p–biharmonic operator*, Electron. J. Differential Equations 2001 (2001), 1-19.

References

Lubyshev **proved in**

- **[L] V.F. Lubyshev**, ***Multiple solutions of an even–order nonlinear problem with convex–concave nonlinearity***, **Nonlinear Anal. 74 (2011), 1345-1354.**

the existence of multiple solutions of a nonlinear Dirichlet problem governed by the scalar operator Δ_p^L **only for** L **even.**

The Model

In the 2–dimensional scalar case (1) arises from the theory of thin plates and describes the deflection $u = u(x_1, x_2)$ of the middle surface of a p–power–like elastic isotropic flat plate of uniform thickness, with non–local flexural rigidity of the plate $M(\|u\|^p)$ depending continuously on $\|u\|^p$ of the deflection u and subject to nonlinear source forces. The coordinates (x_1, x_2) are taken in the plane $x_3 = 0$ of the middle surface of the plate before bending. For other scalar problems modeled by (1) we refer to the Introduction of [CP].

References

For more standard polyharmonic problems we mention the recent monograph

[GGS] **F. Gazzola, H.–C. Grunau, G. Sweers**, ***Polyharmonic boundary value problems. Positivity preserving and nonlinear higher order elliptic equations in bounded domains***, **Lecture Notes in Mathematics, Vol. 1991 (Springer–Verlag, Berlin, 2010) xviii+423 pp.**

The Model

Problem (1) **is a nonlinear perturbation of the natural eigenvalue problem associated to the non–local higher order operator** $M(\|u\|^p)\Delta_p^L u$**. The perturbation** $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ **is a** ***Carathéodory function*****, with growth at infinity** q, $1 < q < p$**. The main assumption** $(\mathcal{F})$ **on** f **is stated later. The weight** w **is** *positive a.e. in* Ω *and*

$$w \in L^{\varpi}(\Omega), \qquad \varpi > \frac{n}{n - \gamma[n - Lp]^+}. \tag{4}$$

Restriction (4) **is meaningful, being** $\gamma \in [1, p_L^*/p)$**.**
The *Kirchhoff function* $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ **is assumed to verify the general structural assumption**

The Kirchhoff function M

$(\mathcal{M})$ *M is continuous, non–decreasing and there exists $s > 0$ such that*

$$s\gamma\tau^{\gamma} \leq \tau M(\tau) \quad \textit{for all } \tau \in \mathbb{R}^{+}.$$

From now on, **we denote by** $\mathscr{M}(\tau) = \displaystyle\int_0^{\tau} M(z)dz$ **for all** $\tau \in \mathbb{R}_0^{+}$**. Problem** (1) **is called** *degenerate* **if** $M(0) = 0$**, otherwise, if** $M(0) > 0$**, it is** *non–degenerate.*

The standard Kirchhoff function introduced in

[K] **G.R. Kirchhoff**, ***Vorlesungen über mathematische Physik: Mechanik***, **Teubner, Leipzig, 1883) 465 pp.**

is

$$M(\tau) = a + b\gamma\tau^{\gamma-1}, \quad a, b \geq 0, \quad a + b > 0, \quad \textbf{with}$$

$$\gamma \begin{cases} \in (1, p_L^*/p), & \textbf{if } b > 0, \\ = 1, & \textbf{if } b = 0, \end{cases} \qquad s = \begin{cases} b, & \textbf{if } b > 0, \\ a, & \textbf{if } b = 0, \end{cases}$$

which clearly verifies condition $(\mathcal{M})$**.**

The Kirchhoff function M

For such M's (1) is *degenerate* if $a = 0$ and $b > 0$, and *non–degenerate* when $a > 0$ and $b \geq 0$. Finally, when $a > 0$ and $b = 0$, the Kirchhoff function M is simply a constant and (1) reduces to a local quasilinear elliptic Dirichlet problem.

The main difficult point of this work is *to cover the more delicate degenerate case*, in which compactness properties are harder to handle. The efforts in treating the degenerate case require a special care and a deeper analysis, as the main proof of Lemma 2.3 shows.

For this reason, even the most recent papers on stationary problems cover only the non–degenerate case, where $\gamma = 1$ in $(\mathcal{M})$, that is when $M(\tau) \geq s > 0$ for all $\tau \in \mathbb{R}_0^+$; see e.g.

Non–degenerate case

[CWL] **B. Cheng, X. Wu, J. Liu**, *Multiple solutions for a class of Kirchhoff type problems with concave nonlinearity*, NoDEA Nonlinear Differential Equations Appl. 19 (2012) 521–537.

[FV] **A. Fiscella, E. Valdinoci**, *A critical Kirchhoff type problem involving a nonlocal operator*, Nonlinear Anal. 94, 156–170. (2014).

[LLS] **Y. Li, F. Li, J. Shi**, *Existence of a positive solution to Kirchhoff type problems without compactness conditions*, J. Differ. Equations 253 (2012) 2285–2294.

[P] **R. Pei**, *On a p–Laplacian Equation of Kirchhoff–Type with a Potential Asymptotically Linear at Infinity*, Int. Journal of Math. Analysis 6 (2012) 1347–1353.

Degenerate case

A preliminary study of Δ_p^L only when $d = 1$ has been first developed in [CP], where a possibly degenerate scalar stationary p–polyharmonic Kirchhoff Dirichlet problem has been considered.

In the higher order vectorial setting several different norms are available for the solution functional space $[W_0^{L,p}(\Omega)]^d$. We prove the equivalence between the standard Sobolev norm and the norm $\|u\| = \|\,|\mathcal{D}_L u|_N\,\|_p$, which is the natural norm arising from the variational structure of problem (1). The proof of the equivalence is based on the *Poincaré* and *Caldéron–Zygmund* inequalities and relies on Proposition A.1 proved in [CP] when $d = 1$.

Proposition A.1 of [CP] when $d = 1$

If $L = 1, 2, \ldots$***, then there exists a positive constant*** $\kappa_L = \kappa_L(n, p)$ ***such that for all*** $u \in W_0^{L,p}(\Omega)$

$$\|u\|_{\mathfrak{D}^{L,p}(\Omega)} \leq \kappa_L \|\mathcal{D}_L u\|_p, \tag{5}$$

where

$$\mathcal{D}_L u = \begin{cases} \Delta^j u \ \textit{if}\ L = 2j, \\ D\Delta^{j-1} u \ \textit{if}\ L = 2j-1, \end{cases} \quad j = 1, 2, \ldots$$

We distinguish two cases depending on whether L **is even or odd.**

Case $L = 2j$***.*** **Proceed by induction on** $j = 1, 2, \ldots$**. The case** $j = 1$ **is true by the consequence of the Caldéron–Zygmund inequality cf. Corollary 9.10 the monograph of Gilbarg and Trudinger. Indeed, there**

Proposition A.1 of [CP] when $d = 1$

exists a constant $\kappa_2 = \kappa_2(n,p) > 0$ such that for all $u \in W_0^{2,p}(\Omega)$

$$\|u\|_{\mathfrak{D}^{2,p}(\Omega)} \le \kappa_2 \|\Delta u\|_p = \kappa_2 \|\mathcal{D}_2 u\|_p.$$

Suppose that the inequality (5) holds for $j \ge 1$ and show that it is true also for $j+1$. Let u be in $C_0^\infty(\Omega)$ and $L = 2(j+1)$, then

$$\begin{aligned}\|u\|^p_{\mathfrak{D}^{L,p}(\Omega)} &= \sum_{|\alpha|=L} \|D^\alpha u\|_p^p = \sum_{|\alpha|=2j}\sum_{|\beta|=2} \|D^\beta(D^\alpha u)\|_p^p = \sum_{|\alpha|=2j} \|D^\alpha u\|^p_{\mathfrak{D}^{2,p}(\Omega)}\\ &\le \kappa_2^p \sum_{|\alpha|=2j} \|\Delta(D^\alpha u)\|_p^p = \kappa_2^p \sum_{|\alpha|=2j} \|D^\alpha(\Delta u)\|_p^p = \kappa_2^p \|\Delta u\|^p_{\mathfrak{D}^{2j,p}(\Omega)}\\ &\le (\kappa_2\kappa_{2j})^p \|\mathcal{D}_{2j}(\Delta u)\|_p^p = \kappa_L^p \|\mathcal{D}_L u\|_p^p,\end{aligned}$$

where $\kappa_L = \kappa_2\kappa_{2j}$, and κ_2 is given above. We conclude the proof using density arguments.

Proposition A.1 of [CP] when $d = 1$

Case $L = 2j - 1$**.** **For** $L = 1$ **inequality** (5) **is trivial, indeed it holds with the equality sign by the definitions of the norms. Proceeding by induction on** $j = 1, 2, \dots,$ **we suppose the inequality holds for** j **and we show that it is true also for** $j + 1$**, namely for** $L = 2j + 1$**. Indeed for all** $u \in C_0^\infty(\Omega)$**,**

$$\begin{aligned}\|u\|^p_{\mathfrak{D}^{L,p}(\Omega)} &= \sum_{|\alpha|=2j-1} \sum_{|\beta|=2} \|D^\beta(D^\alpha u)\|_p^p \le \kappa_2^p \sum_{|\alpha|=2j-1} \|\Delta(D^\alpha u)\|_p^p \\ &= \kappa_2^p \sum_{|\alpha|=2j-1} \|D^\alpha(\Delta u)\|_p^p = \kappa_2^p \|\Delta u\|^p_{\mathfrak{D}^{2j-1,p}(\Omega)} \\ &\le (\kappa_2 \kappa_{2j-1})^p \|\mathcal{D}_{2j-1}(\Delta u)\|_p^p = \kappa_L^p \|D\Delta^{j-1}(\Delta u)\|_p^p = \kappa_L^p \|\mathcal{D}_L u\|_p^p\end{aligned}$$

where $\kappa_L = \kappa_2 \kappa_{2j-1}$ **and** κ_2 **is given above. The proof is completed again using density arguments.**

Degenerate case

However, the space $([W_0^{L,p}(\Omega)]^d, \|\cdot\|)$ is uniformly convex, as shown in this paper by a useful inequality given in Lemma A.1 of

[APV] G. Autuori, P. P., Cs. Varga, ***Existence theorems for quasilinear elliptic eigenvalue problems in unbounded domains***, **Adv. Differential Equ. 18 (2013) 1-48.**

A special case

We conclude the first part of the talk with the easier problem

$$\begin{cases} M(\|u\|^p)\Delta_p^L u = \lambda f(x,u) & \text{in } \Omega, \\ D^\alpha u_k\big|_{\partial\Omega} = 0 \text{ for all } \alpha, \text{ with } |\alpha| \le L-1,\ k = 1, \dots, d, & \end{cases} \tag{6}$$

which is the main model first treated in

[KLV] **A. Kristály, H. Lisei, Cs. Varga**, *Multiple solutions for p–Laplacian type equations*, Nonlinear Anal. 68 (2008) 1375-1381,

when $M \equiv 1$, $L = d = 1$ and $p \ge 2$, and in which the right–hand side of the system presents only the term $\lambda f(x,u)$. The main result for (41), an analogue of the principle theorem for (1), is proved under a simpler and more direct condition on f and without the use of the first eigenfunction of Δ_p^L.

The non–degenerate case $\gamma = 1$

Later we assume that $\gamma = 1$, that is we deal with the *non–degenerate* case of (1), being $M(\tau) \geq s > 0$ for all $\tau \in \mathbb{R}_0^+$ by $(\mathcal{M})$. Hence, we are devoted to the study of the special higher order p–Kirchhoff problem

$$\begin{cases} M(\|u\|^p)\Delta_p^L u = \lambda\{w(x)|u|^{p-2}u + f(x,u)\} \text{ in } \Omega, \\ D^\alpha u_k\big|_{\partial\Omega} = 0 \text{ for all } \alpha, \text{ with } |\alpha| \leq L-1,\ k = 1,\dots,d, \end{cases} \tag{7}$$

where here w satisfies (4), with $\gamma = 1$.

Extensions

We then extend the results of the non–degenerate case $\gamma = 1$ to the $p(x)$–polyharmonic Kirchhoff problem

$$\begin{cases} M(\mathscr{I}_L(u))\Delta^L_{p(x)} u = \lambda\{w(x)|u|^{p(x)-2}u + f(x,u)\} \text{ in } \Omega, \\ D^\alpha u_k\big|_{\partial\Omega} = 0 \text{ for all } \alpha, \text{ with } |\alpha| \le L-1,\ k = 1,\dots,d, \end{cases} \tag{8}$$

where now $\Omega \subset \mathbb{R}^n$ *is a bounded domain with Lipschitz boundary* and

$$\mathscr{I}_L(u) = \int_\Omega \frac{|\mathcal{D}_L u|^{p(x)}}{p(x)} dx \tag{9}$$

is the Dirichlet functional associated to the weak form of $\Delta^L_{p(x)}$, that is related to

$$\langle \Delta^L_{p(x)} u, \varphi \rangle = \int_\Omega |\mathcal{D}_L u|^{p(x)-2} \mathcal{D}_L u \cdot \mathcal{D}_L \varphi dx$$

for all u, $\varphi \in [W_0^{L,p(\cdot)}(\Omega)]^d$.

References for variable exponent Lebesgue and Sobolev spaces

The solution functional space is the vector–valued variable exponent Sobolev space $[W_0^{L,p(\cdot)}(\Omega)]^d$, which in the scalar case $d = 1$ has been extensively studied in the last two decades, see

- [D] **L. Diening**, *Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces* $L^{p(\cdot)}$ *and* $W^{k,p(\cdot)}$, Math. Nachr. 268 (2004) 31-43.
- [DHHR] **L. Diening, P. Harjulehto, P. Hästö and M. Růžička**, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Vol. 2017 (Springer–Verlag, Berlin, 2011) ix+509 pp.
- [ER] **D.E. Edmunds, J. Rákosník**, *Sobolev embeddings with variable exponent*, Studia Math. 143 (2000) 267-293.

The variable exponent case

[KR] **O. Kováčik, J. Rákosník**, *On spaces* $L^{p(x)}$ *and* $W^{1,p(x)}$, Czechoslovak Math. J. 41 (1991) 592-618.

[MOSS] **Y. Mizuta, T. Ohno, T. Shimomura, N. Shioji**, *Compact embeddings for Sobolev spaces of variable exponents and existence of solutions for nonlinear elliptic problems involving the* $p(x)$*–Laplacian and its critical exponent*, Ann. Acad. Sci. Fenn. Math. 35 (2010) 115-130.

Indeed, the variable exponent Lebesgue and Sobolev spaces arouse a great interest not only for the mathematical curiosity, but also for concrete applications. For instance, in models where linear elasticity (Hooke's law) is replaced by $p(x)$–power–like elasticity. Problem (8) can be used in modeling steady electrorheological fluids (that is fluids whose mechanical properties strongly depend on the applied electromagnetic field).

The variable exponent case

See

[DER] **L. Diening, F. Ettwein, M. Růžička**, $C^{1,\alpha}$*–regularity for electrorheological fluids in two dimensions*, NoDEA Nonlinear Differential Equations Appl. 14 (2007) 207–217.

[R] **M. Růžička**, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics Vol. 1748 (Springer–Verlag, Berlin, 2000) xvi+176 pp.

for more specific comments. The range of applications of electrorheological fluids is wide and includes vibration absorbers, engine mounts, earthquake–resistant buildings, clutches, etc.

The variable exponent case

However, the vectorial case does not seem to be so well–known, so that in this lectures we present also the main properties of $[W_0^{L,p(\cdot)}(\Omega)]^d$.

We require that *the variable exponent p is of a specific class $C_+(\overline{\Omega})$* satisfies all the standard assumptions which are natural in this setting. For simplicity, we also *assume that*

$$\textbf{\textit{either}} \qquad \frac{n}{L} > p_+ = \max_{\overline{\Omega}} p \qquad \frac{n}{L} \le p_- = \min_{\overline{\Omega}} p.$$

The weight w is *positive a.e. in Ω and of class $L^{\varpi}(\Omega)$, with $\varpi > n/(n-[n-Lp_-]^+)$*. Furthermore, the most interesting case occurs when $p_- < p_+$, that is in the so–called *nonstandard growth condition of (p_-,p_+) type*; cf.

The variable exponent case

[AS] **S. Antontsev, S. Shmarev**, *Elliptic equations and systems with nonstandard growth conditions: existence, uniqueness, localization properties of solutions*, Nonlinear Anal. 65 (2006) 728-761.

The main reason why the $p(x)$–Laplace operators possess more complicated behavior is the fact that they are no longer homogeneous. Moreover, the first eigenvalue λ_1 of the $p(x)$–Laplace Dirichlet problem could be zero, see

[FZZ] **X. Fan, Q. Zhang, D. Zhao**, *Eigenvalues of $p(x)$–Laplacian Dirichlet problem*, J. Math. Anal. Appl. 302 (2005) 306–317.

When $L = d = 1$, in [FZZ] and then in

[A] **W. Allegretto**, *Form estimates for the $p(x)$–Laplacean*, Proc. Amer. Math. Soc. 135 (2007) 2177-2185.

The first eigenvalue λ_1 of the $p(x)$–polyharmonic Dirichlet problem

[MRS] **M. Mihăilescu, V. Rădulescu, D. Stancu–Dumitru**, *A Caffarelli–Kohn–Nirenberg–type inequality with variable exponent and applications to PDEs*, Complex Var. Elliptic Equ. 56 (2011) 659–669.

the authors give sufficient conditions on the function p in order to have $\lambda_1 > 0$.
We carry on a long discussion on the positivity of the first eigenvalue of the $p(x)$–polyharmonic Dirichlet problem when L, $d \geq 1$, and finally prove the main results of this case.

Preliminaries for (1)

In the scalar setting, by $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ we denote a multi–index, with length $|\alpha| = \sum_{i=1}^n \alpha_i \le L$ and the corresponding partial differentiation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

Throughout the talk we assume that $1 < p < \infty$ and denote by $W_0^{L,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the standard norm $\|\psi\|_{W^{L,p}(\Omega)} = \left(\sum_{|\alpha| \le L} \|D^\alpha \psi\|_p^p\right)^{1/p}$. By the *Poincaré* and the *Caldéron–Zygmund* inequalities, Proposition A.1 in [CP] shows that the standard norm $\| \cdot \|_{W^{L,p}(\Omega)}$ and the norm

$$\|\psi\|_{L,p} = \begin{cases} \|\Delta^j \psi\|_p, & \text{if } L = 2j, \\ \left(\sum_{i=1}^n \|\partial_{x_i} \Delta^{j-1} \psi\|_p^p\right)^{1/p}, & \text{if } L = 2j-1, \end{cases} \qquad j = 1, 2, \ldots \tag{10}$$

are equivalent in $W_0^{L,p}(\Omega)$.

Preliminaries for (1)

As already noted, since we are *in the vectorial setting*, we denote by $\mathcal{D}_L$ the vectorial operator defined in (3). Hence, if $L = 1$, the operator $\mathcal{D}_1$ writes the pointwise Jacobian matrix of u, $Ju(x) \in \mathcal{M}_{d\times n}(\mathbb{R})$, as the dn–row vector $\mathcal{D}_1 u(x) \in \mathbb{R}^{dn}$. Furthermore, for all $L = 1, 2, \ldots$, the norm

$$\|u\|_{d,L,p} = \left(\sum_{k=1}^{d} \|u_k\|_{L,p}^{p}\right)^{1/p}$$

in $[W_0^{L,p}(\Omega)]^d$ is equivalent to the standard norm

$$\|u\|_{[W^{L,p}(\Omega)]^d} = \left(\sum_{k=1}^{d} \|u_k\|_{W^{L,p}(\Omega)}^{p}\right)^{1/p},$$

as a direct consequence of Proposition A.1 in [CP]. Moreover, $\left([W_0^{L,p}(\Omega)]^d, \|\cdot\|_{d,L,p}\right)$ is a uniformly convex Banach space.

$\left([W_0^{L,p}(\Omega)]^d, \|\cdot\|_{d,L,p}\right)$ is a uniformly convex Banach space

Indeed, the vector–valued space $\left([W_0^{L,p}(\Omega)]^d, \|\cdot\|_{d,L,p}\right)$ is the Cartesian product of d copies of the scalar space $W_0^{L,p}(\Omega)$ endowed with the norm $\|u\|_{L,p}$ defined in (10). It is enough to prove that $\left(W_0^{L,p}(\Omega), \|\cdot\|_{L,p}\right)$ is uniformly convex. Indeed, this implies that $([W_0^{L,p}(\Omega)]^d, \|\cdot\|_{d,L,p})$ is uniformly convex, by Theorem 1.22 of

[A] R.A. Adams, *Sobolev spaces*, Pure and Applied Mathematics Vol. 65 (Academic Press, New York–London, 1975) xviii+268 pp.

We distinguish two cases depending on whether L is even or odd.

Case $L = 2j$, $j = 1, 2, \ldots$.

Fix $\varepsilon \in (0,2)$ **and let** u**,** $v \in W_0^{2j,p}(\Omega)$ **be such that** $\|u\|_{L,p} = \|v\|_{L,p} = 1$ **and** $\|u - v\|_{L,p} \geq \varepsilon$**.**
Consider first the case $p \in [2, \infty)$**. By** (35) **of Lemma 2.27 of [A], we have that for all** z**,** $\zeta \in \mathbb{R}$

$$\left|\frac{z+\zeta}{2}\right|^p + \left|\frac{z-\zeta}{2}\right|^p \leq \frac{1}{2}(|z|^p + |\zeta|^p).$$

Hence,

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|_{L,p}^p + \left\|\frac{u-v}{2}\right\|_{L,p}^p &= \int_\Omega \left(\left|\frac{\Delta^j u + \Delta^j v}{2}\right|^p + \left|\frac{\Delta^j u - \Delta^j v}{2}\right|^p\right) dx \\ &\leq \frac{1}{2}\int_\Omega \left(|\Delta^j u|^p + |\Delta^j v|^p\right) dx \\ &= \frac{1}{2}\left(\|u\|_{L,p}^p + \|v\|_{L,p}^p\right) = 1.\end{aligned}$$

Case $L = 2j$, $j = 1, 2, \dots$.

This implies that

$$\left\| \frac{u+v}{2} \right\|_{L,p}^{p} \le 1 - \left(\frac{\varepsilon}{2}\right)^{p}$$

and so, taking $\delta = \delta(\varepsilon)$ such that $1 - (\varepsilon/2)^p = (1-\delta)^p$, the proof of this case is concluded.

If $p \in (1,2)$, then by Theorem 2.7 of [A]

$$\Bigg\| \Big|\mathcal{D}_L\Big(\frac{u+v}{2}\Big)\Big|^{p'} \Bigg\|_{p-1} + \Bigg\| \Big|\mathcal{D}_L\Big(\frac{u-v}{2}\Big)\Big|^{p'} \Bigg\|_{p-1} \le \Bigg\| \Big|\mathcal{D}_L\Big(\frac{u+v}{2}\Big)\Big|^{p'} + \Big|\mathcal{D}_L\Big(\frac{u-}{2}$$

In other words

$$\left\| \frac{u+v}{2} \right\|_{L,p}^{p'} + \left\| \frac{u-v}{2} \right\|_{L,p}^{p'} \le \left\| \left|\mathcal{D}_L\left(\frac{u+v}{2}\right)\right|^{p'} + \left|\mathcal{D}_L\left(\frac{u-v}{2}\right)\right|^{p'} \right\|_{p-1}, \tag{11}$$

since $|\Delta^j \psi|^{p'} \in L^{p-1}(\Omega)$ and $\| |\mathcal{D}_L \psi|^{p'} \|_{p-1} = \|\mathcal{D}_L \psi\|_p^{p'}$ for all $\psi \in W_0^{L,p}(\Omega)$.

Case $L = 2j$, $j = 1, 2, \dots$.

Moreover, being $1 < p < 2$, by (34) of Lemma 2.27 of [A]

$$\left|\frac{z+\zeta}{2}\right|^{p'} + \left|\frac{z-\zeta}{2}\right|^{p'} \le \left[\frac{1}{2}(|z|^p + |\zeta|^p)\right]^{1/(p-1)}$$

for all z, $\zeta \in \mathbb{R}$. Hence,

$$\left\| \left|\mathcal{D}_L\left(\frac{u+v}{2}\right)\right|^{p'} + \left|\mathcal{D}_L\left(\frac{u-v}{2}\right)\right|^{p'} \right\|_{p-1} \le \left[\int_\Omega \frac{1}{2}\left(|\Delta^j u|^p + |\Delta^j v|^p\right) dx\right]^{1/(p-}$$

$$= \left(\frac{1}{2}\|u\|_{L,p}^p + \frac{1}{2}\|v\|_{L,p}^p\right)^{1/(p-1)} = 1 \qquad (12)$$

Combining together (11) and (12), we get

$$\left\|\frac{u+v}{2}\right\|_{L,p}^{p'} \le 1 - \left\|\frac{u-v}{2}\right\|_{L,p}^{p'} \le 1 - \left(\frac{\varepsilon}{2}\right)^{p'}.$$

It is enough to take $\delta = \delta(\varepsilon)$ such that $1 - (\varepsilon/2)^{p'} = (1-\delta)^{p'}$ in order to conclude the proof also

Case $L = 2j - 1$, $j = 1, 2, \dots$

in the case $1 < p < 2$.
Consider the vector–valued space
$[L^p(\Omega)]^n = ([L^p(\Omega)]^n, \|\cdot\|_{[L^p(\Omega)]^n})$**, where**

$$\|g\|_{[L^p(\Omega)]^n} = \left(\sum_{i=1}^{n} \|g_i\|_p^p\right)^{1/p} \qquad \text{for all } g = (g_1, \dots, g_n) \in [L^p(\Omega)]^n.$$

The linear operator $T : W_0^{L,p}(\Omega) \to [L^p(\Omega)]^n$, defined for all $u \in W_0^{L,p}(\Omega)$ by

$$T(u) = \left(\partial_{x_1}\Delta^j u, \dots, \partial_{x_n}\Delta^j u\right),$$

is isometric. Furthermore, the space $[L^p(\Omega)]^n$ is uniformly convex, by Theorem 3 of

[D] M.M. Day, ***Some more uniformly convex spaces***, **Bull. Amer. Math. Soc. 47 (1941) 504–507.**

Case $L = 2j - 1$, $j = 1, 2, \dots$

since $(L^p(\Omega), \|\cdot\|_p)$ **is uniformly convex itself. Hence, also** $\left(W_0^{L,p}(\Omega), \|\cdot\|_{L,p}\right)$ **is uniformly convex, being isometric to a uniformly convex Banach space. This concludes the proof.**

However, since we are interested in the variational problem (1)**,** ***from now on we endow the space*** $[W_0^{L,p}(\Omega)]^d$ ***with the norm***

$$\|u\| = \|\,|\mathcal{D}_L u|_N\,\|_p,$$

where $|\cdot|_N$ **denotes the Euclidean norm in** $\mathbb{R}^N$ **and** $N = d$ **when** L **is even, while** $N = dn$ **when** L **is odd. Also** ***the space*** $\left([W_0^{L,p}(\Omega)]^d, \|\cdot\|\right)$ ***is uniformly convex.***

$\left([W_0^{L,p}(\Omega)]^d, \|\cdot\|\right)$ is uniformly convex

Fix $\varepsilon \in (0,2)$ **and let** $u, v \in [W_0^{L,p}(\Omega)]^d$ **be such that** $\|u\| = \|v\| = 1$ **and** $\|u - v\| \geq \varepsilon$**.**
Consider first the case $p \in [2,\infty)$**. By (A.2) of Lemma A.1 of [APV], we have that for all** z**,** $\zeta \in \mathbb{R}^N$

$$\left|\frac{z+\zeta}{2}\right|_N^p + \left|\frac{z-\zeta}{2}\right|_N^p \leq \frac{1}{2}(|z|_N^p + |\zeta|_N^p).$$

Hence, with $z = \mathcal{D}_L u$**,** $\zeta = \mathcal{D}_L v \in \mathbb{R}^N$**, we get**

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p &= \int_\Omega \left(\left|\frac{\mathcal{D}_L u + \mathcal{D}_L v}{2}\right|_N^p + \left|\frac{\mathcal{D}_L u - \mathcal{D}_L v}{2}\right|_N^p\right) dx \\ &\leq \frac{1}{2}\int_\Omega \left(|\mathcal{D}_L u|_N^p + |\mathcal{D}_L v|_N^p\right) dx \\ &= \frac{1}{2}\left(\|u\|^p + \|v\|^p\right) = 1.\end{aligned}$$

$\left([W_0^{L,p}(\Omega)]^d, \|\cdot\|\right)$ is uniformly convex

This implies that

$$\left\|\frac{u+v}{2}\right\|^p \le 1-\left(\frac{\varepsilon}{2}\right)^p.$$

It is enough to take $\delta=\delta(\varepsilon)$ **such that** $1-(\varepsilon/2)^p=(1-\delta)^p$**, in order to conclude the proof.**
If $p\in(1,2)$**, then by Theorem 2.7 of [A]**

$$\left\|\left|\mathcal{D}_L\left(\frac{u+v}{2}\right)\right|_N^{p'}\right\|_{p-1}+\left\|\left|\mathcal{D}_L\left(\frac{u-v}{2}\right)\right|_N^{p'}\right\|_{p-1} \le \left\|\left|\mathcal{D}_L\left(\frac{u+v}{2}\right)\right|_N^{p'}+\left|\mathcal{D}_L\left(\frac{u-v}{2}\right)\right|_N^{p'}\right\|_{p-1}.$$

In other words

$$\left\|\frac{u+v}{2}\right\|^{p'}+\left\|\frac{u-v}{2}\right\|^{p'} \le \left\|\left|\mathcal{D}_L\left(\frac{u+v}{2}\right)\right|_N^{p'}+\left|\mathcal{D}_L\left(\frac{u-v}{2}\right)\right|_N^{p'}\right\|_{p-1}, \tag{13}$$

$\left([W_0^{L,p}(\Omega)]^d, \|\cdot\|\right)$ is uniformly convex

since $|\mathcal{D}_L\phi|_N^{p'} \in L^{p-1}(\Omega)$ and $\||\mathcal{D}_L\phi|_N^{p'}\|_{p-1} = \||\mathcal{D}_L\phi|_N\|_p^{p'}$ for all $\phi \in [W_0^{L,p}(\Omega)]^d$. Moreover, being $1 < p < 2$, by (A.1) of Lemma A.1 of [APV], we have that for all $z, \zeta \in \mathbb{R}^N$

$$\left|\frac{z+\zeta}{2}\right|_N^{p'} + \left|\frac{z-\zeta}{2}\right|_N^{p'} \le \left(\frac{1}{2}|z|_N^p + \frac{1}{2}|\zeta|_N^p\right)^{1/(p-1)}.$$

Hence, with $z = \mathcal{D}_L u$, $\zeta = \mathcal{D}_L v \in \mathbb{R}^N$, we get

$$\begin{aligned}\left\|\left|\mathcal{D}_L\left(\frac{u+v}{2}\right)\right|_N^{p'} + \left|\mathcal{D}_L\left(\frac{u-v}{2}\right)\right|_N^{p'}\right\|_{p-1} &\le \left[\frac{1}{2}\int_\Omega \left(|\mathcal{D}_L u|_N^p + |\mathcal{D}_L v|_N^p\right) dx\right]^{1/(p-} \\ &= \left(\frac{1}{2}\|u\|^p + \frac{1}{2}\|v\|^p\right)^{1/(p-1)} = 1. \end{aligned} \tag{14}$$

Combining together (13) with (14), we obtain

$$\left\|\frac{u+v}{2}\right\|^{p'} \le 1 - \left\|\frac{u-v}{2}\right\|^{p'} \le 1 - \left(\frac{\varepsilon}{2}\right)^{p'}.$$

Preliminaries for (1)

It is enough to take $\delta = \delta(\varepsilon)$ such that $1 - (\varepsilon/2)^{p'} = (1-\delta)^{p'}$ in order to conclude the proof.

An easy calculation shows that the two norms $\|\cdot\|_{d,L,p}$ and $\|\cdot\|$ are equivalent in $[W_0^{L,p}(\Omega)]^d$. Indeed, for all $u \in [W_0^{L,p}(\Omega)]^d$

$$\min\{1,\, N^{\frac{1}{p}-\frac{1}{2}}\}\|u\| \le \|u\|_{d,L,p} \le \max\{1,\, N^{\frac{1}{p}-\frac{1}{2}}\}\|u\|.$$

In particular, the two norms coincide whenever either $p = 2$, or $N = 1$.

Preliminaries for (1)

The Lebesgue spaces $[L^\sigma(\Omega)]^m$ and $[L^\sigma(\Omega,\omega)]^m$, where $\sigma \geq 1$, ω is any weight on Ω and $m \geq 1$ is any dimension, are endowed with the norms $\|\varphi\|_{m,\sigma} = \|\,|\varphi|_m\|_\sigma$ and $\|\varphi\|_{m,\sigma,\omega} = \|\,|\varphi|_m\|_{\sigma,\omega}$, respectively. When $m = 1$ the norm is denoted by $\|\varphi\|_{\sigma,\omega}$. The dot $\cdot$ indicates the inner product and $|\cdot|_m$ denotes the Euclidean norm in $\mathbb{R}^m$. In what follows, when the dimension is clear from the context, we drop the subscript m and denote the m–Euclidean norm simply by $|\cdot|$.

As already noted, the main assumption

$$1 \leq \gamma < p_L^*/p \quad \text{implies that} \quad \varpi > n/(n-\gamma[n-Lp]^+) \geq 1,$$

by (4). When $n > Lp$ then

$$\varpi' < n/\gamma(n-Lp), \quad \text{that is} \quad \gamma p < p_L^*/\varpi', \tag{15}$$

this will be useful in the next lemma. For simplicity in notation, whenever the embedding operator

Preliminaries for (1)

$$i : [W_0^{L,p}(\Omega)]^d \to [L^\sigma(\Omega, \omega)]^d$$

is continuous, we denote by $\mathcal{S}_{d,\sigma,w} > 0$ the best constant such that $\|u\|_{d,\sigma,\omega} \le \mathcal{S}_{d,\sigma,\omega}\|u\|$ for all $u \in [W_0^{L,p}(\Omega)]^d$, that is $\mathcal{S}_{d,\sigma,w}$ is the operator norm of i. If $d = 1$ and $\omega \equiv 1$, we briefly write $\mathcal{S}_\sigma$. Furthermore, whenever $p_L^* = \infty$, the symbol p_L^*/ϖ' is again ∞.

Preliminaries for (1)

LEMMA

The following embeddings hold.

(i) $[W_0^{L,p}(\Omega)]^d \hookrightarrow\hookrightarrow [L^\sigma(\Omega, w)]^d$ *compactly, if* $\sigma \in [1, \gamma p]$.

(ii) $[W_0^{L,p}(\Omega)]^d \hookrightarrow [L^\sigma(\Omega, w)]^d$ *continuously, if* $\sigma \in (\gamma p, p_L^*/\varpi')$.

Proof. (*i*) **The space** $W_0^{L,p}(\Omega)$ **is compactly embedded into** $L^{\varpi'\sigma}(\Omega)$**, being** $\varpi'\sigma < p_L^*$ **by** (2) **and** (15)**. Similarly,** $L^{\varpi'\sigma}(\Omega)$ **is continuously embedded in** $L^\sigma(\Omega, w)$ **by Hölder's inequality and** (4)**. Hence,** $\Big([W_0^{L,p}(\Omega)]^d, \|\cdot\|\Big)$ **is compactly embedded into** $\big([L^\sigma(\Omega, w)]^d, \|\cdot\|_{d,\sigma,w}\big)$**, being** $\|\cdot\|$ **equivalent to** $\|\cdot\|_{d,L,p}$ **in** $[W_0^{L,p}(\Omega)]^d$ **as observed above.**

Preliminaries for (1)

(ii) **By Hölder's inequality, for all** $\psi \in W_0^{L,p}(\Omega)$

$$\|\psi\|_{\sigma,w}^{\sigma} \leq |\Omega|^{1/\wp}\|w\|_{\varpi}\|\psi\|_{p_L^*}^{\sigma} \leq C\|\psi\|^{\sigma},$$

where $C = \mathcal{S}_{p_L^*}^{\sigma}|\Omega|^{1/\varpi}\|w\|_{\varpi}$ **and** $\wp$ **is the crucial exponent**

$$\wp = \begin{cases} \dfrac{\varpi' p_L^*}{p_L^* - \sigma\varpi'}, & \textbf{if} \qquad n > Lp, \\ \varpi', & \textbf{if } 1 \leq n \leq Lp. \end{cases}$$

Clearly $\wp > 1$**, being** $\sigma < p_L^*/\varpi'$**. The conclusion now follows as in** (i)**.**

Preliminaries for (1)

Let us now turn to the main problem (1) **and let**

$$\lambda_1 = \inf_{\substack{u\in[W_0^{L,p}(\Omega)]^d \\ u\neq 0}} \frac{\|u\|^p}{\|u\|^p_{d,p,w}} \tag{16}$$

be *the first eigenvalue* **of**

$$\begin{cases} \Delta_p^L u = \lambda\, w(x)|u|^{p-2}u \ \textbf{in}\ \Omega, \\ D^\alpha u_k\big|_{\partial\Omega} = 0 \ \textbf{for all}\ \alpha\textbf{, with}\ |\alpha| \le L-1,\ k=1,\dots,d. \end{cases}$$

Clearly, λ_1 **is well defined since the embedding** $[W_0^{L,p}(\Omega)]^d \hookrightarrow\hookrightarrow [L^p(\Omega,w)]^d$ **is compact, as shown in Lemma 1–**(i)**.**

Preliminaries for (1)

PROPOSITION

The infimum λ_1 *in* (16) *is positive and attained at a certain function* $u_1 \in [W_0^{L,p}(\Omega)]^d$, *with* $\|u_1\|_{d,p,w} = 1$.

Proof. For any $u \in [W_0^{L,p}(\Omega)]^d$ define the functionals $\mathcal{I}(u) = \|u\|^p$ and $\mathcal{J}(u) = \|u\|_{d,p,w}^p$. Let

$\lambda_0 = \inf\{\mathcal{I}(u)/\mathcal{J}(u) \,:\, u \in [W_0^{L,p}(\Omega)]^d \setminus \{0\},\ \|u\|_{d,p,w} \leq 1\}$.

Observe that $\mathcal{I}$ and $\mathcal{J}$ are continuously Fréchet differentiable and convex in $[W_0^{L,p}(\Omega)]^d$. Clearly $\mathcal{I}'(0) = \mathcal{J}'(0) = 0$. Moreover, $\mathcal{J}'(u) = 0$ implies $u = 0$. In particular, $\mathcal{I}$ and $\mathcal{J}$ are weakly lower semi–continuous on $[W_0^{L,p}(\Omega)]^d$. Actually, $\mathcal{J}$ is weakly sequentially continuous on $[W_0^{L,p}(\Omega)]^d$. Indeed, if $(u_k)_k \subset [W_0^{L,p}(\Omega)]^d$ and $u_k \rightharpoonup u$ in $[W_0^{L,p}(\Omega)]^d$, then $u_k \to u$ in $[L^p(\Omega, w)]^d$ by Lemma 1–(i). This implies at once that

$\mathcal{J}(u_k) = \|u_k\|_{d,p,w}^p \to \|u\|_{d,p,w}^p = \mathcal{J}(u)$, as claimed.

Preliminaries for (1)

Now, either $W = \{u \in [W_0^{L,p}(\Omega)]^d : \mathcal{J}(u) \leq 1\}$ is bounded in $[W_0^{L,p}(\Omega)]^d$, or not. In the first case we are done, while in the latter $\mathcal{I}$ is coercive in W, being coercive in the reflexive Banach space $[W_0^{L,p}(\Omega)]^d$ as shown above. Therefore, all the assumptions of Theorem 6.3.2 of

[B] M.S. Berger, *Nonlinearity and functional analysis*, Lectures on Nonlinear Problems in Mathematical Analysis, Pure and Applied Mathematics (Academic Press, New York–London, 1977) xix+417 pp.

are fulfilled and λ_0 is attained at a point $u_1 \in [W_0^{L,p}(\Omega)]^d$, with $\|u_1\|_{d,p,w} = 1$.

Preliminaries for (1)

We claim now that $\lambda_0 = \lambda_1$**. Indeed,**

$$\begin{aligned}\lambda_1 &= \inf_{u\in[W_0^{L,p}(\Omega)]^d\setminus\{0\}} \left\| \frac{u}{\|u\|_{d,p,w}} \right\|^p \\ &= \inf_{\substack{u\in[W_0^{L,p}(\Omega)]^d \\ \|u\|_{d,p,w}=1}} \|u\|^p \\ &\geq \inf_{\substack{u\in[W_0^{L,p}(\Omega)]^d \\ 0<\|u\|_{d,p,w}\leq 1}} \frac{\|u\|^p}{\|u\|_{d,p,w}^p} \\ &= \lambda_0 \geq \lambda_1.\end{aligned}$$

Finally, $\lambda_1 = \|u_1\|^p > 0$**. This concludes the proof.**
By Lemma 1–(i)**, there exists** $c_{\gamma p} = \mathcal{S}_{d,\gamma p,w}^{\gamma p} > 0$ **such that**

$$\|u\|_{d,\gamma p,w}^{\gamma p} \leq c_{\gamma p}\|u\|^{\gamma p} \quad \textbf{for all } u \in [W_0^{L,p}(\Omega)]^d. \tag{17}$$

When $\gamma = 1$ **in** $(\mathcal{M})$ **we have** $c_p = 1/\lambda_1 > 0$ **by** (16)**.**

The perturbation f

The nonlinearity f verifies the condition

($\mathcal{F}$) ***Let*** $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, $f = f(x,v) \not\equiv 0$, ***be a Carathéodory function, which admits a potential*** $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $f = D_v F$, ***with*** $F(x,0) = 0$ ***a.e. in*** Ω, ***satisfying the following properties.***

- (a) ***There exist*** $q \in (1, \gamma p)$ ***and*** $C_f > 0$ ***such that***

  $$|f(x,v)| \le C_f w(x)\big(1+|v|^{q-1}\big) \text{ for a.a. } x \in \Omega \text{ and all } v \in \mathbb{R}^d.$$

- (b) ***There exists*** $\mathfrak{p}^\star \in (\gamma p, p^*_L/\varpi')$ ***such that***

  $$\limsup_{|v|\to 0} \frac{|f(x,v)\cdot v|}{w(x)|v|^{\mathfrak{p}^\star}} < \infty, \text{ uniformly a.e. in } \Omega.$$

- (c) $$\int_\Omega F(x,u_1)dx > \frac{1}{p}\left(\frac{\mathscr{M}(\lambda_1)}{s\lambda_1^\gamma} - 1\right).$$

The perturbation f

Note that, in the more familiar and standard setting in the literature in which $L = \gamma = 1$ and $w \in L^\infty(\Omega)$, the exponent $\mathfrak{p}^\star$ in $(\mathcal{F})$–(b) belongs to the open interval (p, p^*). Furthermore, in condition $(\mathcal{F})$–(c), the constant $\dfrac{1}{p}\left(\dfrac{\mathscr{M}(\lambda_1)}{s\lambda_1^\gamma} - 1\right)$ is non–negative thanks to $(\mathcal{M})$. Thus, $(\mathcal{F})$–(c) is automatic when $M \equiv 1$, $s = \gamma = 1$ and $F(x, v) > 0$ a.e. in Ω for all $v \in \mathbb{R}^d \setminus \{0\}$.

An *example* of function $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ verifying $(\mathcal{F})$–(a) and (b) is

$$f(x, v) = w(x) \begin{cases} |v|^{\mathfrak{p}^\star - 2} v, & \text{if } |v| \le 1, \\ |v|^{q-2} v, & \text{if } |v| > 1, \end{cases}$$

with $q \in (1, \gamma p)$, $\mathfrak{p}^\star \in (\gamma p, p_L^*/\varpi')$ and w verifying (4).
More precisely, $(\mathcal{F})$–(a) holds with $C_f = 1$ and $(\mathcal{F})$–(b) is trivially verified. Finally, $F(x, v) > 0$ for a.a. $x \in \Omega$ and all $v \in \mathbb{R}^d \setminus \{0\}$.

The perturbation f

As already noted, this shows that $(\mathcal{F})$–(c) holds when $M \equiv 1$ and $s = \gamma = 1$.
Following [CPV], we later introduce in place of $(\mathcal{F})$–(c) the weaker assumption $(\mathcal{F})$–$(c)'$ much easier to check, which will play the same role for a less involved problem.

PROPOSITION

Assume that $(\mathcal{F})$–(a) *and* (b) *hold. Then* $f(x,0)=0$ *for a.a.* $x \in \Omega$,

$$S_f = \operatorname*{ess\,sup}_{v\neq 0,\, x\in\Omega} \frac{|f(x,v)\cdot v|}{w(x)|v|^{\gamma p}} \in \mathbb{R}^+ \quad \textit{and} \quad \operatorname*{ess\,sup}_{v\neq 0,\, x\in\Omega} \frac{|F(x,v)|}{w(x)|v|^{\gamma p}} \le \frac{S_f}{\gamma p}. \tag{18}$$

Moreover, there exists $K>0$ *such that*

$$|F(x,v)| \le K w(x)|v|^{\mathfrak{p}^\star} \tag{19}$$

The perturbation f

for a.a. $x \in \Omega$ *and all* $v \in \mathbb{R}^d$.

Assume first by contradiction that there exists $A \subset \Omega$, $|A| > 0$, **such that** $|f(x,0)| > 0$ **and** $w(x) > 0$ **for all** $x \in A$. **In particular,**

$$\lim_{|v| \to 0} \frac{|f(x,v) \cdot v|}{w(x)|v|^{\mathfrak{p}^\star}} = \infty$$

for all $x \in A$, **contradicting** $(\mathcal{F})$–(b). **Hence** $f(x,0) = 0$ **for a.a.** $x \in \Omega$.

Since $F(x,0) = 0$ **a.e. in** Ω **by** $(\mathcal{F})$, **we assert that**

$$\limsup_{|v| \to 0} \frac{|F(x,v)|}{w(x)|v|^{\mathfrak{p}^\star}} = \ell_0 < \infty \quad \textbf{uniformly a.e. in } \Omega. \tag{20}$$

Indeed, by $(\mathcal{F})$–(b) **there exist** $\ell > 0$ **and** $\delta > 0$ **such that**

$$\frac{|F(x,v)|}{w(x)|v|^{\mathfrak{p}^\star}} \le \int_0^1 \frac{|f(x,tv) \cdot tv|}{w(x)|tv|^{\mathfrak{p}^\star}} t^{\mathfrak{p}^\star - 1} dt < \frac{\ell}{\mathfrak{p}^\star}$$

The perturbation f

for all $v \in \mathbb{R}^d$, with $0 < |v| < \delta$, and uniformly a.e. in Ω. This implies (20).

Clearly, S_f defined in (18) is positive, being $f \not\equiv 0$. We claim that $S_f < \infty$. Indeed, uniformly a.e. in Ω

$$\lim_{|v|\to 0} \frac{|f(x,v)\cdot v|}{w(x)|v|^{\gamma p}} = \lim_{|v|\to 0} \left[\frac{|f(x,v)\cdot v|}{w(x)|v|^{\mathfrak{p}^\star}}\right] |v|^{\mathfrak{p}^\star - \gamma p} = 0$$

by $(\mathcal{F})$–(b) and the fact that $\gamma p < \mathfrak{p}^\star$. Moreover, $|f(x,v)\cdot v|/w(x)|v|^{\gamma p} \le 2C_f |v|^{q-\gamma p}$ for a.a. $x \in \Omega$ and all $|v| \ge 1$ by $(\mathcal{F})$–(a), that is

$$\lim_{|v|\to \infty} \frac{|f(x,v)\cdot v|}{w(x)|v|^{\gamma p}} = 0 \quad \textbf{uniformly a.e. in } \Omega,$$

since $q < \gamma p$. This shows the claim.

Condition $(18)_1$ implies at once $(18)_2$, since

$$\frac{|F(x,v)|}{w(x)|v|^{\gamma p}} \le \int_0^1 \frac{|f(x,tv)\cdot tv|}{w(x)|tv|^{\gamma p}} t^{\gamma p - 1} dt \le \frac{S_f}{\gamma p}$$

The perturbation f

for a.a. $x \in \Omega$ **and all** $v \in \mathbb{R}^d \setminus \{0\}$**.**
Finally, by (20) **there exists** $\delta > 0$ **such that** $|F(x,v)| \le (\ell_0+1)w(x)|v|^{\mathfrak{p}^\star}$ **for a.a.** $x \in \Omega$ **and all** v**, with** $0 < |v| < \delta$**. Fix** v**, with** $|v| \ge \delta$**, then by** (18) **for a.a.** $x \in \Omega$

$$|F(x,v)| \le \frac{S_f}{\gamma p}|v|^{\gamma p - \mathfrak{p}^\star} w(x)|v|^{\mathfrak{p}^\star} \le \frac{S_f \delta^{\gamma p - \mathfrak{p}^\star}}{\gamma p} w(x)|v|^{\mathfrak{p}^\star},$$

being $\mathfrak{p}^\star \in (\gamma p, p_L^*/\varpi')$ **by** $(\mathcal{F})$–(b)**. Hence, taking** $K = \max\{\ell_0+1, S_f\delta^{\gamma p - \mathfrak{p}^\star}/\gamma p\}$**, we get** (19)**.**

The perturbation f

Now we are ready to introduce *the crucial positive number*

$$\lambda_\star = \frac{s\gamma\lambda_1^\gamma}{\gamma + c_{\gamma p} S_f \lambda_1^\gamma}, \tag{21}$$

which is well defined by (17)**, and Propositions 1 and 2. In passing, we point out that** $\lambda_\star$ **coincides with the same parameter** $\lambda_\star$ **of [CPV], when** $d = s = \gamma = 1$ **and** $M \equiv 1$**, see** (17)**.**

In what follows, the dual space of $[W_0^{L,p}(\Omega)]^d$ **is denoted by** $\left([W_0^{L,p}(\Omega)]^d\right)^\star$**.**

The functional Φ

Lemma

The functional $\Phi : [W_0^{L,p}(\Omega)]^d \to \mathbb{R}$, *defined by*

$$\Phi(u) = \frac{1}{p}\mathscr{M}(\|u\|^p) \tag{22}$$

is convex, weakly lower semi–continuous in $[W_0^{L,p}(\Omega)]^d$ *and of class* $C^1([W_0^{L,p}(\Omega)]^d)$.
Moreover, $\Phi' : [W_0^{L,p}(\Omega)]^d \to \Big([W_0^{L,p}(\Omega)]^d\Big)^\star$ *verifies the* $(\mathscr{S}_+)$ *condition, i.e. for every sequence* $(u_k)_k \subset [W_0^{L,p}(\Omega)]^d$ *such that* $u_k \rightharpoonup u$ *in* $[W_0^{L,p}(\Omega)]^d$ *and*

$$\limsup_{k\to\infty} M(\|u_k\|^p)\int_\Omega |\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k \cdot (\mathcal{D}_L u_k - \mathcal{D}_L u)dx \le 0, \tag{23}$$

then $u_k \to u$ *in* $[W_0^{L,p}(\Omega)]^d$.

The functional Φ

A simple calculation shows that Φ is convex in $[W_0^{L,p}(\Omega)]^d$, since $\mathscr{M}$ is convex and monotone non–decreasing in $\mathbb{R}_0^+$ by $(\mathcal{M})$. Moreover, we claim that $\Phi \in C^1([W_0^{L,p}(\Omega)]^d)$. Indeed, Φ is Gâteaux differentiable in $[W_0^{L,p}(\Omega)]^d$ and for all $u, v \in [W_0^{L,p}(\Omega)]^d$

$$\langle \Phi'(u), v\rangle = M(\|u\|^p) \int_\Omega |\mathcal{D}_L u|^{p-2} \mathcal{D}_L u \cdot \mathcal{D}_L v \, dx.$$

Now, let u, $(u_k)_k \subset [W_0^{L,p}(\Omega)]^d$ be such that $u_k \to u$ in $[W_0^{L,p}(\Omega)]^d$ as $k \to \infty$. We claim that

$$\|\Phi'(u_k) - \Phi'(u)\|_\star = \sup_{\substack{v \in [W_0^{L,p}(\Omega)]^d \\ \|v\|=1}} \left|\langle \Phi'(u_k) - \Phi'(u), v\rangle\right| = o(1)$$

as $k \to \infty$. By Hölder's inequality

$$\begin{aligned}&\left|\langle \Phi'(u_k) - \Phi'(u), v\rangle\right| \\ &\quad \le \left\| M(\|u_k\|^p)|\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u \right\|_{N,p'} \|\mathcal{D}_L v\|_{N,p}.\end{aligned}$$

The functional Φ

Hence

$$\begin{aligned}&\|\Phi'(u_k)-\Phi'(u)\|_\star\\ &\quad\le \left\|M(\|u_k\|^p)|\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\right\|_{N,p'}.\end{aligned} \tag{24}$$

Fix now a subsequence $(u_{k_j})_j$ of $(u_k)_k$. Clearly, $u_{k_j}\to u$ in $[W_0^{L,p}(\Omega)]^d$ and so $\mathcal{D}_L u_{k_j}\to\mathcal{D}_L u$ in $[L^p(\Omega)]^N$ as $j\to\infty$, where as usual $N=d$ if L is even and $N=dn$ if L is odd. By

LEMMA

If $(\varphi_k)_k$ and φ are in $[L^\sigma(\Omega,\omega)]^m$ and $\varphi_k\to\varphi$ in $[L^\sigma(\Omega,\omega)]^m$ as $k\to\infty$, then there exist a subsequence $(\varphi_{k_j})_j$ of $(\varphi_k)_k$ and a function $h\in L^\sigma(\Omega,\omega)$ such that a.e. in Ω

(i) $\varphi_{k_j}\to\varphi$ *as* $j\to\infty$; *(ii)* $|\varphi_{k_j}|\le h$ *for all* $j\in\mathbb{N}$.

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with $m = N$**,** $\sigma = p$ **and** $\omega \equiv 1$**, there exist a subsequence of** $(u_{k_j})_j$**, still denoted by** $(u_{k_j})_j$**, and an appropriate function** $h \in L^p(\Omega)$ **such that a.e. in** Ω **we get that** $\mathcal{D}_L u_{k_j} \to \mathcal{D}_L u$ **as** $j \to \infty$ **and** $|\mathcal{D}_L u_{k_j}| \le h$ **for all** $j \in \mathbb{N}$**. Thus,**

$$\begin{aligned}
&\left| M(\|u_{k_j}\|^p)|\mathcal{D}_L u_{k_j}|^{p-2}\mathcal{D}_L u_{k_j} - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u \right|^{p'} \\
&\quad \le 2^{p'-1}\left\{ \left[M(\|u_{k_j}\|^p)|\mathcal{D}_L u_{k_j}|^{p-1} \right]^{p'} + \left[M(\|u\|^p)|\mathcal{D}_L u|^{p-1} \right]^{p'} \right\} \\
&\quad \le (2K)^{p'} h^p \in L^1(\Omega),
\end{aligned}$$

where $K = \sup_j M(\|u_{k_j}\|^p) < \infty$**, being** $(u_k)_k$ **convergent and so bounded in** $[W_0^{L,p}(\Omega)]^d$**. In particular,** $M(\|u_{k_j}\|^p) \to M(\|u\|^p)$ **by** $(\mathcal{M})$**. Furthermore,** $|\mathcal{D}_L u_{k_j}|^{p-2}\mathcal{D}_L u_{k_j} \to |\mathcal{D}_L u|^{p-2}\mathcal{D}_L u$ **a.e. in** Ω**. Therefore,**

The functional Φ

$$
\begin{aligned}
&\left|M(\|u_{k_j}\|^p)|\mathcal{D}_L u_{k_j}|^{p-2}\mathcal{D}_L u_{k_j} - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\right| \\
&\quad \le K\left||\mathcal{D}_L u_{k_j}|^{p-2}\mathcal{D}_L u_{k_j} - |\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\right| + |M(\|u_{k_j}\|^p) - M(\|u\|^p)| \cdot |\mathcal{D}_L \\
&\quad \to 0 \quad \textbf{a.e. in } \Omega\textbf{, as } j \to \infty.
\end{aligned}
$$

Applying the Lebesgue dominated convergence theorem, we obtain as $j \to \infty$

$$
\|M(\|u_{k_j}\|^p)|\mathcal{D}_L u_{k_j}|^{p-2}\mathcal{D}_L u_{k_j} - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\|_{N,p'} \to 0. \tag{25}
$$

Actually, (25) holds for the entire sequence $(u_k)_k$ and this implies the claim, by virtue of (24).
Therefore Φ is of class $C^1([W_0^{L,p}(\Omega)]^d)$. In particular, Φ is weakly lower semi–continuous in $[W_0^{L,p}(\Omega)]^d$, by Corollary 3.9 of

The functional Φ

[B] **H. Brezis**, ***Functional analysis, Sobolev spaces and partial differential equations***, **Universitext (Springer, New York, 2011) xiv+599 pp.**

Let us now prove the $(\mathscr{S}_+)$ condition. Let $(u_k)_k \subset [W_0^{L,p}(\Omega)]^d$ be such that $u_k \rightharpoonup u$ in $[W_0^{L,p}(\Omega)]^d$ and (23) holds. Since $u_k \rightharpoonup u$, then

$$\lim_{k\to\infty} M(\|u\|^p) \int_\Omega |\mathcal{D}_L u|^{p-2}\mathcal{D}_L u \cdot \mathcal{D}_L(u_k - u)dx = 0, \qquad (26)$$

being $|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u \in [L^{p'}(\Omega)]^N$. Hence, putting $\mathscr{I}_k(x) = \left[M(\|u_k\|^p)|\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k - M(\|u\|^p)|\mathcal{D}_L u|^{p-2}\mathcal{D}_L u\right]\cdot\mathcal{D}_L(u_k - u)$, then (23) is equivalent to

$$\limsup_{k\to\infty} \int_\Omega \mathscr{I}_k(x)dx \le 0.$$

Since $\mathscr{M}(\|\cdot\|^p)$ is convex in $[W_0^{L,p}(\Omega)]^d$, then $\mathscr{I}_k(x) \ge 0$. Therefore

$$\lim_{k\to\infty}\int_\Omega \mathscr{I}_k(x) = 0.$$

This implies by (26)

$$\lim_{k\to\infty} M(\|u_k\|^p)\int_\Omega |\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k\cdot \mathcal{D}_L(u_k-u)\,dx = 0. \qquad (27)$$

Now, two cases arise.

Case $u\neq 0$. **By the weak lower semi–continuity of the norm, we get**

$$0<\|u\|\le \liminf_k \|u_k\| = \ell$$

and consequently there exists $K\in\mathbb{N}$ **such that** $\|u_k\|\ge \ell/2>0$ **for all** $k\ge K$. **Hence, by condition** $(\mathcal{M})$

$$M(\|u_k\|^p)\ge \kappa>0 \quad \textbf{for all } k\ge K, \qquad (28)$$

with $\kappa = s\gamma(\ell/2)^{p(\gamma-1)}$. **Thus, by** (27) **and** (28), **we have**

The functional Φ

$$\lim_{k\to\infty}\int_\Omega |\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k\cdot\mathcal{D}_L(u_k-u)dx=0. \tag{29}$$

By convexity

$$\|\mathcal{D}_L u\|_{N,p}^p + p\int_\Omega |\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k\cdot\mathcal{D}_L(u_k-u)dx \ge \|\mathcal{D}_L u_k\|_{N,p}^p. \tag{30}$$

Therefore, combining together (29), (30) **and the weak lower semi–continuity of the norm, we have**

$$\|\mathcal{D}_L u\|_{N,p}^p \ge \limsup_{k\to\infty}\|\mathcal{D}_L u_k\|_{N,p}^p \ge \liminf_{k\to\infty}\|\mathcal{D}_L u_k\|_{N,p}^p \ge \|\mathcal{D}_L u\|_{N,p}^p.$$

In other words,

$$\lim_{k\to\infty}\|u_k\| = \|u\|. \tag{31}$$

Since $[W_0^{L,p}(\Omega)]^d$ **is uniformly convex as shown above, we immediately get from** (31) **and the weak convergence** $u_k \rightharpoonup u$, **that**

The functional Φ

$$\lim_{k\to\infty} \|u_k - u\| = 0,$$

as required.

Case $u = 0$. **Suppose first by contradiction that**

$$0 = \|u\| < \liminf_{k\to\infty} \|u_k\|. \tag{32}$$

As before, (28) holds. Hence, proceeding as in the previous case, we conclude that $\lim_k u_k = 0$, which contradicts (32). Therefore, the only possible case is

$$0 = \|u\| = \liminf_{k} \|u_k\|.$$

Assume now by contradiction that

$$\mathcal{L} = \limsup_{k\to\infty} \|u_k\| > \liminf_{k\to\infty} \|u_k\| = \|u\| = 0. \tag{33}$$

The functional Φ

In particular, there exist a subsequence $(u_{k_j})_j$ of $(u_k)_k$ and $K \in \mathbb{N}$ such that $\mathcal{L} = \lim_{j\to\infty} \|u_{k_j}\|$ and

$$M(\|u_{k_j}\|^p) \geq \kappa > 0 \qquad \textbf{for all } j \geq K,$$

where $\kappa = s\gamma(\mathcal{L}/2)^{p(\gamma-1)}$. Consequently, the argument from (28) to (31) along the subsequence $(u_{k_j})_j$ implies that $\lim_{j\to\infty} \|u_{k_j}\| = 0$, which contradicts (33).
We have shown also in this case that $\limsup_{k\to\infty} \|u_k\| = \liminf_{k\to\infty} \|u_k\| = \|u\| = 0$. In other words $u_k \to 0$ in $[W_0^{L,p}(\Omega)]^d$, as required.

Variational Structure

Without further mentioning, we assume that $(\mathcal{F})$–(a) and $(\mathcal{F})$–(b) hold. The main result of the section is proved by using the energy functional J_λ associated to (1), which is given by $J_\lambda(u) = \Phi(u) + \lambda\Psi(u)$, where

$$\begin{gathered}\Psi(u) = \Psi_1(u) + \Psi_2(u),\\ \Psi_1(u) = -\frac{1}{p}\|u\|_{d,p,w}^{\gamma p}, \quad \Psi_2(u) = -\int_\Omega F(x,u(x))dx.\end{gathered} \tag{34}$$

Clearly, the functional J_λ is well defined in $[W_0^{L,p}(\Omega)]^d$ and of class $C^1([W_0^{L,p}(\Omega)]^d)$, see the proof of Lemma 2. Furthermore, for all u, $\varphi \in [W_0^{L,p}(\Omega)]^d$

$$\begin{aligned}\langle J_\lambda'(u), \varphi\rangle &= M(\|u\|^p)\int_\Omega |\mathcal{D}_L u(x)|^{p-2}\mathcal{D}_L u(x)\cdot\mathcal{D}_L\varphi(x)\,dx\\ &- \lambda\int_\Omega \Big[\gamma\|u\|_{d,p,w}^{p(\gamma-1)}w(x)|u(x)|^{p-2}u(x) + f(x,u(x))\Big]\cdot\varphi(x)\,dx.\end{aligned}$$

Auxiliary functions

Therefore, the critical points $u \in [W_0^{L,p}(\Omega)]^d$ **of** J_λ **are exactly the weak solutions of** (1)**.**

Given $r \in (\inf_{u\in X} \Psi(u), \sup_{u\in X} \Psi(u))$**, we introduce the two functions**

$$\begin{aligned} \varphi_1(r) &= \inf_{u\in\Psi^{-1}(I_r)} \frac{\displaystyle\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r}, \qquad I_r = (-\infty, r), \\ \varphi_2(r) &= \sup_{u\in\Psi^{-1}(I^r)} \frac{\displaystyle\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r}, \qquad I^r = (r, \infty). \end{aligned} \tag{35}$$

Crucial numbers

If $\Psi(v) < 0$ **at some** $v \in [W_0^{L,p}(\Omega)]^d$**, then** ***the crucial positive number***

$$\lambda^\star = \varphi_1(0) = \inf_{u \in \Psi^{-1}(I_0)} -\frac{\Phi(u)}{\Psi(u)}, \qquad I_0 = (-\infty, 0), \tag{36}$$

is well defined.

Lemma

If f *satisfies also* $(\mathcal{F})$*–*(c)*, then* $\Psi^{-1}(I_0)$ *is non–empty and moreover* $\lambda_\star \le \lambda^\star < s\lambda_1^\gamma$.

Crucial numbers

From $(\mathcal{F})$–(c) it follows that

$$\Psi(u_1) < -\frac{\mathscr{M}(\lambda_1)}{ps\lambda_1^\gamma} < 0, \quad \textbf{i.e. } u_1 \in \Psi^{-1}(I_0). \tag{37}$$

Hence, (36) is meaningful. By (37) and Proposition 1

$$\lambda^\star = \inf_{u\in\Psi^{-1}(I_0)} -\frac{\Phi(u)}{\Psi(u)} \le \frac{\Phi(u_1)}{-\Psi(u_1)} < \frac{\mathscr{M}(\|u_1\|^p)/p}{\mathscr{M}(\lambda_1)/ps\lambda_1^\gamma} = s\lambda_1^\gamma,$$

as required. Finally, by $(\mathcal{M})$, (16), (17), (18) and (34), for all $u \in \Psi^{-1}(I_0)$, we have

$$\frac{\Phi(u)}{|\Psi(u)|} \ge \frac{\mathscr{M}(\|u\|^p)/p}{\dfrac{1}{p}\|u\|_{d,p,w}^{\gamma p} + \dfrac{S_f}{\gamma p}\|u\|_{d,\gamma p,w}^{\gamma p}} \ge \frac{s\|u\|^{\gamma p}/p}{\dfrac{1}{p\lambda_1^\gamma}\|u\|^{\gamma p} + \dfrac{c_{\gamma p}S_f}{\gamma p}\|u\|^{\gamma p}}$$

$$= \frac{s\gamma\lambda_1^\gamma}{\gamma + c_{\gamma p}S_f\lambda_1^\gamma} = \lambda_\star.$$

Hence, in particular $\lambda^\star \ge \lambda_\star$.

Compact operators

LEMMA

The operators $\Psi_1', \Psi_2', \Psi' : [W_0^{L,p}(\Omega)]^d \to \left([W_0^{L,p}(\Omega)]^d\right)^\star$ *are compact and* Ψ_1, Ψ_2, Ψ *are sequentially weakly continuous in* $[W_0^{L,p}(\Omega)]^d$*.*

Of course, $\Psi' = \Psi_1' + \Psi_2'$**, where**

$$\langle \Psi_1'(u), v\rangle = -\gamma \|u\|_{d,p,w}^{p(\gamma-1)} \int_\Omega w(x)|u|^{p-2} u \cdot v dx \text{ and}$$
$$\langle \Psi_2'(u), v\rangle = -\int_\Omega f(x,u) \cdot v dx$$

for all $u, v \in [W_0^{L,p}(\Omega)]^d$**. Since** Ψ_1' **and** Ψ_2' **are continuous, thanks to the reflexivity of** $[W_0^{L,p}(\Omega)]^d$ **it is enough to show that** Ψ_1' **and** Ψ_2' **are weak–to–strong sequentially continuous, i.e.**

Compact operators

if $(u_k)_k$, u **are in** $[W_0^{L,p}(\Omega)]^d$ **and** $u_k \rightharpoonup u$ **in** $[W_0^{L,p}(\Omega)]^d$**, then** $\|\Psi_1'(u_k) - \Psi_1'(u)\|_\star \to 0$ **and** $\|\Psi_2'(u_k) - \Psi_2'(u)\|_\star \to 0$ **as** $k \to \infty$**. To this aim, fix** $(u_k)_k \subset [W_0^{L,p}(\Omega)]^d$**, with** $u_k \rightharpoonup u$ **in** $[W_0^{L,p}(\Omega)]^d$**. First,** $u_k \to u$ **in** $[L^p(\Omega, w)]^d$ **by Lemma 1–**(i)**. Therefore,** $\mathcal{N}_p(u_k) \to \mathcal{N}_p(u)$ **in** $[L^{p'}(\Omega, w)]^d$ **by**

LEMMA

Assume that $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, $f = f(x, v) \not\equiv 0$, *is a Carathéodory function, satisfying* $(\mathcal{F})$–(a) *of Section* **??**. *The Nemytskii operators* $\mathcal{N}_p : [L^p(\Omega, w)]^d \to [L^{p'}(\Omega, w)]^d$ *and* $\mathcal{N}_f : [L^q(\Omega, w)]^d \to [L^{q'}(\Omega, w^{1/(1-q)})]^d$, *defined by* $\mathcal{N}_p(u) = |u|^{p-2}u$ *and* $\mathcal{N}_f(u) = f(\cdot, u(\cdot))$ *respectively, are continuous.*

For all $v \in [W_0^{L,p}(\Omega)]^d$**, with** $\|v\| = 1$**, by Hölder's inequality and** (16)**,**

Compact operators

$$
\begin{aligned}
&|\langle \Psi_1'(u_k)-\Psi_1'(u),v\rangle| \\
&\quad\le \gamma\|u_k\|_{d,p,w}^{p(\gamma-1)}\int_\Omega w(x)^{1/p'}|\mathcal{N}_p(u_k)-\mathcal{N}_p(u)|w(x)^{1/p}|v|dx \\
&\qquad\quad + \gamma\left|\|u_k\|_{d,p,w}^{p(\gamma-1)}-\|u\|_{d,p,w}^{p(\gamma-1)}\right|\|u\|_{d,p,w}^{p-1}\|v\|_{d,p,w} \\
&\quad\le \gamma\mathcal{C}\left\{\|\mathcal{N}_p(u_k)-\mathcal{N}_p(u)\|_{d,p',w}+\left|\|u_k\|_{d,p,w}^{p(\gamma-1)}-\|u\|_{d,p,w}^{p(\gamma-1)}\right|\right\}\|v\|_{d,p,w} \\
&\quad\le \lambda_1^{-1/p}\gamma\mathcal{C}\left\{\|\mathcal{N}_p(u_k)-\mathcal{N}_p(u)\|_{d,p',w}+\left|\|u_k\|_{d,p,w}^{p(\gamma-1)}-\|u\|_{d,p,w}^{p(\gamma-1)}\right|\right\},
\end{aligned}
$$

where $\mathcal{C}=\sup_k\|u_k\|_{d,p,w}^{p(\gamma-1)}$. **Hence,** $\|\Psi_1'(u_k)-\Psi_1'(u)\|_\star\to 0$ **as** $k\to\infty$ **and** Ψ_1' **is compact.**
Finally, for all $v\in[W_0^{L,p}(\Omega)]^d$**, with** $\|v\|=1$**, we have again**

$$
\begin{aligned}
|\langle \Psi_2'(u_k)-\Psi_2'(u),v\rangle| &\le \int_\Omega w(x)^{-1/q}|\mathcal{N}_f(u_k)-\mathcal{N}_f(u)|w^{1/q}|v|dx \\
&\le \|\mathcal{N}_f(u_k)-\mathcal{N}_f(u)\|_{d,q',w^{1/(1-q)}}\|v\|_{d,q,w}
\end{aligned}
$$

Compact operators

$$\leq \mathcal{S}_{d,q,w} \|\mathcal{N}_f(u_k) - \mathcal{N}_f(u)\|_{d,q',w^{1/(1-q)}}.$$

Thus, $\|\Psi_2'(u_k) - \Psi_2'(u)\|_\star \to 0$ **as** $k \to \infty$**, that is** Ψ_2' **is compact.**

Consequently, $\Psi' = \Psi_1' + \Psi_2'$ **is compact, then** Ψ **is sequentially weakly continuous by Corollary 41.9 of**

[Z] **E. Zeidler**, ***Nonlinear functional analysis and its applications, Vol. III, Variational methods and optimization*** **(Springer–Verlag, New York, 1985) xxii+662 pp.**

being $[W_0^{L,p}(\Omega)]^d$ **reflexive.**

Coerciveness

LEMMA

The functional $J_\lambda(u) = \Phi(u) + \lambda\Psi(u)$ *is coercive for all* λ *in the interval* $(-\infty, s\lambda_1^\gamma)$.

Fix $\lambda \in (-\infty, s\lambda_1^\gamma)$**. Then by** $(\mathcal{M})$**,** (16) **and** $(\mathcal{F})$**–**(a) **for all** $u \in [W_0^{L,p}(\Omega)]^d$**, with** $\|u\| \geq 1$**,**

$$
\begin{aligned}
J_\lambda(u) &\geq \frac{1}{p}\mathscr{M}(\|u\|^p) - \frac{\lambda}{p}\|u\|_{d,p,w}^{\gamma p} - |\lambda| \int_\Omega |F(x,u)|dx \\
&\geq \frac{1}{p}\left(\mathscr{M}(\|u\|^p) - \frac{\lambda^+}{\lambda_1^\gamma}\|u\|^{\gamma p}\right) - |\lambda| C_f \int_\Omega \left(w(x)|u| + \frac{w(x)}{q}|u|^q\right) dx \\
&\geq \frac{1}{p}\left(s - \frac{\lambda^+}{\lambda_1^\gamma}\right)\|u\|^{\gamma p} - |\lambda| C_f \left\{\int_{\Omega_1} w(x)dx + \int_{\Omega_2} w(x)|u|^q dx + \frac{1}{q}\int_\Omega w(x)\right. \\
&\geq \frac{1}{p}\left(s - \frac{\lambda^+}{\lambda_1^\gamma}\right)\|u\|^{\gamma p} - |\lambda| C_1 - |\lambda|\, C_2 \|u\|^q,
\end{aligned}
$$

Coerciveness

where $\Omega_1 = \{x \in \Omega : |u(x)| \leq 1\}$**,** $\Omega_2 = \{x \in \Omega : |u(x)| \geq 1\}$**,** $C_1 = C_f \|w\|_1$ **and** $C_2 = C_f \mathcal{S}^q_{d,q,w}(q+1)/q$**. This completes the proof, since** $1 < q < \gamma p$ **by** $(\mathcal{F})$**–**(a)**.**

Thanks to the results above all the structural assumptions $(\mathcal{H}_1)$**–**$(\mathcal{H}_4)$ **of Theorem 2.1 of [CPV] are clearly verified by** J_λ**. Thus we are now able to prove**

The existence and multiplicity results for (1)

THEOREM

Let $(\mathcal{F})$–(a), (b) *hold, and let* $\lambda_\star = \dfrac{s\gamma\lambda_1^\gamma}{\gamma + c_{\gamma p} S_f \lambda_1^\gamma}$, *while*

$\lambda^\star = \varphi_1(0) = \inf_{u \in \Psi^{-1}(I_0)} -\dfrac{\Phi(u)}{\Psi(u)} < s\lambda_1^\gamma$, $I_0 = (-\infty, 0)$.

(i) *If* $\lambda \in [0, \lambda_\star)$, *then*

$$(1) \begin{cases} M(\|u\|^p)\Delta_p^L u = \lambda\{\gamma\|u\|_{p,w}^{p(\gamma-1)} w(x)|u|^{p-2}u + f(x,u)\} \text{ in } \Omega, \\ D^\alpha u_k\big|_{\partial\Omega} = 0 \text{ for all } \alpha, \text{ with } |\alpha| \le L-1,\ k = 1, \dots, d, \end{cases}$$

has only the trivial solution.
(ii) *If furthermore* $(\mathcal{F})$–(c) *holds and* $q \in (1, p)$ *in* $(\mathcal{F})$–(a), *then* (1) *admits at least two nontrivial solutions for every* $\lambda \in (\lambda^\star, s\lambda_1^\gamma)$.

The existence and multiplicity results for (1)

(i) **Let** $u \in [W_0^{L,p}(\Omega)]^d$ **be a nontrivial weak solution of the problem** (1)**, then**

$$\begin{aligned} s\gamma\lambda_1^\gamma \|u\|^{\gamma p} &\le \lambda_1^\gamma M(\|u\|^p)\|u\|^p = \lambda_1^\gamma \lambda \int_\Omega \{\gamma \|u\|_{d,p,w}^{p(\gamma-1)} w(x)|u|^p + f(x,u)\cdot u\} \\ &\le \lambda_1^\gamma \lambda \left(\gamma \|u\|_{d,p,w}^{\gamma p} + \int_\Omega \frac{|f(x,u)\cdot u|}{w(x)|u|^{\gamma p}} w(x)|u|^{\gamma p} dx \right) \\ &\le \lambda_1^\gamma \lambda (\gamma \|u\|_{d,p,w}^{\gamma p} + S_f \|u\|_{d,\gamma p,w}^{\gamma p}) \\ &\le \lambda \left(\gamma + c_{\gamma p} S_f \lambda_1^\gamma\right) \|u\|^{\gamma p} \end{aligned}$$

by $(\mathcal{M})$**,** (16)**,** (17) **and** $(18)_1$**. Therefore** $\lambda \ge \lambda_\star$**, as required.**

The existence and multiplicity results for (1)

(ii) **By Lemmas 2–7 the functional** J_λ **verifies all the structural assumptions** $(\mathcal{H}_1)$**–**$(\mathcal{H}_4)$ **of Theorem 2.1 of [CPV] , with** $I=(-\infty, s\lambda_1^\gamma)$**. It remains to show that there exists**

$$r \in \left(\inf_{u\in X}\Psi(u), \sup_{u\in X}\Psi(u)\right) \quad \textbf{such that} \quad \varphi_1(r)<\varphi_2(r). \quad (38)$$

We claim that $\Psi([W_0^{L,p}(\Omega)]^d)\supset\mathbb{R}_0^-$**. Indeed,** $\Psi(0)=0$ **and arguing as in the proof of Lemma 7, we get the following estimate**

$$\left|\int_\Omega F(x,u(x))dx\right| \le c(\|w\|_1+\|u\|_{d,q,w}^q),$$

where $c=C_f(1+q)/q$**. Furthermore, by Hölder's inequality**

$$\|u\|_{d,q,w}^q \le \|w\|_1^{(p-q)/p}\|u\|_{d,p,w}^q,$$

since $1<q<p$ **and** $w\in L^1(\Omega)$**, being** $\varpi>1$ **and** Ω

The existence and multiplicity results for (1)

bounded. Hence, combining together the previous inequalities, we get

$$\Psi(u) \leq -\frac{1}{p}\|u\|_{d,p,w}^{\gamma p} + c\|w\|_1 + c\|w\|_1^{(p-q)/p}\|u\|_{d,p,w}^{q}.$$

Therefore,

$$\lim_{\substack{u\in[W_0^{L,p}(\Omega)]^d\\ \|u\|_{d,p,w}\to\infty}} \Psi(u) = -\infty,$$

being $q < p \leq \gamma p$**. Hence, the claim follows by the continuity of** Ψ**.**
In particular, $(\inf\Psi, \sup\Psi) \supset \mathbb{R}_0^-$**. Now, for every** $u \in \Psi^{-1}(I_0)$ **we have**

$$\varphi_1(r) \leq \frac{\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r} \leq -\frac{\Phi(u)}{\Psi(u) - r}$$

for all $r \in (\Psi(u), 0)$**, so that**

The existence and multiplicity results for (1)

$$\limsup_{r\to 0^-}\varphi_1(r) \le -\frac{\Phi(u)}{\Psi(u)} \quad \textbf{for all } u\in\Psi^{-1}(I_0),$$

in other words,

$$\limsup_{r\to 0^-}\varphi_1(r) \le \varphi_1(0) = \inf_{u\in\Psi^{-1}(I_0)} -\frac{\Phi(u)}{\Psi(u)} = \lambda^\star.$$

Now, by Lemma 1–(i)**, the fact that** $\mathfrak{p}^\star < p_L^*/\varpi'$ **and** (19)**,**

$$|\Psi(u)| \le \frac{1}{p}\|u\|_{d,p,w}^{\gamma p} + K\|u\|_{d,\mathfrak{p}^\star,w}^{\mathfrak{p}^\star} \le \frac{1}{p}\|u\|_{d,p,w}^{\gamma p} + \mathfrak{K}\|u\|^{\mathfrak{p}^\star} \tag{39}$$

for every $u\in[W_0^{L,p}(\Omega)]^d$**, where** $\mathfrak{K} = K\mathcal{S}_{d,\mathfrak{p}^\star,w}^{\mathfrak{p}^\star} > 0$**.**
Therefore, for $r<0$ **and** $v\in\Psi^{-1}(r)$**,**

$$r = \Psi(v) \ge -\frac{1}{p\lambda_1^\gamma}\|v\|^{\gamma p} - \mathfrak{K}\|v\|^{\mathfrak{p}^\star} \ge -\frac{1}{s\lambda_1^\gamma}\Phi(v) - \mathfrak{K}\left(\frac{p}{s}\right)^{\mathfrak{p}^\star/\gamma p}\Phi(v)^{\mathfrak{p}^\star/\gamma p} \tag{40}$$

by $(\mathcal{M})$**,** (16)**,** (22) **and** (39)**.**

The existence and multiplicity results for (1)

By Lemma 2 the functional Φ is bounded below, coercive and lower semi–continuous in the reflexive Banach space $[W_0^{L,p}(\Omega)]^d$. Hence, it is easy to see that Φ is also coercive in the sequentially weakly closed non–empty set $\Psi^{-1}(r)$. Therefore, by Theorem 6.1.1 of

[B] M.S. Berger, *Nonlinearity and functional analysis*, Lectures on Nonlinear Problems in Mathematical Analysis, Pure and Applied Mathematics (Academic Press, New York–London, 1977) xix+417 pp.

there exists an element

$$u_r \in \Psi^{-1}(r) \quad \textbf{such that } \Phi(u_r) = \inf_{v \in \Psi^{-1}(r)} \Phi(v).$$

By (35) we have

The existence and multiplicity results for (1)

$$\varphi_2(r) = \sup_{u\in\Psi^{-1}(I^r)} \frac{\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r} \geq -\frac{1}{r} \inf_{v\in\Psi^{-1}(r)} \Phi(v) = \frac{\Phi(u_r)}{|r|},$$

being $0 \in \Psi^{-1}(I^r)$, $I^r = (r,\infty)$. **From** (40) **we get**

$$\begin{aligned} 1 &\leq \frac{1}{s\lambda_1^\gamma} \cdot \frac{\Phi(u_r)}{|r|} + \mathfrak{K}\left(\frac{p}{s}\right)^{\mathfrak{p}^\star/\gamma p} |r|^{\mathfrak{p}^\star/\gamma p - 1} \left(\frac{\Phi(u_r)}{|r|}\right)^{\mathfrak{p}^\star/\gamma p} \\ &\leq \frac{\varphi_2(r)}{s\lambda_1^\gamma} + \mathfrak{K}\left(\frac{p}{s}\right)^{\mathfrak{p}^\star/\gamma p} |r|^{\mathfrak{p}^\star/\gamma p - 1} \varphi_2(r)^{\mathfrak{p}^\star/\gamma p}. \end{aligned}$$

There are now two possibilities to be considered: either φ_2 **is locally bounded at** 0^-, **so that the above inequality shows at once that**

$$\liminf_{r\to 0^-} \varphi_2(r) \geq s\lambda_1^\gamma,$$

being $\mathfrak{p}^\star > \gamma p$ **by** $(\mathcal{F})$–(b), **or** $\limsup_{r\to 0^-} \varphi_2(r) = \infty$. **In both cases**

The existence and multiplicity results for (1)

$$\limsup_{r\to 0^-}\varphi_1(r)\le\varphi_1(0)=\lambda^\star<s\lambda_1^\gamma\le\limsup_{r\to 0^-}\varphi_2(r).$$

This yields that for all integers $k\ge k^\star=1+[2/(s\lambda_1^\gamma-\lambda^\star)]$ there exists a number $r_k<0$ so close to zero that

$$\varphi_1(r_k)<\lambda^\star+1/k<s\lambda_1^\gamma-1/k<\varphi_2(r_k),$$

that is

$$r_k\in\left(\inf_{u\in X}\Psi(u),\sup_{u\in X}\Psi(u)\right)\quad \textbf{and}\quad \varphi_1(r_k)<\varphi_2(r_k)$$

holds. Hence, by Theorem 2.1–(ii), Part (a), of [CPV],

The existence and multiplicity results for (1)

being $u \equiv 0$ a critical point of J_λ, problem (1) admits at least two nontrivial solutions for all

$$\lambda \in \bigcup_{k=k^\star}^{\infty} (\varphi_1(r_k), \varphi_2(r_k)) \cap I \supset \bigcup_{k=k^\star}^{\infty} [\lambda^\star + 1/k, s\lambda_1^\gamma - 1/k] = (\lambda^\star, s\lambda_1^\gamma),$$

as claimed.

A simpler problem

We now consider the simpler problem

$$\begin{cases} M(\|u\|^p)\Delta_p^L u = \lambda f(x,u) \ \textbf{in } \Omega, \\ D^\alpha u_k\big|_{\partial\Omega} = 0 \ \textbf{for all } \alpha\textbf{, with } |\alpha| \le L-1,\ k = 1,\dots,d, \end{cases} \tag{41}$$

where f verifies condition $(\mathcal{F})$, with $(\mathcal{F})$–(c) replaced by the less stringent assumption

$(\mathcal{F})$–$(c)'$ ***Assume that there exist*** $x_0 \in \Omega$***,*** $v_0 \in \mathbb{R}^d$ ***and*** $r_0 > 0$ ***so small that the closed ball*** $B_0 = \{x \in \mathbb{R}^n : |x - x_0| \le r_0\}$ ***is contained in*** Ω ***and***

$$\operatorname*{ess\,inf}_{B_0} F(x, v_0) = \mu_0 > 0, \qquad \operatorname*{ess\,sup}_{B_0} \max_{|v|\le|v_0|} |F(x,v)| = M_0 < \infty.$$

Clearly, when f does not depend on x, condition $(\mathcal{F})$–$(c)'$ simply reduces to the request that $F(v_0) > 0$ at a point $v_0 \in \mathbb{R}^d$. This case is interesting also because the unpleasant restriction $q \in (1, p)$ requested in Theorem 8–(ii) can be avoided.

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In this new setting, the next theorem extends the main result of [KLV] to the p–Laplace operator also for $p \in (1,2)$, and Corollary 3.6 of [CPV] to higher order operators, involving the Kirchhoff function.

THEOREM

Let $(\mathcal{F})$–(a), (b) ***hold, and let*** $\ell_\star = s\gamma/c_{\gamma p}S_f$.

- (i) ***If*** $\lambda \in [0, \ell_\star)$, ***then*** (41) ***has only the trivial solution.***
- (ii) ***If furthermore*** $(\mathcal{F})$–$(c)'$ ***holds, then there exists*** $\ell^\star \geq \ell_\star$ ***such that*** (41) ***admits at least two nontrivial solutions for all*** $\lambda \in (\ell^\star, \infty)$.

The part (i) of the statement is proved by using the same argument produced for the proof of Theorem 8–(i), being

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$$
\begin{aligned}
s\gamma\|u\|^{\gamma p} &\le M(\|u\|^p)\|u\|^p = \lambda\int_\Omega f(x,u)\cdot u\,dx \le \lambda S_f\|u\|_{d,\gamma p,w}^{\gamma p}\\
&\le \lambda c_{\gamma p} S_f\|u\|^{\gamma p}.
\end{aligned}
$$

Thus, if u is a nontrivial weak solution of (41), then necessarily $\lambda \ge \ell_\star$, as required.

In order to prove (ii), we consider the energy functional J_λ associated to (41), given by $J_\lambda(u) = \Phi(u) + \lambda\Psi_2(u)$, where Φ is defined in the statement of Lemma 2 and Ψ_2 in (34). We claim that J_λ is coercive for every $\lambda \in \mathbb{R}$. Indeed, as shown in the proof of Lemma 7, for all $u \in [W_0^{L,p}(\Omega)]^d$, with $\|u\| \ge 1$,

$$
J_\lambda(u) \ge \frac{s}{p}\|u\|^{\gamma p} - |\lambda| C_1 - |\lambda|\, C_2\|u\|^q,
$$

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where C_1, C_2 are the constants determined in the proof of Lemma 7. This shows the claim, since $1 < q < \gamma p$ by $(\mathcal{F})$–(a). Hence, here $I = \mathbb{R}$.
Next, we show that there exists $u_0 \in [W_0^{L,p}(\Omega)]^d$ such that $\Psi_2(u_0) < 0$. Note that $v_0 \neq 0$ in $(\mathcal{F})$–$(c)'$. Take $\sigma \in (0,1)$ and put $B = \{x \in \mathbb{R}^n \, : \, |x - x_0| \leq \sigma r_0\}$. Of course, $B \subset B_0$. Consider a function $u_0 \in [C_0^\infty(\Omega)]^d$ such that

$$|u_0| \leq |v_0| \text{ in } \Omega, \quad \operatorname{supp} u_0 \subset B_0 \quad \text{and} \quad u_0 \equiv v_0 \text{ in } B.$$

Clearly, $u_0 \in [W_0^{L,p}(\Omega)]^d$. Now, by $(\mathcal{F})$–$(c)'$,

$$\begin{aligned}\Psi_2(u_0) &= -\int_{B_0 \setminus B} F(x, u_0(x))dx - \int_B F(x, v_0)dx \\ &\leq M_0|B_0 \setminus B| - \mu_0|B| = \omega_n r_0^n \left[M_0(1 - \sigma^n) - \mu_0 \sigma^n\right],\end{aligned}$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$.
Therefore, taking $\sigma \in (0,1)$ so large that

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$\sigma^n > M_0/(\mu_0 + M_0)$, **we get that** $\Psi_2(u_0) < 0$, **as claimed.**
Hence, the crucial number

$$\ell^\star = \varphi_1(0) = \inf_{u \in \Psi_2^{-1}(I_0)} -\frac{\Phi(u)}{\Psi_2(u)}, \qquad I_0 = (-\infty, 0), \tag{42}$$

is well defined.
Furthermore, as in the proof of Lemma 4, for all $u \in \Psi_2^{-1}(I_0)$, **we have**

$$\frac{\Phi(u)}{|\Psi_2(u)|} \geq \frac{\mathscr{M}(\|u\|^p)/p}{c_{\gamma p} S_f \|u\|^{\gamma p}/\gamma p} \geq \frac{s\gamma \|u\|^{\gamma p}}{c_{\gamma p} S_f \|u\|^{\gamma p}} = \frac{s\gamma}{c_{\gamma p} S_f} = \ell_\star.$$

Hence, $\ell^\star \geq \ell_\star$ **by** (42).
In particular, for all $u \in \Psi_2^{-1}(I_0)$, **it results**

$$\varphi_1(r) \leq \frac{\displaystyle\inf_{v \in \Psi_2^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi_2(u) - r} \leq -\frac{\Phi(u)}{\Psi_2(u) - r}$$

for all $r \in (\Psi_2(u), 0)$. **Thus,**

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$$\limsup_{r\to 0^-}\varphi_1(r) \le \varphi_1(0) = \ell^\star. \tag{43}$$

Here (39) **simply reduces to**

$$|\Psi_2(u)| \le \mathfrak{K}\|u\|^{\mathfrak{p}^\star}.$$

Taken $r < 0$ **and** $v \in \Psi_2^{-1}(r)$, **we obtain by** (22)

$$|r| = |\Psi_2(v)| \le \mathfrak{K}\|v\|^{\mathfrak{p}^\star} \le \mathfrak{K}\left(\frac{p}{s}\right)^{\mathfrak{p}^\star/\gamma p}\Phi(v)^{\mathfrak{p}^\star/\gamma p}.$$

Therefore, by (35), **since** $u \equiv 0 \in \Psi_2^{-1}(I^r)$,

$$\varphi_2(r) \ge \frac{1}{|r|}\inf_{v\in\Psi_2^{-1}(r)}\Phi(v) \ge \mathcal{K}|r|^{\gamma p/\mathfrak{p}^\star - 1},$$

where $\mathcal{K} = s\mathfrak{K}^{-\gamma p/\mathfrak{p}^\star}/p$. **This implies that** $\lim_{r\to 0^-}\varphi_2(r) = \infty$, **being** $\mathfrak{p}^\star > \gamma p$ **by** $(\mathcal{F})$–(b).
In conclusion, we have proved that

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$$\limsup_{r\to 0^-} \varphi_1(r) \le \varphi_1(0) = \ell^\star < \lim_{r\to 0^-} \varphi_2(r) = \infty. \qquad (44)$$

This shows that for all integers $k \ge k^\star = 2 + [\ell^\star]$ there exists $r_k < 0$ so close to zero that

$$\varphi_1(r_k) < \ell^\star + 1/k < k < \varphi_2(r_k),$$

that is (38) holds. Hence, since $I = \mathbb{R}$, by Theorem 2.1–(ii) Part (a) of [CPV], being $u \equiv 0$ a critical point of J_λ, problem (41) admits at least two nontrivial solutions for all

$$\lambda \in \bigcup_{k=k^\star}^{\infty} (\varphi_1(r_k), \varphi_2(r_k)) \supset \bigcup_{k=k^\star}^{\infty} [\ell^\star + 1/k, k] = (\ell^\star, \infty),$$

as claimed.

Remark

We conclude this part by noting that $\Phi(u) + \lambda^\star \Psi(u) \geq 0$ **for all** $u \in \Psi^{-1}(I_0)$ **by** (36)**. Hence, if** $\lambda \leq \lambda^\star$ **and** $u \in \Psi^{-1}(I_0)$**, then**

$$J_\lambda(u) = \Phi(u) + \lambda\Psi(u) - \lambda^\star\Psi(u) + \lambda^\star\Psi(u) \geq (\lambda - \lambda^\star)\Psi(u) \geq 0.$$

On the other hand, if $\lambda \geq 0$ **and** $u \in \Psi^{-1}(I^0)$**, then** $J_\lambda(u) \geq 0$**. Combining both inequalities we get that for all** $\lambda \in [0, \lambda^\star]$

$$\inf_{u \in [W_0^{L,p}(\Omega)]^d} J_\lambda(u) = J_\lambda(0) = 0.$$

The special non–degenerate case when $\gamma = 1$

In this final part of the first part we consider the *non–degenerate* problem (7). All the assumptions on f, w and M coincide with the hypotheses required for (1), with $\gamma = 1$. Consequently, *the crucial positive numbers $\lambda_\star$ and $\lambda^\star$ become*

$$\lambda_\star = \frac{s\lambda_1}{1+S_f}, \qquad \lambda^\star = \varphi_1(0) \in (\lambda_\star, s\lambda_1),$$

where $\varphi_1(0)$ is given in (36), see also (35). In this easier setting the proofs of the main Lemmas 2–7 and of Theorem 8 can be reproduced word by word and simplified. In particular, the case $u = 0$ in the proof of Lemma 2 is redundant, being $M(\tau) \geq s > 0$ for all $\tau \in \mathbb{R}_0^+$. Hence,

$$(26) \qquad \lim_{k\to\infty} M(\|u_k\|^p) \int_\Omega |\mathcal{D}_L u_k|^{p-2} \mathcal{D}_L u_k \cdot \mathcal{D}_L(u_k - u)\, dx = 0$$

The special non–degenerate case when $\gamma = 1$

immediately gives

$$\lim_{k\to\infty}\int_\Omega |\mathcal{D}_L u_k|^{p-2}\mathcal{D}_L u_k\cdot\mathcal{D}_L(u_k-u)dx = 0 \tag{28}$$

even when $u = 0$.
In the recent paper [CPV] the following quasilinear problem is studied

$$\begin{cases} -\mathbf{div}\mathbf{A}(x,\nabla u) = \lambda\{w(x)|u|^{p-2}u + f(x,u)\} & \textbf{in } \Omega,\\ \quad u = 0 & \textbf{on } \partial\Omega, \end{cases} \tag{45}$$

when $\mathbf{div}\mathbf{A}(x,\nabla u)$ is essentially the p–Laplacian operator. In the special case $\mathbf{div}\mathbf{A}(x,\nabla u) = s\Delta_p u$, problem (45) coincides with (1), when $d = L = 1$ and $M \equiv s$. Theorem 8 reduces to Theorem 3.4 of [CPV].

The $p(x)$–polyharmonic Kirchhoff problem

In the second part of the talk we extend the results of the first to the $p(x)$–polyharmonic Kirchhoff problem

$$(7)\begin{cases} M(\mathscr{I}_L(u))\Delta^L_{p(x)}u = \lambda\{w(x)|u|^{p(x)-2}u + f(x,u)\} \text{ in } \Omega, \\ D^\alpha u_k\big|_{\partial\Omega} = 0 \text{ for all } \alpha, \text{ with } |\alpha| \le L-1,\ k = 1,\dots,d, \end{cases}$$

We begin by recalling some basic results on the variable exponent Lebesgue and Sobolev spaces. As before, also here $\Omega \subset \mathbb{R}^n$ *is a bounded domain.* **Define for all** $h \in C(\overline{\Omega})$

$$h_+ = \max_{x\in\overline{\Omega}} h(x) \qquad \text{and} \qquad h_- = \min_{x\in\overline{\Omega}} h(x)$$

and put

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h_- > 1\}.$$

Let h be a fixed function in $C_+(\overline{\Omega})$. **The** ***variable exponent Lebesgue space***

$$L^{h(\cdot)}(\Omega) = \left\{\psi : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |\psi(x)|^{h(x)}dx < \infty\right\}$$

is endowed with the so–called *Luxemburg norm*

$$\|\psi\|_{h(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\psi(x)}{\lambda}\right|^{h(x)} dx \le 1\right\}$$

and is a separable, reflexive Banach space; cf. Theorem 2.5 and Corollaries 2.7 and 2.12 of

[KR] O. Kováčik, J. Rákosník, *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J. 41 (1991) 592-618.

Since here $0 < |\Omega| < \infty$, if $\sigma \in C(\overline{\Omega})$ and $1 \le \sigma \le h$ in Ω, then the embedding $L^{h(\cdot)}(\Omega) \hookrightarrow L^{\sigma(\cdot)}(\Omega)$ is continuous and the norm of the embedding operator does not exceed $|\Omega| + 1$; cf. Theorem 2.8 of [KR].
Let h' be the function obtained by conjugating the exponent h pointwise, that is $1/h(x) + 1/h'(x) = 1$ for all $x \in \overline{\Omega}$, then h' belongs to $C_+(\overline{\Omega})$ and $L^{h'(\cdot)}(\Omega)$ is the dual space of $L^{h(\cdot)}(\Omega)$, Corollary 2.7 of [KR].

Preliminaries for the $p(x)$–polyharmonic Kirchhoff problem

For any $h_i \in C_+(\overline{\Omega})$, $\psi_i \in L^{h_i(\cdot)}(\Omega)$ for $i = 1, \dots, m$, with $m \geq 1$ and $1 = \sum_{i=1}^m (1/h_i)$, the following *Hölder type inequality* **holds**

$$\int_\Omega |\psi_1(x) \cdots \psi_m(x)|\, dx \leq c_H \|\psi_1\|_{h_1(\cdot)} \cdots \|\psi_m\|_{h_m(\cdot)}, \tag{46}$$

where $c_H = 1/h_{1-} + \cdots + 1/h_{m-}$, see Theorem 2.1 of [KR] for the case $m = 2$.
Let σ be a function in $C(\overline{\Omega})$. **An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the $\sigma(\cdot)$–*modular* of the $L^{\sigma(\cdot)}(\Omega)$ space, which is the convex function $\rho_{\sigma(\cdot)} : L^{\sigma(\cdot)}(\Omega) \to \mathbb{R}$ defined by**

$$\rho_{\sigma(\cdot)}(\psi) = \int_\Omega |\psi(x)|^{\sigma(x)} dx.$$

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Theorems 1.3 and 1.4 of

[FZ] **X.L. Fan, D. Zhao**, ***On the spaces $L^{p(x)}$ and $W^{m,p(x)}$***, **J. Math. Anal. Appl. 263 (2001) 424–446.**

LEMMA

If ψ, $(\psi_k)_k \subset L^{\sigma(\cdot)}(\Omega)$, ***with*** $1 \le \sigma_- \le \sigma_+ < \infty$, ***then the following relations hold:***

$$\begin{aligned} &\|\psi\|_{\sigma(\cdot)} < 1 \ (= 1; > 1) \Leftrightarrow \rho_{\sigma(\cdot)}(\psi) < 1 \ (= 1; > 1), \\ &\|\psi\|_{\sigma(\cdot)} \ge 1 \quad \Rightarrow \quad \|\psi\|_{\sigma(\cdot)}^{\sigma_-} \le \rho_{\sigma(\cdot)}(\psi) \le \|\psi\|_{\sigma(\cdot)}^{\sigma_+}, \\ &\|\psi\|_{\sigma(\cdot)} \le 1 \quad \Rightarrow \quad \|\psi\|_{\sigma(\cdot)}^{\sigma_+} \le \rho_{\sigma(\cdot)}(\psi) \le \|\psi\|_{\sigma(\cdot)}^{\sigma_-}, \end{aligned} \tag{47}$$

and $\|\psi_k - \psi\|_{\sigma(\cdot)} \to 0 \Leftrightarrow \rho_{\sigma(\cdot)}(\psi_k - \psi) \to 0 \Leftrightarrow \psi_k \to \psi$ ***in measure in*** Ω ***and*** $\rho_{\sigma(\cdot)}(\psi_k) \to \rho_{\sigma(\cdot)}(\psi)$. ***In particular,*** $\rho_{\sigma(\cdot)}$ ***is continuous in*** $L^{\sigma(\cdot)}(\Omega)$, ***and if furthermore*** $\sigma \in C_+(\overline{\Omega})$, ***then*** $\rho_{\sigma(\cdot)}$ ***is weakly lower semi–continuous.***

Preliminaries for the $p(x)$–polyharmonic Kirchhoff problem

Since we are interested in weighted variable exponent Lebesgue spaces, denoted by ω a generic weight on Ω, we put

$$L^{\sigma(\cdot)}(\Omega,\omega)=\{\psi:\Omega\to\mathbb{R}\,\textbf{measurable}:\ \omega^{1/\sigma}|\psi|\in L^{\sigma(\cdot)}(\Omega)\},$$

endowed with the norm

$$\|\psi\|_{\sigma(\cdot),\omega}=\|\omega^{1/\sigma}\psi\|_{\sigma(\cdot)}. \tag{48}$$

If $p\in C_+(\overline{\Omega})$ **and** $L=1,2,\dots$, **the** *variable exponent Sobolev space*

$$W^{L,p(\cdot)}(\Omega)=\Big\{\psi\in L^{p(\cdot)}(\Omega):D^\alpha\psi\in L^{p(\cdot)}(\Omega)\ \textbf{for all}\ \alpha\in\mathbb{N}_0^n\textbf{, with}\ |\alpha$$

is endowed with the standard norm

$$\|\psi\|_{W^{L,p(\cdot)}(\Omega)}=\sum_{|\alpha|\le L}\|D^\alpha\psi\|_{p(\cdot)}.$$

From now on we assume that $p\in C_+^{\log}(\overline{\Omega})$, **where** $C_+^{\log}(\overline{\Omega})$ **is**

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the space of all the functions of $C_+(\overline{\Omega})$ which are *logarithmic Hölder continuous*, **that is there exists $\mathfrak{K} > 0$ such that**

$$|p(x) - p(y)| \leq -\frac{\mathfrak{K}}{\log|x - y|} \tag{49}$$

for all x, $y \in \Omega$, with $0 < |x - y| \leq 1/2$. Indeed, even if the variable exponent Lebesgue and Sobolev spaces have a lot in common with the classical spaces, there are also many fundamental questions left open. For example, it is not known yet, even for "nice" functions p, whether smooth functions are dense in $W^{L,p(\cdot)}(\Omega)$. This is the reason why we assume (49).

The space $W_0^{L,p(\cdot)}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{L,p(\cdot)}(\Omega)}$. As shown in

Corollary 11.2.4 of [DHHR], the space $W_0^{L,p(\cdot)}(\Omega)$ coincides with the closure in $W^{L,p(\cdot)}(\Omega)$ of the set of all $W^{L,p(\cdot)}(\Omega)$–functions with compact support thanks to (49). Moreover $W_0^{L,p(\cdot)}(\Omega)$ is a *separable, uniformly convex, Banach space*, cf. Theorem 8.1.13 of [DHHR].

[DHHR] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Vol. 2017 (Springer–Verlag, Berlin, 2011) ix+509 pp.

[HHKV] P. Harjulehto, P. Hästö, M. Koskenoja, S. Varonen, *The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values*, Potential Anal. 25 (2006) 205-222.

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Also in this context it is possible to prove a *Poincaré* type inequality, so that an equivalent norm for the space $W_0^{L,p(\cdot)}(\Omega)$ is given by

$$\|\psi\|_{\mathfrak{D}^{L,p(\cdot)}(\Omega)} = \sum_{|\alpha|=L} \|D^\alpha \psi\|_{p(\cdot)},$$

see Theorem 2.7 of [FZ] and Theorem 4.3 of [HHKV]. ***In what follows, we require that the bounded domain Ω has Lipschitz boundary.*** **Under this assumption, when $L = 2$, as a consequence of the main *Caldéron–Zygmund* result Theorem 6.4 of [DR]**

[DR] L. Diening, M. Růžička, ***Calderón–Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics***, **J. Reine Angew. Math. 563 (2003) 197–220.**

there exists a constant $\kappa_2 = \kappa_2(n,p) > 0$ such that

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$$\|\psi\|_{\mathfrak{D}^{2,p(\cdot)}(\Omega)} \le \kappa_2\|\Delta\psi\|_{p(\cdot)} = \kappa_2\|\mathcal{D}_2\psi\|_{p(\cdot)} \qquad \textbf{for all } \psi \in W_0^{2,p(\cdot)}(\Omega), \tag{50}$$

where $\mathcal{D}_2$ is defined in (3) when $d = 1$, as already noted and used in [CP]. For another proof of (50) we refer to Theorem 4.4 of

[ZF] A. Zang, Y. Fu, ***Interpolation inequalities for derivatives in variable exponent Lebesgue–Sobolev spaces***, **Nonlinear Anal. 69 (2008) 3629–3636.**

Proposition A.2 of [CP] shows that for all $L = 1, 2, \dots$ there exists a number $\kappa_L = \kappa_L(n,p) > 0$ such that

$$\|\psi\|_{\mathfrak{D}^{L,p(\cdot)}(\Omega)} \le \kappa_L\|\psi\|_{L,p(\cdot)},$$

$$\|\psi\|_{L,p(\cdot)} = \begin{cases} \|\mathcal{D}_L\psi\|_{p(\cdot)}, & \textbf{if } L \textbf{ is even},\\ \sum_{i=1}^n \|(\mathcal{D}_L\psi)_i\|_{p(\cdot)}, & \textbf{if } L \textbf{ is odd},\end{cases}$$

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for all $\psi \in W_0^{L,p(\cdot)}(\Omega)$, where $\mathcal{D}_L$ is the operator given in (3) for $d = 1$. This proves that *the two norms $\|\cdot\|_{\mathfrak{D}^{L,p(\cdot)}(\Omega)}$ and $\|\cdot\|_{L,p(\cdot)}$ are equivalent*. **Hence, also $\left(W_0^{L,p(\cdot)}(\Omega), \|\cdot\|_{L,p(\cdot)}\right)$ is a** *reflexive Banach space*. **Since we study the variational problem (8), when $d = 1$ we actually are interested in the norm**

$$\|\psi\| = \begin{cases} \|\mathcal{D}_L\psi\|_{p(\cdot)}, & \textbf{if } L \textbf{ is even}, \\ \|\,|\mathcal{D}_L\psi|_n\,\|_{p(\cdot)}, & \textbf{if } L \textbf{ is odd}. \end{cases}$$

The two norms $\|\cdot\|_{L,p(\cdot)}$ and $\|\cdot\|$ are exactly the same when L is even. We claim that they are equivalent when L is odd. Let $\psi \in W_0^{L,p(\cdot)}(\Omega)$. First assume that $\|\psi\| > 0$. Then, by definition of the Luxemburg norm, being $|(\mathcal{D}_L\psi)_i| \le |\mathcal{D}_L\psi|_n$ for all $i = 1, \ldots, n$, we have

Preliminaries for the $p(x)$–polyharmonic Kirchhoff problem

$$\int_\Omega \left|\frac{(\mathcal{D}_L\psi)_i}{\|\psi\|}\right|^{p(x)} dx \le \int_\Omega \left|\frac{\big|\mathcal{D}_L\psi\big|_n}{\|\psi\|}\right|^{p(x)} dx \le 1.$$

Hence $\|(\mathcal{D}_L\psi)_i\|_{p(\cdot)} \le \|\psi\|$. **Therefore** $\|\psi\|_{L,p(\cdot)} \le n\|\psi\|$. **Assume now that** $\|\psi\|_{L,p(\cdot)} > 0$. **Then** $\|(\mathcal{D}_L\psi)_k\|_{p(\cdot)} > 0$ **for some** $k \in \{1,\dots,n\}$. **Since** $1 < p_-$, **then** $|\mathcal{D}_L\psi|_n^{p(x)} \le n^{p(x)-1}\sum_{i=1}^n |(\mathcal{D}_L\psi)_i|^{p(x)}$. **In particular,**

$$\begin{aligned}\int_\Omega \left|\frac{|\mathcal{D}_L\psi|_n}{n\|\psi\|_{L,p(\cdot)}}\right|^{p(x)} dx &= \int_\Omega \left|\frac{|\mathcal{D}_L\psi|_n}{n^{1/p'(x)}n^{1/p(x)}\|\psi\|_{L,p(\cdot)}}\right|^{p(x)} dx \\ &\le \frac{1}{n}\sum_{i=1}^n \int_\Omega \left|\frac{(\mathcal{D}_L\psi)_i}{\|\psi\|_{L,p(\cdot)}}\right|^{p(x)} dx \\ &\le \frac{1}{n}\sum_{\substack{i=1\\ \|(\mathcal{D}_L\psi)_i\|_{p(\cdot)}\neq 0}}^n \int_\Omega \left|\frac{(\mathcal{D}_L\psi)_i}{\|(\mathcal{D}_L\psi)_i\|_{p(\cdot)}}\right|^{p(x)} dx \le 1.\end{aligned}$$

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Again, by definition of the Luxemburg norm, $\|\psi\| \le n\|\psi\|_{L,p(\cdot)}$. This completes the proof of the claim.

As already noted here *either* $n > Lp_+$ *or* $n \le Lp_-$. Hence *the critical variable exponent related to* p is defined for all $x \in \overline{\Omega}$ by the pointwise relation

$$p_L^*(x) = \begin{cases} \dfrac{np(x)}{n - Lp(x)}, & \text{if } \quad n > Lp_+, \\ \infty, & \text{if } 1 \le n \le Lp_-. \end{cases} \tag{51}$$

If $n \le Lp_-$, the Sobolev embedding $W_0^{L,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{h(\cdot)}(\Omega)$ is compact for all $h \in C(\overline{\Omega})$ such that $h \ge 1$ in Ω. Indeed, the embedding $W_0^{L,p(\cdot)}(\Omega) \hookrightarrow W_0^{L,p_-}(\Omega)$ is continuous by Lemma 8.1.8 of [DHHR]. Moreover, since $n \le Lp_-$, the embedding $W_0^{L,p_-}(\Omega) \hookrightarrow\hookrightarrow L^{h_+}(\Omega)$ is compact and in turn also $W_0^{L,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{h(\cdot)}(\Omega)$ is compact.

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If $n > Lp_+$ and $h \in C(\overline{\Omega})$, the embedding $W_0^{L,p(\cdot)}(\Omega) \hookrightarrow L^{h(\cdot)}(\Omega)$ is continuous, whenever $1 \le h(x) \le p_L^*(x)$ for all $x \in \Omega$. A proof of this fact, in the case $L = 1$, is given in Theorem 8.3.1–(a) of [DHHR]. The result then follows by induction on L as in Lemma 5.12 of [A]. Moreover, if $1 \le h(x) < p_L^*(x)$ for all $x \in \overline{\Omega}$, then $W_0^{L,p(\cdot)}(\Omega)$ is compactly embedded into $L^{h(\cdot)}(\Omega)$, as proved in Theorem 2.3 of [FZ], Proposition 3.3 of [MOSS] and Theorem 5.7 for $L = 1$ of [D]. Since we are interested in the vectorial variational problem (8), *from now on we endow the space $[W_0^{L,p(\cdot)}(\Omega)]^d$ with the norm*

$$\|u\| = \| \, |\mathcal{D}_L u|_N \, \|_{p(\cdot)},$$

$\left([W_0^{L,p(\cdot)}(\Omega)]^d, \|\cdot\|\right)$ is a uniformly convex Banach space

where $|\cdot|_N$ denotes the Euclidean norm in $\mathbb{R}^N$ and $N = d$ when L is even, while $N = dn$ when L is odd. In particular, $\left([W_0^{L,p(\cdot)}(\Omega)]^d, \|\cdot\|\right)$ is a *uniformly convex* **Banach space.**
Taking inspiration from [CL], we say that the variable exponent $p \in C_+^{\log}(\overline{\Omega})$ belongs to ***the Modular L–Poincaré Inequality Class***, **$p \in \mathcal{MP}_L(\Omega)$, if**

$$\lambda_1 = \inf_{\substack{u\in[W_0^{L,p(\cdot)}(\Omega)]^d \\ u\neq 0}} \frac{\displaystyle\int_\Omega |\mathcal{D}_L u|_N^{p(x)} dx}{\displaystyle\int_\Omega w(x)|u|_d^{p(x)} dx} > 0, \tag{52}$$

where $\mathcal{D}_L$ is given in (3).
We point out that there are exponents $p \in C_+^{\log}(\overline{\Omega}) \setminus \mathcal{MP}_L(\Omega)$ even in the case $d = L = 1$ and

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$w \equiv 1$. **For instance, assuming also that there exists an open set** $U \subset \Omega$ **and a point** $x_0 \in U$ **such that** $p(x_0) < p(x)$ **(or** $p(x_0) > p(x)$**) for all** $x \in \partial U$**, then by Theorem 3.1 [FZZ] the property** (52) **fails, that is** $\lambda_1 = 0$ **and** p **is not of class** $\mathcal{MP}_L(\Omega)$**.**
On the other hand, when $d = L = 1$ **and** $w \equiv 1$ **there are criteria in order that** p **is of class** $\mathcal{MP}_L(\Omega)$**, that is** p **satisfies** (52)**. Indeed, by Theorem 3.3 [FZZ] if there exists** $l \in \mathbb{R}^n \setminus \{0\}$ **such that for all** $x \in \Omega$ **the function** $t \mapsto p(x + tl)$ **is monotone for** $t \in I_x = \{s \in \mathbb{R} : x + sl \in \Omega\}$**, then** (52) **holds. Recently, when** $L = d = 1$ **and** $w \equiv 1$**, Theorem 3.3 of [FZZ] has been extended in [A], assuming the existence of a nonnegative function** $\xi \in C^1(\overline{\Omega})$**, with** $|D\xi| > 0$ **and** $D\xi \cdot Dp \geq 0$ **in** $\overline{\Omega}$**. Another criterium has been proposed in Theorem 1 of [MRS] again when** $L = d = 1$**,** $w \equiv 1$ **and** $p \in C^1(\overline{\Omega})$**, assuming the existence of** $a : \overline{\Omega} \to \mathbb{R}^n$ **such that for all** $x \in \overline{\Omega}$

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$$\operatorname{div}\mathbf{a}(x) \geq a_0 > 0 \quad \textbf{and} \quad \mathbf{a}(x)\cdot Dp(x) = 0.$$

The two results of [A] and [MRS] do not contradict each other but they seem to supplement each other. Furthermore, Theorem 2.2 of [CL] says that if (52) **holds when** $L = d = 1$ **and** $w \equiv 1$**, then** (52) **continues to hold for all** $w \in L^1_{\mathrm{loc}}(\Omega)$**, with** $w_- = \operatorname{ess\,inf}_\Omega w(x) > 0$**. Lately, when** $d = 1$ **and** $w \equiv 1$**, Theorem 3.1 of [FZZ] has been extended to the** $p(x)$**–biharmonic operator under Navier boundary conditions. In particular, when** p **possesses a strict local minimum (or maximum) in** Ω **and** $w_+ = \operatorname{ess\,sup}_\Omega w(x)$ **is finite, then**

$$\lambda_1 \geq \frac{1}{w_+} \inf_{\psi\in X\setminus\{0\}} \frac{\displaystyle\int_\Omega |\Delta\psi|^{p(x)}dx}{\displaystyle\int_\Omega |\psi|^{p(x)}dx} = 0,$$

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where $X = W_0^{1,p(\cdot)}(\Omega) \cap W^{2,p(\cdot)}(\Omega)$. Thus, in principle, (52) could fail. As far as we are aware, there are no criteria in order to have $\lambda_1 > 0$, even when $L = 2$, $d = 1$, $w \equiv 1$ and $p(x) \not\equiv p > 1$. In any case we have these useful results, which seem not to be so well known.

Let $L \in \{1, 2\}$. If $\lambda_1 > 0$ when $d = 1$, then $\lambda_1 > 0$ for all $d \in \mathbb{N}$.

Consider first the case $L = 1$. Clearly, by density, for $d = 1$,

$$\lambda_1 = \inf_{\substack{\psi \in C_0^\infty(\Omega) \\ \psi \neq 0}} \frac{\displaystyle\int_\Omega |D\psi|_n^{p(x)} dx}{\displaystyle\int_\Omega w(x)|\psi|^{p(x)} dx} > 0.$$

Then for all $\varphi = (\varphi_1, \ldots, \varphi_d) \in [C_0^\infty(\Omega)]^d \setminus \{0\}$

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$$\frac{\displaystyle\int_\Omega |D\varphi|_{dn}^{p(x)}dx}{\displaystyle\int_\Omega w(x)|\varphi|_d^{p(x)}dx} \geq \frac{\displaystyle\int_\Omega d^{-p(x)/2}\left(\sum_{i=1}^d |D\varphi_i|_n\right)^{p(x)} dx}{\displaystyle d^{p_+-1}\sum_{i=1}^d \int_\Omega w(x)|\varphi_i|^{p(x)}dx}$$

$$\geq d^{1-3p_+/2}\cdot\frac{\displaystyle\sum_{i=1}^d \int_\Omega |D\varphi_i|_n^{p(x)}dx}{\displaystyle\sum_{i=1}^d \int_\Omega w(x)|\varphi_i|^{p(x)}dx} \geq d^{1-3p_+/2}\lambda_1,$$

that is (52) **holds for all** $d \geq 2$ **when** $L = 1$**. Similarly, when** $L = 2$**, by assumption**

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$$\lambda_1 = \inf_{\substack{\psi\in C_0^\infty(\Omega)\\ \psi\neq 0}} \frac{\displaystyle\int_\Omega |\Delta\psi|^{p(x)}dx}{\displaystyle\int_\Omega w(x)|\psi|^{p(x)}dx} > 0.$$

Hence, for all $\varphi = (\varphi_1,\dots,\varphi_d) \in [C_0^\infty(\Omega)]^d \setminus \{0\}$

$$\frac{\displaystyle\int_\Omega |\Delta\varphi|_d^{p(x)}dx}{\displaystyle\int_\Omega w(x)|\varphi|_d^{p(x)}dx} \geq \frac{\displaystyle\int_\Omega d^{-p(x)/2}\left(\sum_{i=1}^d |\Delta\varphi_i|\right)^{p(x)} dx}{\displaystyle d^{p_+-1}\sum_{i=1}^d \int_\Omega w(x)|\varphi_i|^{p(x)}dx}$$

$$\geq d^{1-3p_+/2}\cdot \frac{\displaystyle\sum_{i=1}^d \int_\Omega |\Delta\varphi_i|^{p(x)}dx}{\displaystyle\sum_{i=1}^d \int_\Omega w(x)|\varphi_i|^{p(x)}dx} \geq d^{1-3p_+/2}\lambda_1,$$

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so that (52) **holds for all** $d \geq 2$ **when** $L = 2$.

Assume that $w_- > 0$. *If* (52) *holds for* $L \in \{1, 2\}$, *then it holds for all* $L \in \mathbb{N}$.

Let us denote for simplicity the number in (52) **by** $\lambda_{1,1}$ **if** $L = 1$ **and** $\lambda_{1,2}$ **if** $L = 2$. **By density,**

$$\lambda_{1,1} = \inf_{\substack{\varphi\in[C_0^\infty(\Omega)]^d \\ \varphi\neq 0}} \frac{\displaystyle\int_\Omega |D\varphi|_{dn}^{p(x)}dx}{\displaystyle\int_\Omega w(x)|\varphi|^{p(x)}dx} > 0,$$

$$\lambda_{1,2} = \inf_{\substack{\varphi\in[C_0^\infty(\Omega)]^d \\ \varphi\neq 0}} \frac{\displaystyle\int_\Omega |\Delta\varphi|_{d}^{p(x)}dx}{\displaystyle\int_\Omega w(x)|\varphi|^{p(x)}dx} > 0.$$

Now, $\mathcal{D}_3\varphi = D\Delta\varphi$ **by** (3). **Hence, for any** $\varphi \in [C_0^\infty(\Omega)]^d \setminus \{0\}$

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$$\frac{\displaystyle\int_\Omega |\mathcal{D}_3\varphi|^{p(x)}_{dn}dx}{\displaystyle\int_\Omega w(x)|\varphi|^{p(x)}_{d}dx} = \frac{\displaystyle\int_\Omega |D\Delta\varphi|^{p(x)}_{dn}dx}{\displaystyle\int_\Omega w(x)|\Delta\varphi|^{p(x)}_{d}dx}\cdot\frac{\displaystyle\int_\Omega w(x)|\Delta\varphi|^{p(x)}_{d}dx}{\displaystyle\int_\Omega w(x)|\varphi|^{p(x)}_{d}dx} \ge \lambda_{1,1}w_-\lambda_{1,2}.$$

In other words, $\lambda_{1,3} \ge \lambda_{1,1}w_-\lambda_{1,2}$.
Similarly, $\mathcal{D}_4\varphi = \Delta^2\varphi$ by (3) and for any $\varphi \in [C^\infty_0(\Omega)]^d \setminus \{0\}$

$$\frac{\displaystyle\int_\Omega |\mathcal{D}_4\varphi|^{p(x)}_{d}dx}{\displaystyle\int_\Omega w(x)|\varphi|^{p(x)}_{d}dx} = \frac{\displaystyle\int_\Omega |\Delta(\Delta\varphi)|^{p(x)}_{d}dx}{\displaystyle\int_\Omega w(x)|\Delta\varphi|^{p(x)}_{d}dx}\cdot\frac{\displaystyle\int_\Omega w(x)|\Delta\varphi|^{p(x)}_{d}dx}{\displaystyle\int_\Omega w(x)|\varphi|^{p(x)}_{d}dx} \ge w_-\lambda^2_{1,2},$$

that is $\lambda_{1,4} \ge w_-\lambda^2_{1,2}$. In conclusion, by induction,

$$\lambda_1 = \lambda_{1,L} \ge \begin{cases} w_-^{j-1}\lambda^j_{1,2}, & L = 2j, \\ \lambda_{1,1}(w_-\lambda_{1,2})^{j-1}, & L = 2j-1, \end{cases} \quad \text{for } j = 2, 3 \ldots.$$

In particular, $\lambda_1 > 0$ for all $L \in \mathbb{N}$, as required.

The main existence result for (8)

We finally turn to problem

$$(8)\ \begin{cases} M(\mathscr{I}_L(u))\Delta^L_{p(x)}u = \lambda\{w(x)|u|^{p(x)-2}u + f(x,u)\} \text{ in } \Omega, \\ D^\alpha u_k\big|_{\partial\Omega} = 0 \text{ for all } \alpha, \text{ with } |\alpha| \le L-1,\ k = 1,\dots,d, \end{cases}$$

recalling the assumptions required. As in (4), the weight w is supposed to be *positive a.e. in Ω and of class $L^{\varpi}(\Omega)$, with*

$$\varpi > \frac{n}{n-[n-Lp_-]^+}, \tag{53}$$

replacing (4), and the variable exponent p is assumed also of class $\mathcal{MP}_L(\Omega)$.
The Kirchhoff function M verifies the assumption $(\mathcal{M})$, with $\gamma = 1$, that is (8) is non–degenerate. The Dirichlet functional $\mathscr{I}_L$ is defined in (9), that is

$$\mathscr{I}_L(u) = \int_\Omega \frac{|\mathcal{D}_L u|^{p(x)}}{p(x)}dx.$$

The main existence result for (8)

The nonlinearity f verifies the foreword in $(\mathcal{F})$, condition $(\mathcal{F})$–$(c)'$ above, while (a) and (b) are replaced by

$(a)'$ ***There exist*** $q \in C_+(\overline{\Omega})$***, with*** $1 < q_+ < p_-$***, and*** $C_f > 0$ ***such that for a.a.*** $x \in \Omega$ ***and all*** $v \in \mathbb{R}^d$

$$|f(x,v)| \leq C_f w(x)\big(1 + |v|^{q(x)-1}\big).$$

$(b)'$ ***There exists*** $\mathfrak{p}^\star \in C_+(\overline{\Omega})$***, with*** $p_+ < \mathfrak{p}^\star_- \leq \mathfrak{p}^\star_+ < (p^*_L)_-/\varpi'$***, such that***

$$\limsup_{|v|\to 0} \frac{|f(x,v)\cdot v|}{w(x)|v|^{\mathfrak{p}^\star(x)}} < \infty, \quad \textit{uniformly a.e. in } \Omega.$$

The main existence result for (8)

Conditions $(\mathcal{F})$–$(a)'$ **and** $(b)'$ **imply that** $f(x,0)=0$ **for a.a.** $x\in\Omega$**,**

$$0<S_f=\operatorname*{ess\,sup}_{v\neq 0,\,x\in\Omega}\frac{|f(x,v)\cdot v|}{w(x)|v|^{p(x)}}<\infty,\qquad 0<\operatorname*{ess\,sup}_{v\neq 0,\,x\in\Omega}\frac{|F(x,v)|}{w(x)|v|^{p(x)}}\leq\frac{S_f}{p_-}$$

and that there exists $K>0$ **such that**

$$|F(x,v)|\leq K\frac{w(x)}{\mathfrak{p}^\star(x)}|v|^{\mathfrak{p}^\star(x)}$$

for a.a. $x\in\Omega$ **and all** $v\in\mathbb{R}^d$**.**

The main existence result for (8)

The energy functional $J_\lambda : [W_0^{L,p(\cdot)}(\Omega)]^d \to \mathbb{R}$ associated to (8) is given by $J_\lambda(u) = \Phi(u) + \lambda\Psi(u)$, where now $\Phi(u) = \mathscr{M}(\mathscr{I}_L(u))$, with $\mathscr{I}_L(u)$ defined in (9) and $\Psi = \Psi_2$, where as usual

$$\Psi_2(u) = -\int_\Omega F(x, u(x))dx.$$

The dual space of $[W_0^{L,p(\cdot)}(\Omega)]^d$ is denoted by $\left([W_0^{L,p(\cdot)}(\Omega)]^d\right)^\star$.

The main existence result for (8)

LEMMA

The functional Φ *is convex, weakly lower semi–continuous in* $[W_0^{L,p(\cdot)}(\Omega)]^d$ *and of class* $C^1([W_0^{L,p(\cdot)}(\Omega)]^d)$*.*
Moreover, $\Phi' : [W_0^{L,p(\cdot)}(\Omega)]^d \to \left([W_0^{L,p(\cdot)}(\Omega)]^d\right)^\star$ *verifies the* $(\mathscr{S}_+)$ *condition, i.e. if* $u_k \rightharpoonup u$ *in* $[W_0^{L,p(\cdot)}(\Omega)]^d$ *and*

$$\limsup_{k\to\infty} M(\mathscr{I}_L(u_k)) \int_\Omega |\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k \cdot (\mathcal{D}_L u_k - \mathcal{D}_L u)dx \le 0, \tag{54}$$

then $u_k \to u$ *in* $[W_0^{L,p(\cdot)}(\Omega)]^d$*.*

The main existence result for (8)

A simple calculation shows that Φ is convex in $[W_0^{L,p(\cdot)}(\Omega)]^d$, being $\mathscr{I}_L$ convex and M non–negative and non–decreasing by $(\mathcal{M})$. Moreover, Φ is Gâteaux differentiable in $[W_0^{L,p(\cdot)}(\Omega)]^d$ and for all $u, v \in [W_0^{L,p(\cdot)}(\Omega)]^d$ it results

$$\langle \Phi'(u), v\rangle = M(\mathscr{I}_L(u)) \int_\Omega |\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u \cdot \mathcal{D}_L v \, dx.$$

Now, let u, $(u_k) \subset [W_0^{L,p(\cdot)}(\Omega)]^d$ be such that $u_k \to u$ as $k \to \infty$. We claim that as $k \to \infty$

$$\|\Phi'(u_k) - \Phi'(u)\|_\star = \sup_{\substack{v \in [W_0^{L,p(\cdot)}(\Omega)]^d \\ \|v\|=1}} \big|\langle \Phi'(u_k) - \Phi'(u), v\rangle\big| = o(1).$$

Put $\mathscr{R}(u_k, u) = \big\| M(\mathscr{I}_L(u_k))|\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k - M(\mathscr{I}_L(u))|\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u\big\|_{N,p'(\cdot)}$. By the Hölder inequality

The main existence result for (8)

$$\left|\langle \Phi'(u_k)-\Phi'(u),v\rangle\right| \leq c_H \mathscr{R}(u_k,u)\|\mathcal{D}_L v\|_{N,p(\cdot)}.$$

Hence

$$\|\Phi'(u_k)-\Phi'(u)\|_\star \leq c_H \mathscr{R}(u_k,u). \tag{55}$$

Let $(u_{k_j})_j$ be a subsequence of $(u_k)_k$. Clearly, $u_{k_j}\to u$ in $[W_0^{L,p(\cdot)}(\Omega)]^d$ and so $\mathcal{D}_L u_{k_j}\to \mathcal{D}_L u$ in $[L^{p(\cdot)}(\Omega)]^N$ as $j\to\infty$, where as usual $N=d$ if L is even and $N=dn$ if L is odd. By Lemma ??, with $m=N$, $\sigma=p$ and $\omega\equiv 1$, there exist a subsequence of $(u_{k_j})_j$, still denoted by $(u_{k_j})_j$, and $h\in L^{p(\cdot)}(\Omega)$ such that for a.a. $x\in\Omega$

$\mathcal{D}_L u_{k_j}(x)\to \mathcal{D}_L u(x)$ as $j\to\infty$ and $|\mathcal{D}_L u_{k_j}(x)|\leq h(x)$ for all $j\in\mathbb{N}$.

Hence,

$$\left|M(\mathscr{I}_L(u_{k_j}))|\mathcal{D}_L u_{k_j}|^{p(x)-2}\mathcal{D}_L u_{k_j} - M(\mathscr{I}_L(u))|\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u\right|^{p'(x)}$$
$$\leq 2^{p'(x)-1}\left\{\left[M(\mathscr{I}_L(u_{k_j}))|\mathcal{D}_L u_{k_j}|^{p(x)-1}\right]^{p'(x)} + \left[M(\mathscr{I}_L(u))|\mathcal{D}_L u|^{p(x)-1}\right]^{p'}\right.$$
$$\leq (2K)^{p'(x)}h^{p(x)}\in L^1(\Omega),$$

The main existence result for (8)

where $K = \sup_j M(\mathscr{I}_L(u_{k_j})) < \infty$, being $(u_{k_j})_j$ convergent and so bounded in $[W_0^{L,p(\cdot)}(\Omega)]^d$. In particular, $M(\mathscr{I}_L(u_{k_j})) \to M(\mathscr{I}_L(u))$ by $(\mathcal{M})$. Furthermore, $|\mathcal{D}_L u_{k_j}|^{p(x)-2}\mathcal{D}_L u_{k_j} \to |\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u$ a.e. in Ω. Hence

$$\begin{aligned}\Big|M(\mathscr{I}_L(u_{k_j}))|\mathcal{D}_L u_{k_j}|^{p(x)-2}\mathcal{D}_L u_{k_j} - M(\mathscr{I}_L(u))|\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u\Big| \\ \leq K\Big||\mathcal{D}_L u_{k_j}|^{p(x)-2}\mathcal{D}_L u_{k_j} - |\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u\Big| \\ + \Big|M(\mathscr{I}_L(u_{k_j})) - M(\mathscr{I}_L(u))\Big| \cdot |\mathcal{D}_L u|^{p(x)-1} \to 0\end{aligned}$$

a.e. in Ω as $j \to \infty$. Thus, applying the Lebesgue dominated convergence theorem, we obtain that the entire sequence $(u_k)_k$ is such that $\mathscr{R}(u_k, u) \to 0$ as $k \to \infty$, which implies the claim by (55). In conclusion, Φ is of class $C^1([W_0^{L,p(\cdot)}(\Omega)]^d)$, as claimed. In particular, Φ is weakly lower semi–continuous in $[W_0^{L,p(\cdot)}(\Omega)]^d$.

The main existence result for (8)

Let $u, (u_k)_k \subset [W_0^{L,p(\cdot)}(\Omega)]^d$ **be such that** $u_k \rightharpoonup u$ **in** $[W_0^{L,p(\cdot)}(\Omega)]^d$ **and** (54) **hold. Then**

$$\lim_{k\to\infty} M(\mathscr{I}_L(u)) \int_\Omega |\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u \cdot \mathcal{D}_L(u_k - u)dx = 0, \quad (56)$$

being $|\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u \in [L^{p'(\cdot)}(\Omega)]^N$. **Hence** (54) **is equivalent to**

$$\limsup_{k\to\infty} \int_\Omega \mathscr{L}(u_k, u) \cdot \mathcal{D}_L(u_k - u)dx \le 0,$$

where $\mathscr{L}(u_k, u) = M(\mathscr{I}_L(u_k))|\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k - M(\mathscr{I}_L(u))|\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u$. **By convexity**

$$\int_\Omega \mathscr{L}(u_k, u) \cdot \mathcal{D}_L(u_k - u)dx \ge 0,$$

therefore

$$\lim_{k\to\infty} \int_\Omega \mathscr{L}(u_k, u) \cdot \mathcal{D}_L(u_k - u)dx = 0.$$

The main existence result for (8)

This implies $\lim_{k\to\infty} M(\mathscr{I}_L(u_k)) \int_\Omega |\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k \cdot \mathcal{D}_L(u_k - u)dx = 0$ **by** (56)**, and in turn**

$$\lim_{k\to\infty} \int_\Omega |\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k \cdot \mathcal{D}_L(u_k - u)dx = 0, \qquad (57)$$

being $M(\mathscr{I}_L(u_k)) \geq s > 0$ **for all** $k \in \mathbb{N}$**, since here** $\gamma = 1$ **in** $(\mathcal{M})$**. Clearly** $\mathscr{I}_L$ **is of class** C^1 **and convex in the Banach space** $[W_0^{L,p(\cdot)}(\Omega)]^d$**, so that** $\mathscr{I}_L(u) \leq \liminf_{k\to\infty} \mathscr{I}_L(u_k)$ **by the weak lower semi–continuity of** $\mathscr{I}_L$ **in** $[W_0^{L,p(\cdot)}(\Omega)]^d$**. By the convexity of** $\mathscr{I}_L$ **for all** k

$$\mathscr{I}_L(u) + \int_\Omega |\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k \cdot \mathcal{D}_L(u_k - u)dx \geq \mathscr{I}_L(u_k),$$

so that $\mathscr{I}_L(u) \geq \limsup_{k\to\infty} \mathscr{I}_L(u_k)$ **by** (57)**. In conclusion,**

$$\lim_{k\to\infty} \mathscr{I}_L(u_k) = \mathscr{I}_L(u). \qquad (58)$$

The main existence result for (8)

Furthermore, by (57) **as** $k \to \infty$

$$\int_\Omega (|\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k - |\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u) \cdot \mathcal{D}_L(u_k - u)dx \to 0,$$

since $u_k \rightharpoonup u$ **in** $[W_0^{L,p(\cdot)}(\Omega)]^d$. **Hence** $(|\mathcal{D}_L u_k|^{p(x)-2}\mathcal{D}_L u_k - |\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u) \cdot \mathcal{D}_L(u_k - u) \geq 0$ **converges to** 0 **in** $L^1(\Omega)$, **and so, up to a subsequence,**

$$(|\mathcal{D}_L u_{k_j}|^{p(x)-2}\mathcal{D}_L u_{k_j} - |\mathcal{D}_L u|^{p(x)-2}\mathcal{D}_L u) \cdot \mathcal{D}_L(u_{k_j} - u) \to 0$$

a.e. in Ω. **Lemma 3 of [DHH] implies that** $\mathcal{D}_L u_{k_j}$ **converges to** $\mathcal{D}_L u$ **a.e. in** Ω, **and in turn** $|\mathcal{D}_L u_{k_j}|^{p(x)}$ **converges to** $|\mathcal{D}_L u|^{p(x)}$ **a.e. in** Ω.

Consider the sequence $(g_j)_j$ **in** $L^1(\Omega)$ **defined pointwise by**

$$g_j(x) = \frac{1}{p(x)}\left\{\frac{|\mathcal{D}_L u_{k_j}|^{p(x)} + |\mathcal{D}_L u|^{p(x)}}{2} - \left|\frac{\mathcal{D}_L u_{k_j} - \mathcal{D}_L u}{2}\right|^{p(x)}\right\}.$$

The main existence result for (8)

By convexity $g_j \geq 0$ and $g_j(x) \to |\mathcal{D}_L u(x)|^{p(x)}/p(x)$ for a.a. $x \in \Omega$. Therefore, by the Fatou lemma and (58) we have

$$\begin{aligned}\mathscr{I}_L(u) &\leq \liminf_{j\to\infty} \int_\Omega g_j dx = \mathscr{I}_L(u) \\ &\qquad - \limsup_{j\to\infty} \int_\Omega \frac{1}{p(x)} \left| \frac{\mathcal{D}_L u_{k_j} - \mathcal{D}_L u}{2} \right|^{p(x)} dx \\ &\leq \mathscr{I}_L(u) - \frac{1}{p_+ 2^{p_+}} \limsup_{j\to\infty} \rho_{p(\cdot)}(\mathcal{D}_L u_{k_j} - \mathcal{D}_L u).\end{aligned}$$

Hence, $\limsup_{j\to\infty} \rho_{p(\cdot)}(\mathcal{D}_L u_{k_j} - \mathcal{D}_L u) = 0$**, that is** $\lim_{j\to\infty} \|u_{k_j} - u\| = 0$ **by Lemma 10. In conclusion, the entire sequence $u_k \to u$, since $u_k \rightharpoonup u$ as $k \to \infty$ in** $[W_0^{L,p(\cdot)}(\Omega)]^d$**. This completes the proof.**

The main existence result for (8)

THEOREM – MULTEPLICITY FOR (8)

Let $(\mathcal{F})$–$(a)'$, $(b)'$ *hold and let* $\ell_\star = \dfrac{p_- s\lambda_1}{p_+ S_f}$.

- (i) *If* $\lambda \in [0, \ell_\star)$, *then* (8) *has only the trivial solution.*
- (ii) *If also* $(\mathcal{F})$–$(c)'$ *holds, then there exists* $\ell^\star \geq \ell_\star$ *such that* (8) *admits at least two nontrivial solutions for all* $\lambda \in (\ell^\star, \infty)$.

The main existence result for (8)

The part (i) of the statement is proved by using a similar argument produced for the proof of Theorem 8–(i), namely if u is a weak solution of (8) we have

$$\begin{aligned} sp_-\lambda_1\mathscr{I}_L(u) &\leq \lambda_1 M(\mathscr{I}_L(u))\int_\Omega |\mathcal{D}_L u|^{p(x)}dx \leq \lambda_1\lambda\int_\Omega |f(x,u)\cdot u|\,dx \\ &\leq \lambda_1\lambda S_f\int_\Omega w(x)|u|^{p(x)}dx \leq p_+\lambda S_f\mathscr{I}_L(u). \end{aligned}$$

Thus, if $u \not\equiv 0$, then necessarily $\lambda \geq \ell_\star$.

The main existence result for (8)

In order to prove (ii), we first show that J_λ is coercive for every $\lambda \in \mathbb{R}$. Indeed, as shown in the proof of Lemma 7, for all $u \in [W_0^{L,p(\cdot)}(\Omega)]^d$, with $\|u\| \geq 1$,

$$\begin{aligned} J_\lambda(u) &\geq s\mathscr{I}_L(u) - |\lambda| \int_\Omega |F(x,u)|dx \\ &\geq \frac{s}{p_+} \int_\Omega |\mathcal{D}_L u|^{p(x)} dx - |\lambda| \int_\Omega \int_0^1 C_f w(x) \left(1 + |tu|^{q(x)-1}\right) |u| dt\, dx \\ &\geq \frac{s}{p_+} \|u\|^{p_-} - |\lambda| C_f \int_\Omega w(x)|u|dx - \frac{|\lambda| C_f}{q_-} \int_\Omega w(x)|u|^{q(x)} dx. \end{aligned} \tag{59}$$

Now, by Hölder's inequality, and the previous lemmas

$$\begin{aligned} \int_\Omega w(x)|u|^{q(x)} dx &\leq \left(\int_\Omega w(x)^\varpi dx\right)^{1/\varpi} \left(\int_\Omega |u|^{\varpi' q(x)} dx\right)^{1/\varpi'} \\ &\leq \|w\|_\varpi \max\left\{\|u\|_{d,\varpi' q(\cdot)}^{q_-}, \|u\|_{d,\varpi' q(\cdot)}^{q_+}\right\} \end{aligned}$$

The main existence result for (8)

$$\leq \|w\|_{\varpi} \max\left\{\mathcal{S}^{q_-}_{d,\varpi' q(\cdot)}\|u\|^{q_-}, \mathcal{S}^{q_+}_{d,\varpi' q(\cdot)}\|u\|^{q_+}\right\}.$$

Hence, for all $u \in [W_0^{L,p(\cdot)}(\Omega)]^d$**, with** $\|u\| \geq 1$**, by** (59)

$$J_\lambda(u) \geq \frac{s}{p_+}\|u\|^{p_-} - |\lambda| C_f \mathcal{S}_{d,1,w}\|u\| - |\lambda| C \|u\|^{q_+},$$

where $C = C_f \|w\|_{\varpi} \max\left\{\mathcal{S}^{q_-}_{d,\varpi' q(\cdot)}, \mathcal{S}^{q_+}_{d,\varpi' q(\cdot)}\right\}/q_-$**. Since** $1 < q_+ < p_-$**, this shows the claim by** $(\mathcal{F})$–$(a)'$**. Thus, here** $I = \mathbb{R}$**.**

The function $u_0 \in [C_0^\infty(\Omega)]^d$ **constructed by using** $(\mathcal{F})$–$(c)'$ **in the proof of Theorem 9 is also in** $[W_0^{L,p(\cdot)}(\Omega)]^d$ **and such that** $\Psi_2(u_0) < 0$**. Hence, also in this setting the crucial number**

$$\ell^\star = \varphi_1(0) = \inf_{u \in \Psi_2^{-1}(I_0)} -\frac{\Phi(u)}{\Psi_2(u)}, \qquad I_0 = (-\infty, 0), \tag{60}$$

is well defined, so that again (43) **continues to hold.**

The main existence result for (8)

Furthermore,

$$\frac{\Phi(u)}{|\Psi_2(u)|} \geq \frac{s\displaystyle\int_\Omega |\mathcal{D}_L u|^{p(x)}dx}{p_+\displaystyle\int_\Omega |F(x,u)|dx} \geq \frac{p_- s\displaystyle\int_\Omega |\mathcal{D}_L u|^{p(x)}dx}{p_+ S_f\displaystyle\int_\Omega w(x)|u|^{p(x)}dx}$$
$$\geq \frac{p_- s\lambda_1}{p_+ S_f} = \ell_\star$$

for all $u \in \Psi_2^{-1}(I_0)$**. Thus,** $\ell^\star \geq \ell_\star > 0$ **by** (60)**.**

The main existence result for (8)

By Hölder's inequality, Lemma 2.1 of [ER] and the continuity of the embedding $[W_0^{L,p(\cdot)}(\Omega)]^d \hookrightarrow [L^{p_L^*(\cdot)}(\Omega)]^d$, we have for every $u \in [W_0^{L,p(\cdot)}(\Omega)]^d$

$$\begin{aligned}|\Psi_2(u)| &\leq \int_\Omega |F(x,u)|\,dx \leq \frac{K}{\mathfrak{p}_-^\star}\int_\Omega w(x)|u|^{\mathfrak{p}^\star(x)}dx \\ &\leq c\,\||u|^{\mathfrak{p}^\star(x)}\|_{p_L^*(\cdot)/\mathfrak{p}^\star(\cdot)} \\ &\leq c\max\left\{\|u\|_{d,p_L^*(\cdot)}^{\mathfrak{p}_+^\star}, \|u\|_{d,p_L^*(\cdot)}^{\mathfrak{p}_-^\star}\right\} \leq \mathfrak{K}\max\{\|u\|^{\mathfrak{p}_+^\star}, \|u\|^{\mathfrak{p}_-^\star}\},\end{aligned} \tag{61}$$

where $c = c_H K\|1\|_{\wp(\cdot)}\|w\|_{\varpi}/\mathfrak{p}_-^\star$, $\mathfrak{K} = c\max\left\{\mathcal{S}_{d,p_L^*(\cdot)}^{\mathfrak{p}_+^\star}, \mathcal{S}_{d,p_L^*(\cdot)}^{\mathfrak{p}_-^\star}\right\}$, and

The main existence result for (8)

$$1 < \wp(x) = \begin{cases} \dfrac{\varpi' p_L^*(x)}{p_L^*(x) - \mathfrak{p}^\star(x)\varpi'}, & \textbf{if } n > Lp_+, \\ \varpi', & \textbf{if } n \le Lp_-. \end{cases}$$

Taken $r < 0$ **and** $v \in \Psi_2^{-1}(r)$**, we obtain by** (47)**,** (61)**, and** $(\mathcal{M})$**, with** $\gamma = 1$**,**

$$|r| = |\Psi_2(v)| \le \mathfrak{K} \max\left\{ \left(p_+ \int_\Omega \frac{|\mathcal{D}_L u|^{p(x)}}{p(x)} dx \right)^{\mathfrak{p}^\star_+/p_-}, \left(p_+ \int_\Omega \frac{|\mathcal{D}_L u|^{p(x)}}{p(x)} dx \right)^{\mathfrak{p}} \right.$$

$$\le (p_+)^{\mathfrak{p}^\star_+/p_-} \mathfrak{K} \max\left\{ \left(\frac{\mathscr{M}(\mathscr{I}_L(u))}{s} \right)^{\mathfrak{p}^\star_+/p_-}, \left(\frac{\mathscr{M}(\mathscr{I}_L(u))}{s} \right)^{\mathfrak{p}^\star_-/p_+} \right\}$$

$$\le \kappa \max\left\{ \Phi(u)^{\mathfrak{p}^\star_+/p_-}, \Phi(u)^{\mathfrak{p}^\star_-/p_+} \right\},$$

where $\kappa = (p_+)^{\mathfrak{p}^\star_+/p_-} \mathfrak{K} / \min\{ s^{\mathfrak{p}^\star_+/p_-}, s^{\mathfrak{p}^\star_-/p_+} \}$**.**

The main existence result for (8)

Therefore, taking r so close to zero that $0<|r|<1$ and putting $\mathcal{K}=\min\left\{\kappa^{-p_-/\mathfrak{p}_+^\star},\kappa^{-p_+/\mathfrak{p}_-^\star}\right\}$, we have by (35) and the facts that $u\equiv 0\in\Psi_2^{-1}(I^r)$ and $\Psi_2(0)=0$,

$$\begin{aligned}\varphi_2(r)&\geq\frac{1}{|r|}\inf_{v\in\Psi_2^{-1}(r)}\Phi(v)\geq\mathcal{K}\min\{|r|^{p_-/\mathfrak{p}_+^\star-1},|r|^{p_+/\mathfrak{p}_-^\star-1}\}\\&=\mathcal{K}|r|^{p_+/\mathfrak{p}_-^\star-1},\end{aligned}$$

being $p_-\leq p_+<\mathfrak{p}_-^\star\leq\mathfrak{p}_+^\star$. This implies that $\lim\limits_{r\to 0^-}\varphi_2(r)=\infty$.
In conclusion, also in this setting, (44) holds, so that the proof can be continued and ended exactly as before.