School on Nonlinear Elliptic Problems

Dipartimento di Matematica e Applicazioni

Università di Milano "Bicocca"

On higher order p–Kirchhoff problems Lifespan estimates for solutions

Patrizia Pucci

University of Perugia

Milan – January 20-24, 2014

- **G. Autuori, F. Colasuonno, P. P.**, *On the existence of stationary solutions for higher order p–Kirchhoff problems via variational methods*, **Commun. Contemp. Math. (2014), pages 42.**
- **G. Autuori, F. Colasuonno, P. P.**, *Lifespan estimates for solutions of polyharmonic Kirchhoff systems*, **Math. Models Methods Appl. Sci. 22, Issue 02 - February 2012 - 1150009, pages 36.**
- **F. Colasuonno, P. P.**, *Multiplicity of solutions for $p(x)$–polyharmonic Kirchhoff equations*, **Nonlinear Anal. 74 (2011), 5962–5974.**
- **G. Autuori, P. P.**, *Local asymptotic stability for polyharmonic Kirchhoff systems*, **Appl. Anal., Special Volume dedicated to Prof. P.L. Butzer, 90 (2011), 493-514.**

Recent work at Perugia on Kirchhoff systems

- **M. Delitala, P. P., M.C. Salvatori**, *From methods of the mathematical kinetic theory for active particles to modelling Virus mutations*, **Math. Models Methods Appl. Sci., 21 (2011), 843-870.**
- **G. Autuori, P. P.**, *Asymptotic stability for Kirchhoff systems in variable exponent Sobolev spaces*, **Complex Var. Elliptic Equ., Special Volume dedicated to Prof. V.V. Zhikov, 56 (2011), 715-753.**
- **G. Autuori, P. P.**, *Kirchhoff systems with dynamic boundary conditions*, **Nonlinear Anal., 73 (2010), 1952-1965.**
- **G. Autuori, P. P., M.C. Salvatori**, *Global Nonexistence for Nonlinear Kirchhoff Systems*, **Archive Rat. Mech. Anal, 196 (2010), 489-516.**

- G. Autuori, P. P., *Kirchhoff systems with nonlinear source and boundary damping terms*, Commun. Pure Appl. Anal., 9 (2010), 1161-1188.
- P. P., V. Radulescu, Remarks of eigenvalue problems for nonlinear polyharmonic equations, Comptes Rendus, Sér. Math., 348 (2010), 161-164.
- G. Autuori, P. P., M.C. Salvatori, *Asymptotic stability for Anisotropic Kirchhoff Systems*, J. Math. Anal. Appl., 352 (2009), 149-165.
- G. Autuori, P. P., M.C. Salvatori, *Asymptotic stability for Nonlinear Kirchhoff Systems*, Nonlinear Anal. Real World Appl., 10 (2009), 889-909.

[ACP] G. Autuori, F.Colasuonno, P. P., *Lifespan estimates for solutions of polyharmonic Kirchhoff systems*, Math. Models Methods Appl. Sci., 22, Issue 02 - February 2012 - 1150009, pages 36.

Let $L \geq 1$ and consider in $\mathbb{R}_0^+ \times \Omega$ the problems

$$(P)\begin{cases} u_{tt} + M(\|\mathcal{D}_L u(t,\cdot)\|_2^2)(-\Delta)^L u + N(\|\mathcal{D}_{L-1} u(t,\cdot)\|_2^2)(-\Delta)^{L-1} u \\ \qquad +\mu u + Q(t,x,u,u_t) = f(t,x,u), \end{cases}$$

$$(KL)\begin{cases} u_{tt} + M(\|\mathcal{D}_L u(t,\cdot)\|_2^2)(-\Delta)^L u + N(\|\mathcal{D}_{L-1} u(t,\cdot)\|_2^2)(-\Delta)^{L-1} u \\ \qquad +\mu u + \varrho(t)(-\Delta)^L u_t + Q(t) u_t = f(t,x,u), \end{cases}$$

Both under $D^\alpha u(t,x) = 0$, $|\alpha| \leq L-1$ on $\mathbb{R}_0^+ \times \partial\Omega$

$\Omega \subset \mathbb{R}^n$ bounded domain, $u = (u_1, \ldots, u_d)$, $d \geq 1$, $\mu \geq 0$

$$(P)\begin{cases} u_{tt} + M(\|\mathcal{D}_L u(t,\cdot)\|_2^2)(-\Delta)^L u + N(\|\mathcal{D}_{L-1} u(t,\cdot)\|_2^2)(-\Delta)^{L-1} u \\ \qquad\qquad + \mu u + Q(t,x,u,u_t) = f(t,x,u), \end{cases}$$

$$(KL)\begin{cases} u_{tt} + M(\|\mathcal{D}_L u(t,\cdot)\|_2^2)(-\Delta)^L u + N(\|\mathcal{D}_{L-1} u(t,\cdot)\|_2^2)(-\Delta)^{L-1} u \\ \qquad\qquad + \mu u + \varrho(t)(-\Delta)^L u_t + Q(t) u_t = f(t,x,u), \end{cases}$$

$$L \geq 0, \quad \mathcal{D}_L u = \begin{cases} \Delta^j u, & \text{if } L = 2j, \\ D\Delta^j u, & \text{if } L = 2j+1, \end{cases} \qquad j = 0, 1, 2, \ldots$$

The models

$$(P)\begin{cases} u_{tt} + M(\|\mathcal{D}_L u(t,\cdot)\|_2^2)(-\Delta)^L u + N(\|\mathcal{D}_{L-1} u(t,\cdot)\|_2^2)(-\Delta)^{L-1} u \\ \qquad\qquad + \mu u + Q(t,x,u,u_t) = f(t,x,u), \end{cases}$$

$$(KL)\begin{cases} u_{tt} + M(\|\mathcal{D}_L u(t,\cdot)\|_2^2)(-\Delta)^L u + N(\|\mathcal{D}_{L-1} u(t,\cdot)\|_2^2)(-\Delta)^{L-1} u \\ \qquad\qquad + \mu u + \varrho(t)(-\Delta)^L u_t + Q(t) u_t = f(t,x,u), \end{cases}$$

$$L \geq 0, \quad \mathcal{D}_L u = \begin{cases} \Delta^j u, & \text{if } L = 2j, \\ D\Delta^j u, & \text{if } L = 2j+1, \end{cases} \qquad j = 0,1,2,\dots$$

The main Kirchhoff function

$$M(\tau) = a + b\gamma\tau^{\gamma-1}, \qquad a, b \geq 0, \quad a+b>0, \quad \gamma \geq 1.$$

$$N \in L^1_{loc}(\mathbb{R}_0^+ \to \mathbb{R}_0^+) : \exists\, \eta \in [1,\gamma] : \quad \eta\mathscr{N}(\tau) \geq \tau N(\tau)$$

$\mathscr{M}, \mathscr{N}$: primitives of M and N

Total energy associated to the system

$$Eu(t) = \frac{1}{2}\|u_t(t,\cdot)\|_2^2 + \mathscr{A}u(t) - \mathscr{F}u(t),$$

where

$$\mathscr{A}u(t) = \frac{1}{2}\left\{\mathscr{M}(\|\mathcal{D}_L u\|_2^2) + \mathscr{N}(\|\mathcal{D}_{L-1}u\|_2^2) + \mu\|u\|_2^2\right\}$$

The functional $\mathscr{A}$ includes the elliptic part of the system

Total energy associated to the system

$$Eu(t) = \frac{1}{2}\|u_t(t,\cdot)\|_2^2 + \mathscr{A}u(t) - \mathscr{F}u(t),$$

where

$$\mathscr{A}u(t) = \frac{1}{2}\left\{\mathscr{M}(\|\mathcal{D}_L u\|_2^2) + \mathscr{N}(\|\mathcal{D}_{L-1}u\|_2^2) + \mu\|u\|_2^2\right\}$$

The functional $\mathscr{A}$ includes the elliptic part of the system

Potential Energy

$$\mathscr{F}u(t) = \int_\Omega \left\{g(t,x)\frac{|u(t,x)|^\sigma}{\sigma} + c(x)\frac{|u(t,x)|^q}{q}\right\} dx$$

Notation

$$T = \sup\{t \in \mathbb{R}_0^+ : u \text{ exists in } [0,t) \times \Omega\} \in [0,\infty)$$

T = lifespan of the solution; $\quad I = [0, T)$.

Solutions and test functions space

$$X = \begin{cases} C(I \to H_0^L(\Omega)) \cap C^1(I \to L^2(\Omega)) & \text{for } (P), \\ C^1(I \to H_0^L(\Omega)) & \text{for } (KL). \end{cases}$$

$$Q \in C(\mathbb{R}_0^+ \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d) \quad \text{external damping}$$
$$(Q(t,x,u,v),v) \geq 0 \quad \text{for all } t,x,u,v$$

$\exists\, \bar{t} >> 1$ such that for all $(t,x,u,v) \in [\bar{t},\infty) \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d$

$$Q(t,x,u,v) = d_1(t,x)|u|^\kappa |v|^{m-2}v + d_2(t,x,u)|v|^{\wp-2}v$$

$$\wp,\, m > 1, \quad m+\kappa \leq \wp < q, \quad \kappa \geq 0$$

$d_1 \in C(\mathbb{R}_0^+ \to L^{\wp_1}(\Gamma_1))$, $d_2 \in C(\mathbb{R}_0^+ \to L^\infty(\Gamma_1))$ non–negative

$$\wp_1 = \begin{cases} \infty, & \text{if } \wp = m+\kappa, \\ \wp/(\wp-\kappa-m), & \text{if } \wp > m+\kappa, \end{cases}$$

$$\delta_1(t) = \|d_1(t,\cdot)\|_{\wp_1}, \quad \delta_2(t) = \sup_{(x,\xi)\in\Omega\times\mathbb{R}^d} d_2(t,x,\xi)$$

$$\delta_1(t)^{1/(m-1)} + \delta_2(t)^{1/(\wp-1)} \leq \mathfrak{K}(1+t)^{\mathfrak{s}/(m-1)}, \ t \geq \bar{t}$$
$$0 \leq \mathfrak{s} \leq m-1, \quad \mathfrak{K} \geq 1$$

$$(P)\begin{cases} u_{tt} + M(\|\mathcal{D}_L u(t,\cdot)\|_2^2)(-\Delta)^L u + N(\|\mathcal{D}_{L-1} u(t,\cdot)\|_2^2)(-\Delta)^{L-1} u \\ \qquad +\mu u + Q(t,x,u,u_t) = f(t,x,u). \end{cases}$$

The most interesting applied problem occurs when $L = 2$, but our results are new also in the more standard case $L = 1$. Problem $(\mathscr{P})$, in a simplified form, has its origin in the canonical model introduced by *Woinowsky–Krieger*, which arises in the dynamic buckling of a hinged extensible beam, subjected to an axial force. The nonlinearities M and N are an approximation, by averaging, of the classical *von Kármán* model, see

- M.M. Cavalcanti, V.N. Domingos Cavalcanti, J.A. Soriano, *Global existence and asymptotic stability for the nonlinear and generalized damped extensible plate equation*, Commun. Contemp. Math., 6 (2004), 705–731.

and references therein for more details.

Recently

- R. Monneau, *Justification of the nonlinear Kirchhoff-Love theory of plates as the application of a new singular inverse method*, Arch. Ration. Mech. Anal. 169 (2003), 1-34.
- R. Monneau, *Some remarks on the asymptotic invertibility of the linearized operator of nonlinear elasticity in the context of the displacement approach*, ZAMM. Z. Angew. Math. Mech. (2006), 1–10.

proved the existence of a solution of the nonlinear Kirchhoff–Love plate model. In particular, we are interested in the strongly damped Kirchhoff–Love model, containing also an intrinsic dissipative term of Kelvin–Voigt type. Global non–existence and a priori estimates for the lifespan of maximal solutions are proved.

The original theory was valid for infinitesimal strains and then was adapted by von Karman to manage also moderate rotations.

- A.E.H. Love, A treatise on the Mathematical Theory of Elasticity - Fourth Edition,Dover Publications, New York, 1944
- P.G. Ciarlet, P. Destuynder, Comput. Methods Appl. Mech. Engrg., 17/18 (1979), 227–258
- P.G. Ciarlet Mathematical elasticity. Vol. I. Three-dimensional elasticity, (Studies in Mathematics and its Applications 20, North-Holland Publishing Co., Amsterdam 1988)
- R. Monneau, Arch. Ration. Mech. Anal., 169 (2003), 1–34

In dimension two, the most usual plate operator is the biharmonic operator and standard simplifications of $(\mathscr{P})$ correspond to various forms of the *Kirchhoff–Love* plate equation, if an internal material damping is added,

$$(KL)\begin{cases} u_{tt} + M(\|\mathcal{D}_L u(t,\cdot)\|_2^2)(-\Delta)^L u + N(\|\mathcal{D}_{L-1} u(t,\cdot)\|_2^2)(-\Delta)^{L-1} u \\ \qquad\qquad\qquad + \mu u + \varrho(t)(-\Delta)^L u_t + Q(t)u_t = f(t,x,u), \end{cases}$$

For these models the external damping depends only on (t,v) and is linear in v, so that for brevity it is simply denoted by $Q(t)v$ and corresponds to the case $m=\wp=2$, $\kappa=0$ in the prototype described above.

$$Q \in C^1(\mathbb{R}_0^+), \qquad Q, -Q' \geq 0 \quad \textbf{in } \mathbb{R}_0^+,$$

which is clearly verified when Q does not depend on t, as it usually happens in literature. Moreover, the above hypothesis corresponds to the linear case of $(\mathscr{P})$, when condition on Q is verified in the whole $\mathbb{R}_0^+$ with $\mathfrak{s}=0$.

$$(KL)\begin{cases} u_{tt} + M(\|\mathcal{D}_L u(t,\cdot)\|_2^2)(-\Delta)^L u + N(\|\mathcal{D}_{L-1} u(t,\cdot)\|_2^2)(-\Delta)^{L-1} u \\ \qquad +\mu u + \varrho(t)(-\Delta)^L u_t + Q(t)u_t = f(t,x,u), \end{cases}$$

It is well known that $\varrho(t)(-\Delta)^L u_t$ represents the internal material damping of *Kelvin–Voigt* type of the body structure, which is always present, even if small, in real material as long as the system vibrates. Also on ϱ we suppose the natural restriction

$$\varrho \in C^1(\mathbb{R}_0^+), \qquad \varrho, -\varrho' \geq 0 \quad \textbf{in} \ \mathbb{R}_0^+,$$

which is automatic in the usually treated standard case in which $\varrho(t) \equiv \varrho_0 > 0$ in $\mathbb{R}_0^+$.

Definition of solution

A solution of the system is $u \in X$:

(A) *Distribution Identity* for all $t \in I$ and $\phi \in X$

$$\langle u_t, \phi \rangle \Big]_0^t = \int_0^t \Big\{ \langle u_t, \phi_t \rangle - \varrho(\tau) \langle \mathcal{D}_L u_t, \mathcal{D}_L \phi \rangle - \langle M(\|\mathcal{D}_L u\|_2^2)(-\Delta)^L u(t,x), \phi \rangle - \langle N(\|\mathcal{D}_{L-1} u\|_2^2)(-\Delta)^{L-1} u(t,x) + \mu u, \phi \rangle + \langle Q(\tau, \cdot, u, u_t) - f(\tau, \cdot, u), \phi \rangle \Big\} d\tau;$$

A solution of the system is $u \in X$:

(A) *Distribution Identity* for all $t \in I$ and $\phi \in X$

$$\langle u_t, \phi \rangle\Big]_0^t = \int_0^t \Big\{ \langle u_t, \phi_t \rangle - \varrho(\tau) \langle \mathcal{D}_L u_t, \mathcal{D}_L \phi \rangle \\ - \langle M(\|\mathcal{D}_L u\|_2^2)(-\Delta)^L u(t,x), \phi \rangle \\ - \langle N(\|\mathcal{D}_{L-1} u\|_2^2)(-\Delta)^{L-1} u(t,x) + \mu u, \phi \rangle \\ + \langle Q(\tau, \cdot, u, u_t) - f(\tau, \cdot, u), \phi \rangle \Big\} d\tau;$$

(B) *Weak Conservation Law*

(*i*) $\mathscr{D}u(t) = \langle Q(t, \cdot, u, u_t), u_t \rangle + \varrho(t) \|\mathcal{D}_L u_t\|_2^2 + \mathscr{F}_t u(t) \in L^1_{\mathrm{loc}}(I)$,

(*ii*) $Eu(t) \le Eu(0) - \int_0^t \mathscr{D}u(\tau)\, d\tau$ per ogni $t \in I$.

$$f(t,x,u) = g(t,x)|u|^{\sigma-2}u + c(x)|u|^{q-2}u \quad \text{source term}$$

$$\boxed{\begin{gathered} 1 \le \sigma < q, \\ 2\gamma < q \le 2_L^* := \frac{2n}{n-2L} \text{ if } n > 2L, \quad 2\gamma < q \text{ if } n \le 2L; \end{gathered}}$$

$$c \in L^\infty(\Omega) \quad \text{non–negative}, \quad \operatorname{ess\,inf}_{\overline{\Omega}} c(x) > 0, \quad \|c\|_\infty > 0;$$

$$g \in C(\mathbb{R}_0^+ \times \Omega) \text{ differentiable in } t \quad \text{and} \quad g_t \in C(\mathbb{R}_0^+ \times \Omega);$$

$$0 \le -g(t,x),\ g_t(t,x) \le h(x) \text{ in } \mathbb{R}_0^+ \times \Omega, \quad h \in L^1(\Omega),$$

$$g(t,\cdot) \in L^{q/(q-p)}(\Omega) \text{ in } \mathbb{R}_0^+$$

the negative contribution in $(f(t,x,u),u)$ perturbs the system againsts the blow up.

The *Distribution Identity* is meaningful provided that $\langle f(t,\cdot,u),\phi\rangle \in L^1_{\rm loc}(I)$ and $\langle Q(t,\cdot,u,u_t),\phi\rangle \in L^1_{\rm loc}(I)$, along the field $\phi \in X$. The first condition is clearly in charge for our prototype, while the latter is assumed. The other terms in the *Distribution Identity* are well defined thanks to the choice of the space X.
In general it is important to consider weak solutions instead of *strong* solutions, namely functions $u \in X$ satisfying (A), (B)–(i), with (B)–(ii) replaced by the *Strong Energy Conservation* (B)$_s$–(ii), that is $Eu(t) = Eu(0) - \int_0^t \mathscr{D}u(\tau)d\tau$ for all $t \in I$. The main reason was first given in Remark 4 at page 199 of [PS1]

[CZ] **W. Chen, Y. Zhou**, *Global nonexistence for a semilinear Petrovsky equation*, Nonlinear Anal., 70 (2009), 3203-3208.

[LS] **H.A. Levine, J. Serrin**, *Global nonexistence theorems for quasilinear evolution equations with dissipation*, Arch. Rational Mech. Anal., 137 (1997), 341–361.

[PS1] **P. P., J. Serrin**, *Asymptotic stability for non–autonomous dissipative wave systems*, Comm. Pure Appl. Math., 49 (1996), 177-216.

[PS2] **P. P., J. Serrin**, *Local asymptotic stability for dissipative wave systems*, Israel J. Math., 104 (1998), 29-50.

[V1] **E. Vitillaro**, *Global nonexistence theorems for a class of evolution equations with dissipation*, Arch. Ration. Mech. Anal., 149 (1999), 155-182.

[WT1] S.T. Wu, L.Y. Tsai, *Blow–up of solutions for some non–linear wave equations of Kirchhoff type with some dissipation*, Nonlinear Anal., 65 (2006), 243-264.

[WT2] S.T. Wu, L.Y. Tsai, *On a system of nonlinear wave equations of Kirchhoff type with a strong dissipation*, Tamkang J. Math., 38 (2007), 1-20.

[WT3] S.T. Wu, L.Y. Tsai, *Existence and nonexistence of global solutions for a nonlinear wave equation*, Taiwanese J. Math., 13 (2009), 2069-2091.

see also Remark 2 at page 49 of [PS2] and the discussion on page 345 [LS]. Of course, if u is a strong solution, then Eu *is non–increasing in* I and this makes the analysis much simpler, see [CZ], [V1] and [WT1]–[WT3].

$$w_1 = \inf_{t \in I} \mathscr{A} u(t), \quad w_2 = \inf_{t \in I} \mathscr{F} u(t)$$

$$E_1 = \left(1 - \frac{2\gamma}{q}\right) w_1, \quad E_2 = \left(\frac{q}{2\gamma} - 1\right) w_2$$

$$w_1 = \inf_{t\in I} \mathscr{A}u(t), \quad w_2 = \inf_{t\in I} \mathscr{F}u(t)$$

$$E_1 = \left(1 - \frac{2\gamma}{q}\right) w_1, \quad E_2 = \left(\frac{q}{2\gamma} - 1\right) w_2$$

Proposition 1

Let u be a solution of either (P) or $(\mathcal{KL})$. Then $\mathscr{A}u$ and $\mathscr{F}u$ are bounded below in I.

(i) If u is a solution of (P), then

$$Eu(0) < E_2 \implies w_2 > 0.$$

(ii) If u is a solution of $(\mathcal{KL})$, then

$$Eu(0) < E_1 \implies w_2 > 0.$$

Let u be as in the statement. Recall that

$$\mathscr{A}u(t) = \frac{1}{2}\left\{\mathscr{M}(\|\mathcal{D}_L u(t,\cdot)\|_2^2) + \mathscr{N}(\|\mathcal{D}_{L-1} u(t,\cdot)\|_2^2) + \mu\|u(t,\cdot)\|_2^2\right\},$$

so that $w_1 \geq 0$.

Let u be as in the statement. Recall that

$$\mathscr{A}u(t) = \frac{1}{2}\left\{\mathscr{M}(\|\mathcal{D}_L u(t,\cdot)\|_2^2) + \mathscr{N}(\|\mathcal{D}_{L-1}u(t,\cdot)\|_2^2) + \mu\|u(t,\cdot)\|_2^2\right\},$$

so that $w_1 \geq 0$.

Moreover, using the definition of E and (B)–(ii), we get

$$\mathscr{F}u(t) \geq w_1 - Eu(0) \geq -Eu(0) \quad \text{for all}, t \in I.$$

Thus

$$w_2 \geq w_1 - Eu(0) > -\infty.$$

Case (i) – u solves (P). Using the definition of E we get

$$\mathscr{F}u(t) \geq w_1 - Eu(0) \quad \text{for all } t \in I.$$

Thus $w_2 \geq w_1 - Eu(0) > w_1 - E_2$ and so

$$w_2 > 2\gamma w_1/q > 0.$$

Case (*i*) – u solves (P). Using the definition of E we get

$$\mathscr{F}u(t) \geq w_1 - Eu(0) \quad \text{for all } t \in I.$$

Thus $w_2 \geq w_1 - Eu(0) > w_1 - E_2$ and so

$$w_2 > 2\gamma w_1/q > 0.$$

Case (*ii*) – u solves ($\mathcal{KL}$). Similarly, we get

$$w_2 \geq w_1 - Eu(0) > w_1 - E_1 = 2\gamma w_1/q > 0.$$

Theorem 1

There are no global solutions u of (P) such that

$$Eu(0) < E_2.$$

Theorem 1

There are no global solutions u of (P) such that

$$Eu(0) < E_2.$$

Remarks. The source term f satisfies:

$\exists\, c_1 = c_1(w_2, u) > 0$ *and* $\varepsilon_0 = \varepsilon_0(w_2, u) > 0$ *such that*

$$(i) \qquad \mathscr{F}u(t) \le c_1 \|u(t,\cdot)\|_q^q \quad \textit{for all } t \in I,$$

and $\forall\, \varepsilon \in (0, \varepsilon_0]\, \exists\, c_2 = c_2(w_2, u, \varepsilon) > 0$ *such that*

$$(ii) \quad \langle f(t,\cdot,u(t,\cdot)), u(t,\cdot)\rangle - (q-\varepsilon)\mathscr{F}u(t) \ge c_2 \|u(t,\cdot)\|_q^q$$

for all $t \in I$.

The main result for (P)

When $Eu(0) \leq 0$, so that clearly $Eu(0) < E_2$, we can take $\varepsilon = \varepsilon_0 = q - 2\gamma$ and *the above condition somehow reduces to the pioneering assumption given in*

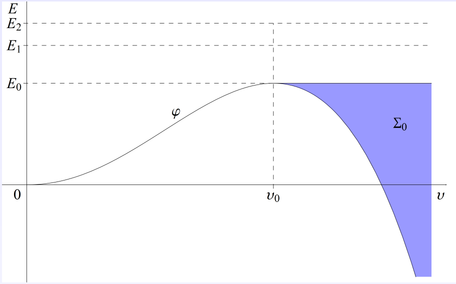
- **H.A. Levine, J. Serrin, Arch. Rational Mech. Anal. 137 (1997) 341–361**

- H.A. Levine, J. Serrin, Arch. Rational Mech. Anal. 137 (1997) 341–361.
- E. Vitillaro, Arch. Ration. Mech. Anal. 149 (1999) 155–182
- S.T. Wu , L.Y. Tsai, Nonlinear Anal. 65 (2006) 243–264

In the above papers the main geometry structure implies

$$Eu(0) < E_1.$$

In this case, $w_2 \geq w_1 - Eu(0) > w_1 - E_1 = 2\gamma w_1/q$ and so $E_2 > E_1$.
From Proposition 1–(i), it follows that if u is a solution of (P) such that $w_2 \leq 0$, then $Eu(0) \geq E_2$. Hence, being $w_1 \leq Eu(0) + w_2 \leq Eu(0)$, the case $Eu(0) < E_1$ can never occur, since $E_1 < w_1$.



$$w_1 = \inf_{t \in I} \mathscr{A} u(t), \quad w_2 = \inf_{t \in I} \mathscr{F} u(t)$$

$$E_1 = \left(1 - \frac{2\gamma}{q}\right) w_1, \quad E_2 = \left(\frac{q}{2\gamma} - 1\right) w_2$$

$$\varphi(v_0) = \left(1 - \frac{2\gamma}{q}\right) w_0 = E_0 > 0, \quad w_0 = \frac{s v_0^{2\gamma}}{(2\mathcal{S}_q^2)^\gamma} > 0$$

$$\Sigma_0 = \{(v, E) \in \mathbb{R}^2 : v > v_0, \; E < E_0\}, \quad v(t) = \|u(t, \cdot)\|_q$$

Lemma 1

Put $v(t) = \|u(t,\cdot)\|_q$. *If* $Eu(0) < E_0$ *then* $v_0 \notin \overline{v(I)}$ *and* $w_1 \neq w_0$. *Moreover, the following conditions are equivalent:*

(i) $w_1 > w_0$;

(ii) $\overline{v(I)} \subset (v_0, \infty)$;

(iii) $w_2 > 2\gamma w_0/q$.

Finally, if one of (i)–(iii) *holds, then* $E_0 < E_1 < E_2$.

In particular, if $v(0) > v_0$ *and* $Eu(0) < E_0$, *then*

$$(v(t), Eu(t)) \in \Sigma_0 \quad \text{for all } t \in I,$$

properties (i)–(iii) *hold*, $E_0 < E_1 < E_2$ *and* $w_2 > 2\gamma w_1/q$.

By contradiction. **Suppose** $v_0 \in \overline{v(I)}$.

Hence *there exists a sequence* $(t_j)_j \subset I$ **such that**

$$\lim_j v(t_j) = v_0.$$

Sketch of the proof

By contradiction. Suppose $v_0 \in \overline{v(I)}$.

Hence *there exists a sequence* $(t_j)_j \subset I$ such that

$$\lim_j v(t_j) = v_0.$$

Put $s = b$ if $b = 0$, otherwise $s = a$. It is

$$Eu(t) \geq \varphi(v(t)) = \frac{s}{(2\mathcal{S}_q^2)^\gamma} v(t)^{2\gamma} - \frac{c_\infty}{q} v(t)^q \quad \text{for all } t \in I,$$

where $\mathcal{S}_q = \mathcal{S}_q(d, L, \Omega)$ derives from $H_0^L(\Omega) \hookrightarrow L^q(\Omega)$.

By contradiction. Suppose $v_0 \in \overline{v(I)}$.

Hence *there exists a sequence* $(t_j)_j \subset I$ such that

$$\lim_j v(t_j) = v_0.$$

Put $s = b$ if $b = 0$, otherwise $s = a$. It is

$$Eu(t) \geq \varphi(v(t)) = \frac{s}{(2\mathcal{S}_q^2)^\gamma} v(t)^{2\gamma} - \frac{c_\infty}{q} v(t)^q \quad \text{for all } t \in I,$$

where $\mathcal{S}_q = \mathcal{S}_q(d, L, \Omega)$ derives from $H_0^L(\Omega) \hookrightarrow L^q(\Omega)$.

Thus $E_0 > Eu(0) \geq Eu(t_j) \geq \varphi(v(t_j))$ and so $E_0 > E_0$ by the continuity of $\varphi \circ v$.
This contradiction proves the claim.

Next, we show that $w_1 \neq w_0$.

Next, we show that $w_1 \neq w_0$.

By contradiction. It results

$$w_1 = w_0 \quad \Rightarrow \quad \mathscr{A}u(t) \geq w_1 = w_0 \quad \text{for all } t \in I$$

Therefore,

$$\begin{aligned}\mathscr{A}u(t) - \frac{2\gamma s}{q(2\mathcal{S}_q^2)^\gamma} v(t)^{2\gamma} &\geq \left(1 - \frac{2\gamma}{q}\right)\mathscr{A}u(t) \\ &\geq E_1 = E_0 > Eu(0) \\ &\geq \mathscr{A}u(t) - \frac{c_\infty}{q} v(t)^q,\end{aligned}$$

so that

$$v(t) > v_0 \quad \text{for all } \ t \in I.$$

Next, we show that $w_1 \neq w_0$.

By contradiction. It results

$$w_1 = w_0 \quad \Rightarrow \quad \mathscr{A}u(t) \geq w_1 = w_0 \quad \text{for all } t \in I$$

Therefore,

$$\begin{aligned} \mathscr{A}u(t) - \frac{2\gamma s}{q(2\mathcal{S}_q^2)^\gamma} v(t)^{2\gamma} &\geq \left(1 - \frac{2\gamma}{q}\right) \mathscr{A}u(t) \\ &\geq E_1 = E_0 > Eu(0) \\ &\geq \mathscr{A}u(t) - \frac{c_\infty}{q} v(t)^q, \end{aligned}$$

so that

$$v(t) > v_0 \quad \text{for all } \; t \in I.$$

Consequently, the first part yields $\overline{v(I)} \subset (v_0, \infty)$.

On the other hand, *there exists a sequence* $(t_j)_j$ such that $\lim_j \mathscr{A} u(t_j) = w_1 = w_0$. Thus

$$\limsup_{j\to\infty} v(t_j) \le \lim_{j\to\infty} [(2\mathcal{S}_q^2)^\gamma \mathscr{A} u(t_j)/s]^{1/2\gamma} = v_0,$$

which *contradicts* the fact that $\overline{v(I)} \subset (v_0, \infty)$.

Hence $w_1 \neq w_0$.

It remains to prove the equivalence of (i)–(iii).

It remains to prove the equivalence of (i)–(iii).

$(i) \Rightarrow (ii)$. It is enough to prove that $v(I) \subset (v_0, \infty)$. In this case $w_1 > w_0$, so that $Eu(0) < E_1$ and

$$\mathscr{A}u(t) \geq \frac{s}{(2\mathcal{S}_q^2)^\gamma} v(t)^{2\gamma} \quad \text{for all}\, t \in I.$$

It remains to prove the equivalence of (i)–(iii).

$(i) \Rightarrow (ii)$. It is enough to prove that $v(I) \subset (v_0, \infty)$. In this case $w_1 > w_0$, so that $Eu(0) < E_1$ and

$$\mathscr{A}u(t) \geq \frac{s}{(2\mathcal{S}_q^2)^\gamma} v(t)^{2\gamma} \quad \text{for all } t \in I.$$

$(ii) \Rightarrow (iii)$. In this case $v(t) > v_0$ for all $t \in I$. Hence,

$$\mathscr{F}u(t) \geq w_0 - Eu(0) > w_0 - E_0 = 2\gamma w_0/q \quad \text{for all } t \in I$$

and in turn

$$w_2 > 2\gamma w_0/q.$$

$(iii) \Rightarrow (i)$. It results

$$\frac{c_\infty}{q} v(t)^q \geq \mathscr{F} u(t) \geq w_2 > \frac{2\gamma}{q} w_0 = \frac{c_\infty}{q} v_0^q$$

which implies that

$$v(t) > v_0 \quad \text{for all}\, t \in I.$$

Combining the last relation with

$$\mathscr{A} u(t) \geq \frac{s}{(2S_q^2)^\gamma} v(t)^{2\gamma} \quad \text{for all}\, t \in I,$$

we get $w_1 \geq w_0$. Hence, $w_1 > w_0$, being $w_1 \neq w_0$.

Finally, if one of (i)–(iii) holds then $E_0 < E_1$.

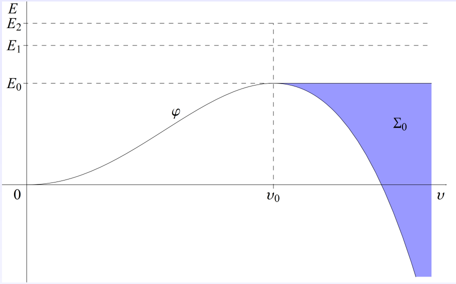
Furthermore,

$$\mathscr{F}u(t) \geq w_1 - Eu(0) > w_1 - E_1 = 2\gamma w_1/q, \quad t \in I.$$

This gives $w_2 > 2\gamma w_1/q$ and so $E_1 < E_2$. In conclusion

$$E_0 < E_1 < E_2$$

as claimed.



$$w_1 = \inf_{t\in I} \mathscr{A} u(t), \quad w_2 = \inf_{t\in I} \mathscr{F} u(t)$$

$$E_1 = \left(1 - \frac{2\gamma}{q}\right) w_1, \quad E_2 = \left(\frac{q}{2\gamma} - 1\right) w_2$$

$$\varphi(v_0) = \left(1 - \frac{2\gamma}{q}\right) w_0 = E_0 > 0, \quad w_0 = \frac{s v_0^{2\gamma}}{(2\mathcal{S}_q^2)^\gamma} > 0$$

$$\Sigma_0 = \{(v, E) \in \mathbb{R}^2 \, : \, v > v_0, \; E < E_0\}$$

$(\mathscr{D})$ There exists $t_* \in (0,T)$ such that one of the following properties holds

(i) $\phi \in X$ and $\langle Q(t,\cdot,\phi,\phi_t),\phi_t\rangle = 0$ in $[0,t_*]$
implies either $\phi(t,\cdot) \equiv 0$ or $\phi_t(t,\cdot) \equiv 0$ for all $t \in [0,t_*]$,

(ii) $g_t(t,x) \geq g_0(t) > 0$ for all $(t,x) \in [0,t_*) \times \Omega$,

(iii) $\varrho(t) > 0$ for all $[0,t_*]$.

$(\mathscr{D})$ There exists $t_* \in (0,T)$ such that one of the following properties holds

(i) $\phi \in X$ and $\langle Q(t,\cdot,\phi,\phi_t),\phi_t\rangle = 0$ in $[0,t_*]$
implies either $\phi(t,\cdot) \equiv 0$ or $\phi_t(t,\cdot) \equiv 0$ for all $t \in [0,t_*]$,

(ii) $g_t(t,x) \geq g_0(t) > 0$ for all $(t,x) \in [0,t_*) \times \Omega$,

(iii) $\varrho(t) > 0$ for all $[0,t_*]$.

Lemma 2

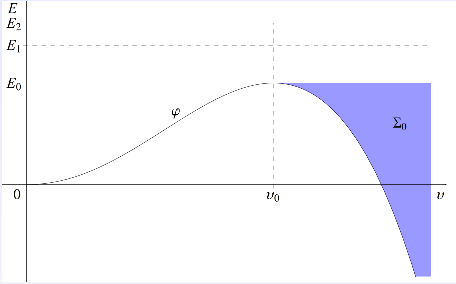
Assume $(\mathscr{D})$. If $v(0) > v_0$ and $Eu(0) = E_0$, then

$$(v(t), Eu(t)) \in \Sigma_0 \quad \textit{for all } t \in I.$$

In particular, $w_1 > w_0$, $w_2 > 2\gamma w_1/q$ and

$$E_0 < E_1 < E_2 .$$

Main picture



$$w_1 = \inf_{t\in I} \mathscr{A}u(t), \quad w_2 = \inf_{t\in I} \mathscr{F}u(t)$$

$$E_1 = \left(1 - \frac{2\gamma}{q}\right) w_1, \quad E_2 = \left(\frac{q}{2\gamma} - 1\right) w_2$$

$$\varphi(v_0) = \left(1 - \frac{2\gamma}{q}\right) w_0 = E_0 > 0, \quad w_0 = \frac{s v_0^{2\gamma}}{(2\mathcal{S}_q^2)^\gamma} > 0$$

$$\Sigma_0 = \{(v, E) \in \mathbb{R}^2 \,:\, v > v_0,\ E < E_0\}$$

$(\mathscr{D})$ There exists $t_* \in (0,T)$ such that one of the following properties holds

(i) $\phi \in X$ and $\langle Q(t,\cdot,\phi,\phi_t),\phi_t\rangle = 0$ in $[0,t_*]$ implies either $\phi(t,\cdot)\equiv 0$ or $\phi_t(t,\cdot)\equiv 0$ for all $t\in[0,t_*]$,

(ii) $g_t(t,x) \geq g_0(t) > 0$ for all $(t,x)\in[0,t_*)\times\Omega$,

(iii) $\varrho(t) > 0$ for all $[0,t_*]$.

THEOREM 2

The are no global solutions u of (P) in $\mathbb{R}_0^+ \times \Omega$, such that

$$\|u(0,\cdot)\|_q > v_0, \qquad Eu(0) \leq E_0,$$

if $(\mathscr{D})$ holds when $Eu(0) = E_0$.

We refine an argument introduced in

- P. Pucci, J. Serrin, J. Diff. Equations 150 (1998) 203–214

for evolution systems with linear damping terms, and *combine in a new way the classical potential well and concavity methods* used for the wave case in

- H.A. Levine, P. Pucci, J. Serrin, Contemporary Math. 208 (1997) 253–263
- H.A. Levine, J. Serrin, Arch. Rational Mech. Anal. 137 (1997) 341–361
- E. Vitillaro, Arch. Rational Mech. Anal. 149 (1999) 155–182

Sketch of the proof

Proof. *By contradiction.* Let u be a global solution of (P) such that

$$\|u(0,\cdot)\|_q > v_0, \qquad Eu(0) < E_0.$$

Proof. *By contradiction.* Let u be a global solution of (P) such that

$$\|u(0,\cdot)\|_q > v_0, \qquad Eu(0) < E_0.$$

By Lemma 1, $v(0) \notin \overline{v(\mathbb{R}_0^+)}$, so that, $\overline{v(\mathbb{R}_0^+)} \subset (v_0, \infty)$, being $v(0) > v_0$. Hence $w_2 > 2\gamma w_0/q$ again *by Lemma 1–(iii)*. Thus $Eu(0) < E_0 < E_2$ and the *contradiction follows* at once by an *application of Theorem 1*.

Proof. *By contradiction.* Let u be a global solution of (P) such that

$$\|u(0,\cdot)\|_q > v_0, \qquad Eu(0) < E_0.$$

By Lemma 1, $v(0) \notin \overline{v(\mathbb{R}_0^+)}$, so that, $\overline{v(\mathbb{R}_0^+)} \subset (v_0, \infty)$, being $v(0) > v_0$. Hence $w_2 > 2\gamma w_0/q$ again *by Lemma 1–(iii)*. Thus $Eu(0) < E_0 < E_2$ and the *contradiction follows* at once by an *application of Theorem 1*.

Assume now also $(\mathscr{D})$ and let u be a global solution of (P) such that

$$\|u(0,\cdot)\|_q > v_0, \qquad Eu(0) = E_0.$$

By Lemma 2 we have that $Eu(0) < E_2$, and *we conclude using Theorem 1.*

- S.T. Wu, L.Y. Tsai, Nonlinear Anal., 65 (2006) 243–264

  $\rightsquigarrow$ $L = 1$, $M \geq a > 0$, $N \equiv 0$, $\varrho \equiv 0$ and $\mu = 0$, $f = f(u)$ and $Q = Q(u_t)$.

- S.A. Messaoudi, B. Said Houari, Appl. Math. Lett., 20 (2007), 866–871

  $\rightsquigarrow$ $N \equiv 0$, $\varrho \equiv 0$ and $\mu = 0$, f and Q special.

- W. Chen, Y. Zhou, Nonlinear Anal., 70 (2009), 3203–3208

  $\rightsquigarrow$ $L = 2$, $M \equiv 1$, $N \equiv 0$, $\varrho \equiv 0$ and $\mu = 0$, f and Q special.

Lifespan estimates for (P)

Take Q as in the prototype. Moreover suppose that $\exists\, k \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+_0)$, $\quad k' \geq 0$ in $\mathbb{R}^+$, $k_0 = k(0) > 0$, verifying

$$\delta_1^{1/(m-1)} + \delta_2^{1/(\wp-1)} \leq k \quad \text{in } \mathbb{R}^+_0$$

and

$$\int^{\infty} k(t)^{-(1+\theta)} dt = \infty, \qquad \theta \in (0, \theta_0],$$

$$\theta_0 = \min\left\{\frac{q-2}{q+2},\ \frac{\bar r}{1-\bar r}\right\},\ \bar r = \frac{1}{\wp} - \frac{1}{q} \in (0,1).$$ Put

$$C_1 = \min\left\{1, \left(\frac{2\gamma w_0}{c_\infty}\right)^{1/q}\right\} \quad \text{and} \quad C_2 = \left(\frac{c_\infty}{2\gamma}\right)^{\bar r} C_1^{\alpha_1-\alpha_2},$$

where $c_\infty = \|c\|_\infty$ and w_0 is defined above.

Theorem 3

Let u be a solution of (P) such that

$$\|u_0\|_q > v_0 \quad \textit{and} \quad Eu(0) < E_0.$$

Consider the numbers

$$\mathscr{H}_0 = E_0 - Eu(0) > 0,$$

$$\varepsilon_0 = \min\{q-p,\, q-2\gamma - \frac{q[Eu(0)]^+}{w_0}\} > 0,$$

$$\mathfrak{L} = \max\left\{1, \frac{4qC_2}{\overline{c}\varepsilon_0}\mathscr{H}_0^{-\overline{r}}\right\}, \quad \overline{r} = \frac{1}{\wp} - \frac{1}{q}, \quad r = \frac{\theta}{1+\theta},$$

$$\lambda = \max\left\{(1+\theta)C_2\mathfrak{L}^{m'/m}\mathscr{H}_0^{r-\overline{r}}, \frac{2\langle u_0, u_1\rangle^-}{k_0\mathscr{H}_0^{1/(1+\theta)}}, \frac{1}{k_0}\right\},$$

$$\mathscr{Z}_0 = \lambda k_0 \mathscr{H}_0^{1/(1+\theta)} + \langle u_0, u_1\rangle,$$

$$K = \frac{2^{2-\alpha}\gamma\min\{1, \overline{c}\varepsilon_0/2q\}}{2\gamma\max\{1, |\Omega|^{(q-2)(1+\theta)/q(1-\theta)}C_1^{2-q}\} + c_\infty}.$$

Then $T \leq T_0$, *where*

$$\int_0^{T_0} k(t)^{-(1+\theta)}dt = \frac{\lambda}{\theta K}\left(\frac{\lambda}{\mathscr{Z}_0}\right)^\theta.$$

$$\mathscr{Z}(t) = \lambda k(t)\left[\mathscr{H}(t)\right]^{1-r} + \langle u(t,\cdot), u_t(t,\cdot)\rangle,$$

$$\mathscr{H}(t) = \mathscr{H}_0 + \int_0^t \mathscr{D}u(\tau)d\tau,$$

$$\mathscr{D}u(t) = \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u_t(t,\cdot)\rangle + \mathscr{F}_t u(t).$$

• The request $\lambda \geq 2\langle u_0, u_1\rangle^- / k_0 \mathscr{H}_0^{1/(1+\theta)}$ guarantees that $\mathscr{Z}_0 > 0$, in the more subtle case $\langle u_0, u_1\rangle < 0$.
On the other hand, if $\langle u_0, u_1\rangle \geq 0$, then $\mathscr{Z}_0 > 0$, being $\lambda > 0$. Hence, for cooperative data the above condition on λ simplifies.

• In the limit case $Eu(0) = E_0$ the expression of $\varepsilon_0 = 0$.
The above theorem provides global non–existence of solutions of (P), but it does not by itself establish that maximal solutions blow up at the lifespan T.
In general global non–existence does not imply the blow up of the solution.
It is possible for the solution to leave the domain of one of the differential operators in the equations before becoming unbounded.

On the other hand, if one can couple the global non–existence with a local continuation argument, then global non–existence will imply finite time blow up.

- H.A. Levine, Trans. Amer. Math. Soc., 192 (1974), 138–146
- J. Ball, Quart. J. Math. Oxford, 28 (1977), 473–486
- V. Gheorgiev, G. Todorova, J. Differential Equations, 109 (1994), 295–308

Global non–existence occurs by blow up when

$$\text{either} \quad T = T_0 \quad \text{or} \quad \lim_{t\to T^-} \mathscr{Z}(t) = \infty,$$

$$\mathscr{Z}(t) = \lambda k(t)\,[\mathscr{H}(t)]^{1-r} + \langle u(t,\cdot), u_t(t,\cdot)\rangle,$$

$$\mathscr{H}(t) = \mathscr{H}_0 + \int_0^t \mathscr{D}u(\tau)d\tau,$$

$$\mathscr{D}u(t) = \langle Q(t,\cdot,u(t,\cdot),u_t(t,\cdot)), u_t(t,\cdot)\rangle + \mathscr{F}_t u(t).$$

Corollary

Under the assumptions of the above theorem, if either $T = T_0$ *or* $\lim_{t\to T^-} \mathscr{Z}(t) = \infty$*, then*

$$\lim_{t\to T^-} \|u(t,\cdot)\|_q = \infty \quad \text{and} \quad \lim_{t\to T^-} \|\mathcal{D}_L u(t,\cdot)\|_2 = \infty.$$

Proof. From the proof of the main Theorem it results that

$$\mathscr{Z}'(t) > 0 \quad \text{for} \quad t \text{ sufficiently large},$$

and so

$$\exists \quad \lim_{t \to T^-} \mathscr{Z}(t).$$

Proof. From the proof of the main Theorem it results that

$$\mathscr{L}'(t) > 0 \quad \text{for} \quad t \text{ sufficiently large},$$

and so

$$\exists \quad \lim_{t \to T^-} \mathscr{L}(t).$$

In both the two cases

$$\lim_{t \to T^-} \mathscr{L}(t) = \infty.$$

Now

$$\mathscr{H}_0 \leq \mathscr{H}_0 + Eu(0) - \tfrac{1}{2}\|u_t(t,\cdot)\|_2^2 + \mathscr{F}u(t), \quad t \in I$$

which gives

$$\|u_t(t,\cdot)\|_2^2 \leq 2[Eu(0) + \mathscr{F}u(t)] < 2[E_0 + \mathscr{F}u(t)].$$

Moreover,

$$\mathscr{H}(t) \leq q\mathscr{F}u(t)/2\gamma \leq c_\infty \|u(t,\cdot)\|_q^q/2\gamma.$$

Consequently

$$\mathscr{Z}(t)^\alpha \leq C[\lambda k(T)]^\alpha \left\{ \left(\frac{E_0}{\gamma w_0} + \frac{2}{q} \right) c_\infty + 1 \right\} \|u(t,\cdot)\|_q^q, \ \alpha > 1,$$

with $C > 0$ appropriate constant. Hence

$$\|u(t,\cdot)\|_q^q \geq \Lambda \mathscr{Z}(t)^\alpha, \quad \Lambda > 0$$

and so

$$\lim_{t \to T^-} \|u(t,\cdot)\|_q = \infty.$$

By the above corollary:
$\lim_{t\to T^-} \mathscr{Z}(t) < \infty \quad \Rightarrow \quad T < T_0$, but it could be

$$\limsup_{t\to T^-} \|\mathcal{D}_L u(t,\cdot)\|_2 < \infty.$$

In this case, or even when $\liminf_{t\to T^-} \|\mathcal{D}_L u(t,\cdot)\|_2 < \infty$ we have

$$\lim_{t\to T^-} \mathscr{H}(t) < \infty.$$

Otherwise, $\lim_{t \to T^-} \mathscr{H}(t) = \infty$ and so $\lim_{t \to T^-} \|u(t, \cdot)\|_q = \infty$, being

$$\mathscr{H}(t) \leq \frac{c_\infty \|u(t, \cdot)\|_q^q}{2\gamma}.$$

But this is impossible, being q subcritical.

Otherwise, $\lim_{t\to T^-} \mathscr{H}(t) = \infty$ and so $\lim_{t\to T^-} \|u(t,\cdot)\|_q = \infty$, being

$$\mathscr{H}(t) \leq \frac{c_\infty \|u(t,\cdot)\|_q^q}{2\gamma}.$$

But this is impossible, being q subcritical.

Therefore, $\mathscr{D}u \in L^1(I)$, $I = [0,T)$, i.e.

the total damping over the entire time interval I is finite

For t sufficiently large

$$Q(t,x,u,v) = d_1(t,x)|u|^{\kappa}|v|^{m-2}v + d_2(t,x,u)|v|^{\wp-2}v$$

$$\wp,\, m > 1, \quad m+\kappa \le \wp < q, \quad \kappa \ge 0$$

$$\delta_1(t) = \|d_1(t,\cdot)\|_{\wp_1}, \;\; \delta_2(t) = \sup_{(x,\xi)\in\Omega\times\mathbb{R}^d} d_2(t,x,\xi)$$

$$\delta_1(t)^{1/(m-1)} + \delta_2(t)^{1/(\wp-1)} \le \mathfrak{K}(1+t)^{\mathfrak{s}/(m-1)}, \; t \text{ large}$$

$$\boxed{0 < \mathfrak{s} < m-1} \quad \text{and} \quad \mathfrak{K} \ge 1$$

Corollary

Let u be a solution of (P) with $\|u_0\|_q > v_0$ and $Eu(0) < E_0$. Then $T \leq T_0$ where

$$T_0 = \begin{cases} \dfrac{\lambda\mathfrak{K}}{\theta K}\left(\dfrac{\lambda\mathfrak{K}}{\mathscr{Z}_0}\right)^{\theta} = \Theta_0, & \mathfrak{s} = 0, \quad 0 < \theta \leq \theta_0, \\ e^{\Theta_0} - 1, & \mathfrak{s} > 0, \quad \theta = \dfrac{m - 1 - \mathfrak{s}}{\mathfrak{s}}, \\ (\mathfrak{m}\Theta_0 + 1)^{1/\mathfrak{m}} - 1, & \mathfrak{s} > 0, \quad \theta < \dfrac{m - 1 - \mathfrak{s}}{\mathfrak{s}}, \end{cases}$$

where $\mathfrak{m} = [m - 1 - \mathfrak{s}(1 + \theta)]/(m - 1) > 0$ and λ and $\mathscr{Z}_0$ are as above.

- S.T. Wu, L.Y. Tsai Nonlinear Anal., 65 (2006), 243–264. $\rightsquigarrow Q(t) \equiv 1$.

Sketch of the proof

Proof. If $\mathfrak{s} = 0$, it is enough to apply formula

$$\int_0^{T_0} k(t)^{-(1+\theta)} dt = \frac{\lambda}{\theta K}\left(\frac{\lambda}{\mathscr{Z}_0}\right)^{\theta}$$

in Theorem 3.

Proof. If $\mathfrak{s} = 0$, it is enough to apply formula

$$\int_0^{T_0} k(t)^{-(1+\theta)} dt = \frac{\lambda}{\theta K} \left(\frac{\lambda}{\mathscr{Z}_0} \right)^{\theta}$$

in Theorem 3.

Consider now the case $0 < \mathfrak{s} < m - 1$. When $0 < \theta \le \min\{\theta_0, \dfrac{m-1-\mathfrak{s}}{\mathfrak{s}}\}$, we get

$$\int^{\infty} k(t)^{-(1+\theta)} dt = \infty.$$

Apply again Theorem 3 and derive the estimates from the formula above.

Clearly, the Corollary above covers all the cases when $Q(t, x, u, u_t) = |u_t|^{m-2}u_t$, as it usually happens in the literature. In particular, it extends the recent Theorems 4.3 and 4.5 of [WT1], in which the authors consider a simplified version of $(\mathscr{P})$ with, in our notations, $L = 1$, $\mu = 0$, $g \equiv 0$ and $c \equiv 1$, and the nonlinear damping function Q as in the Corollary, with $d_1 \equiv 0$ and $d_2 \equiv$ Const.

$$(KL)\begin{cases} u_{tt} + M(\|\mathcal{D}_L u(t,\cdot)\|_2^2)(-\Delta)^L u + N(\|\mathcal{D}_{L-1} u(t,\cdot)\|_2^2)(-\Delta)^{L-1} u \\ \qquad\qquad +\mu u + \varrho(t)(-\Delta)^L u_t + Q(t)u_t = f(t,x,u), \\ \\ D^\alpha u(t,x) = 0, \quad |\alpha| \le L-1 \qquad\qquad \text{on} \quad \mathbb{R}_0^+ \times \partial\Omega, \end{cases}$$

Kirchhoff–Love theory: deformations of thin plates subjected to forces and moments. Extension of the Euler–Bernoulli theory, developed by Love under assumptions proposed by Kirchhoff.

The original theory was valid for infinitesimal strains and then was adapted by von Karman to manage also moderate rotations.

- A.E.H. Love, A treatise on the Mathematical Theory of Elasticity - Fourth Edition,Dover Publications, New York, 1944
- P.G. Ciarlet, P. Destuynder, Comput. Methods Appl. Mech. Engrg., 17/18 (1979), 227–258
- P.G. Ciarlet Mathematical elasticity. Vol. I. Three-dimensional elasticity, (Studies in Mathematics and its Applications 20, North-Holland Publishing Co., Amsterdam 1988)
- R. Monneau, Arch. Ration. Mech. Anal., 169 (2003), 1–34

Denote $u_0 = u(0,\cdot)$ and $u_1 = u_t(0,\cdot)$.

Theorem

Let u a solution of (KL) in $[0,T)\times\Omega$ such that

$$\|u_0\|_q > v_0 \qquad \text{and} \qquad Eu(0) \le E_0,$$

and assume $(\mathscr{D})$ if $Eu(0) = E_0$. Moreover, take $\alpha_0 = 2[Eu(0) - E_0] \ge 0$ and $\beta_0 \ge 0$ such that

$$\begin{cases} \langle u_0, u_1\rangle + \alpha_0\beta_0 > \dfrac{1}{\gamma - 1}\left[Q(0)\|u_0\|_2^2 + \varrho(0)\|\mathcal{D}_L u_0\|_2^2\right], & \gamma > 1, \\[2ex] \langle u_0, u_1\rangle + \alpha_0\beta_0 > \dfrac{2}{q - 2}\left[Q(0)\|u_0\|_2^2 + \varrho(0)\|\mathcal{D}_L u_0\|_2^2\right], & \gamma = 1. \end{cases}$$

Then $T \le T_0$, where

$$T_0 = \begin{cases} \dfrac{\|u_0\|_2^2 + \alpha_0\beta_0^2}{(\gamma-1)[\langle u_0, u_1\rangle + \alpha_0\beta_0] - \left[Q(0)\|u_0\|_2^2 + \varrho(0)\|\mathcal{D}_L u_0\|_2^2\right]} \\ \text{if } \gamma > 1, \\ \dfrac{2\|u_0\|_2^2 + 2\alpha_0\beta_0^2}{(q-2)[\langle u_0, u_1\rangle + \alpha_0\beta_0] - 2\left[Q(0)\|u_0\|_2^2 + \varrho(0)\|\mathcal{D}_L u_0\|_2^2\right]} \\ \text{if } \gamma = 1. \end{cases}$$

N.B. $(\mathscr{D})$ is valid if either $Q(0) > 0$ or $\varrho(0) > 0$.

- M. Guedda, H. Labani, Bull. Belg. Math. Soc. Simon Stevin, 9 (2002), 39–46
- S.A. Messaoudi, B. Said Houari, Appl. Math. Lett., 20 (2007), 866–871
- S.T. Wu, L.Y. Tsai, Taiwanese J. Math., 13 (2009), 2069–2091

THANK YOU

FOR YOUR KIND ATTENTION!