Geometric aspects in competition-diffusion problems

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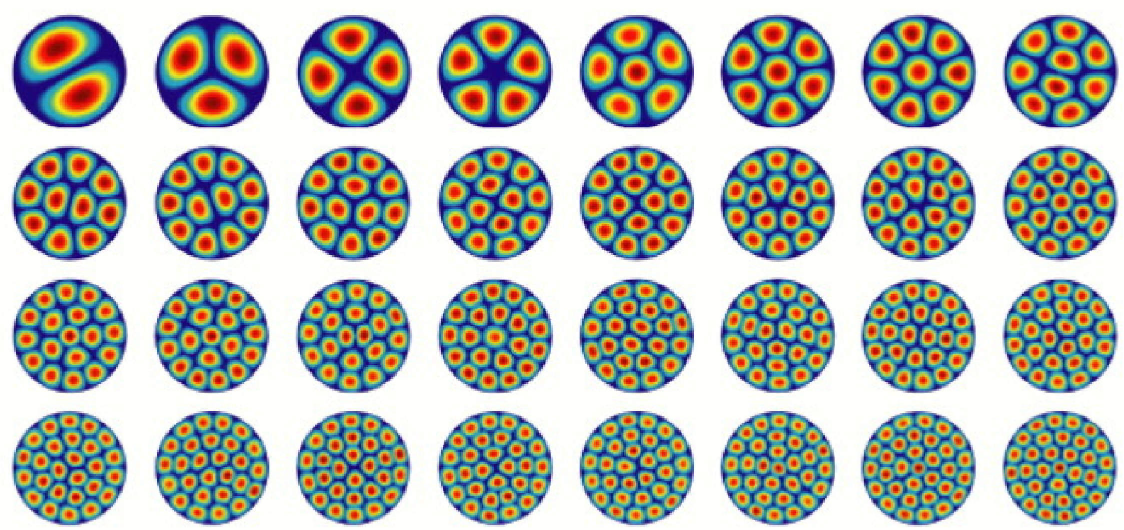

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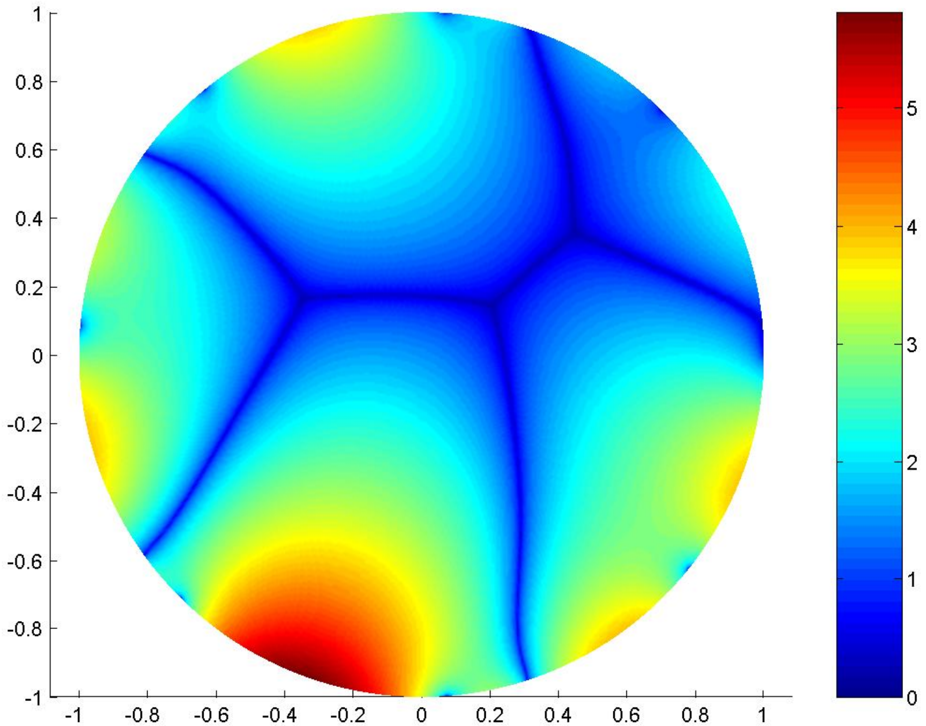
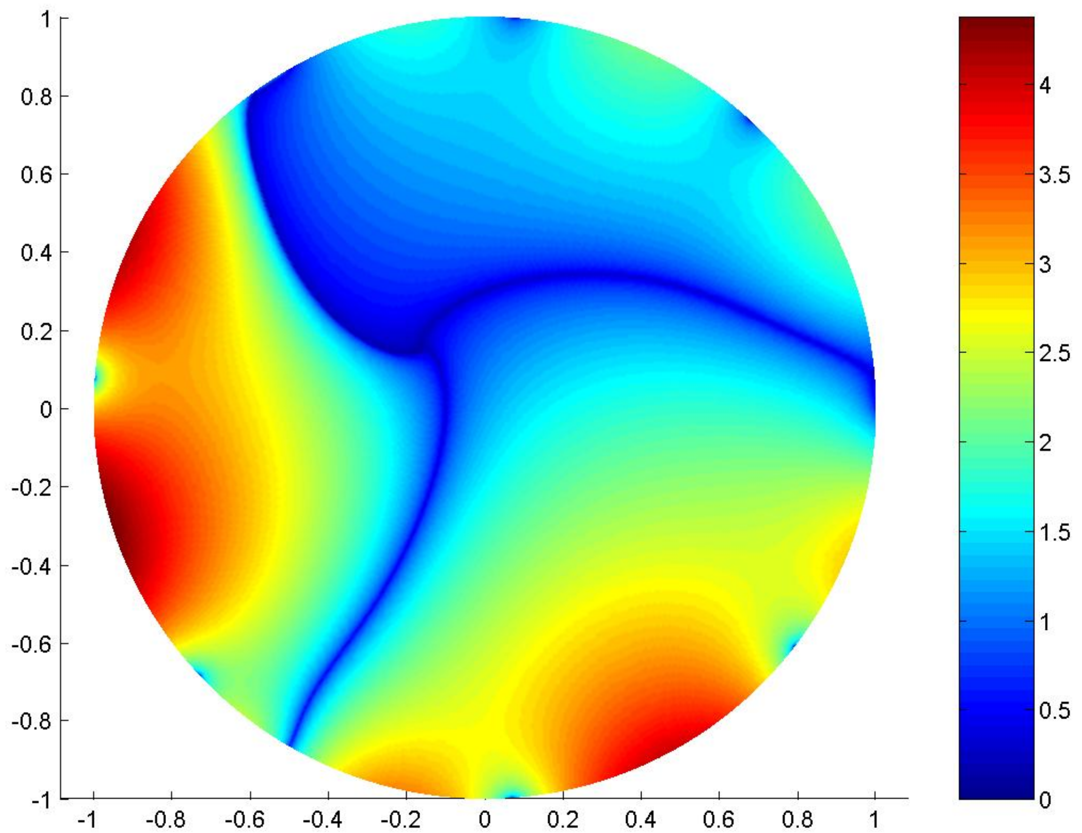
1 Competition diffusion systems with Lotka-Volterra interactions

➡ With symmetric interspecific competition rates $\beta_{i,j} = \beta_{j,i}$ large:

$$-\Delta u_i = f_i(u_i) - u_i \sum_{\substack{j=1 \\ j\neq i}}^{h} \beta_{i,j} u_j \text{ in } \Omega,$$



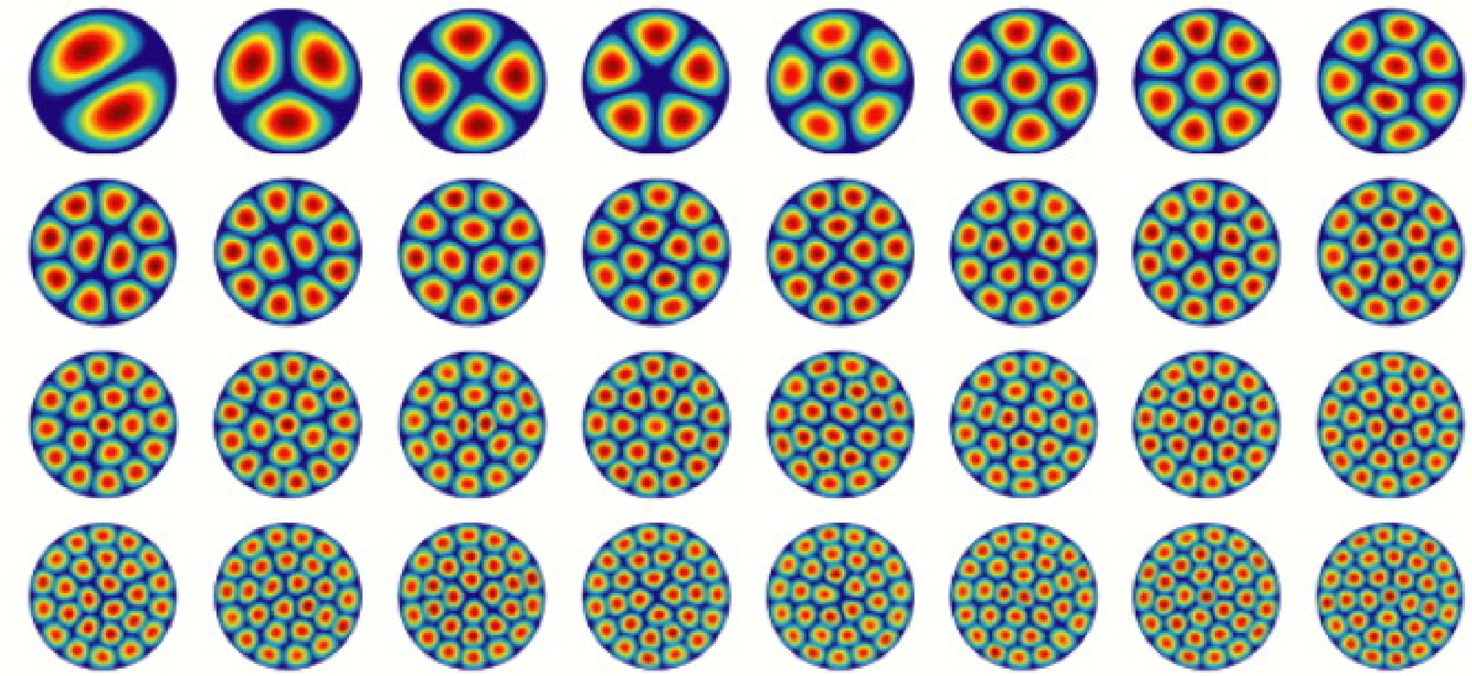
With asymmetric interspecific competition rates $\beta_{i,j} \neq \beta_{j,i}$ large:



2 Energy minimizing configurations of Bose–Einstein condensates in multiple spin–states with repulsive interaction potentials

$$\mathcal{E}(\psi_1, \cdots, \psi_h) = \int \sum_i^h \frac{1}{2} |\nabla \psi_i|^2 + F_i(|\psi_i|^2) + \sum_{j \neq i}^h \beta_{i,j} |\psi_i|^2 |\psi_j|^2 \text{ in } \Omega,$$

$$\int |\psi_i|^2 = m_i \,, \qquad i = 1, \ldots, h$$



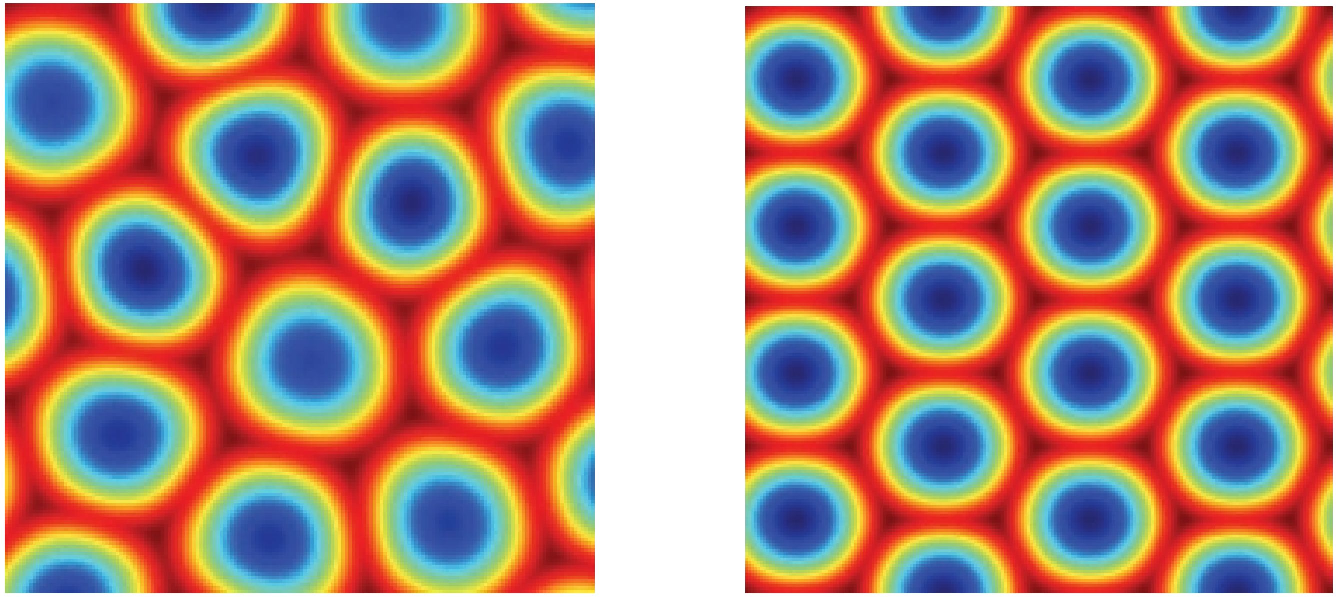
- Defocusing: S.M. Chang, C.S. Lin, T.C. Lin, and W.W. Lin, *Phys. D* **196**, 341–361 (2004)
- Focusing: Conti M., Terracini S., Verzini G., *J. Functional Analysis*, 198 (2003) 160-196

3 Optimal partition problems for Dirichlet eigenvalues

$$\min \left\{ \sum_{i=1}^{h} \lambda_1^p(\omega_i) \ : \ (\omega_1, \cdots, \omega_k) \in \mathfrak{B}_h(\Omega)) \right\}$$

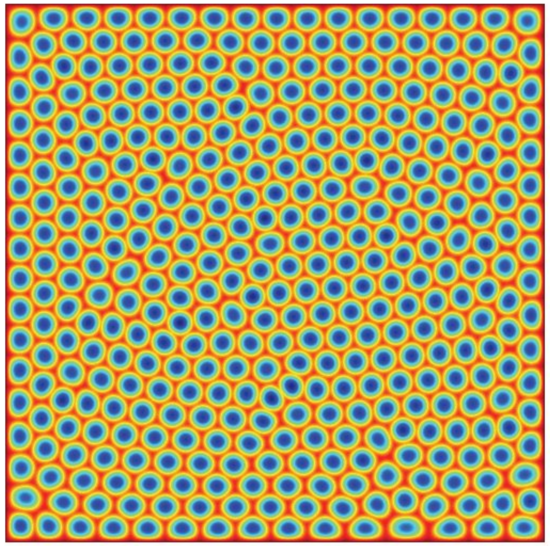
where

$$\mathfrak{B}_h = \{(\omega_1, \ldots, \omega_h) : \ \omega_i \text{ open}, \ |\omega_i \cap \omega_j| = 0 \text{ for } i \neq j \text{ and } \cup_i \omega_i \subseteq \Omega\} \, .$$



B. Bourdin, D. Bucur, and . Oudet, Optimal Partitions for Eigenvalues, *SIAM J. Sci. Comput.* 31, 2009/10 pp. 4100-4114

With more and more nodal components and higher eigenvalues:



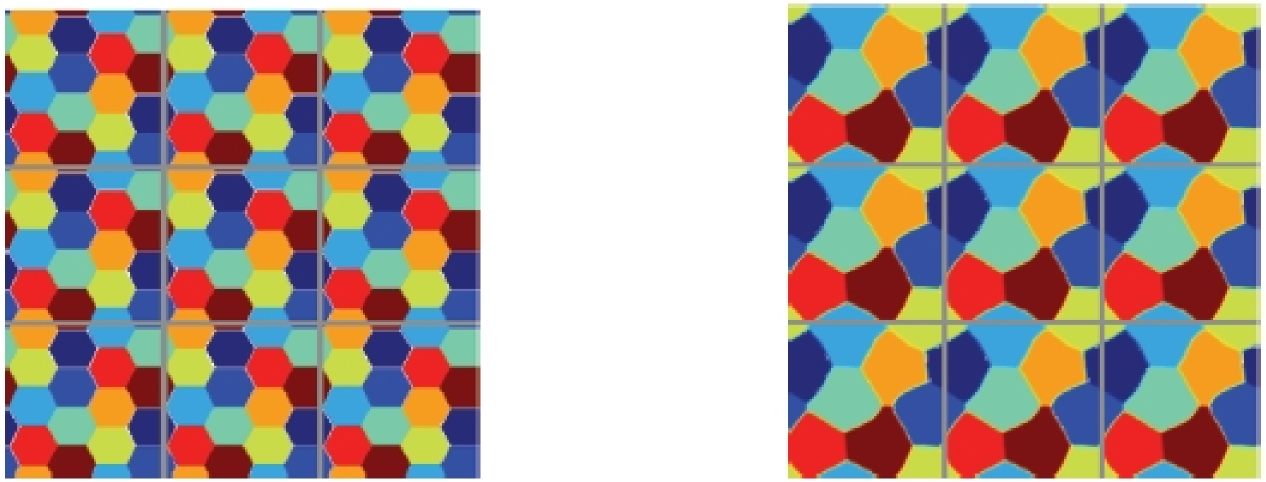


FIG. 3.8. *Optimal partitions of the sum of the second (left) and third (right) eigenvalues of the Dirichlet–Laplacian for* $n = 8$ *cells. The periodicity is highlighted by repeating the unit cell 9 times on a two-dimensional lattice.*

4 Uniform bounds in Hölder spaces

Consider strongly competing systems with either of the Lotka-Volterra or gradient type interactions

$$-\Delta u_i(x) = f(u_i) - \beta u_i(x) \sum_{j \neq i} u_j^2(x)$$

i.e. subject to diffusion, reaction and competitive interaction ($\beta > 0$).

Questions:

- What happens when the competition parameter $\beta \to +\infty$?
- Is there a common regularity shared by all solutions?
- Can we expect convergence to some limiting profile?
- What is the regularity of the limiting profiles?
- What is the equilibrium condition beetween the components?

5 Authors

The singular limit for competition-diffusion system has been studied (also in the parabolic case) by

- Dancer, Du, Hilhorst, Mimura, Peletier
- Conti, Noris, Tavares, Terracini, Verzini
- Wei, Weth
- Caffarelli, Karakhanyan, F. Lin
- Dancer, K. Wang, Z. Zhang
- Berestycki, Lin, Wei, Zhao
- Berestycki, Terracini, Wang, Wei
- Kelei Wang
- Farina, Soave

6 Uniform bounds in Hölder spaces

Theorem 1 (Conti-T-Verzini '06, Noris-Tavares-T-Verzini '10) *Let U_β be a family of H^1–bounded solutions. There exists $L_\alpha > 0$ such that*

$$\sup_{x,y\in\Omega} \frac{u_{i,\beta}(x) - u_{i,\beta}(y)}{|x-y|^\alpha} < L_\alpha$$

for all $i = 1, \ldots, h$ and for all $\beta > 0$. Moreover, the limiting profiles are Lipschitz continuous.

Ingredients of the proof. Assume the contrary, then

- scale and perform a blow-up analysis. We end up with of an entire solution of

$$-\Delta u_i(x) = -u_i(x) \sum_{j\neq i} u_j^2(x)$$

  satisfying a global bound in Hölder norm.

- Prove a Liouville type theorem, i.e. non existence of nontrivial entire solutions satisfying a global Hölder continuity condition. This is achieved through perturbed monitonicity formulæ underlying the rules of the minimal spacial growth for competition systems.

7 A Liouville type theorem

The proof of Theorem 1 rests upon a blow–up argument and the following non existence theorem:

Theorem 2 (Conti-T-Verzini '06, Noris-Tavares-T-Verzini '10) *Let $k \geq 2$, $a_{ij} > 0$, and let $U = (u_1, \ldots, u_k)$ be a solution of*

$$\begin{cases} -\Delta u_i(x) = -u_i(x) \sum_{j \neq i} a_{ij} u_j^2(x) & x \in \mathbb{R}^N \\ u_i(x) \geq 0 & x \in \mathbb{R}^N \end{cases}$$

for every i. Assume that, for some $\alpha \in (0,1)$, there holds

$$\max_{i=1,\ldots,k} \sup_{x \in \mathbb{R}^N} \frac{|u_i(x)|}{1 + |x|^\alpha} < \infty.$$

Then all components (but possibly one) vanish.

8 Classification of entire solutions

vspace-5mm We understand that a key step of the theory is the classification of the solutions of the system, with respect to their spatial growth. To this aim, we have at our disposal two major tools

- The (perturbed) Alt-Caffarelli-Friedman monotonicity formula (for Lotka-Volterra and all other types of interaction): let

$$\Phi(r) := \prod_i \frac{1}{r^{\alpha(k)-\varepsilon}} \int_{B_r(0)} K(x) \left(|\nabla u_i|^2 + \sum_{j \neq i} a_{ij} u_i^2 u_j^2 \right) ,$$

  where K is the fundamental solution of the Laplacian (if $N \geq 2$, $K(x) = |x|^{2-N}$) for $|x| \geq 1$. The exponent $\alpha(k) \geq 2$ depends on a spectral optimal partition problem on the sphere. Then Φ is increasing.

- The Almgren monotonicity formula (only for gradient systems): let

$$N(r) := \frac{r \int_{B_r(0)} \sum_i |\nabla u_i|^2 + \sum_{j \neq i} u_i^2 u_j^2}{\int_{\partial B_r(0)} \sum_i u_i^2}$$

  Then N is increasing. Moreover,

$$\lim_{r \to +\infty} N(r) \geq 1$$

9 Spectral optimal partition problem for the eigenvalues of the laplacian on the sphere

Let $\omega \subset \Sigma^{n-1}$ be open, and consider the first eigenvalue of the Dirichlet Laplace operator on ω:

$$\lambda_1(\omega) := \inf \left\{ \frac{\int_{\Sigma^{N-1}} |\nabla_T u|^2 \, \mathrm{d}\sigma}{\int_{\Sigma^{N-1}} u^2 \, \mathrm{d}\sigma} : u \in H^1(\Sigma^{N-1}) \setminus \{0\}, \, u \equiv 0 \text{ on } \Sigma^{N-1} \setminus \omega \right\}.$$

Consider the following partition problem:

$$\nu^{ACF} := \inf \left\{ \Gamma(\omega_1, \omega_2) : \omega_i \subset \Sigma^{N-1} \text{ open}, \omega_1 \cap \omega_2 = \emptyset \right\}$$

where

$$\Gamma(\omega_1, \omega_2) = \frac{\gamma_1(\lambda_1(\omega_1)) + \gamma_1(\lambda_1(\omega_2))}{2}.$$

and

$$\gamma_1(\lambda) = \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda} - \frac{N-2}{2}.$$

S. Friedland and W.K. Hayman, *E*igenvalue Inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions. Comment. Math. Helvetici 51 (1976), p. 133-161.

Theorem 3 *We have*

$$1 = \nu^{ACF} = \Gamma(\Sigma_+^{N-1}, \Sigma_-^{N-1}) \quad \textit{and} \quad \Gamma(\emptyset, \Sigma^{N-1}) = +\infty.$$

Theorem 4 (Alt-Caffarelli-Friedman) *Let $u, v \in H^1(B) \cap C(B)$ be non negative function such that $uv = 0$. Assume that $-\Delta u \leq 0$ and $-\Delta v \leq 0$ on B and let $x_0 \in B$ be such that $u(x_0) = v(x_0) = 0$. Then the function*

$$J(r) = \frac{1}{r^4} \int_{B_r(x_0)} \frac{|\nabla u|^2}{|x - x_0|^{N-2}} \cdot \int_{B_r(x_0)} \frac{|\nabla v|^2}{|x - x_0|^{N-2}}$$

is monotone (non decreasing) in r.

11 Two components in one space dimension

In order to understand the interplay of two neighboring components, we have to deal with the solutions to the system

$$\begin{cases} \Delta u = uv^2 \\ \Delta v = vu^2 \, , \\ u, v > 0 \quad \text{in} \quad \mathbb{R}^N \end{cases}$$

Of course, there are one variable solutions (depending on the energy $h > 0$):

$$\begin{cases} u'' = uv^2 \\ v'' = vu^2 \, , \\ |u'|^2 + |v'|^2 - u^2 v^2 = h \\ u(x) = v(-x) \; , \qquad u, v > 0 \quad \text{in} \quad \mathbb{R}. \end{cases}$$

All these solution have the lowest possible growth

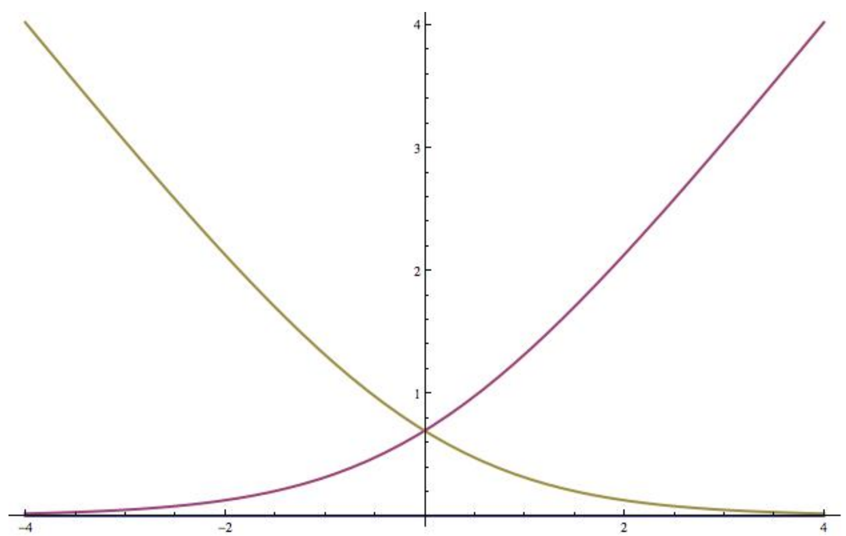
$$\lim_{r \to +\infty} N(r) = 1$$

12 Uniqueness of entire solutions in one space dimension

Theorem 5 (Berestycki-Lin-Wei-Zhao '09, Berestycki-T-Wang-Wei '12)) *Up to translations and reflections, there is only one one-parameter family of solutions to*

$$\begin{cases} u'' = uv^2, \\ v'' = vu^2, \\ u, v > 0 \quad \text{in } \mathbb{R} \end{cases}$$

For this family we have $v(t^* - t) = u(t^* + t)$*. In addition they are all stable.*



Proof:

- Normalize the energy to one;
- observe that u and v are positive convex functions; assume, for instance, that u is increasing and v is decreasing. Moreover u decays superexponentially at $-\infty$, v at $+\infty$
- as $x \to +\infty$, $|u'(x) - 1|$ decays exponentially. This implies the existence of a positive constant A such that

$$|u(x) - x^+| + |v(x) - x^-| \leq A.$$

- exploit a suitable version of the sliding method to prove both symmetry and uniqueness.

13 De Giorgi type conjecture. partial results in two dimensions

Theorem 6 (Berestycki-Lin-Wei-Zhao '09(+BTWW'12)) *Let (u, v) a solution to*

$$\Delta u = uv^2, \quad \Delta v = vu^2, \quad u, v > 0 \quad in \quad \mathbb{R}^2$$

such that

$$u(x) + v(x) \leq C(1 + |x|).$$

and which is monotone in one direction. Then (u, v) is one dimensional , (i.e., there exists $a \in \mathbb{R}^2, |a| = 1, b \in \mathbb{R}$ such that $(u, v) = (u_0(a \cdot x - b), v_0(a \cdot x - b))$ where (u_0, v_0) is the one-dimensional solution).

A quite standard argument shows that monotone $\Longrightarrow$ stable. Recall that a *stable* solution (u, v) is such that the linearization is weakly positive definite. That is, it satisfies

$$\int_{\mathbb{R}^n} [|\nabla\varphi|^2 + |\nabla\psi|^2 + v^2\varphi^2 + u^2\psi^2 + 4uv\varphi\psi] \geq 0, \qquad \forall \varphi, \psi \in C_0^\infty(\mathbb{R}^n).$$

14 Planar stable solutions and a theorem by Kelei Wang

Theorem 7 (Berestycki-T-Wang-Wei '12) *Let* (u, v) *be a stable solution to the system in* $\mathbb{R}^2$ *of*

$$\Delta u = uv^2, \quad \Delta v = vu^2, \quad u, v > 0 \quad in \quad \mathbb{R}^2$$

such that

$$u(x) + v(x) \leq C(1 + |x|).$$

Then (u, v) *is one-dimensional, (i.e., there exists* $a \in \mathbb{R}^2, |a| = 1, b \in \mathbb{R}$ *such that* $(u, v) = (u_0(a \cdot x - b), v_0(a \cdot x - b))$ *where* (u_0, v_0) *is the one-dimensional solution)*

Theorem 8 (Kelei Wang, 2013) *In any space dimensions, let* (u, v) *a solution having at most linear growth and which is minimal (in the sense of Morse): the energy is minimized with respect to compact support variations. Then* (u, v) *is one dimensional .*

Theorem 9 (Farina, Soave 2013) *In any space dimensions, let* (u, v) *a solution having at algebraic growth, such that*

$$\lim_{x_N \to -\infty} u(x', x_N) = 0; \qquad \lim_{x_N \to +\infty} u(x', x_N) = +\infty$$

$$\lim_{x_N \to -\infty} v(x', x_N) = +\infty; \qquad \lim_{x_N \to +\infty} v(x', x_N) = 0 \ ,$$

the limits being uniform in the x'*-variable. Then* (u, v) *is one dimensional .*

Natural questions:

- do all entire solutions to the system have a linear growth?
- are there solutions with polynomial growth?
- are there solutions with exponential growth? (Yes, Soave and Zilio, 2013)

15 Solutions with polynomial growth

For every integer d, there are solutions to the system with polynomial growth $|x|^d$. To describe the behavior at infinity, let us consider the harmonic polynomial Φ of degree d as

$$\boxed{\Phi := \mathrm{Re}(z^d).}$$

Note that Φ has some dihedral symmetry; indeed, let us take its d nodal lines $L_1, \cdots, L_d$ and denote the corresponding reflection with respect to these lines as $T_1, \cdots, T_d$: then there holds $\Phi(T_i z) = -\Phi(z)$.

Theorem 10 (B-T-W-W'12) *For each positive integer $d \geq 1$, there exists a solution (u, v) to the system, satisfying*

(1) $u - v > 0$ *in* $\{\Phi > 0\}$ *and* $u - v < 0$ *in* $\{\Phi < 0\}$*;*

(2) $u \geq \Phi^+$ *and* $v \geq \Phi^-$*;*

(3) $\forall i = 1, \cdots, d$*,* $u(T_i z) = v(z)$*;*

(4) $\forall r > 0$*, the Almgren frequency function satisfies*

$$\boxed{N(r) := \frac{r \int_{B_r(0)} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2}{\int_{\partial B_r(0)} u^2 + v^2} \leq d = \lim_{r \to +\infty} N(r);}$$

16 Asymptotics at infinity

We consider the blow-down sequence

$$(u_R(x), v_R(x)) := (\frac{1}{L(R)}u(Rx)\frac{1}{L(R)}v(Rx)),$$

where $u(0) = v(0)$ and $L(R)$ is chosen so that

$$\int_{\partial B_1(0)} u_R^2 + v_R^2 = 1.$$

We have the following

Theorem 11 *Let (u, v) be a solution of the system such that $d := \lim_{r \to +\infty} N(r) < +\infty$. As $R \to \infty$, (u_R, v_R) defined above (up to a subsequence) converges to (Ψ^+, Ψ^-) uniformly on any compact set of $\mathbb{R}^N$. Here Ψ is a homogeneous harmonic polynomial of degree d. If $d = 1$ then (u, v) is asymptotically linear at infinity.*

17 Systems with many components

Theorem 12 (Berestycki-T-Wang-Wei '12) *There exists a positive solution to the system*

$$\begin{cases} \Delta u_i = u_i \sum_{j \neq i, j=1}^{k} u_j^2, \quad in \ \ \mathbb{C}, i = 1, \ldots, k \end{cases}$$

having the following symmetries (here $\overline{z}$ is the complex conjugate of z)

$$\begin{aligned} u_i(z) &= u_i(G^h z), && on\ \mathbb{C}\,, i = 1, \ldots, k \\ u_i(z) &= u_{i+1}(Gz), && on\ \mathbb{C}\,, i = 1, \ldots, k \\ u_{k+1}(z) &= u_1(z), && on\ \mathbb{C} \\ u_{k+2-i}(z) &= u_i(\overline{z}), && on\ \mathbb{C}\,, i = 1, \ldots, k \end{aligned}$$

such that

$$\lim_{r \to \infty} \frac{1}{r^{1+2d}} \int_{\partial B_r(0)} \sum_{1}^{k} u_i^2 = b \in (0, +\infty) \ ;$$

and

$$\lim_{r \to \infty} \frac{r \int_{B_r(0)} \sum_1^k |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2}{\int_{\partial B_r(0)} \sum_1^k u_i^2} = d \ .$$

18 Segregated critical configuration

Let Ω be an open bounded subset of $\mathbb{R}^N$, with $N \geq 2$. Let $U = (u_1, \dots, u_h) \in (H^1(\Omega))^h$ be a vector of (real, complex, vector-valued)

- nontrivial Lipschitz functions in Ω,
- having mutually disjoint supports: $u_i \cdot u_j \equiv 0$ in Ω for $i \neq j$,
- satisfying

$$-\Delta u_i = f_i(x, u_i) \qquad \text{whenever } u_i \neq 0\,, \quad i = 1, \dots, h,$$

where $f_i : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are C^1 functions such that $f_i(x, s) = O(s)$ when $s \to 0$, uniformly in x.

Our main interest is the study of the regularity of the nodal set of the segregated configurations $U = (u_1, \dots, u_h)$:

$$\Gamma_U = \{x \in \Omega : \; U(x) = 0\}$$

A weak reflection law

Theorem 13 (Tavares-T, 2010) *Let us define, for every* $x_0 \in \Omega$ *and* $r \in (0, dist(x_0, \partial\Omega))$ *the energy*

$$\tilde{E}(r) = \tilde{E}(x_0, U, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla U|^2 \,,$$

and assume that it satisfies the following differential equation

$$\frac{d}{dr}\tilde{E}(x_0, U, r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U)^2 \, d\sigma + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_i f_i(x, u_i)\langle \nabla u_i, x - x_0 \rangle.$$

Then, there exists a set $\Sigma_U \subseteq \Gamma_U$ *the regular part, relatively open in* Γ_U*, such that*

- $\mathcal{H}_{dim}(\Gamma_U \setminus \Sigma_U) \leq N - 2$*, and if* $N = 2$ *then actually* $\Gamma_U \setminus \Sigma_U$ *is a locally finite set;*
- Σ_U *is a collection of hyper-surfaces of class* $C^{1,\alpha}$ *(for every* $0 < \alpha < 1$*). Furthermore for every* $x_0 \in \Sigma_U$

$$\lim_{x \to x_0^+} |\nabla U(x)| = \lim_{x \to x_0^-} |\nabla U(x)| \neq 0,$$

where the limits as $x \to x_0^\pm$ *are taken from the opposite sides of the hyper-surface. Furthermore, if* $N = 2$ *then* Σ_U *consists in a locally finite collection of curves meeting with equal angles at singular points.*

Some remarks

$$\frac{d}{dr}\frac{1}{r^{N-2}}\int_{B_r(x_0)}|\nabla U|^2 = \frac{2}{r^{N-2}}\int_{\partial B_r(x_0)}(\partial_\nu U)^2\,d\sigma + \frac{2}{r^{N-1}}\int_{B_r(x_0)}\sum_i f_i(x,u_i)\langle\nabla u_i, x-x_0\rangle. \quad \text{(WRL)}$$

- It is easy to check that equation (WRL) always holds for balls lying entirely inside one of the component supports, as a consequence of the elliptic equation. Hence, for our class systems, it represents the only interaction between the different components u_i through the common boundary of their supports;
- (WRL) is satisfied by the nodal components of solutions to a single semilinear elliptic equation of the form $-\Delta u = f(u)$.
- equation (WRL) can be seen as a weak form of a reflection property through the interfaces. Consider the following example: take two linear functions on complementary half-spaces:

$$u_1(x) = a_1 x_1^+ \qquad u_2(x) = a_2 x_1^- \, .$$

Then

$$\text{(WRL)} \iff |a_1| = |a_2| \, .$$

More in general, when we have two components with a smooth interface between the supports, then

$$\text{(WRL)} \qquad \Longleftrightarrow \qquad \lim_{x \to x_0^+} |\nabla U(x)| = \lim_{x \to x_0^-} |\nabla U(x)| \,.$$

(WRL) as an extremality condition

Although this hypothesis may look weird and may seem hard to check in applications, it occurs naturally in many situations where the vector U appears as a limit configuration in problems of spatial segregation.

- It has to be noted indeed that a form of (WRL) always holds for solutions of systems of interacting semilinear equations and that it persists under strong H^1 limits.
- In addition, (WRL) holds for vector functions U minimizing Lagrangian functional associated with the system.
- It is fullfilled also for strong limits to competition–diffusion systems, both those possessing a variational structure and those with Lotka-Volterra type interaction.
- Our theorem extends also to sign changing, complex and vector valued functions u_i. Lipschitz continuity can be weakened into Hölder continuity for every $\alpha \in (0, 1)]$.

21 Domain variations and (WRL)

This was brought to my attention by Kelei Wang. Assume U minimizes a Lagrangian energy with a pointwise constraint of the type $U(x) \in \Sigma$, for almost every $x \in \Omega$. Let $Y \in \mathcal{C}_0^\infty(\Omega;\mathbb{R}^N)$. Then, differentiation of the energy with respect to ε with $U(x) \mapsto U_\varepsilon(x) = U(x+\varepsilon Y(x))$ yields the well known identity

$$\int_\Omega \left\{ dY(x)\nabla U(x)\cdot\nabla U(x) - \mathrm{div}Y(x)\left[\frac{1}{2}|\nabla U(x)|^2 - F(U(x))\right]\right\} dx = 0\ , \forall Y \in \mathcal{C}_0^\infty(\Omega;\mathbb{R}^N)\ .$$

By localizing to a regular bounded $\omega \subset \Omega$ this implies that, for every smooth ω and $\forall\ Y \in \mathcal{C}^\infty(\Omega;\mathbb{R}^N)$

$$\int_\omega \left\{ dY(x)\nabla U(x)\cdot\nabla U(x) - \mathrm{div}Y(x)\left[\frac{1}{2}|\nabla U(x)|^2 - F(U(x))\right]\right\} dx$$
$$= \int_{\partial\omega} \left\{ Y(x)\cdot\nabla U(x)\nu(x)\cdot\nabla U(x) - \nu(x)\cdot Y(x)\left[\frac{1}{2}|\nabla U(x)|^2 - F(U(x))\right]\right\} d\sigma\ . \quad (*)$$

Next

$$(*) \quad + \quad \begin{pmatrix} Y(x) &= x - x_0 \\ \omega &= B_r(x_0)\end{pmatrix} \quad \Longrightarrow \quad (WRL)$$

On the other hand, once our regularity theorem is proved, it implies that

$$(WRL) \quad \Longrightarrow \quad (*)$$

$$
\begin{aligned}
E(r) &= \frac{1}{r^{N-2}} \int_{B_r(x_0)} \left(|\nabla U|^2 - \langle F(x,U), U \rangle \right), \\
H(r) &= \frac{1}{r^{N-1}} \int_{\partial B_r} |U|^2,
\end{aligned}
$$

We define the Almgren's quotient as follows:

$$
N(r) = \frac{E(r)}{H(r)}.
$$

Theorem 14 *Assume (WRL). Then, there exist* $\bar{r}, C > 0$*, such that for* $0 < r < \bar{r}$ *we have* $H(r) \neq 0$ *and also* $E(r) \geq 0$*,*

$$
N'(r) \geq -CrN(r)
$$

in particular the function $\tilde{N} = e^{\frac{C}{2}r^2} N(r)$ *is non decreasing and has a limit as* $r \to 0$*, and moreover*

$$
\frac{d}{dr} \log(H(r)) = \frac{2}{r} N(r).
$$

23 Ideas of the proof.

We follow the same general strategy used by L. Caffarelli and F. Lin for energy minimizing vector valued segregated harmonic maps (*Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries*, J. Amer. Math. Soc. (2008)). But now, our configuration needs not to be (even locally) energy minimizing. Define the Almgen's quotient as

$$N(x_0, r) = \frac{\frac{1}{r^{N-2}} \int_{B_r(x_0)} \left(|\nabla U|^2 - \langle F(x,U), U\rangle\right)}{\frac{1}{r^{N-1}} \int_{\partial B_r} |U|^2}$$

Define the regular and singular parts of the boundary

$$\Sigma_U = \{x \in \Gamma_U; : \lim_{r\to 0} N(x,r) = 1\} \qquad \mathcal{S}(U) = \Gamma_U \setminus \Sigma_U = \{x \in \Gamma_U \ : \ \lim_{r\to 0} N(x,r) > 1\}$$

- First we wish to apply Federer's reduction principle:
  - Almgren monotonicity formula at nodal points
  - bounds in Hölder spaces $\Longrightarrow$ convergence of blow-up sequences
  - classification of conic solutions satisfying (WRL)
- Next we analyze the regular part of the nodal set:
  - flatness at regular points of the boundary
  - separation lemma
- Reflection principle:
  - Pohozaev identity $\Longrightarrow$ equality of the gradient norms on the two sides.

Let $\mathcal{F} \subseteq (L^\infty_{\rm loc}(\mathbb{R}^N))^h$, and define, for any given $U \in \mathcal{F}$, $x_0 \in \mathbb{R}^N$ and $t > 0$, the rescaled and translated function

$$U_{x_0,t} := U(x_0 + t\cdot).$$

We say that $U_n \to U$ in $\mathcal{F}$ iff $U_n \to U$ uniformly on every compact set of $\mathbb{R}^N$. Assume that $\mathcal{F}$ satisfies the following conditions:

(A1) (Closure under rescaling, translation and normalization) Given any $|x_0| \leq 1 - t, 0 < t < 1$, $\rho > 0$ and $U \in \mathcal{F}$, we have that also $\rho \cdot U_{x_0,t} \in \mathcal{F}$.

(A2) (Convergence of normalized "blow–up" to a conic element) Given $|x_0| < 1, t_k \downarrow 0$ and $U \in \mathcal{F}$, there exists a sequence $\rho_k \in (0, +\infty)$, a real number $\alpha \geq 0$ and a function $\bar{U} \in \mathcal{F}$ homogeneous of degree α such that, if we define $U_k(x) = U(x_0 + t_k x)/\rho_k$, then

$$U_k \to \bar{U} \qquad \text{in } \mathcal{F}, \qquad\qquad .$$

(A3) (Singular Set hypotheses) There exists a map $\mathcal{S} : \mathcal{F} \to \mathcal{C}$ (where $\mathcal{C} := \{A \subset \mathbb{R}^N : \; A \cap B_1(0)$ is relatively closed in $B_1(0)\}$) such that

(i) Given $|x_0| \leq 1 - t$, $0 < t < 1$ and $\rho > 0$, it holds

$$\mathcal{S}(\rho \cdot U_{x_0,t}) = (\mathcal{S}(U))_{x_0,t} := \frac{\mathcal{S}(U) - x_0}{t}.$$

(ii) Given $|x_0| < 1$, $t_k \downarrow 0$ and $U, \bar{U} \in \mathcal{F}$ such that there exists $\rho_k > 0$ satisfying $U_k := \rho_k U_{x_0,t_k} \to \bar{U}$ in $\mathcal{F}$, the following "continuity" property holds:

$$\forall \varepsilon > 0 \; \exists k(\epsilon) > 0 : \; k \geq k(\varepsilon) \Rightarrow \mathcal{S}(U_k) \cap B_1(0) \subseteq \{x \in \mathbb{R}^N : \; \mathrm{dist}(x, \mathcal{S}(\bar{U})) < \varepsilon\}.$$

Theorem 15 *Define*

$$d = \max\left\{\dim\, L : \; L \text{ is a vector subspace of } \mathbb{R}^N \text{ and there exist } U \in \mathcal{F} \text{ and } \alpha \geq 0 \text{ such that } \mathcal{S}(U) \neq \emptyset \text{ and } U_{y,t} = t^\alpha U \;\forall y \in L,\; t > 0\right\},$$

Then

- *either* $\mathcal{S}(U) \cap B_1(0) = \emptyset$ *for every* $U \in \mathcal{F}$,
- *or else* $\mathcal{H}_{\dim}(\mathcal{S}(U) \cap B_1(0)) \leq d$ *for every* $U \in \mathcal{F}$.

Moreover in the latter case there exist a function $V \in \mathcal{F}$, *a d-dimensional subspace* $L \leq \mathbb{R}^N$ *and a real number* $\alpha \geq 0$ *such that*

$$V_{y,t} = t^\alpha V \qquad \forall y \in L,\; t > 0, \qquad \text{and} \qquad \mathcal{S}(V) \cap B_1(0) = L \cap B_1(0).$$

If $d = 0$ *then* $\mathcal{S}(U) \cap B_\rho(0)$ *is a finite set for each* $U \in \mathcal{F}$ *and* $0 < \rho < 1$.

Lemma 1 *Let $N \geq 2$. Given $\bar{U} = r^\alpha G(\theta) \in Lip_{loc}(\mathbb{R}^N)$ such that $\Delta \bar{U} = 0$ in $\{\bar{U} > 0\}$, and (WRP) holds, then either $\alpha = 1$ or $\alpha \geq 1 + \delta_N$ for some universal constant δ_N depending only on the dimension. Moreover if $\alpha = 1$ then $\Gamma_{\bar{U}}$ is an hyperplane.*

$\bar{U} = r^\alpha(g_1(\theta), \ldots, \gamma_k(\theta))$. Note that for every connected component $A \subseteq \{g_i > 0\} \subset S^{N-1}$ it holds

$$-\Delta_{S^{N-1}} g_i = \lambda g_i \quad \text{in } A, \qquad \text{with } \lambda = \alpha(\alpha + N - 2) \text{ and } \lambda = \lambda_1(A).$$

Lemma 2 *If $\{G > 0\}$ has at least three connected components then there exists an universal constant $\bar{\delta}_N > 0$ such that $\alpha \geq 1 + \bar{\delta}_N$.*

Proof: At least one of the connected components, say C, must have a measure less that one third of the measure of the sphere, and hence $\lambda = \lambda_1(C) \geq \lambda_1(E(\pi/2))$. Moreover it is well know that $\lambda_1(E(\pi/2)) = N - 1$. This implies the existence of $\gamma > 0$ such that $\lambda_1(E(\pi/3)) = N - 1 + \gamma$, and thus $\alpha = \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda} - \frac{N-2}{2} \geq 1 + \bar{\delta}_N$ for some $\bar{\delta}_N > 0$. ∎

Hence we are left with the case of two nodal connected components.

➡ Use an inductive argument on the dimension, starting with dimension $N = 2$. Use upper semicontinuity of the limit $N(x, 0^+)$ with respect to x. The value at the vertex of the cone must be larger than that at the conic surface.

Let x be a regular point:

Lemma 3 *For any given* $0 < \delta < 1$ *there exists* $R > 0$ *such that for every* $x \in \Gamma^\star \cap \tilde{\Omega} = \Gamma_U \cap \tilde{\Omega}$ *and* $0 < r < R$ *there exists an hyper-plane* $H = H_{x,r}$ *containing* x *such that*

$$d_{\mathcal{H}}(\Gamma_U \cap B_r(x), H \cap B_r(x)) \leq \delta r.$$

Proposition 1 (Local Separation Property) *Given* $x_0 \in \Gamma^\star$ *there exists a radius* $R_0 > 0$ *such that* $B_{R_0}(x_0) \cap \Gamma^\star = B_{R_0}(x_0) \cap \Gamma_U$ *and* $B_{R_0}(x_0) \setminus \Gamma_U = B_{R_0}(x_0) \cap \{U > 0\}$ *has exactly two connected components* Ω_1, Ω_2*. Moreover, for sufficiently small* $\delta > 0$*, we have that given* $y \in \Gamma_U \cap B_{R_0}(x_0)$ *and* $0 < r < R - |y|$ *there exist a hyper-plane* $H_{y,r}$ *(passing through* y*) and a unitary vector* $\nu_{y,r}$ *(orthogonal to* $H_{y,r}$*) such that*

$$\{x + t\nu_{y,r} \in B_r(y) : \ x \in H_{y,r}, \ t \geq \delta r\} \subset \Omega_1, \qquad \{x - t\nu_{y,r} \in B_r(y) : \ x \in H_{y,r}, \ t \geq \delta r\} \subset \Omega_2.$$

27 The reflection principle in action

Lemma 4 (Reflection Principle) *Let $u, v \in Lip_{loc}(\mathbb{R}^N)$ be two non zero and non negative functions in $\mathbb{R}^N$ such that $u \cdot v = 0$ and*

$$\begin{cases} -\Delta u = f(x,u) - \lambda \\ -\Delta v = g(x,v) - \mu \end{cases} \qquad \text{in } \mathbb{R}^N$$

for some $\lambda, \mu \in \mathcal{M}_{loc}(\mathbb{R}^N)$, locally non negative Radon measures supported on the common zero set. Suppose moreover that (WRP) holds. Then

$$\boxed{\lambda = \mu}$$

and in particular $-\Delta(u-v) = f(x,u) - g(x,v)$ in $\mathbb{R}^N$. Moreover Γ_U is a regular hyper-surface of codimension 1 at regular points, and Then for every Borel set $E \subseteq \mathbb{R}^N$ it holds

$$\lambda(E) = \int_{E\cap\partial\{u>0\}} -\partial_\nu u \, d\sigma = \int_{E\cap\partial\{v>0\}} -\partial_\nu v \, d\sigma = \mu(E)$$

➪ In the complex or vector valued case, we obtain that $\|\lambda\| = \|\mu\|$. With this and an iterative argument by Caffarelli we deduce the $\mathcal{C}^1$ regularity of the regular part of the boundary.

The main theorem extends to segregated configurations associated with systems of semilinear elliptic equations on Riemannian manifolds, under an appropriate version of the weak reflection law. Following N. Garofalo and F.-H. Lin *Monotonicity properties of variational integrals, A_p weights and unique continuation. Indiana Univ. Math. J.* **35** (1986), we consider with a system of semilinear equations involving the Laplace-Beltrami operator on a Riemannian manifold M:

$$-\Delta_M u_i = f(x, u_i) \qquad \text{where } u_i > 0 \ .$$

We define the "energy" $\tilde{E}$ as

$$\tilde{E}(r) = \tilde{E}(x_0, U, r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla_M U|^2 dV_M,$$

where $B_r(x_0)$ is the geodesic ball of radius r. Let us choose normal coordinates $\tilde{x}^i$ centered at x_0. By Gauss Lemma we know that, denoting by $\rho = \sum_i (\tilde{x}^i)^2$ and θ^i the radial and angular coordinates, it holds

$$g = d\rho^2 + \rho^2 \sum_{i,j} b_{ij}(\rho, \theta) d\theta^i d\theta^j.$$

Notice that the variation with respect to the euclidean metric is purely tangential; moreover, as $\rho \to 0$, the tangential part of the metric converges to the standard metric on the sphere. Moreover the Christoffel symbols vanish at the origin.

29 The reflection law

In normal coordinates, denoting, as usual, $\tilde{g}_{ij} = g(\partial_i, \partial_j)$ the coefficients of the metric, we require that $\tilde{E}$ satisfies the differential equation:

$$\frac{d}{dr}\tilde{E}(x_0, U, r) = \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\rho U)^2 \, d\sigma_M + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \rho \sum_i \left[f_i(x, u_i)\partial_\rho u_i + \frac{1}{\sqrt{\tilde{g}}} \sum_{k,j} \partial_\rho \left(\sqrt{\tilde{g}}\, \tilde{g}^{kj} \right) \partial_k u_i \partial_j u_i \right] \, dV_M.$$

Here $\tilde{g} = |\det(\tilde{g}_{kj})|$ and $(\tilde{g}^{kj})$ is the inverse of the matrix $(\tilde{g}_{kj})$. This identity is satisfied also in the case of Lipschitz metrics , by any solution u of the semilinear equation

$$-\Delta_M u = f(x, u).$$

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Asymptotic limits of a system of Gross-Pitaevskii equations

Consider the following system of nonlinear Schrödinger equations

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \omega_i u_i^3 - \beta u_i \sum_{j \neq i} \beta_{ij} u_j^2 \\ u_i \in H_0^1(\Omega),\ u_i > 0 \text{ in } \Omega. \end{cases} \qquad i = 1, \dots, h,$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$. Such type of systems arises in the theory of Bose-Einstein condensation in multiple spin states. Here we consider $\beta_{ij} = \beta_{ji} \neq 0$ (which gives a variational structure to the problem) and take $\lambda_i, \omega_i \in \mathbb{R}$ and $\beta \in (0, +\infty)$ large. The existence of solutions for β large is still an open problem for some choices of λ_i, ω_i.

One of the many interesting questions about these systems is the asymptotic study of its solutions as $\beta \to +\infty$ (which represents an increasing of the interspecies scattering length) and study of the regularity of the limiting profiles. From the uniform Hölder bounds theorem we know that:

- $C^{0,\alpha}$– bounds (for all $0 < \alpha < 1$) for any given L^∞–bounded family of solutions $U_\beta = (u_{1,\beta}, \dots, u_{h,\beta})$ of the system;
- the possible limit configurations $U = \lim_{\beta \to +\infty} U_\beta$ are Lipschitz continuous.

As a byproduct, we have

Theorem 16 *Let U be a limit as $\beta \to +\infty$ of a family $\{U_\beta\}$ of L^∞–bounded solutions of the system. Then the conclusion of Theorem 13 holds.*

All the required assumptions are satisfied for such limiting profiles, with $f_i(x,s) = f_i(s) = \omega_i s^3 - \lambda_i s$, except for the weak reflection law. The procedure to verify it is the following: defining an approximated energy associated with system - which has a variational structure-,

$$E_\beta(r) = \frac{1}{r^{N-2}} \int_{B_r(x_0)} \left(|\nabla U_\beta|^2 - \langle F(U_\beta), U_\beta\rangle\right) + \int_{B_r(x_0)} 2\beta \sum_{i<j} u_{i,\beta}^2 u_{j,\beta}^2$$

by a direct calculation it holds

$$\begin{aligned} E'_\beta(r) = {} & \frac{2}{r^{N-2}} \int_{\partial B_r(x_0)} (\partial_\nu U_\beta)^2 \, d\sigma + \frac{2}{r^{N-1}} \int_{B_r(x_0)} \sum_i f_i(u_{i,\beta}) \langle \nabla u_{i,\beta}, x - x_0\rangle + \\ & + \frac{1}{r^{N-1}} \int_{B_r(x_0)} (N-2) \langle F(U_\beta), U_\beta\rangle - \frac{1}{r^{N-2}} \int_{\partial B_r(x_0)} \langle F(U_\beta), U_\beta\rangle \, d\sigma + \\ & + \frac{4-N}{r^{N-1}} \int_{B_r(x_0)} \beta \sum_{i<j} u_{i,\beta}^2 u_{j,\beta}^2 + \int_{\partial B_r(x_0)} \beta \sum_{i<j} u_{i,\beta}^2 u_{j,\beta}^2 \, d\sigma. \end{aligned}$$

We know the following facts:

- there holds strong convergence $U_\beta \to U$ in $H^1 \cap C^{0,\alpha}(\Omega)$ for every $0 < \alpha < 1$,
- and $\int_\Omega \beta \sum_{i<j} u_{i,\beta}^2 u_{j,\beta}^2 \to 0$.

Hence, as $\beta \to +\infty$, we prove that U satisfies the weak reflection law.

33 Lotka-Volterra competitive interactions with symmetric competition rates

Consider the following Lotka-Volterra model for the competition between h different species.

$$\begin{cases} -\Delta u_i = f_i(u_i) - \beta u_i \sum_{j\neq i} a_{i,j} u_j \text{ in } \Omega, \\ u_i \geq 0 \text{ in } \Omega, \quad u_i = \varphi_i \text{ on } \partial\Omega. \end{cases}$$

with $\Omega \subset \mathbb{R}^N$ a smooth bounded domain and φ_i positive $W^{1,\infty}(\partial\Omega)$–functions with disjoint supports. We focus on the asymptotic study of solutions as $\beta \to +\infty$. It is not difficult to show that all the possible H^1–limits U of a given sequence of solutions $\{U_\beta\}_{\beta>0}$ (as $\beta \to +\infty$) belong to the class

$$\mathcal{S}(\Omega) = \Big\{ (u_1, \ldots, u_h) \in \big(H^1(\Omega)\big)^h : \ u_i \geq 0 \text{ in } \Omega, \ u_i \cdot u_j = 0 \text{ if } i \neq j \text{ and } -\Delta u_i \leq f_i(u_i), \\ -\Delta\Big(u_i - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} u_j\Big) \geq f_i(x, u_i(x)) - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} f_j(x, u_j) \Big\}.$$

Theorem 17 *Let $U \in \mathcal{S}$, then if $a_{ij} = a_{ji}$, $\forall i, j$ the conclusion of Theorem 13 holds.*

34 Regularity of interfaces in optimal partition problems related to the first eigenvalue

Next we consider some optimal partition problems involving eigenvalues. For any integer $h \geq 0$, we define the set of h–partitions of Ω as

$$\mathfrak{B}_h = \{(\omega_1, \ldots, \omega_h) : \ \omega_i \text{ measurable }, \ |\omega_i \cap \omega_j| = 0 \text{ for } i \neq j \text{ and } \cup_i \omega_i \subseteq \Omega\} .$$

Consider the following optimization problems: for any positive real number $p \geq 1$,

$$\boxed{\mathfrak{L}_{h,p} := \inf_{\mathfrak{B}_h} \left(\frac{1}{h} \sum_{i=1}^{h} (\lambda_1(\omega_i))^p \right)^{1/p},}$$

and, for $p = +\infty$ we find the limiting problem

$$\boxed{\mathfrak{L}_h := \inf_{\mathfrak{B}_h} \max_{i=1,\ldots,h} (\lambda_1(\omega_i)),}$$

where $\lambda_1(\omega)$ denotes the first eigenvalue of $-\Delta$ in $H_0^1(\omega)$ in a generalized sense. We refer to the papers Conti, Verzini, T. and Helffer, Hoffmann-Ostenhof, T., for a more detailed description of these problems.

Our theorem applies to suitable multiples of the eigenfunctions associated with the optimal partition. More precisely, we proved that

- let $p \in [1, +\infty)$ and let $(\omega_1, \dots, \omega_h) \in \mathfrak{B}_h$ be any minimal partition associated with $\mathfrak{L}_{h,p}$ and let $(\phi_i)_i$ be any set of positive eigenfunctions normalized in L^2 corresponding to $(\lambda_1(\omega_i))_i$. Then there exist $a_i > 0$ such that the functions $u_i = a_i \phi_i$ verify in Ω, for every $i = 1, \dots, h$, the differential inequalities (in the distributional sense):

$$-\Delta u_i \leq \lambda_1(\omega_i) u_i \quad \text{and} \quad -\Delta(u_i - \textstyle\sum_{j \neq i} u_j) \geq \lambda_1(\omega_i) u_i - \textstyle\sum_{j \neq i} \lambda_1(\omega_i) u_j;$$

and:

- let $(\tilde{\omega}_1, \dots, \tilde{\omega}_h) \in \mathfrak{B}_h$ be any minimal partition associated with $\mathfrak{L}_h$ and let $(\tilde{\phi}_i)_i$ be any set of positive eigenfunctions normalized in L^2 corresponding to $(\lambda_1(\tilde{\omega}_i))_i$. Then there exist $a_i \geq 0$, not all vanishing, such that the functions $\tilde{u}_i = a_i \tilde{\phi}_i$ verify in Ω, for every $i = 1, \dots, h$, the differential inequalities (in the distributional sense):

$$-\Delta \tilde{u}_i \leq \mathfrak{L}_h \tilde{u}_i \quad \text{and} \quad -\Delta(\tilde{u}_i - \textstyle\sum_{j \neq i} \tilde{u}_j) \geq \mathfrak{L}_h(\tilde{u}_i - \textstyle\sum_{j \neq i} \tilde{u}_j).$$

In particular the functions $\tilde{U} = (\tilde{u}_1, \dots, \tilde{u}_h)$ and $U = (u_1, \dots, u_h)$ belong to $\mathcal{S}(\Omega)$. As consequence, we have the following result:

Theorem 18 *Let $(\omega_1, \dots, \omega_h) \in \mathfrak{B}_h$ be any minimal partition and let Γ be the union of the interfaces; then the conclusion of Theorem 13 holds.*

35 Extremality conditions for partitions involving higher eigenvalues

We would like to attack the optimal partition problem for higher eigenvalues ($k \geq 2$):

$$\mathfrak{L} = \min \frac{1}{h} \left(\sum_{i=1}^{h} \lambda_k(\omega_i) \right) .$$

- What are the extremality conditions? Hadamard domain variation requires simple eigenvalues.
  Introduce the penalized functional:

$$\mathcal{E}_\beta(u_1, \cdots, u_h) = \int_\Omega \sum_i |\nabla u_i|^2 + \beta \sum_{i \neq j} |u_i|^2 |u_j|^2$$

with constraints

$$\int_\Omega |u_i|^2 = 1 \quad \forall\, i = 1, \cdots, h \, .$$

As $\beta + \infty$, critical points of $\mathcal{E}_\beta$ converge to pairs of segregated eigenfunctions.

- Problem: How to define an appropriate critical level for the penalized functional?

Existence of the minimal partition has been proved by Bucur–Buttazzo.